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A possible solution of the G-dwarf problem in the frame-work of closed models with a time-dependent IMF

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Abstract. In this paper we present a method to solve the G-dwarf problem in the frame-work of analytical models (based on the instantaneous recycling approximation, IRA). We consider a one-zone closed model without inflows or outflows. We suppose a time-dependent Initial Mass Function (IMF) and we find an integral-differential equation which must be satisfied in order to honour the G-dwarf metallicity distribution as a function of the oxygen abundance. IMFs with one and two slopes are given and discussed also in the framework of a numerical chemical evolution model without IRA. We conclude that it is difficult to reproduce other observational constraints besides the G-dwarf distribution (such as $[\frac{O}{Fe}]$ vs $[\frac{Fe}{H}]$), and that an IMF with two slopes, with time-dependent shape at the low mass end, would be required. However, even in this case the predicted oxygen gradient along the disk is flat and radial flows would be required to reproduce the observed gradient.

Key words: methods: analytical – stars: abundances – stars: late-type – Galaxy: solar neighbourhood

1. Introduction

It is well known that the metallicity distribution of G-dwarfs in the solar neighbourhood shows a deficit of metal-poor stars relative to the predictions of the Simple Model of chemical evolution. This is the so-called G-dwarf problem, originally noted by van den Bergh (1962) and Schmidt (1963).

Many solutions have been proposed, some in the frame-work of analytical models, based on the IRA. The problem is still present if we adopt the oxygen abundances in place of metallicity.

An important result in the frame-work of IRA models with gas flows was pointed out by Edmunds (1990). He found that the G-dwarf problem cannot be solved by any outflow but it is possible to solve it with particular forms of inflow. Lynden Bell (1975) 'best accretion model' and Clayton's models (Clayton 1988) are models of just this kind.

Of course it is also possible to reduce the number of metal-poor stars with respect to that predicted by the Simple Model by assuming metal dependent stellar yields (Maeder 1992) (more metals are produced at lower metallicity with a consequent Prompt Initial Enrichment (P.I.E.) (Truran & Cameron 1971)). However, it has been shown by several papers (Giovagnoli & Tosi, 1995; Carigi 1996), that in this way shallow gradients along the Galactic disk are produced, which do not agree with the most recent observational estimates (see Matteucci & Chiappini 1999 for a review). Actually, this is an unavoidable problem still present in the models discussed in this paper where we consider time-dependent IMFs.

It is still possible to obtain a Prompt Initial Enrichment by assuming an IMF variable with time and in particular very flat at early times, in order to favour the formation of massive stars.

In this paper we exploit the previous idea to investigate the time behaviour of the IMF. We assume the same hypothesis of the Simple Model (Tinsley 1980) with the exception of adopting a time-dependent IMF. In such a way it is possible to find an equation (Sect. 2.1) which must be satisfied in order to reproduce the observed distribution of G-dwarfs. We then use this equation to investigate the time behaviour of IMFs with one and two slopes. In Sect. 3 we use a numerical model without IRA to test the IMFs on other observational constraints.

2. Recovering the history of the IMF

In the following we need an analytical expression to approximate the data concerning the metallicity distribution of G-dwarfs in the solar neighbourhood. The data are plotted in Fig. 1 where the differential distribution of oxygen abundances for the solar cylinder from Rocha-Pinto & Maciel (1996) is given (triangles).

We have made use of the oxygen abundance rather than the iron abundance, in order to be consistent with chemical evolution models which use the instantaneous recycling approximation. Indeed, IRA is a good approximation for oxygen which is produced on short timescales (few million years) by supernovae (SNe) II as opposed to iron which is produced on long timescales (up to a Hubble time) by SNe Ia (Matteucci & Greggio 1986). In order to compare theory and data we adopt a simple relation between O and Fe reproducing the observed behaviour of these elements in the solar neighbourhood, the same as given by Pagel (1989):

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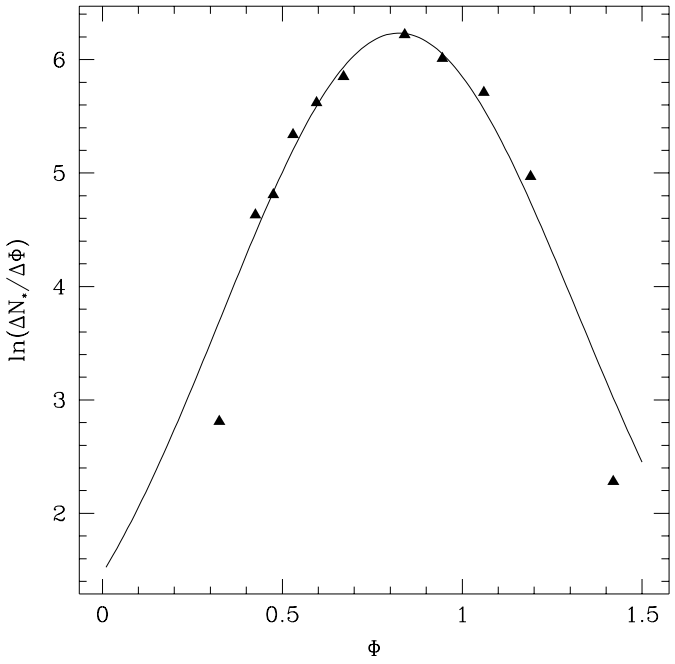

Fig. 1. Metallicity distribution of 287 G-dwarf stars, from Rocha-Pinto and Maciel 1996 (triangles) and the function $f(\Phi)$ (Eq. 2) (continuous line). Φ is the oxygen abundance in units of the solar abundance. ΔN_* is the number of stars with oxygen abundance between Φ and $\Phi + \Delta\Phi$

$$log(\Phi) = \left[\frac{O}{H}\right] = 0.5\left[\frac{Fe}{H}\right] \quad (1)$$

The continuous line in Fig. 1 is the best function approximating data points according to the theory discussed in the appendix.

$$f(\Phi) = ln\left(\frac{\Delta N_*}{\Delta\Phi}\right) = w_1 e^{-\frac{(\Phi-\Phi_1)^2}{2\sigma^2}} + w_2 e^{-\frac{(\Phi-\Phi_2)^2}{2\sigma^2}} \quad (2)$$

with

$$\sigma = 0.454;\ \Phi_1 = 0.690;\ \Phi_2 = 1.06;\ w1 = 4.12;\ w2 = 2.61; \quad (3)$$

We have used the sophisticated method in the appendix to approximate the data because of the strong dependence of our results on the chosen function $f(\Phi)$ and in particular on the value at $\Phi = 0$, $f(0)$. Indeed, the point $\Phi = 0$ is outside of the range of the available data and the metallicity distribution extrapolated to $\Phi = 0$ depends strongly on the adopted method. Since the aim of this paper is to recover the history of the IMF from the data in Fig. 1, we need a method that does not introduce any *a priori* assumption on the specific functional form of the relation $ln(\Delta N_*/\Delta\Phi)$. In other words our approach must be independent of any physical hypothesis on the star formation law at the basis of the observed data, and this is just the essence of the method discussed in the appendix.

2.1. Basic equations

The basic assumptions of our models are the same as those of the Simple Model (Tinsley 1980) with the only exception of adopting a time-dependent IMF. Therefore we consider a model with the following properties:

1. The system is one-zone and closed, namely there are no inflows or outflows
2. The initial gas is primordial (no metals)
3. The gas is well mixed at any time
4. The instantaneous recycling approximation holds (namely the lifetimes of stars above $1M_\odot$ are negligible whereas those of stars below $1M_\odot$ are larger than the age of the Galaxy).
5. The IMF is time-dependent, i.e. $\varphi = \varphi(m,t)$, with the following normalization at any time

$$\int_0^\infty \varphi(m,t) m dm = 1 \quad (4)$$

Under the previous assumptions the oxygen abundance Φ (defined by Eq. (1)) in the interstellar medium is governed by

$$d\Phi = \frac{p}{gZ_0}\alpha ds \quad (5)$$

where $Z_0 = H\frac{O_\odot}{H_\odot} \simeq 9.54 \cdot 10^{-3}$, g is the gas mass, $p\alpha ds = p' ds$ is the mass of oxygen produced and almost immediately returned into the interstellar medium when ds of interstellar material goes into stars. A fraction α of the mass gone into stars is not returned to the interstellar medium, but remains in long-lived stars or stellar remnants. It has also been assumed that $\Phi Z_0 << 1$. We have:

$$\alpha = \int_0^{M_\odot} \varphi(m,t) m dm + \int_{M_\odot}^\infty \varphi(m,t) m_{rem} dm \quad (6)$$

and

$$p'_{oxy} = \int_{M_\odot}^\infty \varphi(m,t) m p_o dm \quad (7)$$

where p_o is the fraction (by mass) of newly produced and ejected oxygen by a star of mass m. We used for p_o the expression given by Woosley & Weaver (1995) whereas for m_{rem} the expression given by Tinsley (1980).

In our model the gas and stellar masses are related by

$$dg = -\alpha ds \quad (8)$$

Now we want to relate the previous quantities to the observed metallicity distribution of G-dwarfs ($f(\Phi)$ in Eq. (2)) in order to find an equation for $\varphi(m,t)$. We have:

$$f(\Phi) = ln\left(\alpha_c \frac{ds}{d\Phi}\right) \quad (9)$$

where

$$\alpha_c = \int_{0.8M_\odot}^{1.1M_\odot} \varphi(m,t) m dm \quad (10)$$

because the stars in our sample are in the range 0.8–$1.1M_\odot$.

Eqs. (5) and (9) give:

$$f(\Phi) = ln\alpha_c - ln\alpha + lng - lnp + lnZ_0 \quad (11)$$

On the other hand we can derive lng from (5) and (8) in the following way:

$$lng = lng_0 - Z_0 \int_0^{\Phi} \frac{d\Phi}{p} \tag{12}$$

where g_0 is the initial gas mass. Hence from (11) we obtain:

$$f(\Phi) = ln\alpha_c - ln\alpha - Z_0 \int_0^{\Phi} \frac{d\Phi}{p} + lng_0 - lnp + lnZ_0 \tag{13}$$

If we evaluate the previous equation at $\Phi = 0$ (that is $t = 0$) we have:

$$f(0) = ln\alpha_{c0} + lng_0 - lnp'_0 + lnZ_0 \tag{14}$$

where p'_0 and α_{c0} are the quantities in (6) and (10), respectively, with $\varphi(m,t) = \varphi(m,0)$.

By differentiating Eq. (13) with respect to Φ we obtain finally

$$\frac{d}{d\Phi}\left[ln\alpha_c - lnp'\right] - \frac{\alpha}{p'} Z_0 = \frac{df(\Phi)}{d\Phi} \tag{15}$$

This is an integro-differential equation for the function $\varphi(m,\Phi)$ with the initial condition given by (14). The previous problem has, of course, infinite solutions. Therefore to proceed further we have to make some assumption on the behaviour of the $\varphi(m,\Phi)$. In the next sections we shall investigate IMF with one slope (Sect. 2.2) and with two slopes (Sect. 2.3).

2.2. Single power-law IMF

Let consider a single power-law IMF, namely

$$\varphi(m,\Phi) = Cm^{-[1+x(\Phi)]} \tag{16}$$

where M_L and M_U are respectively the smallest and the largest stellar mass (that we assume do not depend on the time) and the normalization is performed in the above mass range, i.e. $C = \frac{1-x(\Phi)}{M_U^{1-x(\Phi)} - M_L^{1-x(\Phi)}}$

Substituting Eq. (16) in (15) we find the following equation for $x(\Phi)$:

$$\frac{dx}{d\Phi} = \frac{F_2(x) + F_3(\Phi)}{F_1(x)} \tag{17}$$

which is a nonlinear I order differential equation with initial condition given by substituting (16) in (14). The functions in (17) are:

$$F_1(x) = \frac{d}{dx}\left[ln\alpha_c - lnp'\right] \tag{18}$$

$$F_2(x) = \frac{\alpha}{p'} Z_0 \tag{19}$$

$$F_3(\Phi) = \frac{df}{d\Phi} = -\left[w_1 \frac{\Phi - \Phi_1}{\sigma^2} e^{-\frac{(\Phi-\Phi_1)^2}{2\sigma^2}} + w_2 \frac{\Phi - \Phi_2}{\sigma^2} e^{-\frac{(\Phi-\Phi_2)^2}{2\sigma^2}} \right] \tag{20}$$

Since $F_2(x) > 0$, Eq. (17) tells us that a constant IMF (i.e. with a slope $x = x_0$ at any time) corresponds to a straight line for $f(\Phi)$ with a negative slope (Indeed, in this case Eq. (17) becomes $F_3(\Phi) = \frac{df}{d\Phi} = -F_2(x_0)$). This is the right behaviour since our model becomes the Simple Model when $x = x_0$.

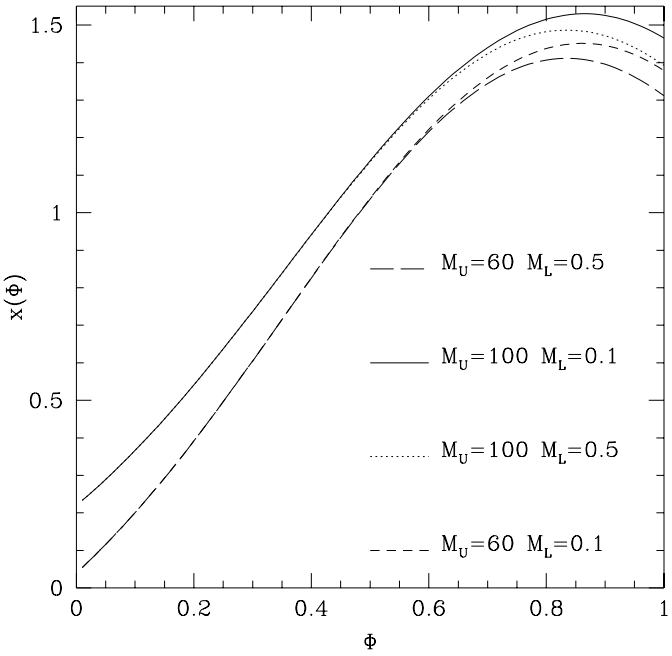

Fig. 2. IMF slope ($x(\Phi)$) as a function of oxygen abundance (Φ) for two choices of the parameters M_U and M_L. The values of M_U and M_L are given in solar mass.

The solutions $x(\Phi)$ are plotted in Fig. 2 for two different values of M_U and M_L. We found very flat IMFs for low oxygen abundances (i.e. at initial times). At the solar metallicity ($\Phi = 1$) the slope is always steeper than a Salpeter (1955) ($x = 1.35$). Decreasing M_U decreases, of course, x, i.e. the IMF becomes flatter. The effect of a change in the M_L value is negligible (especially at low Φ).

2.3. IMF with two slopes

Let us now consider the following IMF:

$$\varphi(m,\Phi) = C \begin{cases} m^{-[1+x_1]} & \text{if } m \le M \\ M^{x_2-x_1} m^{-[1+x_2]} & \text{if } m > M \end{cases}$$

The above IMF depends on the three parameters x_1, x_2 and M. We shall investigate the three following cases:

1. $M = M(\Phi)$, $x_2 = 1.35$ for values of x_1 ranging from -1 to 0.2 (this IMF is similar to the one proposed by Larson (1998));
2. $x_1 = x_1(\Phi)$, $x_2 = 1.35$ for values of M ranging from 2 to $10M_\odot$;
3. $x_2 = x_2(\Phi)$, for values of x_1 ranging from -1 to 1.

In all the previous cases Eq. (15) gives us a nonlinear I order differential equation for the Φ-dependent parameter, with the initial conditions given by the Eq. (14). We consider always $M_U = 100M_\odot$ and $M_L = 0.1M_\odot$ since the dependence on these parameters is negligible.

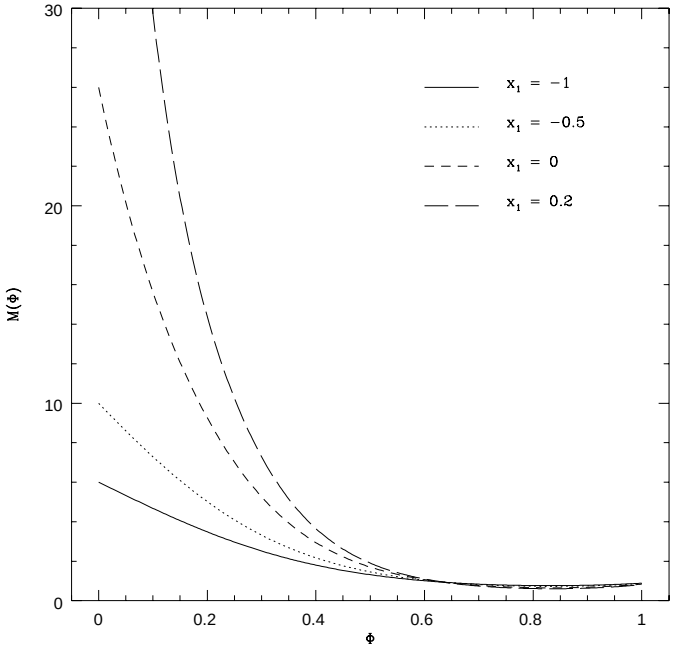

Fig. 3. $M(\Phi)$ in solar masses as a function of Φ for different x_1. The slope x_2 is always 1.35.

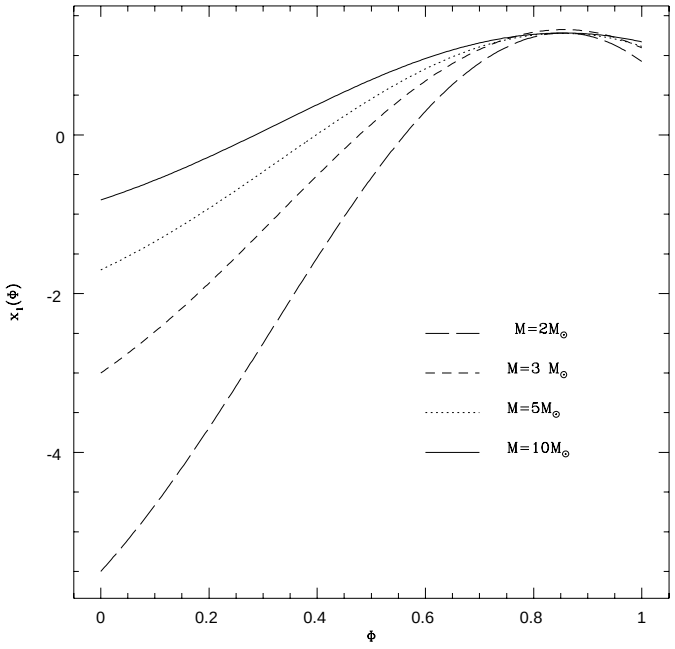

Fig. 4. $x_1(\Phi)$ as a function of Φ for different M. $x_2 = 1.35$.

The function $M(\Phi)$ in the first case considered is shown in Fig. 3. At the initial time ($\Phi = 0$) the mass M increases by increasing the slope at low mass end. In particular when $x_1 = 0.2$ $M(0) \simeq 100 M_\odot = M_U$ and therefore there is no solution for x_1 greater than 0.2.

Fig. 4 shows the slope at the low mass end ($x_1(\Phi)$) for values of M in the range 2–$10 M_\odot$. The initial values of x_1 are always negative and moreover they increase by increasing M, as expected.

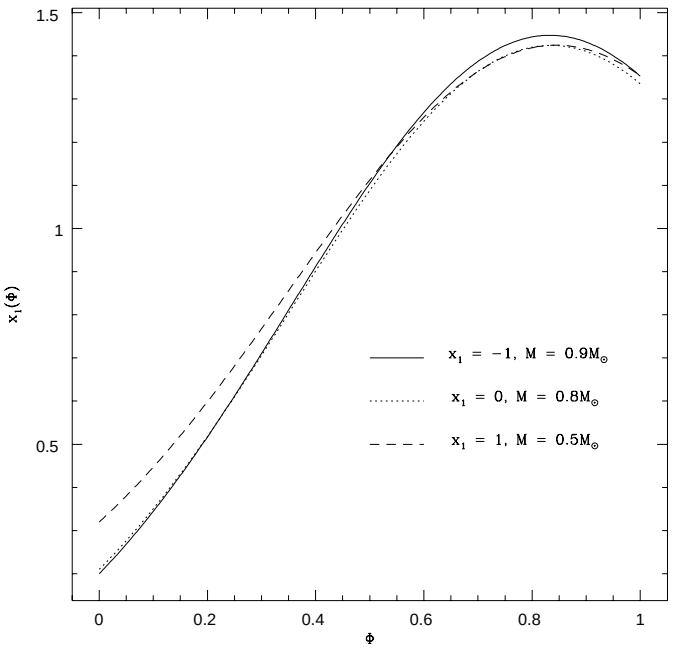

Fig. 5. $x_2(\Phi)$ as a function of Φ for different x_1. M is chosen in order to have a final slope $x_2 \simeq 1.35$

Finally Fig. 5 shows the results for the third case considered. Here the mass M is chosen in order to reproduce a final slope $x_2 = 1.35$.

3. Application of the derived IMFs to a numerical model

Both the proposed IMFs can reproduce the observed G-dwarf distribution and therefore we want to test the validity of the inferred IMF by using numerical models of galactic chemical evolution. This can be done by studying the effect of the above IMFs on the chemical evolution of the solar neighbourhood. The model used is that of Matteucci & Francois (1989) where a detailed description can be found. The main difference with that model is that we assume here a very rapid formation of either the halo and the disk thus simulating a closed model. This is required by the fact that we derived the IMF under the assumption of a closed model. In the original model of Matteucci & Francois (1989) the timescale for the formation of the solar neighbourhood was about 3–4 Gyr and it was chosen in order to best fit the G-dwarf metallicity distribution of Pagel & Patchett (1975), under the assumption of a constant IMF. Recently, Chiappini et al. (1997) presented a more realistic model for the Galaxy where the evolution of the halo-thick disk and the thin disk are decoupled in the sense that the halo-thick disk is formed on a short time scale of the order of 1–2 Gyr, whereas the thin disk is assumed to form very slowly.

In particular, in order to best fit the G-dwarf distribution of Rocha-Pinto & Maciel (1996), Chiappini et al. (1997) found that, with an IMF constant in time, a time scale of about 8 Gyrs is required for the formation of the solar neighbourhood. It is interesting to check if the derived IMFs are able to reproduce other observational constraints besides the G-dwarf distribution,

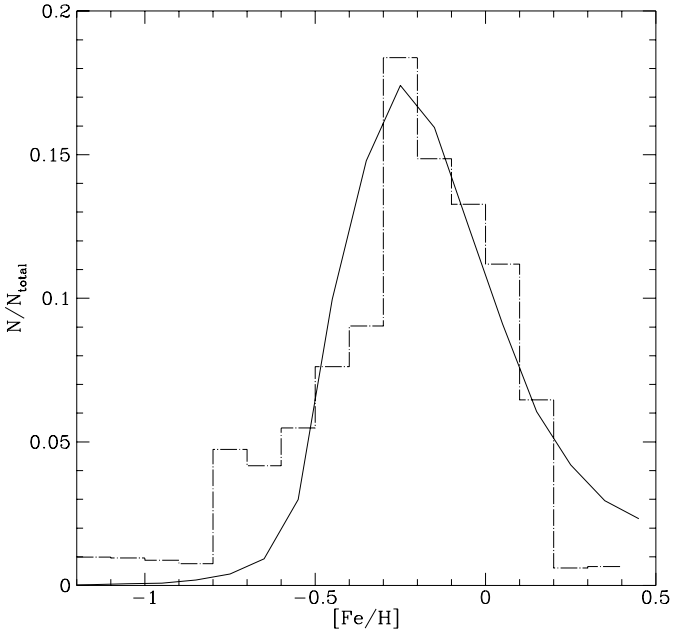

Fig. 6. G-dwarf metallicity distribution as predicted by the numerical model (solid line) with the single power law IMF discussed in Sect. 2.1, and from experimental data (Rocha-Pinto & Maciel 1996)

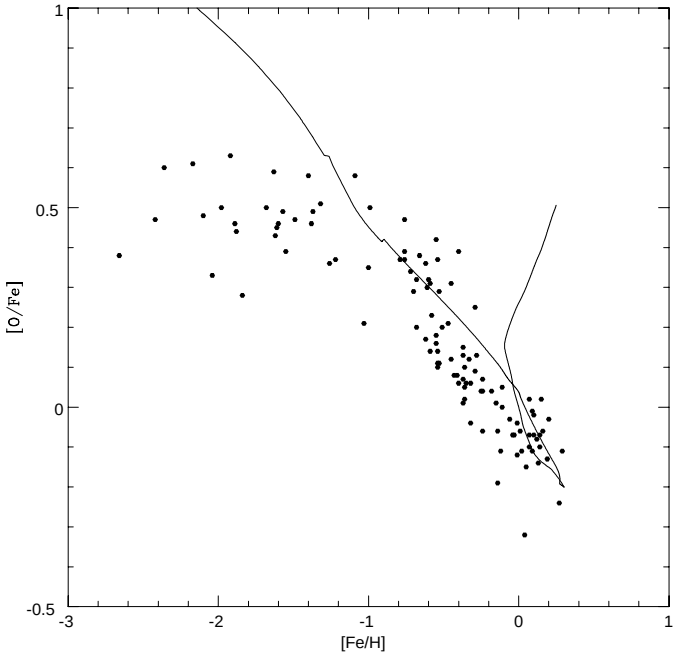

Fig. 7. $[O/Fe]$ vs $[Fe/H]$ as predicted by the numerical model (solid line) with the single power law IMF (Sect. 2.1), and as observed

for example the $[O/Fe]$ vs $[Fe/H]$ trend which is normally very well reproduced by models with constant IMF and infall and taking into account detailed nucleosynthesis from type II and Ia SNe. In this framework, in fact, the plateau shown by the data for $[Fe/H] < -1.0$ is interpreted as due to the pollution by type II SNe which produce an almost constant $[O/Fe]$ ratio. The subsequent decrease of the $[O/Fe]$ ratio for $[Fe/H] \geq -1.0$ is then due to the occurrence of type Ia SNe exploding with a temporal delay relative to SNe II. In the Matteucci & Francois (1989) and Chiappini et al. (1997) model the SNe Ia are supposed to originate from white dwarfs in binary systems following the formalism of Matteucci & Greggio (1986). In Fig. 6 we show the G-dwarf metallicity distribution as predicted by the numerical model and, as expected, the agreement is quite good. However, in Fig. 7 we show the predicted $[O/Fe]$ versus $[Fe/H]$ relation and the agreement with the observations is poor, especially in the domain of metal poor stars where the new IMF predicts an increasing amount of massive stars. On the other hand, the slope for disk stars is well reproduced. Another problem with this IMF is the high O and Fe content reached by the model at the sun birth and at the present time. These high abundances are not evident in Fig. 7 since the abundances are normalized to the predicted solar abundances. Finally, another problem is that Fig. 7 shows that [Fe/H] starts decreasing more than oxygen after having reached the solar abundance. This is due to the large dilution from dying low mass stars and to the fact that the Fe abundance decreases more rapidly than that of oxygen, due to the fact that O is continuously produced, although at a low level, by SNe II on very short timescales. Iron, on the other hand, comes mostly from type Ia SNe born at early times when the IMF was top-heavy favoring massive stars relative to the type Ia SN progenitors. The predicted solar abundances, namely the abundances at 4.5 Gyrs ago, are $X_O = 2.6 \cdot 10^{-2}$ and $X_{Fe} = 6.3 \cdot 10^{-3}$ to be compared with the same abundances from Anders & Grevesse (1989): $X_O = 9.59 \cdot 10^{-3}$ and $X_{Fe} = 1.17 \cdot 10^{-3}$ (mass fractions).

Concerning the IMF with two slopes (discussed in Sect. 2.3), the G-dwarf metallicity distribution as predicted by the numerical model is again in good agreement with the observations, in all the cases considered. On the other hand, the predicted $[O/Fe]$ versus $[Fe/H]$ is still in poor agreement with the observations with the exception of the second case ($x_1 = x_1(\Phi)$) for which we give the G-dwarf metallicity distribution and the $[O/Fe]$ versus $[Fe/H]$ in Figs. 8 and 9, respectively, for $M = 5M_\odot$ (we found results very similar to the ones shown in the Figs. 8 and 9 for values of M in the range 2–$10M_\odot$). This time-dependent IMF fits the $[O/Fe]$ vs $[Fe/H]$ relation much better than the other time dependent IMFs, since the variation in the number of type II SNe at early times is less than in the other cases. However, it is clear that the best agreement with the data (especially for $[Fe/H] < -1.0$) is achieved with a constant IMF.

We also investigated the case of an IMF with only the lower mass limit dependent on Φ (($M_L = M_L(\Phi)$). However, in this case, the Eqs. (14) and (15) give solutions which are not physical. In fact, in order to lower the number of stars in the range of mass 0.8–$1.1M_\odot$ at early times, to solve the G-dwarf problem, our equations fix the value of $M_L(\Phi)$ (for $\Phi \simeq 0$) very close to $1.1M_\odot$. This result does not have any physical meaning because it is strongly dependent on the particular mass range of interest. This is the reason why the inferred solution, when used in a numerical model of chemical evolution, it is not able to reproduce the observational G-dwarf distribution.

The predicted solar abundances related to the IMF adopted in Figs. 8 and 9 are still larger than the observed ones ($X_O = 3.1 \cdot$

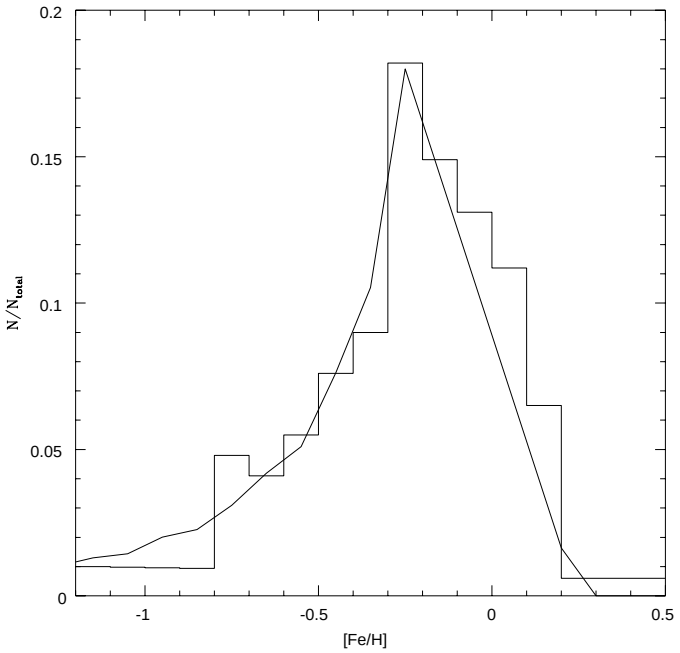

Fig. 8. G-dwarf metallicity distribution as predicted by the numerical model (solid line) and from experimental data (Rocha-Pinto & Maciel 1996). The IMF adopted is with two slopes (discussed in Sect. 2.2) with $x_1 = x_1(\Phi)$ $x_2 = 1.35$ and $M = 5M_\odot$

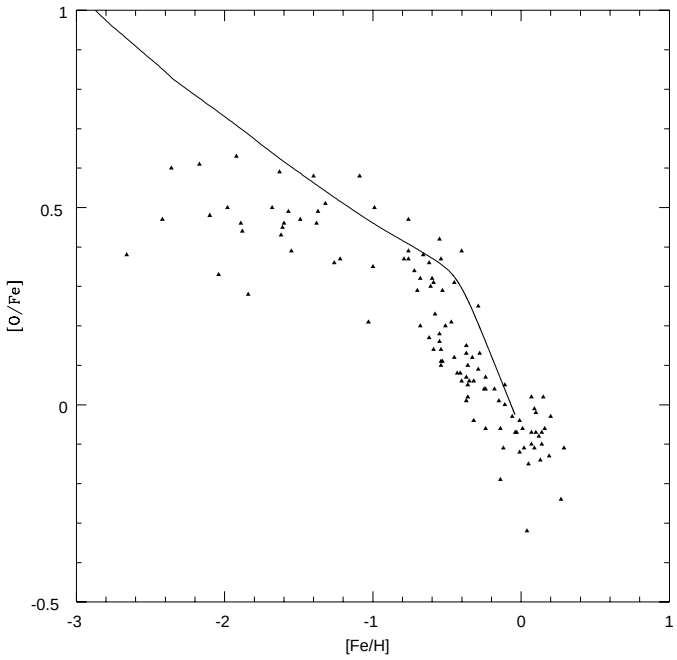

Fig. 9. $[O/Fe]$ vs $[Fe/H]$ as predicted by the numerical model (solid line) and as observed. The adopted IMF is the same as in Fig. 8.

10^{-2} and $X_{Fe} = 8.6 \cdot 10^{-3}$). However, these absolute values are strongly model dependent and in particular it is possible to lower these abundances by increasing the infall.

4. Conclusions

We proposed a method to solve the G-dwarf problem in a closed box model with a time-dependent IMF, based on the IRA. The method gives us an equation (Eq. 15) which has infinite solutions. Therefore, in order to use this equation, we had to make some assumptions on the behaviour of the IMF. In particular, we considered both a single power-law IMF and an IMF with two slopes.

We tested the validity of the inferred IMF by using numerical models of galactic chemical evolution, namely by relaxing the IRA, and we find the following results:

1. all the IMFs investigated can reproduce the observed G-dwarf distribution;
2. a single power-law IMF fails in reproducing the behaviour of abundances (in particular the $[O/Fe]$ vs $[Fe/H]$ relation);
3. in order to reproduce the behaviour of abundances besides the G-dwarf problem, an IMF with a time dependence at the low mass end is required. However, the fit produced by such an IMF of the $[O/Fe]$ vs $[Fe/H]$, is not as good as that produced by a constant IMF. Moreover, this IMF, like most of the variable IMF proposed insofar, fails in reproducing the oxygen gradient along the Galactic disk, unless other assumptions such as increasing star formation efficiency with galactocentric distance and/or radial flows are introduced. We computed the expected gradients of oxygen along the disk by adopting this IMF and the model of Chiappini et al. (1997). We found that the O gradient disappears. A variable efficiency of star formation could recover the gradient but at expenses of the gas distribution along the disk. We did not try to include radial flows but the conclusion that a variable IMF of this kind worsen the agreement with the disk propeties relative to a constant IMF seems unavoidable.
4. an IMF similar to the one proposed recently by Larson (1998), fails in reproducing the $[O/Fe]$ vs $[Fe/H]$ relation because of the strong time dependence of the number of type II SNe at early times.
5. In summary, from the analysis of models with variable IMF done in this paper, we are tempted to conclude that infall models with a constant IMF are the best solution of the G-dwarf problem, since variable IMFs could in principle solve it but they are not able to reproduce the properties of the galactic disk.

Appendix

We give a method to approximate observational data in Fig. 1, based on the regularization theory of Tikonhov (1963).

The method is general and it applies whenever one wants to approximate some data by an analytical function, and nothing is known about the physics at the basis of these data.

Accordingly to Tikonhov's regularization theory (1963) the function $f(\Phi)$ which approximates the data in Fig. 1 is determined by minimizing a cost functional $E[f]$, so-called because it maps functions (in some suitable function space) to the real line. $E[f]$ is the sum of two terms

$$E[f] = E_s[f] + E_c[f] \tag{21}$$

where $E_s[f]$ is the standard error term and $E_c[f]$ the regularizing term. The first-one measures the standard error (distance) between the desired response f_i and the actual response $f(\Phi_i)$,

$$E_s[f] = \frac{1}{2}\sum_{i=1}^{N}(f_i - f(\Phi_i))^2 \tag{22}$$

where N is the total number of available data (in our case $N = 12$). The second-one depends on the geometric properties of the approximating function $f(\Phi)$,

$$E_c[f] = \frac{1}{2}\|Pf\|^2 \tag{23}$$

where P is a differential operator. As suggested by Poggio and Girosi (1990) the best choice for P consists in a differential operator invariant under both rotations and translations. This is defined by

$$\|Pf\|^2 = \sum_{k=0}^{\infty} a_k \|D^k f\|^2 \tag{24}$$

where $a_k = \frac{\sigma^{2k}}{k!2^k}$ with σ a constant associated with the data point Φ_i and

$$\|D^k f\|^2 = \int_{-\infty}^{\infty}\left(\frac{\partial^k f}{\partial \Phi^k}\right)^2 d\Phi \tag{25}$$

The function which solves the variational problem given by Eq. (21) (when the regularizing term is specified by Eq. (24)) is:

$$f(\Phi) = \sum_{j=1}^{M} w_j e^{-\frac{(\Phi-\Phi_j)^2}{2\sigma^2}} \tag{26}$$

which consists of a linear superposition of multivariate Gaussian basis functions with centers Φ_j. The above theory establishes only the form of the function $f(\Phi)$, but does not solve completely our problem. We do not know for example the number M of gaussians in (25).

As suggested by Guyon et al. (1992) and Vapnik (1992), the method to fix the parameters in the expression (26) is to minimize the sum of a pair of competing terms. The former is again the standard error given by Eq. (22), and decreases monotonically as the number of parameters is increased. The latter measures the complexity of the model and increases by increasing the number of parameters. Therefore there is an optimal compromise which minimizes the sum. It is possible to demonstrate (Amari et al. 1997) that we obtain this compromise by using $N-k$ data and the average error made on the patterns left out. This method is called *leave-k-out cross validation*. When N is small, the most reasonable choice is $k = 1$ (*leave-one-out*).

Since in our case the value of N is small we apply the *leave-one-out cross-validation method* in order to fix the parameters in (26). The average errors made on the left data point are reported in Table 1 related to several values of M.

Table 1. Average errors made on the left data points following the *leave-one-out cross validation* method for several values of M.

M	$Error$
1	0.9131
2	0.4988
3	0.7545
4	0.8080
5	0.9097

Consequently, the best model occurs for $M = 2$ with values of the parameters appearing in Eq. (26) given by Eq. (3).

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