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# Liouville's equation in post Newtonian approximation

## II. The post Newtonian modes

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**Abstract.** We use the post-Newtonian ($pn$) order of Liouville's equation to study the normal modes of oscillation of a spherically symmetric relativistic system. Perturbations that are neutral in Newtonian approximation develop into a new sequence of normal modes. In the first $pn$ order; a) their frequency is an order $q$ smaller than the classical frequencies, where $q$ is a $pn$ expansion parameter; b) they are not damped, for there is no gravitational wave radiation in this order; c) they are not coupled with the classical modes in $q$ order; d) because of the spherical symmetry of the underlying equilibrium configuration, they are designated by a pair of angular momentum eigennumbers, $(j, m)$, of a pair of phase space angular momentum operators $(J^2, J_z)$. The eigenfrequencies are, however, $m$-independent. Hydrodynamics of these modes is also investigated; a) they generate oscillating macroscopic toroidal motions that are neutral in the classical case; and b) they give rise to an oscillatory $g_{0i}$ component of the metric tensor that otherwise is zero in the unperturbed system. The conventional classical modes, which in their hydrodynamic behaviour emerge as $p$ and $g$ modes are, of course, perturbed to order $q$. These, however, have not been of concern in this paper.

**Key words:** methods: numerical – stars: general – stars: oscillations

### 1. Introduction

Chandrasekhar's (1965a, b) formulation of post-Newtonian hydrodynamics is among the pioneering ones. He generalized Eulerian equations of Newtonian hydrodynamics to $pn$ order consistent with Einstein's field equations, and applied them to obtain the $pn$ corrections to the equilibrium and stability of uniformly rotating homogeneous masses. Blanchet, Damour and Schäfer (1990) studied the gravitational wave generation of a self gravitating fluid by adding an appropriate term to $pn$ equation of hydrodynamics. Cutler (1991) employed the $pn$ hydrodynamics and a perturbation technique to derive an expression for the $pn$ correction to Newtonian eigenfrequencies. Cutler & Lindblom (1992) adopted Cutler's method to calculate numerically the oscillation frequencies of the $l = m$ $f$-modes of rapidly rotating polytropic neutron stars.

In this paper we study normal modes of a non-rotating relativistic stars in $pn$ approximation through the relativistic Liouville's equation rather than the relativistic hydrodynamics. The reason for doing so is to avoid thermodynamic concepts being incorporated into hydrodynamics. Liouville's equation is a purely dynamical theory and free from such complexities. One, of course, pays the price by having to dwell in the six dimensional phase space, an elaborate mathematical task, but not obtrusive. In compiling this work we have relied heavily on the following studies dealing with various aspects of Liouville's, Liouville-Poisson's and Antonov's equations.

O(3) symmetry and mode classification of classical Liouville's equation for spherically symmetric potentials was studied by Sobouti (1989a,b). GL(3, c) symmetry, its subgroups, and eigenmodes of $r^2$ potential and O(4) symmetry of $r^{-1}$ potential were obtained by Sobouti (1989a,b), Sobouti & Dehghani (1992) and Dehghni & Sobouti (1993). Dynamical symmetry of Liouville's equation for $r^2$ potential was worked out by Dehghani & Sobouti (1995). Dynamical symmetry group of general relativistic Liouville's equation was discussed by Dehghani & Rezania (1996). In particular they found that in de Sitter's space-time the group is SO(4,1) $\bigotimes$ SO(4,1).

In applications to self-gravitating systems the pioneering work was done by Antonov (1962). He reduced the linearized Liouville-Poisson equations to a self adjoint operation in phase space. Further elaborations on Antonov's equation were made by Lynden-Bell (1966), Milder (1967), Lynden-Bell & Sanitt (1969), Ipser & Thorne (1968). Attempts to solve the linearized Liouville-Poisson equation for eigenfrequencies and eigenmodes of oscillations were made by Sobouti (1984, 1985, 1986). Further and more transparent exposition of mode classification and mode calculations were given by Sobouti & Samimi (1989) and Samimi & Sobouti (1995).

In Sect. 2 we give the $pn$ order of the linearized Liouville equation that governs the evolution of small perturbations from an equilibrium state. In Sects. 3 and 4 we extract the equation for a sequence of new modes that are generated solely by $pn$ force but are absent in a classical regime. In Sect. 5 we explore the O(3) symmetry of the modes and classify them on the basis of this symmetry. In Sect. 6 we study hydrodynamics of these modes. In Sect. 7 we seek a variational approach to the calculation of $pn$ modes and give numerical values for polytropes. Sect. 8 is devoted to concluding remarks.

## 2. Formulation of the problem

Liouville's equation in the post-Newtonian approximation ($pnl$) for the one particle distribution of a gas of collisionless particles maybe written as

$$\left(-i\frac{\partial}{\partial t}+\mathcal{L}\right)F(\mathbf{x},\mathbf{u},t) = \left(-i\frac{\partial}{\partial t}+\mathcal{L}^{cl}+q\mathcal{L}^{pn}\right)F(\mathbf{x},\mathbf{u},t)=0, \quad (1)$$

where $(\mathbf{x},\mathbf{u})$ are phase space coordinates, $q$ is a small post-Newtonian expansion parameter, the ratio of Schwarzschild radius to a typical spatial dimension of the system, for example. The classical and post-Newtonian operators, $\mathcal{L}^{cl}$ and $\mathcal{L}^{pn}$, respectively, are

$$\mathcal{L}^{cl} = -i(u^i\frac{\partial}{\partial x^i}+\frac{\partial\theta}{\partial x^i}\frac{\partial}{\partial u^i}), \quad (2a)$$

$$\mathcal{L}^{pn} = -i\left[(\mathbf{u}^2-4\theta)\frac{\partial\theta}{\partial x^i}-u^iu^j\frac{\partial\theta}{\partial x^j}-u^i\frac{\partial\theta}{\partial t}+\frac{\partial\Theta}{\partial x^i} + u^j(\frac{\partial\eta_i}{\partial x^j}-\frac{\partial\eta_j}{\partial x^i})+\frac{\partial\eta_i}{\partial t}\right]\frac{\partial}{\partial u^i}. \quad (2b)$$

The imaginary factor $i$ is included for later convenience. The potentials $\theta(\mathbf{x},t)$, $\Theta(\mathbf{x},t)$ and $\boldsymbol{\eta}(\mathbf{x},t)$, solutions of Einstein's equations in $pn$ approximation, are

$$\theta(\mathbf{x},t) = \int\frac{F(\mathbf{x}',t,\mathbf{u}')}{|\mathbf{x}-\mathbf{x}'|}d\Gamma', \quad (3a)$$

$$\boldsymbol{\eta}(\mathbf{x},t) = 4\int\frac{\mathbf{u}'F(\mathbf{x}',t,\mathbf{u}')}{|\mathbf{x}-\mathbf{x}'|}d\Gamma', \quad (3b)$$

$$\Theta(\mathbf{x},t) = -\frac{1}{4\pi}\int\frac{\partial^2F(\mathbf{x}'',t,\mathbf{u}'')/\partial t^2}{|\mathbf{x}-\mathbf{x}'||\mathbf{x}'-\mathbf{x}''|}d^3x'd\Gamma'' +2\int\frac{\mathbf{u}'^2F(\mathbf{x}',t,\mathbf{u}')}{|\mathbf{x}-\mathbf{x}'|}d\Gamma' -2\int\frac{F(\mathbf{x}',t,\mathbf{u}')F(\mathbf{x}'',t,\mathbf{u}'')}{|\mathbf{x}-\mathbf{x}'||\mathbf{x}'-\mathbf{x}''|}d\Gamma'd\Gamma'', \quad (3c)$$

where $d\Gamma=d^3xd^3u$. See Rezania & Sobouti (2000, hereafter paper I) for details. In an equilibrium state, $F(\mathbf{x},\mathbf{u})$ is time-independent. If, further, it is isotropic in $\mathbf{u}$, macroscopic velocities along with the vector potential $\boldsymbol{\eta}$ vanish. It is also shown in paper I that the following generalizations of the classical energy and classical angular momentum are integrals of $pnl$:

$$e = e^{cl}+qe^{pn}=\frac{1}{2}u^2-\theta+q(2\theta^2-\Theta), \quad (4a)$$

$$l_i = \varepsilon_{ijk}x^ju^k exp(q\theta)\approx l_i^{cl}(1+q\theta),$$
for spherically symmetric $\theta(r)$ and $\Theta(r)$. (4b)

Equilibrium distribution functions in $pn$ approximation can be constructed as appropriate functions of these integrals. In paper I the $pn$ models of polytrope were studied in this spirit.

Here we are interested in the time evolution of small deviations from a static solution. Let $F\to\ F(e)+\delta F(\mathbf{x},\mathbf{u},t)$, $|\ \delta F\ |\ll F,\quad \forall\ \mathbf{x},\mathbf{u},t$. Accordingly, the potentials split into large and small components, $\theta(r)+\delta\theta(\mathbf{x},t),\ \ \Theta(r)+\delta\Theta(\mathbf{x},t)$ and $\delta\boldsymbol{\eta}(\mathbf{x},t)$ where $r=|\mathbf{x}|$. Both the large and small components, can be read out from Eqs. (3). Substituting this splitting in Eq. (1) and keeping terms linear in $\delta F$ gives

$$i\frac{\partial}{\partial t}\delta F=\mathcal{L}\delta F+\delta\mathcal{L}F(e), \quad (5)$$

where $\mathcal{L}$ is now calculated from Eqs. (2) with $\theta(r),\quad\Theta(r)$ and $\boldsymbol{\eta}=0$. Thus

$$\mathcal{L} = \mathcal{L}^{cl}+q\mathcal{L}^{pn}, \quad (6a)$$

$$\mathcal{L}^{cl} = -i\left(u^i\frac{\partial}{\partial x^i}+\frac{\theta'}{r}x^i\frac{\partial}{\partial u^i}\right) \qquad \theta'=d\theta/dr, \quad (6b)$$

$$\mathcal{L}^{pn} = -\frac{i}{r}\left\{[(u^2-4\theta)\theta'+\Theta']x^i-\theta'(\mathbf{x}\cdot\mathbf{u})u^i\right\}\frac{\partial}{\partial u^i}. \quad (6c)$$

For $\delta\mathcal{L}$ Eqs. (2), similarly, give

$$\delta\mathcal{L} = \delta\mathcal{L}^{cl}+q\delta\mathcal{L}^{pn}, \quad (7a)$$

$$\delta\mathcal{L}^{cl}F(e) = -iF_eu^i\frac{\partial\delta\theta}{\partial x^i} \qquad F_e=dF/de, \quad (7b)$$

$$\delta\mathcal{L}^{pn}F(e) = -iF_e\left[u^i\frac{\partial}{\partial x^i}(\delta\Theta-4\theta\delta\theta) -u^2\frac{\partial\delta\theta}{\partial t}+u^i\frac{\partial\delta\eta_i}{\partial t}\right]. \quad (7c)$$

Eqs. (5)-(7) are the generalizations of the linearized classical Liouville-Poisson equations to $pn$ order. The classical case was studied briefly by Antonov (1962). He separated $\delta F$ into even and odd components in $\mathbf{u}$ and extracted an eigenvalue equation for $\delta F_{odd}$. Sobouti (1984, 1985, 1986, 1989a,b) elaborated on this eigenvalue problem, studied some of its symmetries and approaches to its solution. Sobouti & Samimi (1989), and Samimi & Sobouti (1995) showed that Antonov's equation has an O(3) symmetry and its oscillation modes can be classified by a pair of eigennumbers $(j,m)$ of a pair phase space angular momentum operators $(J^2,J_z)$. In analysing Eqs. (5)-(7) we have heavily relied on these studies.

## 3. The Hilbert space

Let $\mathcal{H}$ be the space of complex square integrable functions of phase coordinates $(\mathbf{x},\mathbf{u})$ that vanish at the phase space boundary of the system:

$$\mathcal{H}: f(\mathbf{x},\mathbf{u});\int f^*f\sqrt{-g}d\Gamma=\text{finite},\ f(\text{boundary})=0, \quad (8)$$

where $\sqrt{-g}=1+2q\theta$ in $pn$ order. Integrations in $\mathcal{H}$ are over the volume of the phase space available to the system. In particular the boundedness of the system sets the upper limit of $u$ at the escape velocity $\sqrt{2\theta}$, where $= \theta(\mathbf{x})$ is the gravitational potential at $\mathbf{x}$. Thus, $f(\mathbf{x}, \sqrt{2\theta(\mathbf{x})}) = 0$.

*Theorem* : $\mathcal{L} = \mathcal{L}^{cl} + q\mathcal{L}^{pn}$ of Eqs. (6) is Hermitian in $\mathcal{H}$,

$$\int g^*(\mathcal{L}f)\ (1+2q\theta)d\Gamma$$
$$= \int (\mathcal{L}g)^* f\ (1+2q\theta)d\Gamma; \qquad g, f \in \mathcal{H} \tag{9}$$

The proof is a matter of substitution of Eqs. (6) in (9), carrying out some integrations by parts over the $\mathbf{x}$ and $\mathbf{u}$ coordinates and letting the integrated parts vanish on the phase space boundary.

The term $\delta\mathcal{L}$ is not, in general, Hermitian. Nonetheless, one may proceed as Antonov did with the classical case and obtain a second order differential operator (almost square of $\mathcal{L} + \delta\mathcal{L}$) in some subspace of $\mathcal{H}$. We are, however, pursuing a much simpler problem here in which $\delta\mathcal{L}$ term vanishes identically leaving Eq. (5) as an eigenvalue problem governed with the Hermitian operator $\mathcal{L}$ alone.

## 4. The post-Newtonian modes of oscillations

The effect of $pn$ corrections on the classical solutions of Eq. (5) can be analyzed by the usual perturbation techniques. Whatever the procedure, the first order corrections on the known modes will be small and will not change their nature. We will not pursue such issues here. The main interest of this paper is to study a new class of solutions of Eq. (5) that originate solely from the $pn$ terms and have no precedence in classical theories. It is not difficult to anticipate the existence of such modes. Perturbations on an equilibrium state, that are functions of classical integrals (energy and angular momentum, say) do not disturb the equilibrium of the system at classical level. That is they do not induce restoring forces in the system. They, however, do so in the $pn$ regime, and make the system oscillate about the $pn$ equilibrium state. Such perturbations may be considered as a class of infinitely degenerate zero frequency modes of the classical system. The $pn$ forces unfold this degeneracy and turn them into a sequence of non zero frequency modes distinct and uncoupled from the other classical modes. We have termed them as $pn$ modes.

A hydrodynamic interpretation of $pn$ modes is the following. In spherically symmetric fluids, toroidal motions are neutral. Sliding one spherical shell of fluid over the other is not opposed by a restoring force. The $pn$ forces or for that matter a small magnetic field or a slow rotation (mainly through Coriolis forces) gives rigidity to the system. The fluid resists against such displacements and a sequence of well defined toroidal modes of oscillation develop. See Sobouti (1980), Hasan & Sobouti (1987), Nasiri & Sobouti (1989), and Nasiri (1992) for examples and typical calculations in the case weak magnetic fields and slow rotations.

In the Fourier time transform of Eq. (5),

$$\mathcal{L}\delta F + \delta\mathcal{L}F(e) = \omega\delta F, \tag{10a}$$

we split $\delta F$ into even and odd terms in $\mathbf{u}$. Thus,

$$\delta F(\mathbf{x},\mathbf{u})\ =\ G_-(\mathbf{x},\mathbf{u}) + G_+(\mathbf{x},\mathbf{u}),$$
$$G_\pm(\mathbf{x},\mathbf{u})\ =\ \pm G_\pm(\mathbf{x},\pm\mathbf{u}). \tag{10b}$$

Considering the fact that both $\mathcal{L}$ and $\delta\mathcal{L}$ are odd in $\mathbf{u}$, Eq. (10a) splits accordingly:

$$\mathcal{L}G_-\ +\ q\omega F_e u^2\delta\theta = \omega G_+, \tag{11a}$$

$$\mathcal{L}G_+\ -\ iF_e u^i\frac{\partial}{\partial x^i}\left[\delta\theta + q(\delta\Theta - 4\theta\delta\theta)\right]$$
$$-\ q\omega F_e u^i\delta\eta_i = \omega G_-, \tag{11b}$$

where

$$\delta\theta\ =\ \int \frac{G_+(\mathbf{x}',\mathbf{u}')}{|\mathbf{x}-\mathbf{x}'|}d\Gamma', \tag{12b}$$

$$\boldsymbol{\eta}\ =\ 4\int \frac{\mathbf{u}' G_-(\mathbf{x}',\mathbf{u}')}{|\mathbf{x}-\mathbf{x}'|}d\Gamma', \tag{12b}$$

$$\Theta(\mathbf{x},t) = \frac{\omega^2}{4\pi}\int \frac{G_+(\mathbf{x}'',\mathbf{u}'')}{|\mathbf{x}-\mathbf{x}'||\mathbf{x}'-\mathbf{x}''|}d^3x'd\Gamma''$$
$$+\,2\int \frac{{u'}^2 G_+(\mathbf{x}',\mathbf{u}')}{|\mathbf{x}-\mathbf{x}'|}d\Gamma' \tag{12b}$$
$$-\,2\int \frac{G_+(\mathbf{x}',\mathbf{u}')F(e'') + F(e')G_+(\mathbf{x}'',\mathbf{u}'')}{|\mathbf{x}-\mathbf{x}'||\mathbf{x}'-\mathbf{x}''|}d\Gamma' d\Gamma'',$$

Operating on Eq. (11a) by $\mathcal{L}$ and substituting for $\mathcal{L}G_+$ from Eq. (11b) gives a second order differential equation for $G_-$:

$$\mathcal{L}^2G_-\ =\ \omega^2G_- + i\omega F_e u^i\frac{\partial}{\partial x^i}\left[\delta\theta + q(\delta\Theta - 4\theta\delta\theta)\right]$$
$$+\,q\omega^2F_e u^i\delta\eta_i - q\omega F_e\mathcal{L}(u^2\delta\theta). \tag{13a}$$

We now seek a solution of Eq. (13a) in the form of classical energy and angular momentum integrals, $G_-(\mathbf{x},\mathbf{u}) = G_-(e^{cl}, l_i^{cl})$. In the next section, after we discuss the O(3) of Eq. (13a), we show that such solutions can be chosen from among the eigenfunctions of a pair of phase space angular momentum operators, $(J^2, J_z)$. We also show that for such solutions $\delta\theta$ and $\delta\Theta$ vanish identically reducing Eq. (13a) to

$$\mathcal{L}^2G_- = \omega^2\left(G_- + qF_e u^i\delta\eta_i\right). \tag{13b}$$

Multiplying Eq. (13b) by $G_-^*$, integrating over the phase space volume of the system, and considering the facts that $\mathcal{L} = \mathcal{L}^{cl} + q\mathcal{L}^{pn}$ is Hermitian and $\mathcal{L}^{cl}G_-(e^{cl}, l_i^{cl}) = 0$, gives

$$\int (\mathcal{L}G_-)^*\mathcal{L}G_-(1+2q\theta)d\Gamma$$
$$=\ q^2\int (\mathcal{L}^{pn}G_-)^*\mathcal{L}^{pn}G_-(1+2q\theta)d\Gamma$$
$$=\ \omega^2\left[\int G_-^*G_-(1+2q\theta)d\Gamma\right.$$
$$\left. +\,q\int G_-^*F_e u^i\delta\eta_i(1+2q\theta)d\Gamma\right]. \tag{14a}$$

Eq. (14a) shows that $\omega$ is of the same order of smallness as $q$. Thus, eliminating the terms of order $q^3$, $\omega^2 q$ and higher reduces Eq. (14a) to

$$\int (\mathcal{L}^{pn}G_-)^*\mathcal{L}^{pn}G_- d\Gamma = \frac{\omega^2}{q^2}\int G_-^*G_- d\Gamma. \tag{14b}$$

Eq. (14b) provides a variational expression for $\omega^2$ and will be used as such to calculate the allowable $\omega^2$. The frequencies, $\omega$, are real meaning that the corresponding deviations from the equilibrium state are stable oscillation modes. Furthermore, these perturbations will be different from the conventional classical modes, for they are excited by $pn$ terms in the equations of motion that are absent at classical level.

## 5. O(3) symmetry of $\mathcal{L} = \mathcal{L}^{cl} + q\mathcal{L}^{pn}$

For spherically symmetric potentials, $\theta(r)$ and $\Theta(r)$, both $\mathcal{L}^{cl}$ and $\mathcal{L}^{pn}$ depend on the angle between $\mathbf{x}$ and $\mathbf{u}$ and their magnitudes. Simultaneous rotations of the $x$ and $u$ coordinates about the same axis by the same angle leaves these operators form invariant. The generator of such simultaneous infinitesimal rotations on the function space $\mathcal{H}$ is

$$J_i = J_i^\dagger = -i\varepsilon_{ijk}\left(x^j\frac{\partial}{\partial x^k} + u^j\frac{\partial}{\partial u^k}\right), \quad (15)$$

which has the angular momentum algebra

$$[J_i, J_j] = i\varepsilon_{ijk}J_k. \quad (16)$$

Commutation of $J_i$ with $\mathcal{L}^{cl}$ was first established by Sobouti (1989a,b). Here we confine the discussion to the symmetry of $\mathcal{L}^{pn}$. Straightforward calculations reveal that

$$[\mathcal{L}^{pn}, J_i] = 0. \quad (17)$$

Thus, it is possible to choose the eigensolutions, $G_-$ of Eq. (14b) simultaneously with those of $J^2$ and $J_z$. The eigensolutions of the latter pair of operators are worked out in the Appendix. They are of the form $f(e^{cl}, l_i^{cl})\Lambda_{jm}; j, m$ integers, where $f$ is an arbitrary function of the classical integrals and $\Lambda_{jm}$ is a complex polynomial of order $j$ of the components of the classical angular momentum, $l_i^{cl}$. The $x$ and $u$ parity of $\Lambda_{jm}$ is that of $j$. See Appendix for proofs this statement.

We are now in a position to point out an interesting feature of the eigenmodes. Both $\omega^2$ and $\mathcal{L}^2$ in Eq. (13b) and the integrals in Eq. (14b) are real. Thus, $G_-$ can be chosen real or purely imaginary. By Eq. (11a), then $G_+$ will be purely imaginary or real. That is, an eigensolution $\delta F = G_- + G_+$ belonging to a nonzero $\omega$ is a complex function of phase coordinates in which both the $x$ and $u$ parities of the real and imaginary parts are opposite to each other. This feature is shared by the classical modes of the classical Liouville's and Antonov's equation.

In Sect. 7 we will take a variational approach to solutions of Eq. (14b). As variational trial functions we will consider the following

$$\begin{aligned} G_- &= f_{jm} = f(e)Re\,\Lambda_{jm} \\ &= [\sum_{n=j+1}^{N} c_n(-e)^n]Re\Lambda_{jm},\ j = \mathrm{odd}, c_n = \mathrm{consts.} \end{aligned} \quad (18)$$

Combining this with its corresponding even counterpart from Eq. (10a) we obtain

$$\delta F_{jm}(\mathbf{x}, \mathbf{u}, t) = (1 + \frac{q}{\omega}\mathcal{L}^{pn})f_{jm}e^{-i\omega t}. \quad (19)$$

At this stage let us note an important property of Liouville's equation. If a pair $(\omega, \delta F)$ is an eigensolution of Liouville's equation, $(-\omega, \delta F^*)$ is another eigensolution. This can be verified by taking the complex conjugate of Eq. (10a). These solutions, being complex quantities, cannot serve as physically meaningful distribution functions. Their real or imaginary parts, however, can. With no loss of generality we will adopt the real part. Thus,

$$\begin{aligned} &Re\ \delta F_{jm}(\mathbf{x}, \mathbf{u}, t) \\ &= f(e)Re\ \Lambda_{jm}\cos\omega t + i\frac{q}{\omega}\mathcal{L}^{pn}(f(e)Re\ \Lambda_{jm})\sin\omega t. \end{aligned} \quad (20)$$

The eigenmodes of Eq. (10a) are $m$-independent. By $m$-independence we mean a) the eigenvalues $\omega$ do not depend on $m$ and are $2j+1$ fold degenerate, and b) the expansion coefficients, $c_n$, of Eq. (12) do not depend on $m$. *Proof:* From the Appendix, Eq. (A. 4), $J_\pm = J_x \pm iJ_y$ are ladder operators for $\Lambda_{jm}$. Operating on $f_{jm}$ of Eq. (18) by $J_\pm$ will give the mode $f_{j,m\pm1}$ without changing the expansion coefficients. Secondly, substituting $J_\pm f_{jm} = \sqrt{(j \mp m)(j \pm m + 1)}f_{j,m\pm1}$ in Eq. (14a) instead of $f_{jm}$, and noting that $f_{jm}$'s can be normalized for all $m$'s, $\omega^2$ will remain unchanged.

## 6. Hydrodynamics of $pn$ modes

In this section we calculate the density fluctuations, macroscopic velocities, and the perturbations in the space-time metric generated by a $pn$ mode.

It was pointed out earlier that for $j$ an odd integer, $f_{jm}(\mathbf{x}, \mathbf{u})$ of Eq. (18) is odd while $\mathcal{L}^{pn}f_{jm}$ is even in both $\mathbf{x}$ and $\mathbf{u}$. The macroscopic velocities are obtained by multiplying Eq. (20) by $\mathbf{u}$ and integrating over the u-space. Only the odd component of $\delta F_{jm}$ contributes to this bulk motion,

$$\rho\mathbf{v} = \int f(e)Re\ \Lambda_{jm}\mathbf{u}d^3u\ \cos\omega t. \quad (21)$$

In Appendix, Eqs. (A. 11), we show that $\rho\mathbf{v}$ is a toroidal spherical harmonic vector field. In spherical polar coordinates it has the following form

$$\begin{aligned} \rho(v_r,\ v_\vartheta,\ v_\varphi) = r^jG(v_{es})\Big(0,\ Re\ \frac{-1}{\sin\vartheta}\frac{\partial}{\partial\varphi}Y_{jm}(\vartheta,\varphi), \\ Re\ \frac{\partial Y_{jm}}{\partial\vartheta}(\vartheta,\varphi)\Big)\ \cos\omega t, \end{aligned} \quad (22a)$$

where

$$G(v_{es}) = \int_0^{v_{es}} f(e)u^{j+3}du, \quad (22b)$$

and $v_{es} = \sqrt{2\theta}$ is the escape velocity from the potential $\theta(r)$. The macroscopic density, generated by the even component of Eq. (20), is

$$\begin{aligned} \delta\rho(\mathbf{x}, t) &= i\frac{q}{\omega}\int \mathcal{L}^{pn}(f(e)Re\ \Lambda_{jm})d^3u\sin\omega t \\ &= 2\frac{q}{\omega}\frac{\theta'}{r}\mathbf{x}\cdot\int f(e)Re\ \Lambda_{jm}\mathbf{u}d^3u\ \sin\omega t = 0. \end{aligned} \quad (23)$$

The second integral is obtained by an integration by parts. The vanishing of it comes about because of the fact that the radial vector $\mathbf{x}$ is orthogonal to the toroidal vector $\rho\mathbf{v}$. One also notes that $\nabla\cdot(\rho\mathbf{v})=0$. It can further be verified that, the continuity equation is satisfied at both classical and $pn$ level.

To complete the reduction of Eqs. (13) we should also show that $\delta\theta$ and $\delta\Theta$ vanish. The former is zero because $\delta\rho=0$. For the latter, from Eq. (3c) and Eq. (20) for $\delta F$, one has

$$\begin{aligned}\delta\Theta \;=\; & \frac{\omega^2}{4\pi}\int\frac{\delta\theta(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \\ & -2\int\frac{\rho(r')\delta\rho(\mathbf{x}'')+\delta\rho(\mathbf{x}')\rho(r'')}{|\mathbf{x}-\mathbf{x}'||\mathbf{x}'-\mathbf{x}''|}d^3x'd^3x'' \\ & +2\int\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}\int\mathbf{u}'^2\delta F(\mathbf{x}',\mathbf{u}')d^3u'=0. \qquad (24)\end{aligned}$$

The vanishing of the first two terms is obvious. The third term vanishes because the integral over $\mathbf{u}'$ has the same form as in $\delta\rho$ except for the additional scalar factor $\mathbf{u}'^2$. Like $\delta\rho$ it can be reduced to the inner product of the radial vector $\mathbf{x}$ and a toroidal vector. QED.

The toroidal motion described here slides one spherical shell of the fluid over the other without perturbing the density, the Newtonian gravitational field and, therefore, the hydrostatic equilibrium of the classical fluid. In doing so, it does not affect and is not affected by the conventional classical modes of the fluid at this first $pn$ order.

Nonetheless, the $pn$ modes are associated with space time perturbations. From Eq. (8c) of paper I and Eq. (3b) of this paper, $g_{0i}$ component of the metric tensor is

$$g_{0i}=\eta_i=4\int\frac{\rho v_i(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'. \qquad (25)$$

In spherical polar coordinates, one obtains

$$\eta_r \;=\; 0, \qquad (26a)$$

$$\eta_\vartheta \;=\; -a_j Re\,\frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}Y_{jm}(\vartheta,\varphi)\cos\omega t, \qquad (26b)$$

$$\eta_\varphi \;=\; a_j Re\,\frac{\partial Y_{jm}}{\partial\vartheta}(\vartheta,\varphi)\cos\omega t, \qquad (26c)$$

where

$$a_j \;=\; \frac{16\pi}{2j+1}\begin{cases}(r/R)^j y_j(R) & \text{for } r<R \\ \quad +(2j+1)r^j\int_r^R r'^{-j-1}y_j(r')dr' & \\ (R/r)^{j+1}y_j(R) & \text{for } r>R,\end{cases} \qquad (26d)$$

$$y_j(r) \;=\; r^{-j-1}\int_0^r r'^{2j+2}G(\theta(r'))dr', \qquad (26e)$$

$$\begin{aligned}G(\theta(r)) \;=\; & \int_0^{v_{es}} f(e)u^{j+3}du \\ =\; & \frac{2^{j/2+1}\Gamma(j/2+2)\Gamma(n+1)\theta(r)^{n+j/2+2}}{\Gamma(n+j/2+3)}, \qquad (26f)\end{aligned}$$

where $R$ is the radius of the system and $\Gamma(n)$ is the gamma function. The remaining components of the metric tensor remain unperturbed.

## 7. Variational solutions of $pn$ modes

We substitute the trial function of Eq. (18) in Eq. (14b) and turn it into a matrix equation. Thus

$$C^\dagger WC=\frac{\omega^2}{q^2}C^\dagger SC, \qquad (27)$$

where $C=[c_n]$ is the column matrix of the variational coefficients of Eq. (18), and the elements of $S$ and $W$ matrices are

$$S_{pq}=\int(-e)^{p+q}|Re\ \Lambda_{jm}|^2d\Gamma, \qquad (28a)$$

$$W_{pq}=\int(\mathcal{L}^{pn}(-e)^pRe\ \Lambda_{jm})^*(\mathcal{L}^{pn}(-e)^qRe\ \Lambda_{jm})d\Gamma. \quad (28b)$$

Minimizing $\omega^2$ with respect to variations of $C$ gives the following matrix equation

$$WC=\frac{\omega^2}{q^2}SC. \qquad (29)$$

Eigen $\omega$'s are the roots of the characteristic equation

$$|W-\frac{\omega^2}{q^2}S|=0. \qquad (30)$$

For each $\omega$, Eq. (29) can then be solved for the eigenvector C. This completes the Rayleigh-Ritz variational formalism of solving Eq. (14a). In what follows we present some numerical values for polytropes.

### 7.1. *pn Modes of polytropes belonging to* $(j,m)=(1,m)$

We analyse the case $m=0$, only. From the $m$-independence of eigenmodes (see theorem of Sect. 5) the eigenvalue and the expansion coefficients, $c_n$, for $m=\pm1$ will be the same. From Eqs. (A. 9), $\Lambda_{1\,0}=l_z=ru\sin\vartheta\sin\alpha\sin(\beta-\varphi)$, where $(\vartheta,\varphi)$ and $(\alpha,\beta)$ are the polar angles of $\mathbf{x}$, of $\mathbf{u}$, respectively. Substituting this in Eqs. (28) and integrating over directions of $\mathbf{x}$ and $\mathbf{u}$ vectors and over $0<u<\sqrt{2\theta}$ gives

$$S_{pq} \;=\; \int_0^1\theta^{p+q+2.5}x^4dx, \qquad (31a)$$

$$\begin{aligned}W_{pq} \;=\; \pi G\rho_c\Big\{ & (16a_{pq}-b_{pq})\int_0^1\theta'^2\theta^{p+q+3.5}x^4dx \\ & +(1-8a_{pq})\int_0^1\Theta'\theta'\theta^{p+q+2.5}x^4dx \\ & +a_{pq}\int_0^1\Theta'^2\theta^{p+q+1.5}x^4dx\Big\}, \qquad (31b)\end{aligned}$$

$$a_{pq} \;=\; \frac{pq(p+q+2.5)}{(p+q)(p+q-1)},$$

$$b_{pq} \;=\; \frac{4(p+q)^2+9(p+q)-13}{(p+q-1)(p+q+3.5)},\ p,\ q\ =2,3,\cdots. \qquad (31c)$$

Polytropic potentials $\theta$ and $\Theta$ were obtained from integrations of Lane Emden equation and Eqs. (28) of paper I, respectively. Eventually, the matrix elements of Eqs. (31), the characteristic

**Table 1.** $pn$ modes of polytrope n=2, belonging to $(j, m) = (1, 0)$. Eigenvalues are in units $\pi G\rho_c q^2$, $c_n$'s are the linear variational parameters of Eq. (18). A number $a \times 10^{\pm b}$ is written as $a \pm b$. To appraise the accuracy of the computations two sets of data with six and seven variational parameters are given. The first three eigenvalues are reliable up to three figures. Characteristically, the accuracy deteriorates as one goes to higher order modes.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $\omega^2$ | .1825+01 | .4973+01 | .6448+01 | .1216+02 | .3425+02 | .1686+03 | |
| $c_1$ | .3113+02 | -.8912+02 | .1663+03 | .1344+03 | .7545+01 | -.1399+04 | |
| $c_2$ | .3908+02 | .1045+04 | -.3234+04 | -.9746+03 | -.2392+04 | .8484+04 | |
| $c_3$ | -.1420+03 | -.6649+04 | .1801+05 | .4514+04 | .7952+04 | -.9647+04 | |
| $c_4$ | .5803+03 | .1804+05 | -.4351+05 | -.7014+04 | -.2607+03 | -.2251+05 | |
| $c_5$ | -.9110+03 | -.2210+05 | .4724+05 | .8324+03 | -.1811+05 | .5188+05 | |
| $c_6$ | .5252+03 | .1020+05 | -.1874+05 | .2882+04 | .1317+05 | -.2717+05 | |
| $\omega^2$ | .1823+01 | .4865+01 | .5895+01 | .9113+01 | .1465+02 | .4228+02 | .3226+03 |
| $c_1$ | .3028+02 | -.7086+02 | .1529+03 | -.3129+02 | .1561+03 | -.4624+02 | .2042+04 |
| $c_2$ | .4812+02 | .6908+03 | -.2810+04 | .1313+04 | -.1513+04 | -.2762+04 | -.1461+05 |
| $c_3$ | -.1305+03 | -.3993+04 | .1702+05 | -.5686+04 | .6685+04 | .1077+05 | .2271+05 |
| $c_4$ | .2576+03 | .8181+04 | -.4788+05 | .3425+04 | -.3673+04 | .1875+04 | .4154+05 |
| $c_5$ | .1303+03 | -.3086+04 | .6823+05 | .2433+05 | -.2910+05 | -.4718+05 | -.1496+06 |
| $c_6$ | -.7534+03 | -.7924+04 | -.4771+05 | -.4855+05 | .5132+05 | .5873+05 | .1425+06 |
| $c_7$ | .5475+03 | .6707+04 | .1302+05 | .2568+05 | -.2386+05 | -.2120+05 | -.4423+05 |
| | $pn_1$ | $pn_2$ | $pn_3$ | $pn_4$ | $pn_5$ | $pn_6$ | $pn_7$ |

Eq. (30) and the eigenvalue Eq. (29) were numerically solved in succession. Tables 1-4 show some sample calculations for polytropes 2, 3, 4, and 4.9. Eigenvalues are displayed in lines marked by an asterisks. The column following an eigenvalue is the corresponding eigenvector, i.e. the values of $c_1,\ c_2,\ \cdots$, of Eq. (18). To demonstrate the accuracy of the procedure, calculations with six and seven variational parameter are given for comparison. The first three eigenvalues can be trusted up to two to four figures. Convergence improves as the polytropic index, i.e. the central condensation, increases. Eigenvalues are in units of $\pi G\rho_c q^2$ and increase as the mode order increases.

## 8. Concluding remarks

Linear perturbations of phase space distribution functions have been studied. Their evolution in both classical and $pn$ order takes place through an eigenvalue equation. The eigensolutions of the latter are the normal modes of oscillation of the system. If the underlying potentials are spherically symmetric, the evolution equation is O(3) symmetric. The modes can be characterized by a pair of angular momentum eigennumbers, $(j, m)$. The eigenvalues $\omega_j$ are, however, $(2j + 1)$ fold degenerate.

Perturbations that are functions of classical energy and classical angular momentum are neutral in classical approximation, but not in $pn$ order. Neutral, here, means to belong to zero frequency modes. The weak $pn$ forces generate a sequence of low frequency modes from such perturbations. In their hydrodynamic behavior, they constitute a sequence of low frequency toroidal modes. There is an oscillatory $g_{0i}$ component of the metric tensor associated with these modes. From a conceptual point of view, they are similar to toroidal modes of slowly rotating fluids generated by Coriolis forces or to the standing Alfven waves of a weakly magnetized fluids.

## Appendix A: eigensolutions of $J^2$ and $J_z$

As pointed out earlier, $J_i$'s of Eq. (15) have the angular momentum algebra,

$$[J_i, J_j] = i\varepsilon_{ijk} J_k. \tag{A.1}$$

Therefore, the simultaneous eigensolutions of $J^2$ and $J_z$, $\Lambda_{jm}(\mathbf{x}, \mathbf{u})$, obey the following

$$J^2\Lambda_{jm} = j(j+1)\Lambda_{jm}, \quad j = 0, 1, \cdots, \tag{A.2}$$

$$J_z\Lambda_{jm} = m\Lambda_{jm}, \quad -j \le m \le j. \tag{A.3}$$

The ladder operators, $J_\pm = J_x \pm iJ_y$, raise and lower the $m$ values:

$$J_\pm\Lambda_{jm} = \sqrt{(j \mp m)(j \pm m + 1)}\Lambda_{jm\pm1}. \tag{A.4}$$

In particular

$$J_\pm\Lambda_{j,\pm j} = 0. \tag{A.4a}$$

The effect of $J_i$ on classical energy integral, $e = u^2/2 - \theta(r)$, and the classical angular momentum integral, $l_i = \varepsilon_{ijk}x_j u_k$, are as follows

$$J_i e \ = \ J_i l^2 = J_i f(e, l^2) = 0, \tag{A.5a}$$

$$J_i l_j \ = \ i\varepsilon_{ijk} l_k. \tag{A.5b}$$

*Theorem 1:*

$$\Lambda_{j,\pm j} = l^j_\pm = (\frac{1}{2})^j (l_x \pm i l_y)^j. \tag{A.6}$$

*Proof:*

$$J_z l^j_\pm \ = \ j l^{j-1}_\pm (J_z l_\pm) = \pm j l^j_\pm, \quad \text{by (A.5b)}, \tag{A.7a}$$

$$J^2 l^j_+ \ = \ (J_- J_+ + J_z^2 + J_z) l^j_+ = j(j+1) l^j_+, \quad \text{by (A.4a) and (A.7a)}, \tag{A.7b}$$

$$J^2 l^j_- \ = \ (J_+ J_- + J_z^2 - J_z) l^j_- = j(j+1) l^j_-, \tag{A.7c}$$

**Table 2.** Same as Table 1. $n = 3$ and $(j, m) = (1, 0)$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $\omega^2$ | .1534+01 | .4836+01 | .9473+01 | .1938+02 | .4083+02 | .1128+03 | |
| $c_1$ | .9752+02 | -.6975+02 | .2464+03 | -.2246+03 | -.9102+03 | .3169+04 | |
| $c_2$ | .3284+02 | -.8725+03 | -.1121+04 | -.2590+04 | .1713+05 | -.2631+05 | |
| $c_3$ | .2096+03 | .3859+04 | .5591+04 | .1444+05 | -.1023+06 | .6390+05 | |
| $c_4$ | -.5354+03 | -.5728+04 | -.1216+05 | -.9903+04 | .2599+06 | -.3406+05 | |
| $c_5$ | .3941+03 | .2528+04 | .5215+04 | -.2221+05 | -.2933+06 | -.4814+05 | |
| $c_6$ | .1803+01 | .1125+04 | .3307+04 | .2153+05 | .1208+06 | .4268+05 | |
| $\omega^2$ | .1533+01 | .4688+01 | .7993+01 | .9068+01 | .1124+02 | .1909+02 | .1093+03 |
| $c_1$ | .9318+02 | -.1440+03 | -.1202+03 | -.1069+04 | -.5706+03 | -.5482+02 | .3703+04 |
| $c_2$ | .1121+03 | .6997+03 | .5482+04 | .1856+05 | .7685+04 | -.5626+04 | -.3381+05 |
| $c_3$ | -.2118+03 | -.4506+04 | -.2955+05 | -.1063+06 | -.4112+05 | .3078+05 | .1007+06 |
| $c_4$ | .2709+03 | .9777+04 | .5298+05 | .2726+06 | .7791+05 | -.4371+05 | -.1109+06 |
| $c_5$ | .1206+03 | -.9309+03 | -.6283+04 | -.3375+06 | -.1278+05 | -.7049+03 | .1239+05 |
| $c_6$ | -.7005+03 | -.1574+05 | -.7154+05 | .1894+06 | -.9027+05 | .3228+05 | .4581+05 |
| $c_7$ | .5309+03 | .1200+05 | .5087+05 | -.3511+05 | .5945+05 | -.1218+05 | -.1722+05 |
| | $pn_1$ | $pn_2$ | $pn_3$ | $pn_4$ | $pn_5$ | $pn_6$ | $pn_7$ |

**Table 3.** Same as Table 1. $n = 4$ and $(j, m) = (1, 0)$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $\omega^2$ | .7569+00 | .2822+01 | .5661+01 | .8814+01 | .1519+02 | .6952+02 | |
| $c_1$ | .6291+03 | -.1067+04 | .2143+04 | -.1949+04 | -.6870+04 | .1400+05 | |
| $c_2$ | -.9217+02 | .1770+04 | -.1693+05 | .1131+05 | .8373+05 | -.2337+06 | |
| $c_3$ | .4162+03 | .2808+04 | .5682+05 | -.3654+04 | -.3195+06 | .1293+07 | |
| $c_4$ | -.3883+04 | .5860+04 | -.1184+06 | -.2807+05 | .4791+06 | -.3112+07 | |
| $c_5$ | .6427+04 | -.2303+05 | .1257+06 | -.4668+04 | -.2545+06 | .3371+07 | |
| $c_6$ | -.3089+04 | .1612+05 | -.4514+05 | .3416+05 | .1251+05 | -.1344+07 | |
| $\omega^2$ | .7569+00 | .2813+01 | .5021+01 | .8747+01 | .1272+02 | .3322+02 | .7683+02 |
| $c_1$ | .5590+03 | -.8716+03 | .2653+03 | -.2421+04 | .1881+04 | .1412+05 | .3376+05 |
| $c_2$ | .1189+04 | -.2018+04 | .1406+05 | .1926+05 | -.7436+04 | -.2356+06 | -.5191+06 |
| $c_3$ | -.6377+04 | .2349+05 | -.1057+06 | -.4732+05 | -.5363+05 | .1298+07 | .2528+07 |
| $c_4$ | .9376+04 | -.3509+05 | .2059+06 | .6165+05 | .2228+06 | -.3112+07 | -.4750+07 |
| $c_5$ | .5449+03 | -.4645+04 | -.2977+05 | -.4272+05 | -.7106+05 | .3356+07 | .2298+07 |
| $c_6$ | -.1192+05 | .4364+05 | -.2533+06 | -.3854+05 | -.4046+06 | -.1333+07 | .2455+07 |
| $c_7$ | .7228+04 | -.2275+05 | .1775+06 | .5845+05 | .3227+06 | -.1382+03 | -.2085+07 |
| | $pn_1$ | $pn_2$ | $pn_3$ | $pn_4$ | $pn_5$ | $pn_6$ | $pn_7$ |

**Table 4.** Same as Table 1. $n = 4.9$ and $(j, m) = (1, 0)$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $\omega^2$ | .4481+00 | .1827+01 | .4078+01 | .6515+01 | .1170+02 | .1391+03 | |
| $c_1$ | -.2888+02 | .1663+03 | -.2794+03 | .1593+03 | .1405+03 | .1081+05 | |
| $c_2$ | -.2440+03 | -.7593+04 | .2050+05 | -.2099+05 | .2665+05 | -.2129+06 | |
| $c_3$ | .4933+05 | -.2772+04 | -.1400+06 | .1883+06 | -.3467+06 | .1344+07 | |
| $c_4$ | -.1722+06 | .1443+06 | .2902+06 | -.5138+06 | .1372+07 | -.3583+07 | |
| $c_5$ | .2124+06 | -.2675+06 | -.2194+06 | .4871+06 | -.2092+07 | .4207+07 | |
| $c_6$ | -.8916+05 | .1394+06 | .5712+05 | -.1179+06 | .1073+07 | -.1790+07 | |
| $\omega^2$ | .4380+00 | .1805+01 | .4006+01 | .6190+01 | .7980+01 | .1439+02 | .8964+02 |
| $c_1$ | -.1701+02 | .1379+03 | -.3341+03 | .3427+03 | -.3020+03 | .7695+03 | .8642+04 |
| $c_2$ | -.6649+03 | -.6322+04 | .2326+05 | -.3097+05 | .2196+05 | -.1111+05 | -.1534+06 |
| $c_3$ | .5135+05 | -.1143+05 | -.1601+06 | .2940+06 | -.2552+06 | .1349+06 | .8174+06 |
| $c_4$ | -.1667+06 | .1599+06 | .3264+06 | -.9227+06 | .1022+07 | -.9712+06 | -.1565+07 |
| $c_5$ | .1694+06 | -.2551+06 | -.1784+06 | .1132+07 | -.1574+07 | .3018+07 | .4968+06 |
| $c_6$ | -.1770+05 | .8656+05 | -.9582+05 | -.4879+06 | .7432+06 | -.3959+07 | .1421+07 |
| $c_7$ | -.3646+05 | .3341+05 | .9586+05 | .2938+05 | .8318+05 | .1819+07 | -.1047+07 |
| | $pn_1$ | $pn_2$ | $pn_3$ | $pn_4$ | $pn_5$ | $pn_6$ | $pn_7$ |

QED. Combining Eqs. (A. 6), (A. 4) and (A. 5) one obtains

$$\Lambda_{jm} = af(e,l^2)J_+^{j+m}l_-^j = bf(e,l^2)J_-^{j-m}l_+^j, \quad \text{(A.8)}$$

where $f(e,l^2)$ is an arbitrary function of its arguments, and $a$ and $b$ are normalization constants. Examples: Aside from an arbitrary factor of classical constants of motion, one has

$$\Lambda_{1\,0} = l_z, \quad \text{(A.9a)}$$
$$\Lambda_{1\,\pm1} = l_\pm, \quad \text{(A.9b)}$$
$$\Lambda_{2\,0} = 2l_+l_- - l_z^2 = \frac{1}{2}(3l_z^2 - l^2), \quad \text{(A.9c)}$$
$$\Lambda_{2\,\pm1} = l_\pm l_z, \quad \text{(A.9d)}$$
$$\Lambda_{2\,\pm2} = l_\pm^2. \quad \text{(A.9e)}$$

*Theorem 2:* The vector field $\mathbf{V}^{jm} = \int \Lambda_{jm}\mathbf{u}d\Omega$ is a toroidal vector field belonging to the spherical harmonic numbers $(j,m)$, where integration is over the directions of $\mathbf{u}$.

*Preliminaries:* Let $(\vartheta,\varphi)$ and $(\alpha,\beta)$ denote the polar angles of $\mathbf{x}$, of $\mathbf{u}$, respectively, and $\gamma$ be the angle between $(\mathbf{x},\mathbf{u})$. Also choose magnitudes of $\mathbf{x}$ and $\mathbf{u}$ to be unity, for only integrations over the direction angles are of concern. One has $\cos\gamma = \cos\vartheta\ \cos\alpha\ +\ \sin\vartheta\ \sin\alpha\ \cos(\varphi-\beta)$

$$u_r = \cos\gamma, \quad \text{(A.10a)}$$
$$u_\vartheta = -\sin\vartheta\ \cos\alpha\ +\ \cos\vartheta\ \sin\alpha\ \cos(\varphi-\beta), \quad \text{(A.10b)}$$
$$u_\varphi = -\sin\alpha\ \sin(\varphi-\beta), \quad \text{(A.10c)}$$
$$l_+ = i(\sin\vartheta\ \cos\alpha\ e^{i\varphi} - \cos\vartheta\ \sin\alpha\ e^{i\beta}). \quad \text{(A.10d)}$$

*Proof:* By induction, we show that a) $\mathbf{V}^{jj}$ is a toroidal field and b) if $\mathbf{V}^{jm}$ is a toroidal field, so is $\mathbf{V}^{j\ m-1}$.

a) Direct integrations over $\alpha$ and $\beta$ gives

$$V_r^{jj} = \int l_+^j u_r d\Omega = 0, \qquad d\Omega = \sin\alpha\ d\alpha\ d\beta, \quad \text{(A.11a)}$$
$$V_\vartheta^{jj} = \int l_+^j u_\vartheta d\Omega = -\frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}Y_{jj}(\vartheta,\varphi), \quad \text{(A.11b)}$$
$$V_\varphi^{jj} = \int l_+^j u_\varphi d\Omega = \frac{\partial}{\partial\vartheta}Y_{jj}(\vartheta,\varphi).\ \ QED. \quad \text{(A.11c)}$$

b) Suppose $\mathbf{V}^{jm}$ is a toroidal vector field and calculate $\mathbf{V}^{j\ m-1} = \int(J_-\Lambda_{jm})\mathbf{u}d\Omega$, where $J_\pm = L_\pm + K\pm$, $L_\pm = \pm e^{\pm i\varphi}(\frac{\partial}{\partial\vartheta} \pm i\text{cotg}\vartheta\frac{\partial}{\partial\varphi})$, $K_\pm = \pm e^{\pm i\beta}(\frac{\partial}{\partial\alpha} \pm i\text{cotg}\alpha\frac{\partial}{\partial\beta})$.

Again direct integrations gives

$$V_r^{j\ m-1} = L_-V_r^{jm} = 0, \quad \text{if } V_r^{jm} = 0, \quad \text{(A.12a)}$$
$$V_\vartheta^{j\ m-1} = -\frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}Y_{j\ m-1}(\vartheta,\varphi), \quad \text{if } V_\vartheta^{j\ m} = -\frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}Y_{j\ m}(\vartheta,\varphi), \quad \text{(A.12b)}$$
$$V_\varphi^{j\ m-1} = \frac{\partial}{\partial\vartheta}Y_{j\ m-1}(\vartheta,\varphi), \quad \text{if } V_\varphi^{j\ m} = \frac{\partial}{\partial\vartheta}Y_{j\ m}(\vartheta,\varphi). \quad \text{(A.12c)}$$

QED.

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