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Slow surface wave damping in plasmas with anisotropic viscosity and thermal conductivity

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Abstract. This paper studies the damping of slow surface MHD waves propagating along the equilibrium magnetic field on a finite-thickness magnetic interface. The plasma is assumed to be strongly magnetised, and the full Braginskii's expressions for viscosity and the heat flux are used. The primary focus of the paper is on the competition between resonant absorption in the thin dissipative layer embracing the ideal resonant position and the bulk wave damping due to viscosity and thermal conductivity as damping mechanisms for surface MHD waves. The dependence of the wave damping decrement on the wave length and the dissipative coefficients is studied. Application of the obtained results to the surface MHD wave damping in the solar chromosphere is discussed.

Key words: Magnetohydrodynamics (MHD) – waves – methods: analytical – Sun: chromosphere – Sun: corona – Sun: oscillations

1. Introduction

The solar atmosphere is strongly magnetically structured (see, e.g., Acton et al. 1992, Brekke et al. 1997, Fludra et al. 1997, Kjeldseth-Moe & Brekke 1998, Schrijver et al. 1997). The presence of magnetic structuring drastically changes the character of MHD wave propagation in plasmas. For instance, surface MHD waves can exist in magnetically structured plasmas. Such waves can propagate wherever there is a sharp change of plasma parameters across a surface called 'magnetic interface'. The surface MHD waves on magnetic interfaces have been intensively studied (see, e.g., Miles & Roberts 1992; Miles et al. 1992; Roberts 1981). In the solar atmosphere the surface MHD waves can propagate, e.g., along the boundaries of sunspots, coronal holes, coronal loops, and in the canopy regions in the chromosphere.

When a magnetic interface is a true discontinuity the surface waves are eigenmodes of the ideal linear MHD equations. Dissipation in the solar atmosphere (e.g. viscosity, thermal conductivity, and electrical resistivity) causes surface wave damping. The surface wave damping in the solar corona was considered by, e.g., Gordon & Hollweg (1983) and Ruderman (1991). It was found that for typical coronal conditions dissipation in the solar coronal plasma is not enough to cause substantial damping of surface waves in the inner part of the solar corona unless wave periods are very short (of the order of ten seconds or less).

In nature there are no true discontinuities. Instead there are magnetic plasma configurations with plasma and magnetic field parameters that rapidly vary in a thin layer. Such configurations are very often called finite-thickness magnetic interfaces. Then surface MHD waves are no longer eigenmodes of linear ideal MHD. However there are non-stationary solutions of ideal MHD that closely resemble the surface waves on true magnetic interfaces. These solutions are called quasi-modes or global modes. Away from the inhomogeneous layer they behave like exponentially damped surface waves. In the inhomogeneous layer there is a resonant magnetic surface where the phase velocity of the surface quasi-mode matches either the local Alfvén frequency or the local slow frequency. In the vicinity of this resonant magnetic surface there is strong coupling between the global plasma oscillation represented by the quasi-mode and the local Alfvén or slow oscillations. This coupling results in the conversion of the global oscillation energy into the energy of the local small-scale Alfvén or slow oscillations. It is this energy conversion that causes damping of the global surface wave. In the vicinity of the resonant magnetic surface spatial gradients linearly grow with time, so that they eventually tend to infinity (see, e.g., Mann et al. 1995; Zorzan & Cally 1992). The decrease of the spatial scale is stopped by dissipation. However, when dissipation is weak, this happens only when the spatial scale in the vicinity of the ideal resonant surface is extremely small. Dissipation only operates in a narrow dissipative layer containing large spatial gradients. This dissipative layer embraces the ideal resonant magnetic surface. In the dissipative layer the small-

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scale oscillations are strongly damped and as a result the energy of the global motion is converted into heat. Dissipation of the global motion energy due to energy conversion in the dissipative layer is called resonant absorption. Resonant absorption of quasi-modes was studied by, e.g., Ofman et al. (1994a, 1995), Ofman & Davila (1995, 1996), Poedts et al. (1989), Poedts et al. (1994), Steinolfson & Davila (1993), and Tirry & Goossens (1996).

Resonant absorption makes it possible to effectively damp MHD waves even in weakly dissipative plasmas. For typical coronal conditions the damping rate of resonant MHD waves can be a few orders of magnitude larger than the damping rate of non-resonant MHD waves. This property of resonant MHD waves to be relatively strongly damped even in weakly dissipative plasmas enabled Ionson (1978) to suggest resonant MHD waves as a possible mechanism for the heating of magnetic loops in the solar corona. Since this original work resonant absorption has grown into a popular mechanism for explaining the heating of the solar corona (see, e.g., Davila 1987; Hollweg 1991; Ionson 1985; Goossens 1991; Kuperus et al. 1981; also papers cited at the end of the previous paragraph).

The account of dissipation makes it possible to find the solutions as eigenmodes of the linear dissipative MHD equations even in case of finite-thickness magnetic interfaces. These solutions describe damped surface waves on a finite-thickness interface. When dissipation is weak they differ from ideal quasi-modes only in a thin dissipative layer. Such solutions were studied by, e.g., Mok & Einaudi (1985) and Ruderman et al. (1995) in the approximation of incompressible plasmas.

It is commonly accepted that the solar corona is strongly magnetically dominated in the sense that the magnetic pressure is much larger than the plasma pressure. This prompted many researchers to use the approximation of cold plasma in order to describe propagation and damping of MHD waves in the solar corona. However, the assumption of cold plasmas results in the loss of slow MHD waves which can be also important for physical processes in the solar corona. As for importance of slow waves in application to the solar photosphere and chromosphere, it has never been questioned. Recently Čadež & Ballester (1996) have shown that slow resonant surface waves can propagate along magnetic arcade boundaries. The conclusion that the resonant damping strongly dominates the bulk dissipative damping is only justified for resonant surface Alfvén waves on the basis of the cold plasma model. It is not obvious at all what the relative importance of these two damping mechanisms is in the case of slow surface waves.

For typical coronal conditions the collisional frequency of protons is much smaller than the proton gyrofrequency, and the inverse electron collisional time is much smaller than the electron gyrofrequency. The first statement is also valid for the upper part of the chromosphere, while the second is valid for the whole chromosphere. As a result the first term of the Braginskii's tensorial expression for viscosity (which describes the compressional viscosity, see Braginskii 1965) strongly dominates all other terms being at least five orders of magnitude larger than the other terms in the corona and at least two orders of magnitude in the upper chromosphere, and thermal conductivity along magnetic field lines is much larger than that in the directions perpendicular to the magnetic field lines. Dissipation related to finite resistivity and the Hall effect can be neglected (see, e.g., discussion in Ruderman et al. 1996). However, these conclusions may be not valid in dissipative layers embracing ideal resonant positions. We calculate both contributions of resonant absorption and the bulk viscosity and thermal conductivity into the wave damping decrement.

The paper is organised as follows. In the next section we describe the main assumptions and basic equations. In Sect. 3 we obtain the dispersion equation for surface waves in the long-wavelength approximation under the assumption that the compressional viscosity and the parallel thermal conductivity dominate all other dissipative processes in the slow dissipative layer. In Sect. 4 this dispersion equation is solved with the use of the regular perturbation method and the decrement of the wave damping is calculated. In Sect. 5 the dispersion equation is re-derived in the case where dissipative mechanisms other than the compressional viscosity and parallel thermal conductivity are important in the dissipative layer. In Sect. 6 the expression for the damping decrement is used to study in detail the slow surface wave damping in isothermal equilibrium states. Sect. 7 contains our conclusions.

2. Basic equations

2.1. Evaluation of dissipative terms

For typical conditions in the solar chromosphere and corona the coronal plasma can be considered as collision-dominated for waves with periods larger than a few seconds (see, e.g., Hollweg 1985). For waves with periods shorter than tens of minutes dissipation in the solar chromosphere and corona is due to viscosity, thermal conductivity, and resistivity. Viscosity, thermal conductivity, and resistivity are strongly anisotropic in the corona and the upper part of the chromosphere. In what follows we consider a steady equilibrium state where there is no equilibrium flow. We use the one-fluid description of the plasma. Then the linearised Braginskii's expression for the viscosity tensor S is given by (see Braginskii 1965)

$$\mathsf{S} = \eta_0 \mathsf{S}_0 + \eta_1 \mathsf{S}_1 + \eta_2 \mathsf{S}_2 - \eta_3 \mathsf{S}_3 - \eta_4 \mathsf{S}_4 \,, \tag{1}$$

where

$$\mathsf{S}_0 = \left(\boldsymbol{b}\otimes\boldsymbol{b} - \tfrac{1}{3}\mathsf{I}\right) Q, \tag{2}$$

$$\begin{aligned}\mathsf{S}_1 \;=\;& \nabla\otimes\boldsymbol{v} + (\nabla\otimes\boldsymbol{v})^T - \boldsymbol{b}\otimes\boldsymbol{W} - \boldsymbol{W}\otimes\boldsymbol{b} \\ &+\; (\boldsymbol{b}\otimes\boldsymbol{b} - \mathsf{I})\nabla\cdot\boldsymbol{v} + (\boldsymbol{b}\otimes\boldsymbol{b} + \mathsf{I})\boldsymbol{b}\cdot\nabla(\boldsymbol{b}\cdot\boldsymbol{v}),\end{aligned} \tag{3}$$

$$\mathsf{S}_2 = \boldsymbol{b}\otimes\boldsymbol{W} + \boldsymbol{W}\otimes\boldsymbol{b} - 4(\boldsymbol{b}\otimes\boldsymbol{b})\boldsymbol{b}\cdot\nabla(\boldsymbol{b}\cdot\boldsymbol{v}), \tag{4}$$

$$\begin{aligned}\mathsf{S}_3 \;=\;& \tfrac{1}{2}\{(\boldsymbol{b}\times\nabla)\otimes\boldsymbol{v} + [(\boldsymbol{b}\times\nabla)\otimes\boldsymbol{v}]^T + \nabla\otimes(\boldsymbol{b}\times\boldsymbol{v}) \\ &+\; [\nabla\otimes(\boldsymbol{b}\times\boldsymbol{v})]^T - \boldsymbol{b}\otimes(\boldsymbol{b}\times\boldsymbol{W}) - (\boldsymbol{b}\times\boldsymbol{W})\otimes\boldsymbol{b}\},\end{aligned} \tag{5}$$

$$\mathsf{S}_4 = \boldsymbol{b} \otimes (\boldsymbol{b} \times \boldsymbol{W}) + (\boldsymbol{b} \times \boldsymbol{W}) \otimes \boldsymbol{b}, \quad (6)$$

$$Q = 3\boldsymbol{b} \cdot \nabla(\boldsymbol{b} \cdot \boldsymbol{v}) - \nabla \cdot \boldsymbol{v}, \quad \boldsymbol{W} = \nabla(\boldsymbol{b} \cdot \boldsymbol{v}) + (\boldsymbol{b} \cdot \nabla)\boldsymbol{v}. \quad (7)$$

Here $\boldsymbol{v} = (u, v, w)$ is the velocity, $\boldsymbol{b} = \boldsymbol{B}_0/B_0$, $\boldsymbol{B}_0$ is the equilibrium magnetic field (which is assumed to be unidirectional, so that $\boldsymbol{b}$ is constant), I is the unit tensor, and $\otimes$ indicates the tensorial product of two vectors. The superscript 'T' indicates a transposed tensor, i.e., if the tensor S has the components S_{jl}, then the tensor S^T has the components S_{lj}. Note that the terms proportional to η_0, η_1 and η_2 in Eq. (1) describe viscous dissipation, while the terms proportional to η_3 and η_4 are non-dissipative and describe wave dispersion related to the finite ion gyroradius. For the first viscosity coefficient η_0 the following approximate expression is valid

$$\eta_0 = m_p^{-1} \rho_0 k_B T_0 \tau_p \,, \quad (8)$$

where ρ and T are the density and temperature, m_p is the proton mass, k_B the Boltzmann constant, τ_p the proton collisional time, and the subscript '0' indicates an equilibrium quantity. For typical conditions in the solar corona $\eta_0 \approx 5 \times 10^{-2}\ \mathrm{kg\,m^{-1}\,s^{-1}}$ (see, e.g., Hollweg 1985). For the upper chromosphere the corresponding estimate is $\eta_0 \approx 2 \times 10^{-7}\ \mathrm{kg\,m^{-1}\,s^{-1}}$. The other viscosity coefficients depend on the quantity $\omega_p \tau_p$, where ω_p is the proton gyrofrequency. When $\omega_p \tau_p \gg 1$ these coefficients are given by the approximate expressions

$$\eta_1 = \tfrac{1}{4}(\omega_p \tau_p)^{-2} \eta_0, \quad \eta_2 = 4\eta_1, \quad (9)$$

$$\eta_3 = \tfrac{1}{2}(\omega_p \tau_p)^{-1} \eta_0, \quad \eta_4 = 2\eta_3. \quad (10)$$

Since the first term in Eq. (1) determined by Eqs. (2) and (7) contains the term proportional to $\nabla \cdot \boldsymbol{v}$, we call the viscosity described by this term 'the compressional viscosity'. In spite that this name does not completely reflect the physical nature of the first term, we use it for the sake of brevity. Similarly, we call the viscosity described by the sum of the second and third terms in Eq. (1) 'the shear viscosity'. Resonant absorption of the wave energy in Alfvén resonant layer in cold viscous plasmas with the viscosity described by the full Braginskii's expression was numerically studied by Erdélyi & Goossens (1995) and Ofman et al. (1994b).

For typical coronal conditions $\omega_p \tau_p$ is of the order $10^5 - 10^6$, and it is of the order $10^2 - 10^3$ for typical conditions in the upper chromosphere, so that the first term in Eq. (1) is much larger than the other four terms. This seems to imply that all terms in Eq. (1) can be neglected in comparison with the first one. However this is not completely correct. In what follows we consider magnetic plasma configurations where all equilibrium quantities depend on x only in the Cartesian coordinates x, y, z. We assume that the equilibrium magnetic field is perpendicular to the x-direction and we adopt a coordinate system with the z-axis parallel to this field. Hence $\boldsymbol{b}$ is the unit vector in the z-direction. The aim of the present paper is to study the damping of surface waves propagating along an inhomogeneous slab of plasma. We shall see that the ideal resonant position can be present in the slab. Solutions describing surface waves are singular at this position. Dissipation removes the singularity. Instead the dissipative layer embracing the ideal singular position appears. However, when dissipative coefficients are small this dissipative layer is very thin. As a result gradients in the x-direction can be much larger than those in the y- and z-direction. The viscous term in the momentum equation is $\nabla \cdot \mathsf{S}$. It is straightforward to check that $\nabla \cdot \mathsf{S}_0$ and $\nabla \cdot \mathsf{S}_4$ contain only the first order derivatives with respect to x. Since $\eta_4/\eta_0 = \mathcal{O}[(\omega_p \tau_p)^{-1}]$ and $\omega_p \tau_p \gg 1$ we can neglect the fifth term, which is the term proportional to η_4, in the expression for $\nabla \cdot \mathsf{S}$ in comparison with the first term, which is the term proportional to η_0. However, the quantities $\nabla \cdot \mathsf{S}_1$, $\nabla \cdot \mathsf{S}_2$, and $\nabla \cdot \mathsf{S}_3$ contain the second order derivatives with respect to x. This fact implies that the contribution of the second, third, and fourth terms, which are terms proportional to η_1, η_2, and η_3 respectively, in the expression for the viscous force can be of the order or even larger than the contribution of the first term in spite that η_1, η_2, and η_3 are much smaller than η_0. In what follows we retain only terms containing the second order derivatives with respect to x in expressions for $\nabla \cdot \mathsf{S}_1$, $\nabla \cdot \mathsf{S}_2$, and $\nabla \cdot \mathsf{S}_3$, and rewrite them in the simplified form

$$\nabla \cdot \mathsf{S}_1 = \frac{\partial^2 \boldsymbol{v}}{\partial x^2} - \boldsymbol{b} \frac{\partial^2 w}{\partial x^2}, \ \nabla \cdot \mathsf{S}_2 = \boldsymbol{b} \frac{\partial^2 w}{\partial x^2}, \ \nabla \cdot \mathsf{S}_3 = \boldsymbol{e}_y \frac{\partial^2 u}{\partial x^2}, \quad (11)$$

where $\boldsymbol{e}_y$ is the unit vector in the y-direction.

The linearised Braginskii's expression for the heat flux is given by (see Braginskii 1965)

$$\begin{aligned} \boldsymbol{q} \ &= \ -\kappa_\parallel \boldsymbol{b} \left(\boldsymbol{b} \cdot \nabla T' + \frac{B_x'}{B_0} \frac{dT_0}{dx} \right) \\ &- \ \kappa_\perp \left[\nabla T' - \boldsymbol{b} \left(\boldsymbol{b} \cdot \nabla T' + \frac{B_x'}{B_0} \frac{dT_0}{dx} \right) \right] \\ &- \ \kappa_\wedge \left[\boldsymbol{b} \times \nabla T' - \frac{1}{B_0} \frac{dT_0}{dx} (\boldsymbol{e}_x \times \boldsymbol{B}') \right], \end{aligned} \quad (12)$$

where $\boldsymbol{e}_x$ is the unit vector in the x-direction, $\boldsymbol{B} = (B_x, B_y, B_z)$ the magnetic field, and the prime indicates the Eulerian perturbation of a quantity. In what follows we call the thermal conductivity described by the first and the second term of this expression 'the parallel thermal conductivity' and 'the perpendicular thermal conductivity', respectively. Once again the terms proportional to $\kappa_\parallel$ and $\kappa_\perp$ in Eq. (12) describe dissipation, while the term proportional to $\kappa_\wedge$ is non-dissipative and describes wave dispersion related to the finite electron gyroradius.

The coefficient $\kappa_\parallel$ is given by

$$\kappa_\parallel = \frac{3\rho_0 k_B^2 T_0 \tau_e}{m_p m_e} \,, \quad (13)$$

where τ_e is the electron collisional time and m_e is the electron mass. For typical coronal conditions $\kappa_\parallel \approx 5 \times 10^4\ \mathrm{m\,s^{-3}\,kg\,K^{-1}}$. The corresponding estimate for the upper chromosphere is $\kappa_\parallel \approx 0.2\ \mathrm{m\,s^{-3}\,kg\,K^{-1}}$. In strongly magnetised plasmas the coefficients $\kappa_\perp$ and $\kappa_\wedge$ are given by the approximate expressions

$$\kappa_\perp = \frac{2\kappa_\parallel}{3(\omega_p \tau_p)(\omega_e \tau_e)}, \quad \kappa_\wedge = \frac{2\kappa_\parallel}{3\omega_e \tau_e}, \quad (14)$$

where ω_e is the electron gyrofrequency. Since for typical coronal conditions $\omega_e\tau_e$ is of the order 10^7, and it is of the order 10^4 in the upper chromosphere, the coefficients $\kappa_\perp$ and $\kappa_\wedge$ are much smaller than $\kappa_\parallel$. The dissipative term in the energy equation related to thermal conductivity is proportional to $\nabla\cdot\boldsymbol{q}$. The terms proportional to $\kappa_\parallel$ and $\kappa_\wedge$ in the expression for $\nabla\cdot\boldsymbol{q}$ contain only the first order derivatives with respect to x, while the term proportional to $\kappa_\perp$ contains the second order derivatives with respect to x. These facts enable us to neglect the term proportional to $\kappa_\wedge$ in comparison with the term proportional to $\kappa_\parallel$ and retain only the part containing the second order derivatives with respect to x in the expression for the term proportional to $\kappa_\perp$. As a result we arrive at the simplified expression

$$\nabla\cdot\boldsymbol{q} = -\boldsymbol{b}\cdot\nabla\left[\kappa_\parallel\left(\boldsymbol{b}\cdot\nabla T' + \frac{B'_x}{B_0}\frac{dT_0}{dx}\right)\right] - \kappa_\perp\frac{\partial^2 T'}{\partial x^2}. \quad (15)$$

In the one-fluid approximation the linear generalised Ohm's law for fully ionised plasmas is (see, e.g., Priest 1982)

$$\boldsymbol{E}' = -\boldsymbol{v}\times\boldsymbol{B}_0 + \boldsymbol{E}'_{\rm non}, \quad (16)$$

where the non-ideal terms are given by

$$\boldsymbol{E}'_{\rm non} = \frac{\boldsymbol{j}'}{\sigma} + \frac{m_p}{e\rho_0}\left(\boldsymbol{j}'\times\boldsymbol{B}_0 + \boldsymbol{j}_0\times\boldsymbol{B}' - \tfrac{1}{2}\nabla p'\right). \quad (17)$$

Here $\boldsymbol{E}$ is the electrical field, $\boldsymbol{j}$ the density of the electrical current, p the pressure, e the elementary charge, and σ the electrical conductivity. The first and second term on the left-hand side of Eq. (17) describe resistivity and the Hall effect, respectively. For typical chromospheric and coronal conditions these non-ideal effects are small in comparison with non-ideal effects due to viscosity and thermal conductivity (see, e.g., the discussion in Ruderman et al. 1996). However, once again this conclusion is only valid when the spatial derivatives of the perturbations are of the same order in all directions. The non-ideal effects are represented in the induction equation by the term $\nabla\times\boldsymbol{E}'_{\rm non}$. Similarly to the cases of viscosity and thermal conductivity we retain only terms containing the second derivatives with respect to x in the expression for this quantity and arrive at

$$\nabla\times\boldsymbol{E}'_{\rm non} = -\lambda\frac{\partial^2\boldsymbol{B}'}{\partial x^2}, \quad (18)$$

where $\lambda = (\mu\sigma)^{-1}$ is the coefficient of magnetic diffusion and μ is the magnetic permeability of vacuum. When deriving this equation we have used Ampere's law $\nabla\times\boldsymbol{B} = \mu\boldsymbol{j}$. Note that expression (18) does not contain terms describing the Hall effect.

2.2. *Derivation of equations for u and $\tilde{P}'$*

The equilibrium quantities satisfy the equation of the total pressure balance and the Clapeyron equation

$$p_0 + \frac{B_0^2}{2\mu} = {\rm const}, \quad p_0 = \frac{2k_B}{m_p}\rho_0 T_0. \quad (19)$$

The equilibrium quantities vary in the layer $|x| < a$, while they take constant values for $x < -a$ and $x > a$.

With the aid of Eqs. (1), (2), (7), (9)–(11), (15), and (18) the linear equations of viscous resistive thermal conductive MHD can be written as

$$\frac{\partial\rho'}{\partial t} + \rho_0\nabla\cdot\boldsymbol{v} + u\frac{d\rho_0}{dx} = 0, \quad (20)$$

$$\rho_0\frac{\partial\boldsymbol{v}}{\partial t} = -\nabla\tilde{P}' + \frac{1}{\mu}(\boldsymbol{B}_0\cdot\nabla)\boldsymbol{B}' + \frac{B'_x}{\mu}\frac{d\boldsymbol{B}_0}{dx} + \eta_0\boldsymbol{b}(\boldsymbol{b}\cdot\nabla Q) + \eta_1\frac{\partial^2}{\partial x^2}(\boldsymbol{v} + 3\boldsymbol{b}w) + \eta_3\boldsymbol{e}_y\frac{\partial^2 u}{\partial x^2}, \quad (21)$$

$$\frac{\partial\boldsymbol{B}'}{\partial t} = (\boldsymbol{B}_0\cdot\nabla)\boldsymbol{v} - u\frac{d\boldsymbol{B}_0}{dx} - \boldsymbol{B}_0\nabla\cdot\boldsymbol{v} + \lambda\frac{\partial^2\boldsymbol{B}'}{\partial x^2}, \quad (22)$$

$$\frac{\partial p'}{\partial t} + \gamma p_0\nabla\cdot\boldsymbol{v} + u\frac{dp_0}{dx} = (\gamma-1)\kappa_\perp\frac{\partial^2 T'}{\partial x^2} + \kappa_\parallel(\gamma-1)\boldsymbol{b}\cdot\nabla\left(\boldsymbol{b}\cdot\nabla T' + \frac{B'_x}{B_0}\frac{dT_0}{dx}\right), \quad (23)$$

$$\frac{p'}{p_0} = \frac{\rho'}{\rho_0} + \frac{T'}{T_0}. \quad (24)$$

Here γ is the adiabatic exponent and the Eulerian perturbation of the total pressure modified by viscosity, $\tilde{P}'$, is given by

$$\tilde{P}' = p' + \frac{B_0}{\mu}(\boldsymbol{b}\cdot\boldsymbol{B}') + \frac{\eta_0}{3}Q. \quad (25)$$

We look for solutions to the set of Eqs. (20)–(25) that are eigenmodes. The time dependence of these solutions is given by the factor $\exp(-i\omega t)$ with complex ω. Since the equilibrium quantities depend on x only, we can Fourier-analyse the perturbations and take them proportional to $\exp[i(k_y y + k_z z)]$. Then we rewrite Eqs. (20)–(24) as

$$\omega\rho' - \rho_0\boldsymbol{k}\cdot\boldsymbol{v}_{\rm pl} + i\frac{d(\rho_0 u)}{dx} = 0, \quad (26)$$

$$\omega\rho_0 u = -i\frac{d\tilde{P}'}{dx} - \frac{B_0(\boldsymbol{k}\cdot\boldsymbol{b})}{\mu}B'_x + i\eta_1\frac{\partial^2 u}{\partial x^2}, \quad (27)$$

$$\omega\rho_0\boldsymbol{v}_{\rm pl} = \boldsymbol{k}\tilde{P}' - \frac{B_0(\boldsymbol{k}\cdot\boldsymbol{b})}{\mu}\boldsymbol{B}'_{\rm pl} + \frac{i\boldsymbol{b}}{\mu}\frac{dB_0}{dx}B'_x - \eta_0(\boldsymbol{k}\cdot\boldsymbol{b})\boldsymbol{b}Q + i\eta_1\frac{d^2}{dx^2}(\boldsymbol{v}_{\rm pl} + 3\boldsymbol{b}w) + i\eta_3\boldsymbol{e}_y\frac{d^2 u}{dx^2}, \quad (28)$$

$$\omega B'_x = -B_0(\boldsymbol{k}\cdot\boldsymbol{b})u + i\lambda\frac{d^2 B'_x}{dx^2}, \quad (29)$$

$$\omega\boldsymbol{B}'_{\rm pl} = B_0[\boldsymbol{b}(\boldsymbol{k}\cdot\boldsymbol{v}_{\rm pl}) - (\boldsymbol{k}\cdot\boldsymbol{b})\boldsymbol{v}_{\rm pl}] - i\boldsymbol{b}\frac{d}{dx}(B_0 u) + i\lambda\frac{d^2\boldsymbol{B}'_{\rm pl}}{dx^2}, \quad (30)$$

$$\omega p' + \gamma p_0\left[i\frac{du}{dx} - (\boldsymbol{k}\cdot\boldsymbol{v}_{\rm pl})\right] + iu\frac{dp_0}{dx} = -\frac{\gamma\chi_\parallel(\boldsymbol{k}\cdot\boldsymbol{b})}{\gamma-1}\left[\frac{i(\boldsymbol{k}\cdot\boldsymbol{b})}{\rho_0}(\rho_0 p' - p_0\rho') + \frac{\rho_0 B'_x}{B_0}\frac{d}{dx}\left(\frac{p_o}{\rho_0}\right)\right] + \frac{i\gamma\chi_\perp}{\rho_0(\gamma-1)}\frac{d^2}{dx^2}(\rho_0 p' - p_0\rho'). \quad (31)$$

In these equations $\boldsymbol{k} = (0, k_y, k_z)$, and we have introduced the components of the velocity $\boldsymbol{v}_{\rm pl} = (0, v, w)$ and the Eulerian perturbation of the magnetic field $\boldsymbol{B}'_{\rm pl} = (0, B'_y, B'_z)$, that are parallel to the yz-plane. The quantity Q has the form

$$Q = 3i(\boldsymbol{k}\cdot\boldsymbol{b})(\boldsymbol{b}\cdot\boldsymbol{v}_{\rm pl}) - i\boldsymbol{k}\cdot\boldsymbol{v}_{\rm pl} - \frac{du}{dx}\,. \tag{32}$$

The coefficients $\chi_\parallel$ and $\chi_\perp$ are given by

$$\chi_\parallel = \frac{m_p(\gamma-1)^2\kappa_\parallel}{2\gamma k_B\rho_0}, \quad \chi_\perp = \frac{m_p(\gamma-1)^2\kappa_\perp}{2\gamma k_B\rho_0}. \tag{33}$$

The effect of viscosity in Eqs. (25)–(31) is characterised by the Reynolds numbers

$$R_e^{(0)} = \frac{\rho_0|\omega|}{\eta_0 k^2}, \quad R_e^{(1)} = \frac{\rho_0|\omega|}{\eta_1 k^2}, \quad R_e^{(3)} = \frac{\rho_0|\omega|}{\eta_3 k^2}. \tag{34}$$

The effect of thermal conductivity is characterised by the parallel and perpendicular Pecklet numbers

$$P_e^\parallel = \frac{|\omega|}{\chi_\parallel k^2}, \quad P_e^\perp = \frac{|\omega|}{\chi_\perp k^2}. \tag{35}$$

The effect of resistivity is characterised by the magnetic Reynolds number

$$R_m = \frac{|\omega|}{\lambda k^2}. \tag{36}$$

The dissipative coefficients depend on x, so that we choose the representative values for them. In what follows we assume that

$$R_e^{(0)} \gg 1, \quad P_e^\parallel \gg 1. \tag{37}$$

Since for typical conditions in the upper chromosphere and corona $R_e^{(1)},\ R_e^{(3)},\ R_m \gg R_e^{(0)}$ and $P_e^\perp \gg P_e^\parallel$, conditions (37) imply that the dissipative effects are weak. Consequently, when calculating the dissipative terms in Eqs. (25)–(31), which are terms proportional to η_0, η_1, η_3, $\chi_\parallel$, $\chi_\perp$, and λ, we can use results obtained on the basis of ideal MHD. As a result we arrive at the approximate equations

$$Q = -\frac{A}{D}[\omega^2 - 3c_S^2(\boldsymbol{k}\cdot\boldsymbol{b})^2]\frac{du}{dx}, \tag{38}$$

$$\begin{aligned}\frac{d^2}{dx^2}(\boldsymbol{v}_{\rm pl} + 3\boldsymbol{b}w) \;=\; & \frac{i}{D}[\boldsymbol{b}(\boldsymbol{k}\cdot\boldsymbol{b})(\omega^2 v_A^2 - 3c_S^2 A) \\ & - \boldsymbol{k}(c_S^2+v_A^2)C]\frac{d^3u}{dx^3},\end{aligned} \tag{39}$$

$$\frac{d^2B'_x}{dx^2} = -\frac{B_0(\boldsymbol{k}\cdot\boldsymbol{b})}{\omega}\frac{d^2u}{dx^2}, \tag{40}$$

$$\frac{d^2\boldsymbol{B}'_{\rm pl}}{dx^2} = \frac{iB_0}{\omega D}[(c_S^2+v_A^2)C(\boldsymbol{k}\cdot\boldsymbol{b})\boldsymbol{k} - \omega^4\boldsymbol{b}]\frac{d^3u}{dx^3}, \tag{41}$$

$$\begin{aligned}\frac{i(\boldsymbol{k}\cdot\boldsymbol{b})}{\rho_0}(\rho_0 p' - p_0\rho') \;+\; & \frac{\rho_0 B'_x}{B_0}\frac{d}{dx}\left(\frac{p_0}{\rho_0}\right) \\ = \; & \frac{(\gamma-1)(\boldsymbol{k}\cdot\boldsymbol{b})\omega\rho_0 c_S^2 A}{\gamma D}\frac{du}{dx}.\end{aligned} \tag{42}$$

In these equations

$$A = \omega^2 - \omega_A^2,\ \ C = \omega^2 - \omega_c^2,\ \ D = \omega^4 - (c_S^2+v_A^2)k^2 C, \tag{43}$$

the squares of the Alfvén and slow frequencies are given by

$$\omega_A^2 = v_A^2(\boldsymbol{k}\cdot\boldsymbol{b})^2\,,\ \ \omega_c^2 = c_T^2(\boldsymbol{k}\cdot\boldsymbol{b})^2, \tag{44}$$

and the squares of the Alfvén, sound, and cusp velocities are determined by

$$v_A^2 = \frac{B_0^2}{\mu\rho_0}\,,\ \ c_S^2 = \frac{\gamma p_0}{\rho_0}\,,\ \ c_T^2 = \frac{c_S^2 v_A^2}{c_S^2+v_A^2}\,. \tag{45}$$

We use Eqs. (38)–(42) to eliminate all variables but u and $\tilde{P}'$ from Eqs. (25)–(31) and obtain

$$\frac{d\tilde{P}'}{dx} = \frac{i\rho_0 A}{\omega}u + \left(\eta_1 + \rho_0\lambda\frac{\omega_A^2}{\omega^2}\right)\frac{d^2u}{dx^2}, \tag{46}$$

$$\alpha\frac{d^3u}{dx^3} - i\eta_3\frac{k_y\omega(c_S^2+v_A^2)C}{\rho_0}\frac{d^2u}{dx^2} + iA\Phi\frac{du}{dx} = -\frac{\omega D}{\rho_0}\tilde{P}', \tag{47}$$

where

$$\begin{aligned}\alpha \;=\; & -\chi_\perp\frac{c_S^2\omega^3A^2}{D} - \frac{\eta_1\omega}{\rho_0 D}\left[k^2(c_S^2+v_A^2)^2C^2\right. \\ & + \left.(c_S^2A - v_A^2\omega^2)(3c_S^2A + v_A^2\omega^2)(\boldsymbol{k}\cdot\boldsymbol{b})^2\right] \\ & - \frac{\lambda v_A^2}{\omega D}\left[\omega^8 - C(c_S^2+v_A^2)(\omega^4+D)(\boldsymbol{k}\cdot\boldsymbol{b})^2\right],\end{aligned} \tag{48}$$

$$\begin{aligned}\Phi \;=\; & (c_S^2+v_A^2)C \\ & - \frac{i\omega A}{D}\left\{\frac{\eta_0}{3\rho_0}\left[\omega^2 - 3c_S^2(\boldsymbol{k}\cdot\boldsymbol{b})^2\right]^2 + \chi_\parallel\omega^2c_S^2(\boldsymbol{k}\cdot\boldsymbol{b})^2\right\}.\end{aligned} \tag{49}$$

Note that we have neglected small terms proportional to products of dissipative coefficients when deriving Eq. (47).

The set of Eqs. (46) and (47) for u and $\tilde{P}'$ is similar to that used by Ruderman & Goossens (1996) for studying driven slow resonant waves in plasmas with the compressional viscosity and parallel thermal conductivity, i.e., in the case where all dissipative coefficients but η_0 and $\kappa_\parallel$ are zero (see their Eqs. (24) and (30)). If we take $\eta_1 = \eta_3 = \chi_\perp = \lambda = 0$ then Eq. (46) coincides with Eq. (24) of Ruderman & Goossens (1996), while Eq. (47) differs from their Eq. (30) in that Eq. (30) contains an additional term proportional to dc_S^2/dx in its right-hand side. This difference is due to an erroneous expression for the heat flux $\boldsymbol{q}$ used by Ruderman & Goossens (1996) (see their Eq. (2)). This expression misses the term proportional to dT_0/dx. Fortunately, in their analysis Ruderman & Goossens (1996) neglected the erroneous term in the right-hand side of their Eq. (30) because it was small in comparison to the other term. Hence, the error in the linear expression for the heat flux $\boldsymbol{q}$ did not affect the results obtained by Ruderman & Goossens (1996).

In the next section the set of Eqs. (46) and (47) is used to derive the dispersion equation for the surface waves propagating on finite-thickness magnetic interfaces.

3. Derivation of dispersion equation for weakly damped surface waves when compressional viscosity and parallel thermal conductivity are strong

In this section we consider the case where the compressional viscosity and parallel thermal conductivity everywhere dominate all other dissipative processes, so that all dissipative terms but the terms proportional to η_0 and $\chi_\|$ in Eq. (47) can be neglected. We conventionally refer to this case as the case where the compressional viscosity and parallel thermal conductivity are strong.

3.1. Solutions in homogeneous regions

In this subsection we obtain the solutions to the set of Eqs. (46) and (47) in the regions $x < -a$ and $x > a$, where the equilibrium quantities are constant. In these regions there are no characteristic spatial scales except the wavelength. This implies that there are no large spatial gradients and, in accordance with the discussion in Sect. 2.1, the first and second terms on the left-hand side of Eq. (47) can be neglected in comparison with the third term, and the second term on the right-hand side of Eq. (46) can be also neglected. Then we eliminate $\tilde{P}'$ from Eqs. (46) and (47) to arrive at

$$\frac{d^2u}{dx^2} - \Gamma^2 u = 0, \qquad (50)$$

where $\Gamma^2 = -D/F$. The solution to Eq. (50) in the region $x < -a$ has to vanish as $x \to -\infty$, while the solution in the region $x > a$ has to vanish as $x \to \infty$. The solutions satisfying these conditions are

$$u_1 = U_1 e^{\Gamma_1 x}\ , \quad u_2 = U_2 e^{-\Gamma_2 x}, \qquad (51)$$

where U_1 and U_2 are arbitrary constants, $\Re(\Gamma_{1,2}) > 0$, $\Re$ indicates the real part of a quantity, and subscripts '1' and '2' refer to quantities at $x < -a$ and $x > a$, respectively. The solutions for $\tilde{P}'_1$ and $\tilde{P}'_2$ follow immediately from Eq. (46):

$$\tilde{P}'_1 = \frac{i\rho_{01}A_1U_1}{\omega\Gamma_1}e^{\Gamma_1 x}, \quad \tilde{P}'_2 = -\frac{i\rho_{02}A_2U_2}{\omega\Gamma_2}e^{-\Gamma_2 x}. \qquad (52)$$

3.2. Solution in the inhomogeneous region

In what follows we restrict our analysis to the long wavelength approximation and assume that $ak \ll 1$. In addition we assume *ad hoc* that the first and second terms on the left-hand side of Eq. (47) can be neglected in comparison with the third term. This *ad hoc* assumption will be discussed in Sect. 4 where we obtain the solution to the dispersion equation. In Sect. 5 we shall consider the case where this assumption is not valid.

In what follows we consider solutions to the set of Eqs. (46) and (47) that possess the slow resonant position in the approximation of ideal plasmas. In the vicinity of the ideal slow resonant position gradients of perturbations are large and $C \approx 0$, so that the dissipative terms on the right-hand side of Eq. (47) are important. However, in this vicinity the quantity A is not small at all, so that we can neglect the second term on the right-hand side of Eq. (46), which is the dissipative term, in comparison with the first term.

Now we neglect the first and second terms on the left-hand side of Eq. (47) and eliminate $\tilde{P}'$ from the set of Eqs. (46) and (47) to obtain

$$\frac{d}{dx}\frac{\rho_0 A\Phi}{D}\frac{du}{dx} = -\rho_0 A u. \qquad (53)$$

The ratio of the right-hand side of this equation to the left-hand side is of the order $(ak)^2 \ll 1$. In what follows we only retain the terms of the order ak, $1/R_e^{(0)}$, and $1/P_e^{\|}$, while we neglect smaller terms. In particular, we neglect the right-hand side of Eq. (53) in comparison with the left-hand side. Then it is straightforward to obtain the following approximate solution to Eq. (53):

$$u = W\int^x \frac{D(\bar{x})\,d\bar{x}}{\rho_0(\bar{x})A(\bar{x})\Phi(\bar{x})}, \qquad (54)$$

where W is an arbitrary constant. The lower limit of integration in Eq. (54) is arbitrary, so that we do not show it explicitly. Now it follows from Eq. (47) that

$$\tilde{P}' = -\frac{iW}{\omega}, \qquad (55)$$

so that in the long-wavelength approximation the Eulerian perturbation of the total pressure modified by viscosity, $\tilde{P}'$, does not vary across the inhomogeneous layer.

At the boundary of the inhomogeneous layer determined by the equations $x = \pm a$ the quantities u and $\tilde{P}'$ have to be continuous. It follows from the condition of continuity of $\tilde{P}'$ and Eqs. (52) and (55) that

$$U_1 = -\frac{W\Gamma_1}{\rho_{01}A_1}e^{a\Gamma_1}, \quad U_2 = \frac{W\Gamma_2}{\rho_{02}A_2}e^{a\Gamma_2}. \qquad (56)$$

Now we use the condition of continuity of u and Eqs. (51), (54), and (56) to obtain

$$\frac{\Gamma_1}{\rho_{01}A_1} + \frac{\Gamma_2}{\rho_{02}A_2} = \int_{-a}^{a} \frac{D(x)\,dx}{\rho_0(x)A(x)\Phi(x)}. \qquad (57)$$

This is the dispersion equation which determines the dependence of ω on k. In the next sections we find the solution to this equation.

4. Solving the dispersion equation by perturbation method

We use the regular perturbation method to find the solution to the dispersion Eq. (57). In this method we look for the solution in the form $\omega = \bar{\omega} + \omega'$, where $|\omega'| \ll |\bar{\omega}|$.

4.1. The first order approximation

When calculating $\bar{\omega}$ we neglect terms of the order R_e^{-1}, P_e^{-1}, and ak in dispersion Eq. (57). As a result we obtain the following equation for $\bar{\omega}$:

$$\frac{\Gamma_1}{\rho_{01}A_1} + \frac{\Gamma_2}{\rho_{02}A_2} = 0. \qquad (58)$$

Eq. (58) is the dispersion equation for surface waves on a true magnetic discontinuity in an ideal plasma. It is straightforward to show by means of squaring that Eq. (58) is equivalent to the set of equation

$$\rho_{01}^2(c_{S1}^2 + v_{A1}^2)\frac{A_1^2 C_1}{D_1} = \rho_{02}^2(c_{S2}^2 + v_{A2}^2)\frac{A_2^2 C_2}{D_2}, \tag{59}$$

and the two inequalities

$$C_1 D_1 < 0, \quad A_1 A_2 < 0, \tag{60}$$

where the quantities $A_{1,2}$, $C_{1,2}$, and $D_{1,2}$ are calculated at $\omega = \bar\omega$ and $\bar\omega$ is assumed to be real.

It follows from the second inequality (60) that $A = 0$ at a position $x = x_A$, called the Alfvén singularity, inside the inhomogeneous layer, so that the integral in the right-hand side of Eq. (57) is divergent. However, when the wave propagates along the equilibrium magnetic field, $D = A(\omega^2 - c_S^2 k^2)$ and the Alfvén singularity is not present in Eq. (57). In what follows we only consider waves propagating along the equilibrium magnetic field. Then Eq. (59) is reduced to

$$\frac{\rho_{01}^2(c_{S1}^2 + v_{A1}^2)(\bar\omega^2 - v_{A1}^2 k^2)(\bar\omega^2 - c_{T1}^2 k^2)}{(\bar\omega^2 - c_{S1}^2 k^2)} = \frac{\rho_{02}^2(c_{S2}^2 + v_{A2}^2)(\bar\omega^2 - v_{A2}^2 k^2)(\bar\omega^2 - c_{T2}^2 k^2)}{(\bar\omega^2 - c_{S2}^2 k^2)}. \tag{61}$$

We can assume without loss of generality that $v_{A1} < v_{A2}$. Then inequalities (60) reduce to

$$v_{A1}^2 k^2 < \bar\omega^2 < v_{A2}^2 k^2, \quad c_{T1}^2 k^2 < \bar\omega^2 < c_{S1}^2 k^2. \tag{62}$$

These inequalities can be simultaneously satisfied only if $c_{S1} > v_{A1}$. Hence surface waves can propagate along the equilibrium magnetic field only if

$$v_{A1} < c_{S1}. \tag{63}$$

In what follows we assume that inequality (63) is satisfied. This is a very important inequality. It, in particular, shows that slow surface waves cannot propagate along the magnetic field in low-beta plasmas. Since the coronal plasma is a low-beta plasma, the results that will be obtained in what follows are only applicable to the upper chromosphere, but they are not applicable to the corona.

It follows from Eq. (59) and the first inequality (60) that $C_2 D_2 < 0$, which gives

$$C_2(\bar\omega^2 - c_{S2}^2 k^2) > 0. \tag{64}$$

This inequality is satisfied if either $\bar\omega^2 > c_{S2}^2 k^2$ or $\bar\omega^2 < c_{T2}^2 k^2$. The first inequality can be satisfied only if $c_{S2} < v_{A2}$.

Summarising we state that $\bar\omega^2$ has to satisfy either

$$v_{A1}^2 k^2 < \bar\omega^2 < \min(c_{S1}^2 k^2,\, c_{T2}^2 k^2), \tag{65}$$

or

$$\max(v_{A1}^2 k^2,\, c_{S2}^2 k^2) < \bar\omega^2 < \min(c_{S1}^2 k^2,\, v_{A2}^2 k^2). \tag{66}$$

Inequality (65) can be satisfied only when

$$c_{T2} > v_{A1}, \tag{67}$$

while inequality (66) can be satisfied only when

$$\min(c_{S1},\, v_{A2}) > c_{S2}. \tag{68}$$

Roberts (1981) considered a non-magnetic plasma in the region $x < -a$ $(v_{A1} = 0)$ and called solutions satisfying Eqs. (65) and (66) slow and fast surface waves, respectively. We generalise Roberts' definition and use the same names for the two types of surface waves in the general case where $v_{A1} \neq 0$. In what follows we assume that inequality (67) is satisfied and consider only the slow surface waves. A graphical investigation of Eq. (61) shows that under conditions (63) and (67) Eq. (61) has exactly one solution that lies in the interval determined by (65), i.e., there is exactly one solution to Eq. (58) that corresponds to a slow surface wave. The consideration of symmetry enables us to restrict the analysis to $\bar\omega > 0$.

$\Phi = (c_S^2 + v_A^2)C$ in an ideal plasma ($\eta_0 = \chi = 0$) and the integral on the right-hand side of Eq. (57) is singular at the slow resonant position x_c where $C = 0$. This position is determined by

$$\bar\omega^2 = c_T^2(x_c)k^2. \tag{69}$$

Since $c_{T1}k^2 < \bar\omega^2 < c_{T2}k^2$ there is at least one solution to Eq. (69). In what follows we assume that $c_T(x)$ is a monotonic function, so that there is exactly one resonant position.

4.2. *The second order approximation*

In the second order approximation we calculate ω'. In accordance with the general rule we have to substitute $\bar\omega$ for ω and take $\nu = \chi = 0$ when calculating the integral in the right-hand side of dispersion Eq. (57). However, in this case the integrand is singular at the slow resonant position $x = x_c$. Hence we have to retain terms proportional to η_0, $\chi_\parallel$, and ω' in expression (49) for Φ in the dissipative layer that embraces the resonant position. Since $|\omega'| \ll |\bar\omega|$ we use the approximate formula

$$C = \bar C + 2\bar\omega\omega'. \tag{70}$$

In what follows we use the rule that a quantity with the bar is obtained from the same quantity without the bar by substituting $\bar\omega$ for ω. For large values of $R_e^{(0)}$ and $P_e^\parallel$ the thickness of the dissipative layer is very small. This enables us to use the Taylor expansion for the quantity $\bar C$ in the vicinity of the resonant position and to retain only the first non-zero term in this expansion. As a result we obtain

$$\bar C = \Delta(x - x_c), \quad \Delta = \left.\frac{d\bar C}{dx}\right|_{x=x_c}. \tag{71}$$

We can substitute $\bar\omega$ for ω when calculating the second term in expression (49) for Φ in the vicinity of the ideal resonant position. Then with the aid of Eqs. (70) and (71) we arrive at the approximate expression

$$\Phi = (c_S^2 + v_A^2)[\Delta(x - x_c) + 2\bar\omega\omega_r'] - i\bar\omega\left[2(c_S^2 + v_A^2)\gamma_d - \frac{\bar\omega^3}{k^2 R_\parallel}\right]. \tag{72}$$

Here ω_r' and ω_i' are the real and imaginary part of ω', $\gamma_d = -\omega_i'$ is the wave damping decrement, and the total parallel Reynolds number $R_\parallel$ is given by

$$\frac{1}{R_\parallel} = \frac{(3c_S^2 + 2v_A^2)^2}{3c_S^2 v_A^2 R_e^{(0)}} + \frac{c_S^2 + v_A^2}{c_S^2 P_e^\parallel}. \tag{73}$$

All equilibrium quantities in Eqs. (72) and (73) are calculated at $x = x_c$. The thickness of the dissipative layer $\delta_\parallel$ is determined by the condition that the real and imaginary parts of Φ are of the same order. This condition leads to

$$\delta_\parallel = \left(1 - \frac{2\bar{\omega}\gamma_d}{\tilde{\delta}_\parallel |\Delta|}\right)\tilde{\delta}_\parallel, \quad \tilde{\delta}_\parallel = \frac{\bar{\omega}^4}{k^2|\Delta| R_\parallel (c_S^2 + v_A^2)}, \tag{74}$$

where $\tilde{\delta}_\parallel$ is the thickness of the slow dissipative layer in the steady state of driven resonant oscillations (i.e., when ω is real, see Ruderman & Goossens 1996). Once again all equilibrium quantities in Eq. (74) are calculated at $x = x_c$. In this section we assume that $\delta_\parallel > 0$, whereas the case $\delta_\parallel < 0$ will be considered in the next section.

Following Sakurai et al. (1991) and Goossens et al. (1995) we assume that there is a quantity s_c such that $\delta_\parallel \ll s_c \ll a$ and the Taylor expansion (71) is valid in the interval $[x_c - s_c,\, x_c + s_c]$. Then we use Eqs. (72) and (74) to rewrite the integral on the right-hand side of Eq. (57) as

$$\begin{aligned}\int_{-a}^{a} &\frac{D(x)\,dx}{\rho_0(x)A(x)\Phi(x)} \\ &= \left(\int_{-a}^{x_c - s_c} + \int_{x_c + s_c}^{a}\right)\frac{\bar{D}(x)\,dx}{\rho_0(x)[c_S^2(x) + v_A^2(x)]\bar{A}(x)\bar{C}(x)} \\ &- \frac{\bar{\omega}^4}{\rho_{0c} v_{Ac}^4 k^2}\int_{x_c - s_c}^{x_c + s_c}\frac{dx}{\Delta(x - x_c) + 2\bar{\omega}\omega_r' + i\delta_\parallel|\Delta|},\end{aligned} \tag{75}$$

where the subscript ‘c’ indicates that a quantity is calculated at $x = x_c$. We make the substitution $x - x_c = \delta_\parallel \tau$ in the last integral in this equation. The variable τ is of the order one in the dissipative layer while $x \to x_c \pm s_c$ corresponds to $\tau \to \pm\infty$. In the first two integrals on the right-hand side of Eq. (75) we take the limit $s_c \to +0$. As a result we arrive at

$$\begin{aligned}\int_{-a}^{a} \frac{D\,dx}{\rho_0 A\Phi} &= \mathcal{P}\int_{-a}^{a}\frac{\bar{D}\,dx}{\rho_0(c_S^2 + v_A^2)\bar{A}\bar{C}} \\ &- \frac{\bar{\omega}^4}{\rho_{0c} v_{Ac}^4 k^2}\mathcal{P}\int_{-\infty}^{\infty}\frac{d\tau}{\Delta\tau + 2\bar{\omega}\omega_r'/\delta_\parallel + i|\Delta|}.\end{aligned} \tag{76}$$

Here $\mathcal{P}$ indicates the principal Cauchy part of an integral. We use the symbol $\mathcal{P}$ for the first integral in Eq. (76) because of the singularity in the integrand at $x = x_c$, and in the second integral because it diverges logarithmically as $|\tau| \to \infty$. Taking into account that, in accordance with our assumption, $\delta_\parallel > 0$ it is straightforward to obtain

$$\mathcal{P}\int_{-\infty}^{\infty}\frac{d\tau}{\Delta\tau + 2\bar{\omega}\omega_r'/\delta_\parallel + i|\Delta|} = -\frac{\pi i}{|\Delta|}. \tag{77}$$

With the use of this result we finally derive the approximate expression

$$\int_{-a}^{a} \frac{D\,dx}{\rho_0 A\Phi} = \mathcal{P}\int_{-a}^{a}\frac{\bar{D}\,dx}{\rho_0(c_S^2 + v_A^2)\bar{A}\bar{C}} + \frac{\pi i\bar{\omega}^4}{\rho_{0c} v_{Ac}^4 k^2|\Delta|}. \tag{78}$$

We use this result and the dispersion equation of the first order approximation (58) to get from Eq. (57) in the second order approximation

$$\gamma_d = \gamma_\eta + \gamma_\chi + \gamma_r, \tag{79}$$

where γ_η, γ_χ and γ_r are the contributions to the wave damping decrement due to viscosity, thermal conductivity and resonant absorption, respectively. These quantities are given by

$$\begin{aligned}\gamma_\eta = \frac{1}{6H}\Bigg[&\frac{\eta_{02}(\bar{\omega}^2 - 3c_{S2}^2k^2)^2}{\rho_{02}(c_{S2}^2 + v_{A2}^2)(\bar{\omega}^2 - c_{S2}^2k^2)\bar{C}_2} \\ &- \frac{\eta_{01}(\bar{\omega}^2 - 3c_{S1}^2k^2)^2}{\rho_{02}(c_{S1}^2 + v_{A1}^2)(\bar{\omega}^2 - c_{S1}^2k^2)\bar{C}_1}\Bigg],\end{aligned} \tag{80}$$

$$\gamma_\chi = \frac{\bar{\omega}^2k^2}{2H}\left[\frac{\chi_{\parallel 2}c_{T2}^2}{v_{A2}^2\bar{C}_2(\bar{\omega}^2 - c_{S2}^2k^2)} - \frac{\chi_{\parallel 1}c_{T1}^2}{v_{A1}^2\bar{C}_1(\bar{\omega}^2 - c_{S1}^2k^2)}\right], \tag{81}$$

$$\gamma_r = \frac{\pi\rho_{01}\bar{A}_1\bar{\omega}^3}{H\bar{\Gamma}_1|\Delta|\rho_{0c}v_{Ac}^4k^2}, \tag{82}$$

where

$$H = \frac{1}{\bar{C}_1} - \frac{(c_{S1}^2 - v_{A1}^2)k^2}{\bar{A}_1(\bar{\omega}^2 - c_{S1}^2k^2)} - \frac{\bar{\omega}^4 - 2c_{S2}^2k^2\bar{C}_2}{\bar{A}_2\bar{C}_2(\bar{\omega}^2 - c_{S2}^2k^2)}. \tag{83}$$

It is straightforward to show that $\gamma_\eta > 0$, $\gamma_\chi > 0$ and $\gamma_r > 0$. The quantities γ_η and γ_χ describe the effect of viscosity and thermal conductivity in the outer regions determined by the condition $|x| > a$. We see that viscosity and thermal conductivity in the outer regions and resonant absorption in the dissipative layer lead to wave damping as can be anticipated from a physical argumentation.

4.3. *Comparison of dissipative terms*

Let us now check the *ad hoc* assumption adopted in this section that the first and second terms on the left-hand side of Eq. (47) can be neglected in comparison with the third term. Since we only consider waves propagating along the equilibrium magnetic field ($k_y = 0$), we have to compare the first and third terms only. Far away from the ideal slow resonant position there are no large gradients and the terms describing the compressional viscosity and parallel thermal conductivity, which are terms proportional to η_0 and $\chi_\parallel$, dominate the other dissipative terms in Eq. (47). Hence, we have to compare the terms on the left-hand side of this equation in the dissipative layer only.

It is straightforward to obtain the following approximate expression for the quantity α valid in the dissipative layer:

$$\frac{\alpha}{A} = \frac{c_T^2 c_S^2}{R_\perp}, \quad \frac{1}{R_\perp} = \frac{1}{Rm} + \frac{v_A^2}{c_S^2 P_e^\perp} + 4\frac{c_S^2 + v_A^2}{c_S^2 R_e^{(1)}}, \tag{84}$$

where $R_\perp$ is the perpendicular total Reynolds number. All equilibrium quantities in Eq. (84) are calculated at $x = x_c$. Since in the dissipative layer the characteristic scale in the x-direction is

$\delta_\parallel$, and the real and imaginary parts of Φ are of the same order, it follows that

$$\frac{\alpha}{A}\frac{d^3u}{dx^3} \sim \frac{c_T^2 c_S^2 u}{R_\perp \delta_\parallel^3}, \quad |\Phi| \sim \frac{c_T^4 k^2 \delta_\parallel}{R_\parallel \tilde{\delta}_\parallel}. \tag{85}$$

Using these estimates we get

$$\frac{\alpha d^3u/dx^3}{A\Phi du/dx} \sim \frac{\tilde{\delta}_\parallel R_\parallel}{k^2 \delta_\parallel^3 R_\perp}. \tag{86}$$

We have seen in Sect. 4.1 that slow surface waves can propagate along the equilibrium magnetic field only when $v_A \lesssim c_S$. Therefore we assume that c_S, v_A, and c_T are of the same order in the dissipative layer and use the estimate $|\Delta| \sim \bar{\omega}^2/a$ and Eq. (74) to obtain $k\tilde{\delta}_\parallel \sim ka/R_\parallel$. Then it follows from Eq. (86) that the first term on the left-hand side of Eq. (47) can be neglected in comparison with the third term only when the condition

$$\left(\frac{\delta_\parallel}{\tilde{\delta}_\parallel}\right)^3 \gg \frac{(R_\parallel)^3}{(ak)^2 R_\perp} \tag{87}$$

is satisfied. For typical conditions in the upper chromosphere we obtain the very crude estimates $R_\parallel \sim 10^4 k^{-1}$, $R_\perp \sim 10^{10} k^{-1}$, where k is measured in km^{-1}. Then condition (87) yields

$$\left(\frac{\delta_\parallel}{\tilde{\delta}_\parallel}\right)^3 \gg \frac{100}{k^2(ak)^2}. \tag{88}$$

Taking $ak \leq 0.1$ we obtain that even for very short waves with $k \sim 1\ \mathrm{km}^{-1}$ corresponding to the wave period smaller than 1 s the right-hand side of Eq. (88) is much larger than unity and this condition cannot be satisfied. Hence we conclude that for typical conditions in the upper chromosphere the shear viscosity, perpendicular thermal conductivity and resistivity dominate in the dissipative layer in spite that they are negligible in comparison with the compressional viscosity and parallel thermal conductivity far away from the dissipative layer.

It is instructive to compare the contributions of different dissipative mechanisms in the wave damping decrement γ_d. It will be shown in the next section that expression (79) for γ_d remains valid even when condition (87) is not satisfied. It is straightforward to obtain the estimates $\gamma_\eta + \gamma_\chi = \mathcal{O}(\bar{\omega}R_\parallel^{-1})$ and $\gamma_r = \mathcal{O}(\bar{\omega}ak)$, so that $\gamma_r/(\gamma_\eta+\gamma_\chi) = \mathcal{O}(akR_\parallel)$. Thus, the damping decrement is mainly determined by resonant absorption when $akR_\parallel \gg 1$, while it is mainly determined by dissipation in the outer regions when $akR_\parallel \ll 1$. On the other hand, $2\bar{\omega}\gamma_d/\tilde{\delta}_\parallel|\Delta| = \mathcal{O}(\gamma_d R_\parallel/\bar{\omega})$, so that $2\bar{\omega}\gamma_d/\tilde{\delta}_\parallel|\Delta| = \mathcal{O}(akR_\parallel)$ when $akR_\parallel \gg 1$ and resonant absorption dominates, while $2\bar{\omega}\gamma_d/\tilde{\delta}_\parallel|\Delta| = \mathcal{O}(1)$ when $\gamma_r \lesssim \gamma_\eta + \gamma_\chi$ and either the dissipation in the outer regions dominates or the two dissipative processes give contributions of the same order of magnitude. These estimates, together with Eq. (74), show that $\delta_\parallel < 0$ in the case where resonant absorption dominates over dissipation in the outer regions. For typical conditions in the upper chromosphere $akR_\parallel \sim 10^4(ak)k^{-1}$, so $akR_\parallel \gg 1$ for $k \lesssim 1\ \mathrm{km}^{-1}$ and $ak \gtrsim 10^{-3}$, and the damping decrement is mainly determined by resonant absorption.

The corresponding estimates for the solar corona are quite different and show that for realistic wave parameters the structure of the dissipative layer and the damping can be determined by the compressional viscosity and parallel thermal conductivity. However, since we restricted our analysis to the finite beta plasma, we do not embark on a further discussion of this problem.

In the next section we consider the case where the first term on the left-hand side of Eq. (47) has to be retained.

5. Calculation of the wave damping for weak compressional viscosity and parallel thermal conductivity

In this section we consider the case where either condition (87) is not satisfied or $\delta_\parallel < 0$. However, we first define more precisely the meaning of the phrase 'the compressional viscosity and parallel thermal conductivity are strong'. Let us fix all parameters involved in the problem and vary the quantities η_0 and $\chi_\parallel$ in the dissipative layer only. Then the wave damping decrement γ_d remains fixed and the quantity $\tilde{\delta}_\parallel$ varies. When η_0 and $\chi_\parallel$ are large enough in the dissipative layer we obtain $\delta_\parallel \sim \tilde{\delta}_\parallel$, condition (87) is satisfied and the analysis in Sects. 3 and 4 is valid. Hence, the phrase 'the compressional viscosity and parallel thermal conductivity are strong' exactly means that these dissipative processes are strong enough to ensure that condition (87) is satisfied. Let us now decrease η_0 and $\chi_\parallel$ in the dissipative layer. Then the ratio $\delta_\parallel/\tilde{\delta}_\parallel$ decreases and $R_\parallel$ increases and, for small enough values of η_0 and $\chi_\parallel$, condition (87) is not satisfied. This consideration gives sense to the phrase 'the compressional viscosity and parallel thermal conductivity are weak'.

While it is clear from the analysis in the previous section that the first term on the left-hand side of Eq. (47) has to be retained when $\delta_\parallel > 0$ and $\delta_\parallel/\tilde{\delta}_\parallel$ is small, it seems at first sight that this term can be neglected when $\delta_\parallel < 0$ and $|\delta_\parallel| \sim \tilde{\delta}_\parallel$. However, we shall see in what follows that this is not correct. When $\delta_\parallel < 0$ the thickness of the dissipative layer is $|\delta_\parallel|$. However, the solution in the dissipative layer is strongly oscillatory with the characteristic spatial scale in the x-direction much smaller than $|\delta_\parallel|$. This characteristic spatial scale is determined by the first term on the left-hand side of Eq. (47).

5.1. Solution in the dissipative layer and connection formulae

The concept of connection formulae was first introduced by Sakurai et al. (1991) and then further developed by Erdélyi et al. (1995), Erdélyi (1997), Goossens et al. (1995) (see also the review by Goossens & Ruderman 1995). This concept turned out to be very useful in studying resonant MHD waves. It can be described as follows. Since in weakly dissipative plasmas dissipative layers embracing ideal resonant positions are very narrow, we can use the method of matched asymptotic expansions. In accordance with this method we look for the solution inside and outside the dissipative layer separately. Since dissipation is only important in the dissipative layer we use ideal MHD when looking for the outer solution. When we are not interested in the inner solution, which is the solution in the dissipative layer, we can consider the dissipative layer as a surface of discontinuity. Then all we need from the solution in the dissipative layer are expressions for the jumps in the normal component of the velocity and in the perturbation of the total pressure across the dissipative layer. These jumps are given by the connection formulae.

To derive the connection formulae we first note that, in accordance with Eq. (79), dissipation due to viscosity and thermal conductivity inside the slab containing the inhomogeneous plasma and outside the dissipative layer does not contribute to the wave damping. This conclusion is based on the analysis in Sects. 3 and 4, which is only valid when condition (87) is satisfied. However, the conclusion itself is also valid when condition (87) is not satisfied. To show this we denote the contribution of dissipation in the region $|x| < a$ and outside the dissipative layer as $\gamma_{\rm in}$. The contribution of dissipation in the regions $x < -a$ and $x > a$ to the wave damping decrement is $\gamma_\eta + \gamma_\chi$. It is straightforward to show that $\gamma_{\rm in}/(\gamma_\eta + \gamma_\chi) \sim ak$, so that $\gamma_{\rm in}$ can be neglected in comparison with $\gamma_\eta + \gamma_\chi$. This fact enables us to use ideal MHD to describe the plasma motion in the slab outside the dissipative layer.

Let us introduce the characteristic scale in the x-direction in the dissipative layer l_d. In accordance with the analysis in Sect. 4.3, $l_d = \delta_\parallel$ when condition (87) is satisfied and, anyway, $l_d \ll a$. Following Sect. 4.2 we adopt the assumption that there is a quantity s_c such that $l_d \ll s_c \ll a$ and all coefficient functions in Eqs. (46) and (47) can be approximated in the interval $[x_c - s_c, x_c + s_c]$ by the first non-zero terms of their Taylor expansions in the vicinity of x_c. This assumption enables us to use expression (72) for Φ and take all other coefficient functions in Eqs. (46) and (47) equal to their values at $x = x_c$ when solving these equations in the dissipative layer. In addition, since $|\omega'| \ll \bar{\omega}$, we substitute $\bar{\omega}$ for ω. Eliminating $\tilde{P}'$ from the set of Eqs. (46) and (47) and taking into account that $k_y = 0$ we obtain

$$\alpha \frac{d^4 u}{dx^4} + i \frac{d}{dx} \Phi \frac{du}{dx} = -iDu - \frac{\omega D}{\rho_0 A} \left(\eta_1 + \rho_0 \lambda \frac{\omega_A^2}{\omega^2} \right) \frac{d^2 u}{dx^2}. \quad (89)$$

It is straightforward to show that the ratio of the last term on the right-hand side of this equation to the first term on the left-hand side is of the order $(kl_d)^2 < (ak)^2 \ll 1$, and the ratio of the first term on the right-hand side to the second term on the left-hand side is of the order $ak^2 l_d < (ak)^2 \ll 1$. These estimates enable us to neglect the right-hand side of Eq. (89) in comparison to the left-hand side and reduce this equation to

$$\alpha \frac{d^3 u}{dx^3} + i\Phi \frac{du}{dx} = U_{\rm const}, \quad (90)$$

where $U_{\rm const}$ is the constant of integration. Using Eq. (47) and taking into account that all coefficient functions in Eqs. (46) and (47) are calculated at $x = x_c$ and $\omega = \bar{\omega}$ we obtain

$$U_{\rm const} = \frac{c_S^3 c_T^2 k^2}{\rho_0 v_A^2} \tilde{P}'. \quad (91)$$

This result, in particular, implies that $\tilde{P}'$ is approximately constant in the dissipative layer. The approximate constancy of the total pressure in the dissipative layer was first used for studying the resonant damping of surface waves on a finite-thickness magnetic interface by Hollweg (1987a, b) and Hollweg & Yang (1988). It was subsequently used by Hollweg (1988) and Sakurai et al. (1991) for studying driven resonant MHD waves in one-dimensional planar and cylindrical equilibria. Goossens et al. (1995) gave the rigorous mathematical derivation of the property of perturbation of the total pressure to be constant in the dissipative layer for Alfvén resonance, while Erdélyi (1997) showed it for slow resonance.

Since far away from the dissipative layer the quantity $\tilde{P}'$ tends to the perturbation of the total pressure P' given by Eq. (25) with $\eta_0 = 0$ and $\tilde{P}'$ is constant in the dissipative layer, we obtain for the jump in P' across the dissipative layer

$$[P'] = 0, \quad (92)$$

where the square brackets indicate the jump in a quantity. This is the first connection formula.

Let us make the substitution of the independent variable

$$\tau = \frac{1}{\delta_\perp} \left(x - x_c + \frac{2\bar{\omega}\omega_r'}{\Delta} \right), \quad \delta_\perp = \left(\frac{c_T^4}{v_A^2 |\Delta| R_\perp} \right)^{1/3} \quad (93)$$

in Eq. (90). Then with the use of Eqs. (72), (84), and (91) we rewrite this equation as

$$\frac{d^3 u}{d\tau^3} + \left(i\tau\, {\rm sign}\Delta - \frac{\delta_\parallel}{\delta_\perp} \right) \frac{du}{d\tau} = \frac{k^3 c_T^5}{\rho_0 v_A^4 |\Delta|} \tilde{P}'. \quad (94)$$

This equation coincides with the corresponding equation obtained by Tirry & Goossens (1996) when studying quasi-modes in one-dimensional axisymmetric equilibria in compressible plasmas (see their Eq. (8)). Using Eq. (11) of Tirry & Goossens (1996) we can immediately give the solution to Eq. (94)

$$u = \frac{ik^3 c_T^5 \tilde{P}'}{\rho_0 v_A^4 \Delta} G(\tau) + {\rm const}, \quad (95)$$

$$G(\tau) = \int_0^\infty \frac{\exp(i\tau r\, {\rm sign}\Delta) - 1}{r} \exp\left(-\frac{\delta_\parallel r}{\delta_\perp} - \frac{r^3}{3} \right) dr. \quad (96)$$

Note that this solution differs from that in Tirry & Goossens (1996) by a constant. This solution also coincides with that obtained by Ruderman et al. (1995) when studying the propagation of surface waves on a finite-thickness magnetic interface in an incompressible plasma (see their Eqs. (59) and (60)).

The jump in the quantity u is given by

$$[u] = \lim_{\tau \to \infty} \{u(\tau) - u(-\tau)\}. \quad (97)$$

It is straightforward to obtain $[G] = \pi i\, {\rm sign}\Delta$. With the use of this result we get

$$[u] = -\frac{\pi k^3 c_T^5 \tilde{P}'}{\rho_0 v_A^4 \Delta}. \quad (98)$$

This is the second connection formula. Note that this formula coincides with that obtained by Sakurai et al. (1991) and Erdélyi (1997) in the case where $\delta_\parallel = 0$.

5.2. Behaviour of the solutions in the dissipative layer

In this subsection we study how the solutions in the dissipative layer depend on the parameter $\delta_\parallel/\delta_\perp$. In addition to the behaviour of the x-component of the velocity, u, we also study the behaviour of the z-component, w, because, as we shall see in what follows, this is the dominant component of the velocity in the dissipative layer. Note, that for the parallel propagation $(k_y = 0)$ $v = 0$. Using Eqs. (28)–(30), (95), and (96), and retaining only the largest terms we obtain the following approximate expression for w in the dissipative layer,

$$w = -\frac{ik^2 c_T^3 \tilde{P}'}{\rho_0 v_A^2 |\Delta| \delta_\perp} F(\tau), \tag{99}$$

$$F(\tau) = \int_0^\infty \exp\left(i\tau r\,\mathrm{sign}\Delta - \frac{\delta_\parallel r}{\delta_\perp} - \frac{r^3}{3}\right) dr. \tag{100}$$

Let us first consider the case where condition (87) is satisfied. It is straightforward to show with the use of the estimate $k\tilde{\delta}_\parallel \sim ka/R_\parallel$ and Eq. (93) that this condition is equivalent to $\delta_\parallel \gg \delta_\perp$. With the use of the asymptotic formulae (A.2) and (A.3) we obtain that the quantities w and u are given by the approximate expressions

$$w = \frac{k^2 c_T^3 \tilde{P}'}{\rho_0 v_A^2 \Delta \delta_\parallel} \frac{\theta - i\,\mathrm{sign}\Delta}{1+\theta^2}, \tag{101}$$

$$u = -\frac{k^3 c_T^5 \tilde{P}'}{\rho_0 v_A^4 |\Delta|} \left\{\arctan\theta + \frac{i}{2}\,\mathrm{sign}\Delta\,\log(1+\theta^2)\right\} + \mathrm{const}, \tag{102}$$

with

$$\theta = \frac{\delta_\perp \tau}{\delta_\parallel} = \frac{1}{\delta_\parallel}\left(x - x_c + \frac{2\bar{\omega}\omega_r'}{\Delta}\right). \tag{103}$$

One can easily check that these expressions coincide with the corresponding expressions for the parallel and normal components of the velocity in the dissipative layer in a plasma with strongly anisotropic viscosity and thermal conductivity obtained by Ruderman & Goossens (1996) for the driven problem (see their Eqs. (34) and (40)). The real and imaginary parts of u and w given by the exact expressions (95) and (99) for $\delta_\parallel/\delta_\perp = 5$ are shown in Fig. 1 (solid lines). The rectangles show the asymptotic approximation given by formulae (101) and (102). We see that the asymptotic formulae are actually very accurate. Note that $u/w \sim k\delta_\parallel \ll 1$, so w is the dominant component of the velocity in the dissipative layer.

When $|\delta_\parallel| \lesssim \delta_\perp$ the behaviour of u and w is similar to that for $\delta_\parallel = 0$. In accordance with Eq. (93) the characteristic thickness of the dissipative layer is $\delta_\perp$, which is proportional to $R_\perp^{-1/3}$. In Fig. 2 the real and imaginary parts of u and w are shown for $\delta_\parallel = 0$. This figure coincides with Figs. 1 and 2 in Goossens et al. (1995). Since $F(\tau)$ and $G(\tau)$ are of order 1 when $|\tau| \sim 1$ and $|\delta_\parallel| \lesssim \delta_\perp$, we obtain $u/w \sim k\delta_\perp \ll 1$, so that w is once again the dominant velocity component.

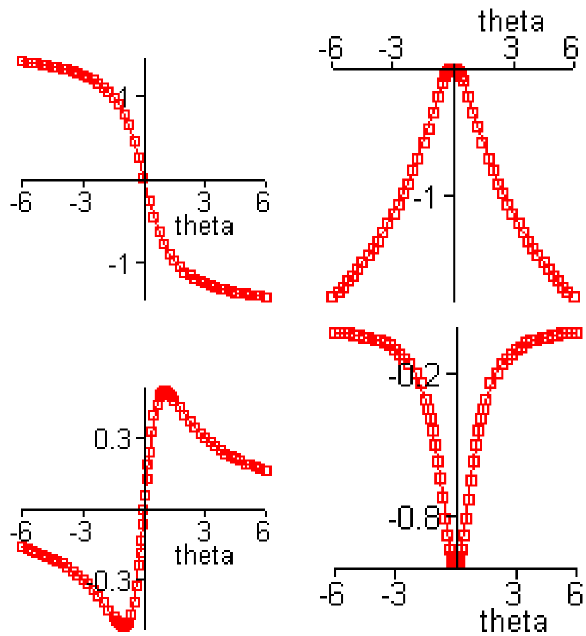

Fig. 1. Dependence of the real (left panels) and imaginary (right panels) parts of u and w on θ for $\Delta > 0$. The upper and lower panels correspond to u and w, respectively. The solid curves show the exact solution with $\delta_\parallel = 5\delta_\perp$ and the rectangles show the asymptotic approximation. Note that u and w are given in arbitrary units, however the same units are used for the real and imaginary parts of a quantity.

Now we proceed to the case where $\delta_\parallel < 0$ and $|\delta_\parallel| \gg \delta_\perp$. We use Eqs. (A.8) and (A.14) to obtain the asymptotic expressions

$$w = \frac{ik^2 c_T^3 \tilde{P}'}{\rho_0 v_A^2 |\Delta| \delta_\perp} \left\{\frac{\epsilon}{1+i\epsilon\tau\,\mathrm{sign}\Delta} - \left(\frac{\pi^2\epsilon}{1+i\epsilon\tau\,\mathrm{sign}\Delta}\right)^{1/4} \exp\left[\tfrac{2}{3}(\epsilon^{-1} + i\tau\,\mathrm{sign}\Delta)^{3/2}\right]\right\}, \tag{104}$$

$$u = -\frac{ik^3 c_T^5 \tilde{P}'}{\rho_0 v_A^4 |\Delta|} \left\{\log(1+i\epsilon\tau\,\mathrm{sign}\Delta) + (\pi^2\epsilon^3)^{1/4} \exp\left(\tfrac{2}{3}\epsilon^{-3/2}\right) - \frac{(\pi\epsilon^3)^{1/2}(1+\epsilon^2\tau^2)^{3/4}}{(1+i\epsilon\tau\,\mathrm{sign}\Delta)^{3/2}} \times \exp\left[\tfrac{2}{3}(\epsilon^{-1} + i\tau\,\mathrm{sign}\Delta)^{3/2}\right]\right\}, \tag{105}$$

We use these asymptotic expressions to obtain the estimate $u/w \sim k(\delta_\perp^3/|\delta_\parallel|)^{1/2} \ll 1$, so that once again w is the dominant component of the velocity. It is straightforward to get, with the use of the asymptotic formula (105), that at $\tau = 0$

$$\frac{w}{w_{\mathrm{out}}} \sim \frac{a}{\delta_\perp}\epsilon^{1/4} \exp\left(\tfrac{2}{3}\epsilon^{-3/2}\right), \tag{106}$$

where w_{out} is the value of w far away from the dissipative layer. Even for the very moderate value $\epsilon = \frac{1}{4}$ this formula gives $w/w_{\mathrm{out}} \sim 150\, a/\delta_\perp$, so that the velocity in the dissipative layer can reach huge values even when it is very small far away from the dissipative layer. It is instructive to compare this estimate with the similar estimate $w/w_{\mathrm{out}} \sim a/\delta_\perp$ valid for $|\delta_\parallel| \lesssim \delta_\perp$.

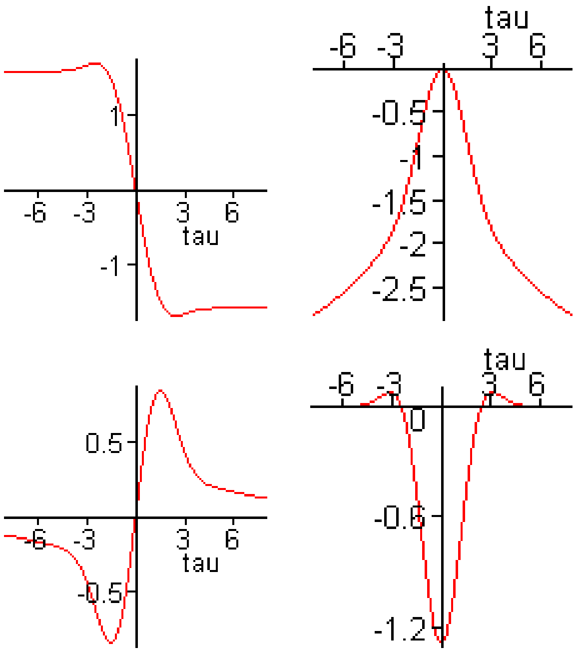

Fig. 2. Dependence of the real (left panels) and imaginary (right panels) parts of u and w on τ for $\delta_\parallel = 0$ and $\Delta > 0$. The upper and lower panels correspond to u and w, respectively. Note that u and w are given in arbitrary units, however the same units are used for the real and imaginary parts of a quantity.

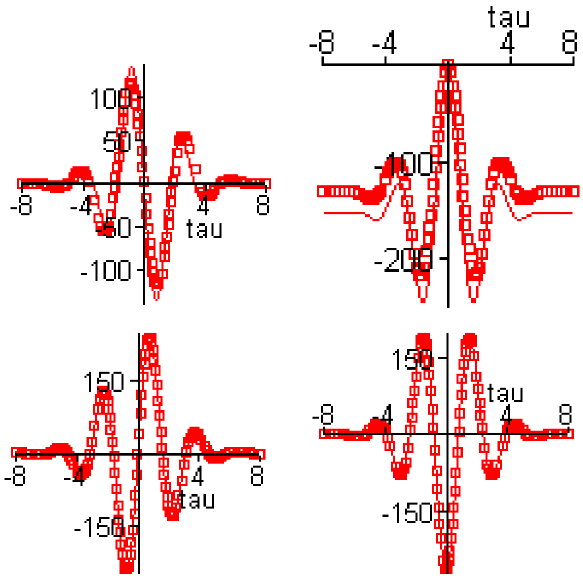

Fig. 3. Dependence of the real (left panels) and imaginary (right panels) parts of u and w on τ. The upper and lower panels correspond to u and w, respectively. The solid curves show the exact solution with $\delta_\parallel = -4\delta_\perp$ and $\Delta > 0$, and the rectangles show the asymptotic approximation. Note that u and w are given in arbitrary units, but the same units are used for the real and imaginary parts of a quantity.

The real and imaginary parts of u and w given by exact expressions (95) and (99) for $\epsilon = \frac{1}{4}$ are shown in Fig. 3 (solid lines). The rectangles show the asymptotic values given by formulae (105) and (105). We see that the asymptotic formulae give a very good approximation for w. Although the asymptotic approximation reproduces the main properties of the real and imaginary parts of u fairly well, it noticeably differs from the exact solution. If we recall how the asymptotic expressions for u were derived, this difference is not surprising at all. The asymptotic expression (A.9) is accurate enough, but the asymptotic formulae (A.11) and (A.13) give not very good approximations for $\epsilon = \frac{1}{4}$. For example, the difference between the left-hand side and the write-hand side of Eq. (A.11) is about 12% for this value of ϵ. It is worth to note that the dependences of u and w on τ have the form of wave packets with quasi-sinusoidal carrying waves. The width of the wave packets is of the order $|\delta_\parallel|$, and the wavelength is of the order $(\delta_\perp^3/|\delta_\parallel|)^{1/2} \ll \delta_\perp$, which, according to Eq. (93), is proportional to $R_\perp^{-1/2}$. Let us recall that the characteristic thickness of the dissipative layer is proportional to $R_\perp^{-1/3}$ when $|\delta_\parallel| \lesssim \delta_\perp$.

The dependence of u and w on τ for different values of ϵ can be found in Ruderman et al. (1995) and Tirry & Goossens (1996).

5.3. *Calculation of the wave damping*

In case where either $\delta_\parallel \lesssim \delta_\perp$ or $\delta_\parallel < 0$ the derivation of expression (79) for γ_d given in Sect. 4 is not valid. Here we give a different derivation valid in this case and show that expression (79) remains the same.

Dissipation is only important in the dissipative layer. Therefore we can use Eqs. (46) and (47) with all dissipative coefficients equal to zero to describe the plasma motion outside the dissipative layer. We use the connection formulae (92) and (98) to connect solutions to the left and the right of the dissipative layer.

The analysis in Sect. 3 and, in particular, Eq. (55), is always valid in the region $|x| < a$ outside the dissipative layer. Hence, $\tilde{P}'$ is constant to the left and the right to the dissipative layer. Since, in accordance with connection formula (92), $\tilde{P}'$ does not vary across the dissipative layer, this quantity is constant in the whole region $|x| < a$.

It follows from Eqs. (51), (54) and (55), and the continuity conditions at $x = \pm a$ that u in the region $|x| < a$ outside the dissipative layer is given by

$$u = \begin{cases} i\omega\tilde{P}' \int_{-a}^{x} \Upsilon(\bar{x})\, d\bar{x} + U_1 e^{-a\Gamma_1}, & x < x_c, \\ -i\omega\tilde{P}' \int_{x}^{a} \Upsilon(\bar{x})\, d\bar{x} + U_2 e^{a\Gamma_2}, & x > x_c, \end{cases} \tag{107}$$

where

$$\Upsilon(x) = \frac{D(x)}{\rho_0(x)[c_S^2(x) + v_A^2(x)]A(x)C(x)}.$$

The jump in u can be calculated as

$$[u] = \lim_{\epsilon \to +0} \{u(x_c + \epsilon) - u(x_c - \epsilon)\}.$$

Then we obtain from Eq. (107)

$$[u] = U_2 e^{a\Gamma_2} - U_1 e^{-a\Gamma_1} - i\omega\tilde{P}'\mathcal{P}\int_{-a}^{a} \frac{D\, dx}{\rho_0[c_S^2 + v_A^2]AC}. \tag{108}$$

On the other hand, $[u]$ is given by Eq. (98). Comparing Eqs. (98) and (108) and using Eq. (52) we arrive at

$$\frac{\Gamma_1}{\rho_{01}A_1}+\frac{\Gamma_2}{\rho_{02}A_2}=\mathcal{P}\int_{-a}^{a}\frac{D\,dx}{\rho_0[c_S^2+v_A^2]AC}+\frac{\pi i k^3 c_T^5}{\rho_0 v_A^4|\Delta|\omega}. \quad (109)$$

The ratio of the right-hand side of this equation to the left-hand side is of the order ak, so that we can substitute $\bar\omega$ for ω in the right-hand side when calculating ω with the use of the regular perturbation method. Then Eq. (109) coincides with the dispersion Eq. (57) with the right-hand side given by Eq. (78). This implies that the wave damping decrement γ_d is given by the same Eq. (79).

6. Wave damping in isothermal plasmas

To further simplify the analysis and give numerical examples we consider the case where the equilibrium plasma is isothermal, so that $c_S = \text{const}$. In particular, $c_{S1} = c_{S2} = c_S$. Now the first order approximation (61) of the dispersion equation is reduced to

$$K_1\bar\omega^4 + 2K_2 c_S^2\bar\omega^2 k^2 - 4K_3 c_S^4 k^4 = 0, \quad (110)$$

where

$$K_1 = 4(\gamma-1)c_S^4 + \gamma^2 c_S^2(v_{A1}^2+v_{A2}^2) + \gamma^2 v_{A1}^2 v_{A2}^2, \quad (111)$$

$$K_2 = 4c_S^4 + 2c_S^2(v_{A1}^2+v_{A2}^2) + \gamma(2-\gamma)v_{A1}^2 v_{A2}^2, \quad (112)$$

$$K_3 = c_S^2(v_{A1}^2+v_{A2}^2) + \gamma v_{A1}^2 v_{A2}^2. \quad (113)$$

When deriving Eq. (110) we have used the condition of total pressure balance (19), which can be re-written as

$$\rho_0(x)[2c_S^2+\gamma v_A^2(x)] = \text{const}. \quad (114)$$

The positive root of Eq. (110) is given by

$$\left(\frac{\bar\omega}{kc_S}\right)^2 = \frac{(K_2^2+4K_1K_3)^{1/2}-K_2}{K_1}. \quad (115)$$

It can be checked that this root satisfies the inequalities (65). When $v_{A1} = 0$ the expression (115) for $\bar\omega^2$ coincides with the corresponding expression obtained by Roberts (1981) (see his Eq. (29)).

In what follows we assume that the dissipative coefficients η_0/ρ_0 and $\chi_\parallel$ are constant. Then the expressions for γ_η and γ_χ reduce to

$$\gamma_\eta = \frac{J(v_{A1}^2-v_{A2}^2)}{R_e^{(0)}}, \quad \gamma_\chi = \frac{3\chi_\parallel J c_S^2\bar\omega^2 k^2}{P_e^\parallel}, \quad (116)$$

where

$$J = \frac{\bar\omega(v_{A1}^2-v_{A2}^2)}{6Hk^2(c_S^2+v_{A1}^2)(c_S^2+v_{A2}^2)(\bar\omega^2-c_{T1}^2k^2)(\bar\omega^2-c_{T2}^2k^2)}.$$

Let us assume that v_A^2 is a linear function of x. Then it is straightforward to calculate x_c, Δ, v_{Ac} and ρ_{01}/ρ_{0c}, and obtain the expression for γ_r

$$\gamma_r = \frac{2\pi a c_S^2[2c_S^2k^2-(2-\gamma)\bar\omega^2]}{H\bar\omega(v_{A2}^2-v_{A1}^2)(2c_S^2+\gamma v_{A1}^2)} \times \left[\frac{(c_S^2+v_{A1}^2)(\bar\omega^2-c_{T1}^2k^2)(\bar\omega^2-v_{A1}^2k^2)}{(c_S^2k^2-\bar\omega^2)^3}\right]^{1/2}. \quad (117)$$

For our numerical examples we take $P_e^\parallel \ll R_e^{(0)}$ and $v_{A2} = 2c_S$, which is realistic for the upper chromosphere, and $\gamma = 5/3$. We follow Roberts (1981) and Čadež & Ballester (1996) and assume that the dynamical pressure at one side of the inhomogeneous layer is much larger than the magnetic pressure, so that we can take $v_{A1} = 0$ (nevertheless, the magnetic field is assumed to be strong enough to cause anisotropy of viscosity and thermal conductivity). Then $\bar\omega = 0.717\,kc_S$ and

$$P_e^\parallel = 2.06\,R_\parallel, \quad \frac{\gamma_r}{\gamma_\eta+\gamma_\chi} = 1.6\,akR_\parallel. \quad (118)$$

Using the same estimate as in Sect. 4.3, $R_\parallel \sim 10^4k^{-1}$, and taking $k \lesssim 1\ \text{km}^{-1}$, we obtain $\gamma_r \gg \gamma_\eta+\gamma_\chi$ for $ak \lesssim 10^{-3}$, in complete agreement with the analysis in Sect. 4.3. For the damping decrement γ_d we have

$$\gamma_d \approx \gamma_r \approx 0.48\,(ak)c_Sk = 0.67\,(ak)\bar\omega, \quad (119)$$

so the wave damping is independent of the dissipative coefficients and completely determined by the parameter ak.

Let us now place our results in the context of the solar chromosphere. Recent observations obtained during the SOHO mission clearly show that small amplitude oscillations with periods of a few minutes exist in the solar chromosphere (Carlson et al. 1997; Curdt & Heinzel 1998; Doyle et al. 1999; Gallagher et al. 1999; Judge et al. 1997). MDI observations (Schrijver et al. 1997) indicate that the chromospheric network magnetic field is constantly evolving, which, in particular, leads to the creation of tangential discontinuities in the overlying field. The presence of tangential discontinuities gives the possibility that at least some of oscillations, observed in the chromosphere, are surface MHD waves. Now it is almost commonly accepted that the lower internetwork chromosphere is heated by dissipation of non-magnetic acoustic shocks (Carlson et al. 1997; Carlson & Stein 1997; Doyle et al. 1999). However in the network the heating required must be in excess of that provided by acoustic shocks (Gallagher et al. 1999). Damping of surface MHD waves can be considered as an additional source of heating.

To give a numerical example we consider the chromosphere as an isothermal fully ionised plasma with a temperature $T_0 \approx 10^4$ K, and a corresponding density scale hight 6×10^5 m and sound speed $c_S \approx 10^4$ m/s. In our analysis we have used the local approximation and assumed that the magnetic interface is planar and the equilibrium quantities are constant in the outer regions. This local approximation is only valid when the wave length $2\pi/k$ is much smaller than the density scale hight. This condition can be written as $k \gg 10^{-5}\ \text{m}^{-1}$, which corresponds to waves with periods smaller than 60 s. However, we do not expect that the account of stratification would drastically change our analysis. In addition, for horizontally propagating surface waves, e.g. waves on the magnetic canopy, the criterion of applicability of the local approximation can be reduced. The point is that the wave amplitude exponentially decreases with the distance from the surface on the scale of the order k^{-1}, which is about 6 times smaller than the wave length. As a result we obtain the restriction $k \gg 2 \times 10^{-6}\,\mathrm{m}^{-1}$, which corresponds to wave with periods smaller than 300 s. This discussion enables us to consider waves with the period equal to 3 min., which is one of the most pronounced periods in the observations.

So, let us consider waves with the period 3 min., which corresponds to $k = 3.5 \times 10^{-6}\ \mathrm{m}^{-1}$ and the wavelength approximately $1800\ \mathrm{km}$. We take $ak = 0.1$, so that $a \approx 30\ \mathrm{km}$. In accordance with Eq. (119) we obtain for the characteristic damping time of surface waves $\gamma_d^{-1} \approx 7$ min. We see that the resonant absorption provides very efficient wave damping in the chromosphere.

7. Conclusions

In the present paper we have considered slow surface wave damping in plasmas with strongly anisotropic viscosity, thermal conductivity, and resistivity, as in the upper part of the solar chromosphere and in the solar corona. Far away from the dissipative layer we neglected electrical resistivity and we only retained the first term in the Braginskii's expression for the viscosity tensor and for the heat flux. These are terms that describe the compressional viscosity and the parallel thermal conductivity. For typical coronal conditions the retained terms are much larger than other terms in the Braginskii's expressions. However, when studying the motion in the dissipative layer, we used the full Braginskii's expressions for viscosity and the heat flux and took electrical resistivity into account.

The analysis has been restricted to waves propagating along the equilibrium magnetic field. This condition removes the ideal Alfvén singularity which is otherwise present in the region where the equilibrium quantities are inhomogeneous. Slow surface waves cannot propagate along the magnetic field in a low-beta plasma, so our analysis is only applicable to the upper chromosphere. The dispersion equation has been derived in the long-wavelength approximation and the decrement of the wave damping has been calculated. This decrement has been written as a sum of three terms. The first term is due to viscosity in the two outer regions where the equilibrium quantities are constant. The second term is due to thermal conductivity in the two outer regions. The third term is due to resonant absorption related to the ideal slow resonant position that is present in the layer with inhomogeneous equilibrium quantities. The qualitative argumentation shows that resonant absorption dominates over the viscous and thermal conductive damping when $akR_\parallel \gg 1$, while the viscous and thermal conductive damping dominates over resonant absorption when $akR_\parallel \ll 1$. We recall that a is the thickness of the inhomogeneous layer, k is the wavenumber, and $R_\parallel$ is the parallel total Reynolds number characterising dissipation due to the compressional viscosity and parallel thermal conductivity. Estimates show that for typical conditions in the chromosphere $akR_\parallel \gg 1$, so the surface wave damping is mainly due to resonant absorption.

The behaviour of the velocity in the slow dissipative layer has been studied. It has been shown that this behaviour is determined by the compressional viscosity and parallel thermal conductivity when these two dissipative processes are strong enough. The characteristic spatial scale of the velocity variation is of the order of the thickness of the dissipative layer, which is proportional to $R_\parallel^{-1}$. In case where the compressional viscosity and parallel thermal conductivity are relatively weak the velocity behaviour is determined by the shear velocity, perpendicular thermal conductivity, and resistivity. When these three dissipative processes are strong enough the characteristic scale of the velocity variation is of the order of the thickness of the dissipative layer, which is proportional to $R_\perp^{-1/3}$. Here $R_\perp$ is the perpendicular Reynolds number characterising dissipation due to the shear viscosity, perpendicular thermal conductivity, and resistivity. When these three dissipative processes are weak the dependence of the velocity on the spatial coordinate normal to the dissipative layer takes the form of a wavepacket. The width of the wavepacket is proportional to the wave damping decrement γ_d, and the wavelength of the carrying wave to $R_\perp^{-1/2}$.

The example for isothermal equilibrium state has been considered. The equilibrium state was taken to be non-magnetic at one side of the inhomogeneous region. This example confirms the conclusions obtained on the basis of the qualitative consideration of the general expression for the increment of the wave damping that, for typical chromospheric conditions, the wave damping is mainly due to resonant absorption. For equilibrium quantities typical for the solar chromosphere and for the waves with the period 3 min. the characteristic damping time is 7 min. Hence, resonant absorption provides very efficient wave damping in the solar chromosphere.

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Appendix A: asymptotic behaviour of $F(\tau)$ and $G(\tau)$

In this Appendix we study the asymptotic behaviour of the functions $F(\tau)$ and $G(\tau)$ for two cases: $\delta_\parallel \gg \delta_\perp$ and $-\delta_\parallel \gg \delta_\perp$. We start with the case where $\delta_\parallel \gg \delta_\perp$ and introduce the small parameter $\epsilon = \delta_\perp/\delta_\parallel$. The second term in the exponent in expression (100) for $F(\tau)$ dominates over the third term when $r \ll \epsilon^{-1/2}$. Let us take any large quantity that is much smaller than $\epsilon^{-1/2}$, for example, $\epsilon^{-1/3}$. Then we have the asymptotic expression

$$F(\tau) \simeq \int_0^{\epsilon^{-1/3}} \exp(i\tau r\,\mathrm{sign}\Delta - \epsilon^{-1} r)\,dr + \int_{\epsilon^{-1/3}}^{\infty} \exp\left(i\tau r\,\mathrm{sign}\Delta - \epsilon^{-1} r - \tfrac{1}{3} r^3\right) dr. \quad \text{(A.1)}$$

It is straightforward to show that the second integral on the right-hand side of this expression is exponentially small (it is smaller than $\exp(-\epsilon^{-4/3})$). We can substitute ∞ for the large upper limit $\epsilon^{-1/3}$ in the first integral. As a result we obtain

$$F(\tau) \simeq \frac{\epsilon}{1 - i\epsilon\tau\,\mathrm{sign}\Delta}. \tag{A.2}$$

To obtain the asymptotic expression for the function $G(\tau)$ we can either use the same approach as for $F(\tau)$, or simply note that $dG/d\tau = iF(\tau)\,\mathrm{sign}\Delta$ and use expression (A.2). In both ways we arrive at

$$G(\tau) \simeq -\log(1 - i\epsilon\tau\,\mathrm{sign}\Delta), \tag{A.3}$$

where the fact that $G(0) = 0$ was used.

By using partial integration we can show that $F(\tau) \simeq i\tau^{-1}\,\mathrm{sign}\Delta$ as $|\tau| \to \infty$. Integrating this asymptotic relation we obtain $G(\tau) \simeq -\log|\tau|$ as $|\tau| \to \infty$. It is interesting to note that, although asymptotic formulae (A.2) and (A.3) are derived under the assumption $|\tau| \ll \epsilon^{-1}$, they actually give the correct asymptotic behaviour for $F(\tau)$ and $G(\tau)$ as $|\tau| \to \infty$. Moreover, Eq. (A.3) correctly reproduces the jump of $G(\tau)$ across the dissipative layer (see Sect. 5.1).

Let us now obtain the asymptotic expression for $F(\tau)$ in the case $\delta_\| < 0$ and $|\delta_\|| \gg \delta_\perp$. We introduce the small parameter as $\epsilon = -\delta_\perp/\delta_\|$ and rewrite expression (100) as

$$F(\tau) = \epsilon^{-1/2}\int_0^\infty \exp[\epsilon^{-3/2}h(r)]\,dr \tag{A.4}$$

with $h(r) = (i\epsilon\tau\,\mathrm{sign} + 1)r - \frac{1}{3}r^3$. In what follows we assume that $\epsilon|\tau| \lesssim 1$.

To obtain the asymptotics of $F(\tau)$ for $\epsilon \to +0$ we use the method of steepest descent (see, e.g., Nayfeh 1981; Bender & Ország 1987). Since the change of sign of either τ or Δ causes the substitution of $F(\tau)$ by its complex conjugate, we assume in what follows that $\tau > 0$ and $\Delta > 0$. The function $h(r)$ has the stationary point r_0 in the complex r-plane given by $r_0 = (1+i\epsilon\tau)^{1/2}$, where we take a branch of the square root satisfying the condition $\Re(r_0) > 0$. In Fig. A.1, the left panel, we show the contours of steepest descent that pass through the points $r = r_0$ and $r = 0$. Since the integrand in the expression (A.4) is an analytical function of r, this expression can be rewritten as

$$\epsilon^{1/2}F(\tau) = I_1 + I_2 \equiv \left(\int_{\gamma_1} + \int_{\gamma_2}\right)\exp[\epsilon^{-3/2}h(r)]\,dr, \tag{A.5}$$

where the integration along the contours of the steepest descent γ_1 and γ_2 is carried out in the direction shown by arrows near these contours in Fig. A.1.

According to the method of the steepest descent the asymptotics of the first integral in Eq. (A.5) is given by the contribution of the infinitesimal vicinity of $r = 0$, and the asymptotics of the second integral by the infinitesimal vicinity of $r = r_0$. The calculation of these contributions is straightforward and yields the asymptotic expressions

$$I_1 \simeq -\epsilon^{3/2}(1 + i\epsilon\tau)^{-1}. \tag{A.6}$$

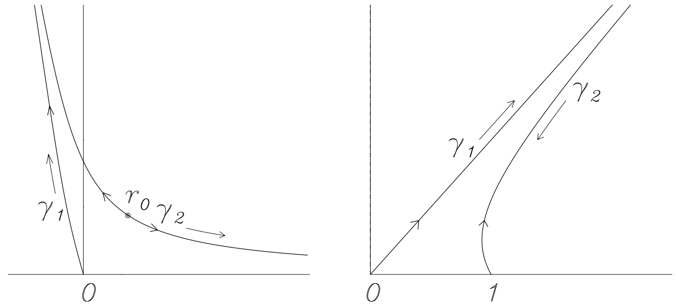

Fig. A.1. Contours of the steepest descent γ_1 and γ_2 in the complex r-plane. The arrows on the contours show the directions of the steepest descent, while arrows near the contours show the direction of integration.

$$I_2 \simeq \left(\frac{\pi^2\epsilon^3}{1 + i\epsilon\tau}\right)^{1/4}\exp\left[\tfrac{2}{3}\epsilon^{-3/2}(1 + i\epsilon\tau)^{3/2}\right]. \tag{A.7}$$

Recalling the dependence of $F(\tau)$ on the signs of τ and Δ we finally arrive at

$$F(\tau) \simeq -\frac{\epsilon}{1 + \epsilon\tau\,\mathrm{sign}\Delta} + \left(\frac{\pi^2\epsilon}{1 + i\epsilon\tau\,\mathrm{sign}\Delta}\right)^{1/4}\exp\left[\tfrac{2}{3}\epsilon^{-3/2}(1 + i\epsilon\tau\,\mathrm{sign}\Delta)^{3/2}\right]. \tag{A.8}$$

It is instructive to compare the first and the second term on the right-hand side of Eq. (A.8). When $|\epsilon\tau| < 3^{1/2}$, $\Re(1 + i\epsilon\tau\,\mathrm{sign}\Delta)^{3/2} > 0$, the first term is exponentially small in comparison with the second term, i.e. the first term is subdominant, and the asymptotic behaviour of $F(\tau)$ is given by the second term. On the other hand, when $|\epsilon\tau| > 3^{1/2}$, $\Re(1 + i\epsilon\tau\,\mathrm{sign}\Delta)^{3/2} < 0$, the second term is subdominant, and the asymptotic behaviour of $F(\tau)$ is given by the first term. This exchange of identities of the dominant and subdominant terms is called the Stokes phenomenon (see, e.g., Bender & Ország 1987).

The asymptotic formula (A.8) has been derived under the assumption that $|\epsilon\tau| \lesssim 1$. However, it is interesting to note that it correctly reproduces the behaviour of the function $F(\tau)$ as $|\tau| \to \infty$.

To obtain the asymptotic expression for $G(\tau)$ we once again use the relations $dG/d\tau = iF(\tau)\,\mathrm{sign}\Delta$ and $G(0) = 0$ and, as a result, arrive at

$$G(\tau) \simeq -\log(1 + i\epsilon\tau\,\mathrm{sign}\Delta) + i\,\mathrm{sign}\Delta\int_0^\tau \left(\frac{\pi^2\epsilon}{1 + i\epsilon\tilde{\tau}\,\mathrm{sign}\Delta}\right)^{1/4} \times \exp\left[\tfrac{2}{3}(\epsilon^{-1} + i\tilde{\tau}\,\mathrm{sign}\Delta)^{3/2}\right]d\tilde{\tau}. \tag{A.9}$$

With the use of obvious transforms of the integration variable, in a few steps we arrive at

$$G(\tau) \simeq -\log(1 + i\epsilon\tau\,\mathrm{sign}\Delta) - \tfrac{4}{3}\pi^{1/2}\epsilon^{-3/4} \times \left[\int_0^1 \exp\left(\tfrac{2}{3}\epsilon^{-3/2}z^2\right)dz - r_0^{3/2}\int_0^1 \exp\left(\tfrac{2}{3}\epsilon^{-3/2}r_0^3 z^2\right)dz\right], \tag{A.10}$$

where now $r_0 = (1 + i\epsilon\tau\,\mathrm{sign}\Delta)^{1/2}$. Using integration by parts we immediately obtain

$$\int_0^1 \exp\left(\tfrac{2}{3}\epsilon^{-3/2}z^2\right)dz \simeq \tfrac{3}{4}\epsilon^{3/2}\exp\left(\tfrac{2}{3}\epsilon^{-3/2}\right). \qquad \text{(A.11)}$$

To calculate the asymptotics of the second integral on the right-hand side of Eq. (A.11) we once again use the method of the steepest descent. In Fig. A.1, the right panel, the contours of the steepest descent γ_1 and γ_2 passing through $z = 0$ and $z = 1$ respectively in the complex z-plane are shown. The second integral in the square brackets in Eq. (A.11) can be written as a sum of two integrals along the contours γ_1 and γ_2, where the integration is carried out in the direction shown by the arrows near γ_1 and γ_2. The asymptotic behaviour of the integral along γ_1 is determined by an infinitesimal vicinity of $z = 0$, and the integral along γ_2 by the infinitesimal vicinity of $z = 1$. The straightforward calculation gives

$$\int_{\gamma_1} \exp\left(\tfrac{2}{3}\epsilon^{-3/2}r_0^3z^2\right)dz \simeq i\epsilon^{3/4}(3\pi)^{1/2}\left(\frac{r_0^*}{2|r_0|^2}\right)^{3/2}, \qquad \text{(A.12)}$$

$$\int_{\gamma_2} \exp\left(\tfrac{2}{3}\epsilon^{-3/2}r_0^3z^2\right)dz \simeq \frac{3\epsilon^{3/4}}{4}\left(\frac{r_0^*}{|r_0|}\right)^3 \exp\left(\tfrac{2}{3}\epsilon^{-3/2}r_0^3\right). \qquad \text{(A.13)}$$

With the aid of Eqs. (A.11)–(A.13) we eventually arrive at

$$G(\tau) \simeq -\log(1 + i\epsilon\tau\,\mathrm{sign}\Delta) - (\pi^2\epsilon^3)^{1/4}\exp\left(\tfrac{2}{3}\epsilon^{-3/2}\right) + \frac{(\pi^2\epsilon^3)^{1/4}(1+\epsilon^2\tau^2)^{3/4}}{(1 + i\epsilon\tau\,\mathrm{sign}\Delta)^{3/2}}\exp\left[\tfrac{2}{3}(\epsilon^{-1} + i\tau\,\mathrm{sign}\Delta)^{3/2}\right]. \qquad \text{(A.14)}$$

Once again the first term on the right-hand side of Eq. (A.14) is subdominant and can be neglected in comparison with the third term when $|\epsilon\tau| < 3^{1/2}$. When $|\epsilon\tau| > 3^{1/2}$ the third term is subdominant and can be neglected. We do not compare the first and the third term with the second one because only the first and the third term determine the dependence on τ.

Once again, in spite that the asymptotic formula (A.14) is derived under the assumption that $|\epsilon\tau| \lesssim 1$, it correctly reproduces the behaviour of the function $G(\tau)$ in the main order approximation, $G(\tau) \simeq -\log|\tau|$, as $|\tau| \to \infty$. However, it gives a wrong sign for the quantity $[G]$.

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