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ASTRONOMY AND ASTROPHYSICS

Caustics in special multiple lenses

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Abstract. Despite its mathematical complexity, the multiple gravitational lens can be studied in detail in every situation where a perturbative approach is possible. In this paper, we examine the caustics of a system with a lens very far from the others with respect to their Einstein radii, and a system where mutual distances between lenses are small compared to the Einstein radius of the total mass. Finally we review the case of a planetary system adding some new information (area of caustics, duality and higher order terms).

Key words: cosmology: gravitational lensing – stars: binaries: close – stars: planetary systems

1. Introduction

The investigation of a gravitational lens composed by a discrete set of point-like masses was opened by Schneider & Weiß in 1986. They considered a binary lens, deriving analytical expressions for caustics in the case of lenses with the same mass. Erdl & Schneider (1993) extended their analysis to an arbitrary mass ratio, confirming the presence of three possible topologies in the caustic structure.

Besides these two fundamental papers, many other contributions have been given to this field by several authors studying particular aspects of this matter. It has often been considered the relation between the binary lens and the Chang–Refsdal (1979; 1984) model describing the behaviour of a point-like lens immersed in a uniform gravitational field (Schneider & Weiß 1986; Dominik 1999). In the special case of very asymmetric lens, as that of a planet orbiting a star, the Chang–Refsdal approximation can give very reliable results (Gould & Loeb 1992; Gaudi & Gould 1997). In the search for good approximate models, also the quadrupole lens was given a role in close binary systems (Dominik 1999).

However, while the picture for the binary lens is somewhat well defined at present day, the same cannot be said about general multiple lenses. The attempts to enlighten this kind of lenses have really been very few. This is because of the mathematical complexity of the lens equation. Yet, multiple lenses can play an important role in several cases and therefore are not just mathematical curiosities. Planetary systems are likely to be composed by a star with more than one planet, so it is important to study them in their full complexity in order to interpret planetary microlensing events correctly, especially those with high magnification (Wambsganss 1997; Gaudi, Naber & Sackett 1998; Bozza 1999). N-body systems made of similar masses are hard to be found as gravitationally stable systems, yet the projected distances between far stable subsystems could be compatible with the appearance of "interaction" in the gravitational lensing behaviour; some considerations about microlensing light curves by multiple lenses have been done by Rhie (1996).

Another context where multiple lensing is surely of great interest is cosmological lensing by rich galactic clusters: as a first approximation, all lensing galaxies can be considered as point masses, then the multiple Schwarzschild lens constitutes a good starting point for the analysis of caustics and images produced by such systems.

The aim of this work is to give a systematic investigation of the caustics of multiple lenses whenever they can be referred to well–defined situations to be employed as starting points in perturbative expansions. The three cases where this happens have been pointed out by Dominik (1999) and are resumed here.

In Sect. 2 the lens equation for multiple lenses and the basic definitions are given; Sect. 3 deals with the case of a lensing mass that is very far from the others; in Sect. 4 a system formed by near (with respect to the Einstein radius of the total mass) lenses is studied; Sect. 5 reconsiders the case of a multiple planetary system, already examined in a former work (Bozza 1999); Sect. 6 contains the conclusions.

2. Basics of multiple lensing

Let

$$R_{\rm E}^0 = \sqrt{\frac{4GM_0}{c^2}\frac{D_{\rm LS}D_{\rm OL}}{D_{\rm OS}}} \tag{1}$$

be the Einstein radius of a mass M_0 placed at distance $D_{\rm OL}$ from the observer and at distance $D_{\rm LS}$ from a source having distance $D_{\rm OS} = D_{\rm OL} + D_{\rm LS}$ from the observer. The coordinates in the lens plane normalized to $R_{\rm E}^0$ will be denoted by $\mathbf{x} = (x_1; x_2)$, while the coordinates in the source plane normalized to $R_{\rm E}^0 \frac{D_{\rm OS}}{D_{\rm OL}}$

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are $\mathbf{y} = (y_1; y_2)$. All masses are meant to be measured in units of M_0 which is a typical mass of the problem we are considering. For example it can be the solar mass if the lensing objects are stars, or a typical galaxy mass if we are considering extragalactic lensing.

According to the standard theory of gravitational lensing, the deviation induced by a system formed by n masses $m_1, ..., m_n$ whose projections in the lens plane are at positions $\mathbf{x}_1, ..., \mathbf{x}_n$ is the sum of the deviations produced by the single objects. So the lens equation reads:

$$\mathbf{y} = \mathbf{x} - \sum_{i=1}^{n} \frac{m_i (\mathbf{x} - \mathbf{x}_i)}{|\mathbf{x} - \mathbf{x}_i|^2} \tag{2}$$

Given a source position $\mathbf{y}$, the $\mathbf{x}$'s solving this equation are called images. Their properties can be studied through the Jacobian matrix of the lens application. In particular, the determinant of this matrix is:

$$\begin{aligned}\det J = \ & 1 - \left[\sum_{i=1}^{n} \frac{m_i \left(\Delta_{i1}^2 - \Delta_{i2}^2\right)}{\left(\Delta_{i1}^2 + \Delta_{i2}^2\right)^2}\right]^2 + \\ & - 4\left[\sum_{i=1}^{n} \frac{m_i \Delta_{i1} \Delta_{i2}}{\left(\Delta_{i1}^2 + \Delta_{i2}^2\right)^2}\right]^2\end{aligned} \tag{3}$$

where $\Delta_i = (\Delta_{i1}; \Delta_{i2}) = \mathbf{x} - \mathbf{x}_i$.

The magnification of an image is given by the absolute value of the inverse of $\det J$ calculated at the image position. Whenever the Jacobian determinant vanishes, an image at that point would be (in ray optics) infinitely amplified. These points are thus said to be critical and their corresponding points in the source plane, found by use of Eq. (2), constitute the caustics.

Caustics appear as closed cusped curves and split the source plane in several domains. The number of images corresponding to a single source only depends on the domain where the source is located. In neighboring domains the number of images always differs by two units. The caustics of a given lens model provide very useful hints for the description of the general properties of the system so that their study is essential to understand the behaviour of a lens.

To find the critical curves and then the caustics of a multiple lens, one has to solve the equation $\det J = 0$, which is very complicated. However, if it is possible to single out some parameters which are very large or small with respect to the others, a perturbative approach can be tried (Bozza 1999). The situations we are to explore in this paper always refer to the single Schwarzschild lens as the zero order solution. The critical curve of a lens with mass m placed at the origin is a circle with radius $\sqrt{m}$, while the caustic is reduced to the origin itself: $\mathbf{y} = 0$.

3. Single lens perturbed by far masses

Let's start considering an isolated point-lens. The presence of other masses that are very far from the first one (with respect to all Einstein radii) warps the circular critical curve, giving rise to an extended caustic.

Let's consider a mass m_1 placed at the origin of the lens plane. The other masses $m_2, ..., m_n$ are at the positions $\mathbf{x}_2, ..., \mathbf{x}_n$. In polar coordinates, we put $\mathbf{x}_i = (\rho_i \cos\varphi_i; \rho_i \sin\varphi_i)$ and the generic coordinate in the lens plane is $\mathbf{x} = (r\cos\vartheta; r\sin\vartheta)$. The hypothesis we start from is that $\rho_i \gg \sqrt{m_j}$ for each i and j. This allows us to expand the Jacobian determinant (3) in series of inverse powers of ρ_i. If we stop at the fourth order we have:

$$\begin{aligned}\det J \ = \ & 1 - \frac{m_1^2}{r^4} + \\ & -\frac{2m_1}{r^2} \sum_{i=2}^{n} \frac{m_i \cos(2\vartheta - 2\varphi_i)}{\rho_i^2} + \\ & -\frac{4m_1}{r} \sum_{i=2}^{n} \frac{m_i \cos(3\vartheta - 3\varphi_i)}{\rho_i^3} + \\ & -6m_1 \sum_{i=2}^{n} \frac{m_i \cos(4\vartheta - 4\varphi_i)}{\rho_i^4} + \\ & - \sum_{i,j=2}^{n} \frac{m_i m_j \cos(2\varphi_i - 2\varphi_j)}{\rho_i^2 \rho_j^2} + \\ & + o\left(\frac{1}{\rho_i^4}\right)\end{aligned} \tag{4}$$

We see that a linear superposition principle is valid at the second and the third order. In the fourth order there are "interaction" terms that make the calculations more difficult.

The zero order solution is simply the Einstein radius $r = \sqrt{m_1}$. We build the general solution as an infinite series in $1/\rho_i$:

$$r = \sqrt{m_1}\,(1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + ...) \tag{5}$$

where $\varepsilon_j \sim 1/\rho_i^j$. Substituting this expression in Eq. (4) and expanding again, we can easily solve the equation of critical curves at each order to find the perturbations. The first three orders are:

$$\begin{aligned}\varepsilon_1 &= 0 \\ \varepsilon_2 &= \sum_{i=2}^{n} \frac{m_i \cos(2\vartheta - 2\varphi_i)}{2\rho_i^2} \\ \varepsilon_3 &= \sum_{i=2}^{n} \frac{\sqrt{m_1} m_i \cos(3\vartheta - 3\varphi_i)}{\rho_i^3}\end{aligned} \tag{6}$$

Recalling that the shear produced by one mass in a Chang–Refsdal approximation is $\gamma_i = \frac{m_i}{\rho_i^2}$, the second order can be easily recognized as the sum of the Chang–Refsdal expansions for each mass to the first order in the γ_i. Actually, the full second order can be identified with a Chang–Refsdal lens expanded to the first order in this total shear:

$$\gamma = \sqrt{\left[\sum_{i=2}^{n} \frac{m_i \sin(2\varphi_i)}{\rho_i^2}\right]^2 + \left[\sum_{i=2}^{n} \frac{m_i \cos(2\varphi_i)}{\rho_i^2}\right]^2} \tag{7}$$

and oriented along the direction at the angle:

$$\varphi = \frac{1}{2} \arctan\left[\frac{\sum_{i=2}^{n} \frac{m_i \sin(2\varphi_i)}{\rho_i^2}}{\sum_{i=2}^{n} \frac{m_i \cos(2\varphi_i)}{\rho_i^2}}\right] \tag{8}$$

So, the Chang–Refsdal lens provides a first approximation for the critical curves of a far lens not only in the binary case (Dominik 1999) but even for an arbitrary number of lenses, as already stated in (Chang & Refsdal 1979).

Now we exploit the perturbative results just shown to find the caustic formed in this case. It suffices to apply the lens equation (2) and expand it to the third order again:

$$\begin{aligned} y_1\ (\vartheta) &= \sum_{i=2}^{n} \frac{m_i \cos(\varphi_i)}{\rho_i} + \\ &+\sqrt{m_1} \sum_{i=2}^{n} \frac{m_i}{\rho_i^2} \left[\cos\vartheta \cos(2\vartheta - 2\varphi_i) + \cos(\vartheta - 2\varphi_i)\right] + \\ &+m_1 \sum_{i=2}^{n} \frac{m_i}{\rho_i^2} \left[2\cos\vartheta \cos(3\vartheta - 3\varphi_i) + \cos(2\vartheta - 3\varphi_i)\right] \end{aligned} \tag{9}$$

$$\begin{aligned} y_2\,(\vartheta) &= \sum_{i=2}^{n} \frac{m_i \sin(\varphi_i)}{\rho_i} + \\ &+\sqrt{m_1} \sum_{i=2}^{n} \frac{m_i}{\rho_i^2} \left[\sin\vartheta \cos(2\vartheta - 2\varphi_i) - \sin(\vartheta - 2\varphi_i)\right] + \\ &+m_1 \sum_{i=2}^{n} \frac{m_i}{\rho_i^2} \left[2\sin\vartheta \cos(3\vartheta - 3\varphi_i) - \sin(2\vartheta - 3\varphi_i)\right] \end{aligned} \tag{10}$$

The first term is independent of ϑ and only fixes the position of the caustic which is displaced towards the other masses; the two second order terms describe the shape of the caustic reproducing the Chang–Refsdal limit; the third order terms correct this approximation.

The Chang–Refsdal model describes the gravitational lensing by a mass embedded in a uniform gravitational field. For $\gamma > 1$, the caustics of this model are diamond–shaped curves with four cusps which are independent on the versus of the gravitational field. The approximation of uniform field is reasonable in our case. However, it breaks down quite soon when the distances are not so high. It happens that our lens begins to feel the non uniformity of the field and the caustics lose their symmetry, elongating in the direction of the other masses. The third order describes this elongation very well, considerably extending the range of applicability of the perturbative results.

The breakdown of this perturbative expansion rises when the Einstein ring of the first mass is too close to the critical curves produced by the other masses. A particular case is obtained when several small masses are very close each other. The critical curve generated by such a subsystem has a radius of the order of the square root of its total mass. So even if the condition $\rho_i \gg \sqrt{m_j}$ is satisfied for each single mass, the distance of the first mass to the subsystem could be smaller than the radius of its total critical curve, causing the failure of the perturbative hypothesis.

In Fig. 1, we see an example of the power of the perturbative approach in reproducing the numerically found exact results with great accuracy not taking care of the number of masses.

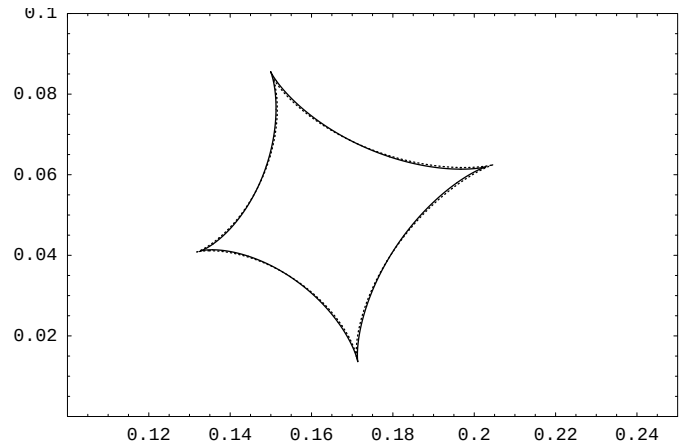

Fig. 1. The caustic of a lens perturbed by two other bodies. The masses are $m_1 = 0.25$, $m_2 = 0.25$, $m_3 = 0.5$. The positions of m_2 and m_3 are $\rho_2 = 4$, $\varphi_2 = 0$ and $\rho_3 = \sqrt{20}$, $\varphi_3 = \arctan 0.5$ respectively. The solid line is the perturbative caustic to be compared with the dashed line that is the exact one, almost completely hidden behind the solid line.

Fig. 2 shows an interesting comparison among the different orders approximations and the exact caustic for different distances between the two lenses. We see that even at separations about $10\sqrt{2}$ times the Einstein radius of each lens, the second order approximation (Chang–Refsdal) seems quite poor, while the third order caustic is practically coincident with the exact one up to five Einstein radii. As previously said the inadequacy of the Chang–Refsdal approximation is in its impossibility in giving account of the asymmetry of the caustic, which is already evident at quite far separations. This justifies us in considering the third order that is the lowest order reproducing this asymmetry.

Now let's turn to the cusps of these caustics. These points are characterized by the vanishing of the tangent vector:

$$\begin{cases} y_1'\,(\vartheta) = 0 \\ y_2'\,(\vartheta) = 0 \end{cases} \tag{11}$$

At the second order we have a Chang–Refsdal caustic rotated at an angle φ given by Eq. (8) from the y_1–axis. Without loss of generality we can put this angle to zero. Then it is well known (Dominik 1999) that the curve is symmetric for reflections on two axes passing through the center of the caustic and has four cusps at the intersections with these axes, corresponding to $\vartheta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.

At the third order, summing and subtracting the two Eqs. (11), we get:

$$\begin{cases} (\cos\vartheta + \sin\vartheta)\, F\,(\vartheta) = 0 \\ (\cos\vartheta - \sin\vartheta)\, F\,(\vartheta) = 0 \end{cases} \tag{12}$$

with

$$F\,(\vartheta) = \sum_{i=2}^{n} m_i \left[\frac{3\cos(2\vartheta - 2\varphi_i)}{\rho_i^2} + \frac{8\sqrt{m_1}\cos(3\vartheta - 3\varphi_i)}{\rho_i^3}\right] \tag{13}$$

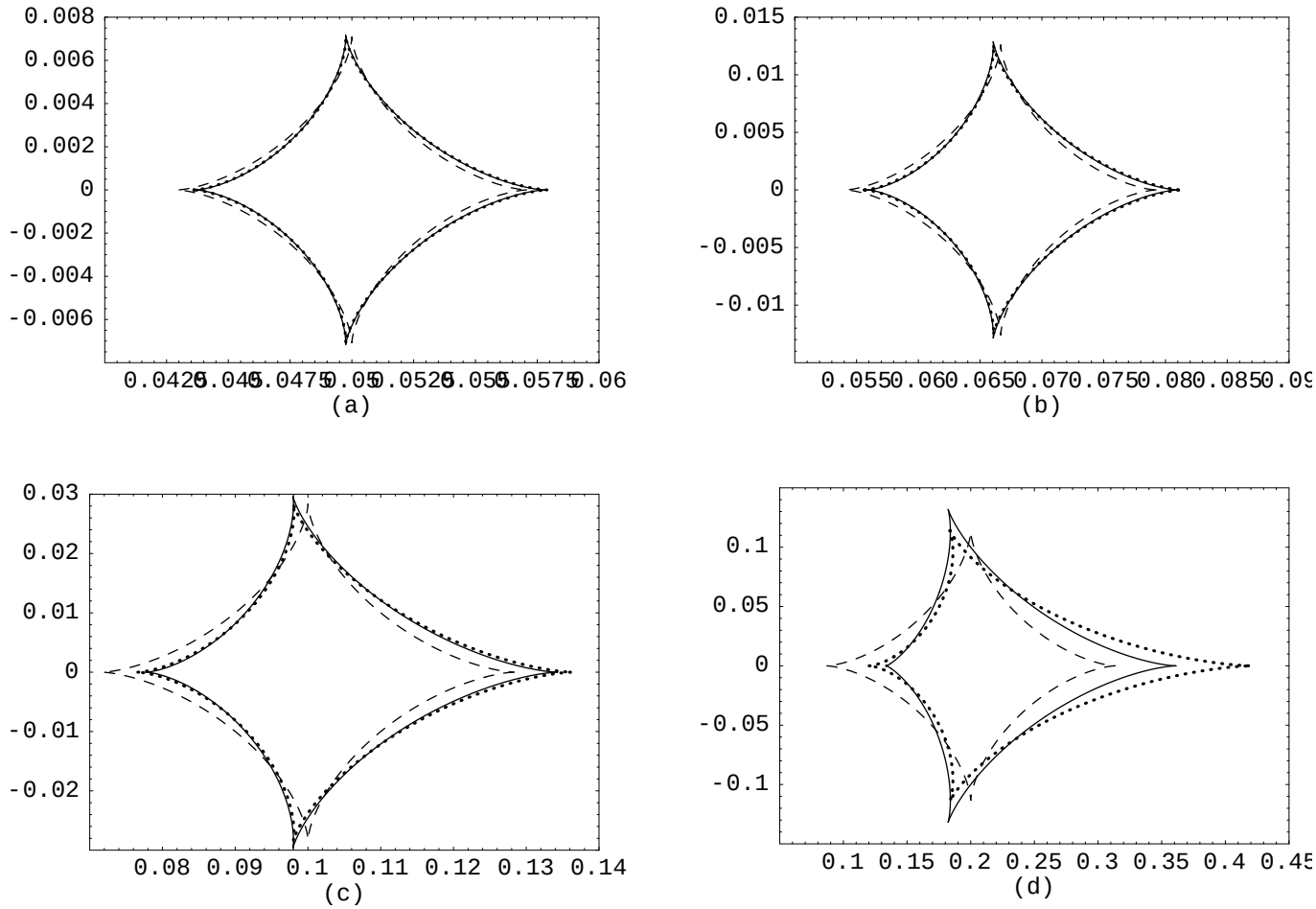

Fig. 2a–d. Caustic of a lens with a far companion of the same mass (0.5), along the direction $\varphi_2 = 0$. The dotted curves are the exact caustics, the dashed curves are the second order perturbative ones (Chang–Refsdal) and the continuous lines are the third order ones. (a) $\rho_2 = 10$, (b) $\rho_2 = 7.5$, (c) $\rho_2 = 5$, (d) $\rho_2 = 2.5$.

The two equations can be simultaneously satisfied only if $F(\vartheta)$ vanishes. For a binary system, this equation has six roots:

$$\begin{aligned}
&\vartheta = 0\\
&\vartheta = \pi\\
&\vartheta = \arccos\frac{-3\rho_2+\sqrt{256m_1+9\rho_2^2}}{32\sqrt{m_1}}\\
&\vartheta = -\arccos\frac{-3\rho_2+\sqrt{256m_1+9\rho_2^2}}{32\sqrt{m_1}}\\
&\vartheta = \arccos\frac{-3\rho_2-\sqrt{256m_1+9\rho_2^2}}{32\sqrt{m_1}}\\
&\vartheta = -\arccos\frac{-3\rho_2-\sqrt{256m_1+9\rho_2^2}}{32\sqrt{m_1}}
\end{aligned} \tag{14}$$

The first two are the cusps along the y_1–axis already present at the second order. The second two are the modification of the other two Chang–Refsdal cusps (in fact, when ρ_2 tends to infinity, they approach $\pm\frac{\pi}{2}$). The last two cusps are imaginary for $\rho_2 > 4\sqrt{m_1}$, but become real for lower distances, giving rise to a "butterfly" geometry around $\vartheta = \pi$ (Fig. 3). For binary systems, this is not what happens in exact results which always yield four cusps; yet, with more than two lenses, an increment in the number of cusps is effectively present when the separation between the lenses is not very high.

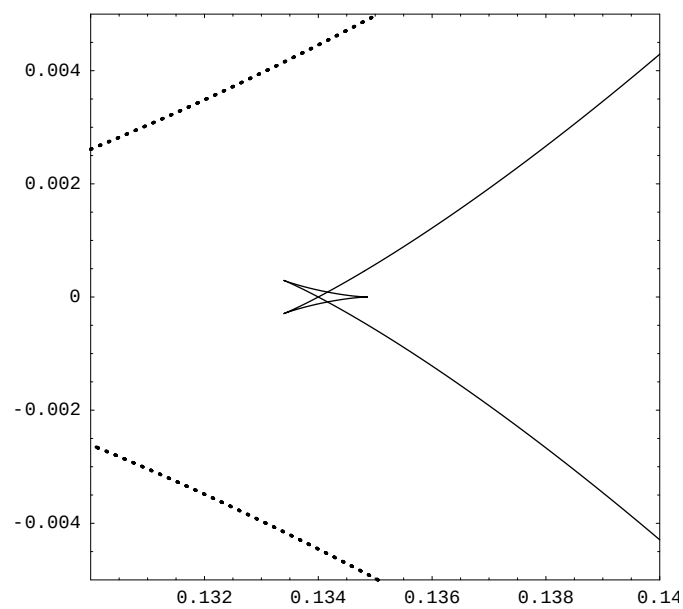

Fig. 3. A particular of Fig. 2d showing the two additional cusps yielded by the perturbative expansion.

Having at our disposal good approximate expressions for the caustics of multiple lenses, an interesting quantity to compute is the area covered by these curves in the source plane. We do this calculation for an arbitrary multiple lens at the second order and for a binary system at the third order.

For the Chang–Refsdal approximation, again we assume that the angle φ defined in Eq. (8) is zero. Then the caustic is oriented along the y_1–axis and its area is given by:

$$A = 2 \int_{y_1^{\min}}^{y_1^{\max}} y_2 \mathrm{d}y_1 \tag{15}$$

We have previously expressed y_2 and y_1 as functions of ϑ. Considering that, when $\vartheta = 0$, y_1 is $y_1^{\max}$ and, when $\vartheta = \pi$, y_1 is $y_1^{\min}$, also considering that y_2 is negative for $0 < \vartheta < \pi$ and positive for $\pi < \vartheta < 2\pi$, the area can be calculated by the following integral:

$$A = \int_0^{2\pi} y_2(\vartheta) \frac{\mathrm{d}y_1}{\mathrm{d}\vartheta} \mathrm{d}\vartheta \tag{16}$$

The result is:

$$A = \frac{3}{2}\pi m_1 \gamma^2 \tag{17}$$

Remembering the expression (7) for γ in the physical system we are analyzing, we see that the main dependences of the area of the caustic on its parameters are quadratic in the masses and inverse fourth power type in the distances.

For the third order calculation, a direction of symmetry can no longer be determined unless we limit ourselves to the binary system. The procedure just exposed can be followed for $\rho_2 > 4\sqrt{m_1}$, when the geometry of the caustic remains unaltered. Then, we obtain a correction to the previous result:

$$\Delta A = 4\pi \frac{m_1^2 m_2^2}{\rho_2^6} \tag{18}$$

So, not only the third order stretches the caustic but also adds a positive contribution to its area.

4. Close multiple lenses

Now we consider a set of n point-like lenses whose mutual distances are small compared to the total Einstein radius of the system. The starting point for our analysis is the Schwarzschild lens we would obtain if the total mass M were concentrated at the barycentre. The separations between the masses modulate the deviations from the Schwarzschild lens and then constitute the perturbative parameters in our expansion.

Now there is no privileged mass, so the most reasonable choice of the coordinates origin would be the centre of mass. We derive our results in the general case but we will choose the centre of mass system for some particular considerations.

Besides the main critical curve, which is a slight modification of the Einstein ring of the barycentral lens, some secondary critical curves are present near the centre of mass. They give rise to very small and far caustics which must be treated separately.

4.1. *Main critical curve and central caustic*

We consider our masses $m_1, ..., m_n$ placed at the positions $\mathbf{x}_1, ..., \mathbf{x}_n$. In polar coordinates, we put $\mathbf{x}_i = (\rho_i \cos\varphi_i; \rho_i \sin\varphi_i)$ and the generic coordinate in the lens plane is $\mathbf{x} = (r\cos\vartheta; r\sin\vartheta)$. Now we assume that $\rho_i \ll \sqrt{M}$ for each i. We carry the expansion of the Eq. (3) in series of powers of ρ_i up to the second order:

$$\begin{aligned}\det J \;=\; & 1 - \frac{M^2}{r^4} - \frac{4M}{r^5}\sum_{i=1}^{n} \rho_i m_i \cos(\vartheta - \varphi_i) + \\ & -\frac{2M}{r^6}\left[3\sum_{i=1}^{n} m_i M \rho_i^2 \cos(2\vartheta - 2\varphi_i) + \right. \\ & \left. +2\sum_{i,j=1}^{n} m_i m_j \rho_i \rho_j \cos(\varphi_i - \varphi_j)\right]\end{aligned} \tag{19}$$

The zero order terms give the Einstein ring for a point mass M at the origin. We see that interference terms arise at the second order and this is a good reason to stop our expansion to avoid long calculations at higher orders.

We can consider an expression analogue to Eq. (5) for the radial coordinate r:

$$r = \sqrt{M}\,(1 + \varepsilon_1 + \varepsilon_2 + ...) \tag{20}$$

where now $\varepsilon_j \sim \rho_i^j$. Substituting in Eq. (19) and expanding again, we can solve for the ε_j at each order:

$$\begin{aligned}\varepsilon_1 \;&=\; \frac{1}{M^{3/2}}\sum_{i=1}^{n} m_i \rho_i \cos(\vartheta - \varphi_i) \\ \varepsilon_2 &= \frac{1}{4M^3}\left\{6\sum_{i=1}^{n} M m_i \rho_i^2 \cos(2\vartheta - 2\varphi_i) + \right. \\ & \left. -\sum_{i,j=1}^{n} m_i m_j \rho_i \rho_j \left[5\cos(2\vartheta - \varphi_i - \varphi_j) + \cos(\varphi_i - \varphi_j)\right]\right\}\end{aligned} \tag{21}$$

Now that the perturbations to the critical curve are known (to the second order), we can find the caustic by use of the lens equation:

$$\begin{aligned}y_1(\vartheta) = & \frac{1}{M}\sum_{i=1}^{n} m_i \rho_i \cos\varphi_i + \frac{1}{2M^{5/2}}\sum_{i=1, j\neq i}^{n} m_i m_j \cdot \\ & \cdot \left\{3\rho_i^2 \cos(\vartheta - 2\varphi_i) + \rho_i^2 \cos(3\vartheta - 2\varphi_i) + \right. \\ & \left. -\rho_i \rho_j \left[3\cos(\vartheta - \varphi_i - \varphi_j) + \cos(3\vartheta - \varphi_i - \varphi_j)\right]\right\}\end{aligned} \tag{22}$$

$$\begin{aligned}y_2(\vartheta) = & \frac{1}{M}\sum_{i=1}^{n} m_i \rho_i \sin\varphi_i - \frac{1}{2M^{5/2}}\sum_{i=1, j\neq i}^{n} m_i m_j \cdot \\ & \cdot \left\{3\rho_i^2 \sin(\vartheta - 2\varphi_i) - \rho_i^2 \sin(3\vartheta - 2\varphi_i) + \right. \\ & \left. -\rho_i \rho_j \left[3\sin(\vartheta - \varphi_i - \varphi_j) - \sin(3\vartheta - \varphi_i - \varphi_j)\right]\right\}\end{aligned} \tag{23}$$

The first order perturbations only displace the caustic to the centre of mass. Choosing this point as the origin of the coordinate system, we can eliminate these terms. The second order

perturbations then yield the first non trivial deviations from the Schwarzschild lens. These perturbations are in the form of products between the masses forming the set of lenses, so there is no superposition principle here.

This perturbative expansion is tightly related to the quadrupole lens, as pointed out by Dominik (1999) for the binary lens. The quadrupole lens equation has the form:

$$\mathbf{y} = \mathbf{x} - \frac{M\mathbf{x}}{|\mathbf{x}|^2} + \frac{|\mathbf{x}|^2 \hat{Q}\mathbf{x} - 2\left(\mathbf{x}^{\mathrm{T}}\hat{Q}\mathbf{x}\right)\mathbf{x}}{|\mathbf{x}|^6} \tag{24}$$

where

$$\hat{Q} = \begin{pmatrix} Q_1 & Q_2 \\ Q_2 & -Q_1 \end{pmatrix} \tag{25}$$

is the quadrupole moment.

A careful analysis of the second order Jacobian of the close multiple system (19) in the centre of mass system reveals its coincidence with the Jacobian of the quadrupole lens expanded to the first order in the quadrupole moment, in polar coordinates. The elements of the matrix (25) are:

$$\begin{aligned} Q_1 &= M \sum_{i=1}^{n} m_i \rho_i^2 \cos(2\varphi_i) \\ Q_2 &= M \sum_{i=1}^{n} m_i \rho_i^2 \sin(2\varphi_i) \end{aligned} \tag{26}$$

The two eigenvalues of the quadrupole matrix have the same absolute value

$$Q = \sqrt{Q_1^2 + Q_2^2} \tag{27}$$

and opposite signs. We can define the orientation of the lens as the direction of the eigenvector corresponding to the positive eigenvalue, because it reduces to the positive x_1 direction in the case of a binary system with the masses aligned on the x_1–axis. The angle that this eigenvector forms with the x_1–axis is:

$$\varphi = \arctan \frac{Q - Q_1}{Q_2} \tag{28}$$

The caustic of a quadrupole lens is a diamond–shaped figure symmetric for all reflections on the eigenvalues axes. If we choose the coordinates so as to have the caustic oriented along the x_1–axis, the four cusps are at the positions $\vartheta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$. Fig. 4 shows some examples of central caustics for a binary system with a mass ratio $q = 3$. The second order approximation works well up to separations of tenths of the total Einstein radius. It is interesting to remark that the second order result is always symmetric even if the system of lenses has no symmetry at all. The symmetry is only lost when the perturbative hypothesis begins to be more forced, as we see in Fig. 4a.

For the extension of the central caustic of a system of close multiple lenses, we can directly refer to a well oriented quadrupole lens caustic expanded to the first order in Q. The area of such a caustic can be calculated with the same procedure explained in the previous section for Chang–Refsdal caustics. The result is:

$$A = \frac{3\pi Q^2}{2M^3} \tag{29}$$

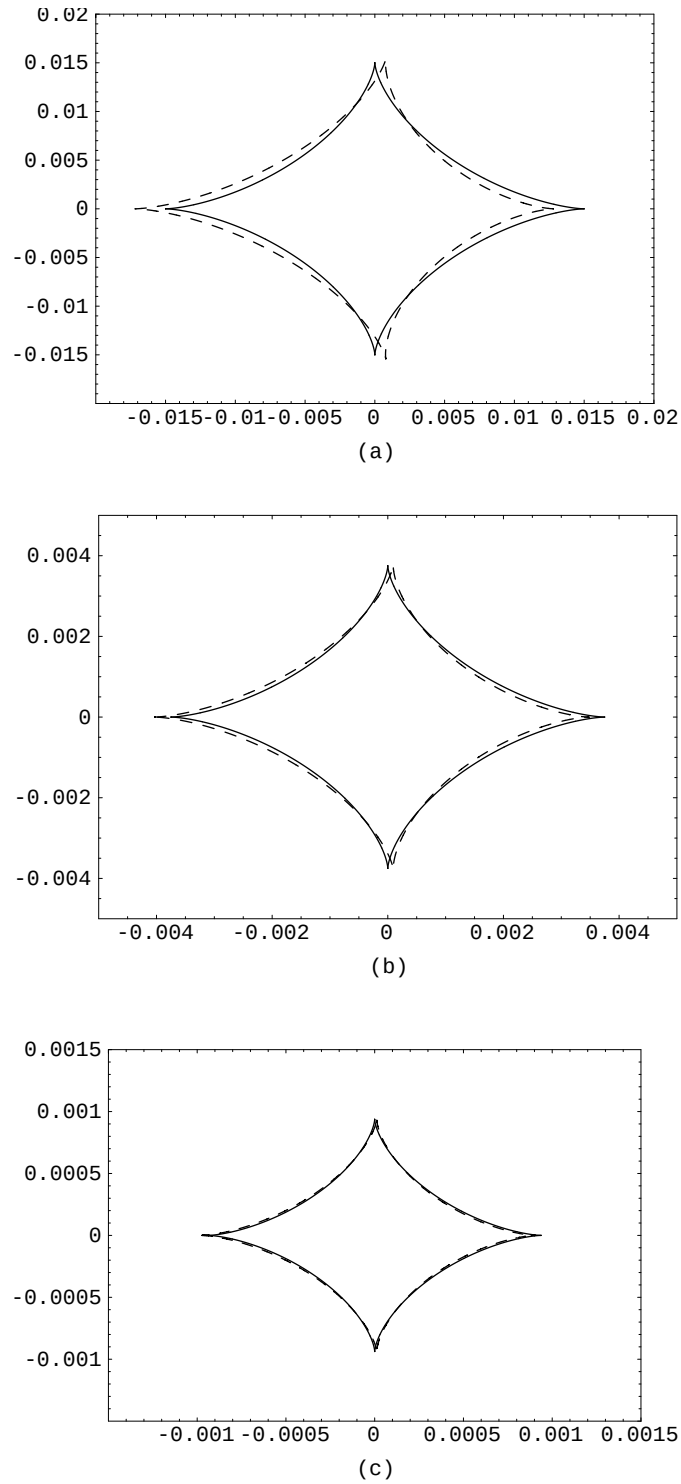

Fig. 4a–c. Central caustic of a binary lens with $m_1 = 0.75$ and $m_2 = 0.25$ and separation (a) 0.2, (b) 0.1, (c) 0.05. The dashed curve is the exact caustic and the solid is the perturbative one.

Recalling Eqs. (26) and (27), we see that the area goes as the fourth power of the distances between the masses, while the mass dependence is not so immediate because of the presence of the cube of the total mass in the denominator.

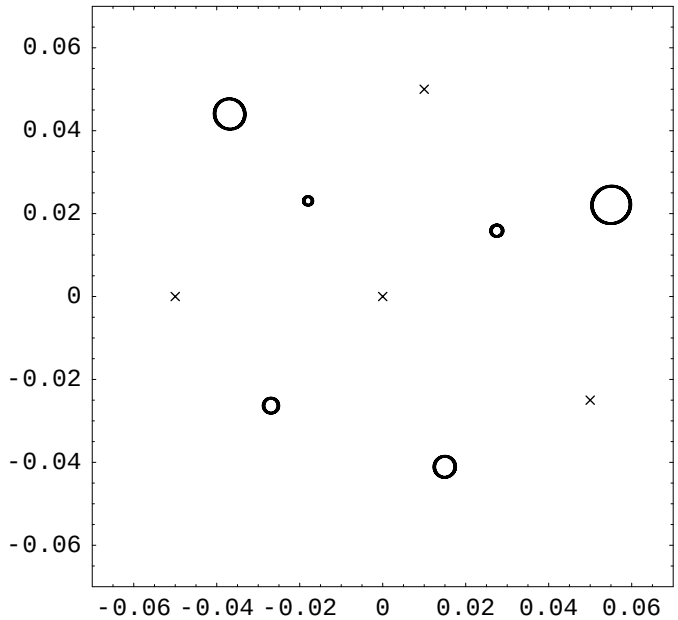

Fig. 5. Secondary critical curves generated by four equal masses, represented by the small crosses in the figure.

4.2. Secondary caustics

In the neighbourhood of the centre of mass of the system, some critical curves in the form of small ovals can appear (see Fig. 5). The number of these curves is not fixed by the number of lenses but depends on the position and the values of the masses composing the system. The binary lens is the only case where the number of critical curves is always fixed to two.

The perturbative approach can be used in the case of the binary lens to find compact formulae for the positions of the secondary critical curves and consequently the positions of the caustics. The general multiple lens involves high degree equations which do not allow any analytical considerations.

Let's consider two masses m_1 and m_2 separated by a distance a along the x_1–axis. In the centre of mass system, the equation $\det J = 0$ can be rationalized in the following form:

$$\begin{aligned}&\left[\left(x_1-\frac{am_2}{M}\right)^2+x_2^2\right]^2\left[\left(x_1+\frac{am_1}{M}\right)^2+x_2^2\right]^2+\\&-m_2^2\left[\left(x_1-\frac{am_2}{M}\right)^2+x_2^2\right]^2-m_1^2\left[\left(x_1+\frac{am_1}{M}\right)^2+x_2^2\right]^2+\\&-2m_1m_2\left[\left(x_1-\frac{am_2}{M}\right)^2+x_2^2\right]\left[\left(x_1+\frac{am_1}{M}\right)^2+x_2^2\right]+\\&-8m_1m_2x_2^2\left(x_1-\frac{am_2}{M}\right)\left(x_1+\frac{am_1}{M}\right)=0\end{aligned} \tag{30}$$

where $M = m_1 + m_2$.

We search for solutions of the first order in the separation a. So we put $x_1 = a\Delta x_1$, $x_2 = a\Delta x_2$ and expand in powers of a. The first non trivial order is the fourth, so the equation is of the fourth order in Δx_1 and Δx_2. Introducing the polar coordinates $\Delta x_1 = \Delta r \cos\vartheta$, $\Delta x_2 = \Delta r \sin\vartheta$, our equation can be solved for Δr. The four solutions

$$\begin{aligned}\Delta r \;=\; &\pm\frac{1}{M}\left[(m_2-m_1)\cos\vartheta+\sqrt{m_1m_2}\sin\vartheta+\right.\\&\left.\pm i\left(\sqrt{m_1m_2}\cos\vartheta+(m_1-m_2)\sin\vartheta\right)\right]\end{aligned} \tag{31}$$

are real only if

$$\sqrt{m_1m_2}\cos\vartheta+(m_1-m_2)\sin\vartheta=0 \tag{32}$$

This equation is satisfied by the angles:

$$\vartheta=\pm\arccos\frac{\pm(m_2-m_1)}{\sqrt{m_1^2+m_2^2-m_1m_2}} \tag{33}$$

Inserting these values in Eq. (31), discarding the negative solutions, and returning to the cartesian coordinates, we have:

$$\Delta x_1 \;=\; \pm\frac{m_1-m_2}{M} \tag{34}$$

$$\Delta x_2 \;=\; \pm\frac{\sqrt{m_1m_2}}{M} \tag{35}$$

If we put these values in the original equation, there are only two acceptable solutions:

$$x_1 \;=\; -a\frac{m_1-m_2}{M} \tag{36}$$

$$x_2 \;=\; \pm a\frac{\sqrt{m_1m_2}}{M} \tag{37}$$

These two points represent the centre of the small ovals at the first order in the separation between the two masses. The shape of the curves can be studied only at higher orders but high degree polynomials heavily complicate the calculations.

Through the lens equation, we can find the positions of the corresponding caustics:

$$y_1 \;=\; \frac{m_1-m_2}{a} \tag{38}$$

$$y_2 \;=\; \pm\frac{2\sqrt{m_1m_2}}{a} \tag{39}$$

We notice that when a tends to zero, the two caustics become infinitely far.

These formulae are compatible with those for the positions of the couple of planetary caustics of planets internal to the Einstein ring, given in (Bozza 1999), in the limit of planets very close to the star.

5. Planetary systems

A particularly interesting case of multiple lensing is that of planetary systems. Here a stellar mass is surrounded by planets having masses much smaller. Their effects can usually be treated as slight perturbations to the main lensing object.

The structure of the caustics of this lens has been widely explored by perturbative methods in a previous work (Bozza 1999). Here we shall recall the main formulae to complete the picture of perturbative results in multiple lensing and make some additional considerations.

We can distinguish between the central caustic which is generated by the distortion of the star's Einstein ring and the planetary caustics which are the images of the planetary critical curves and can be well approximated by Chang–Refsdal critical curves.

5.1. Central caustic

The central caustic of a star with mass m_1 placed at the origin, surrounded by planets with masses $m_2, ..., m_n$ placed at $\mathbf{x}_2, ..., \mathbf{x}_n$ respectively is expressed by the following parametric form:

$$y_1(\vartheta) = 2\sqrt{m_1}\varepsilon(\vartheta)\cos\vartheta - \sum_{i=2}^{n} \frac{m_i \Delta_{i1}^0}{\left[\left(\Delta_{i1}^0\right)^2 + \left(\Delta_{i2}^0\right)^2\right]} \tag{40}$$

$$y_2(\vartheta) = 2\sqrt{m_1}\varepsilon(\vartheta)\sin\vartheta - \sum_{i=2}^{n} \frac{m_i \Delta_{i2}^0}{\left[\left(\Delta_{i1}^0\right)^2 + \left(\Delta_{i2}^0\right)^2\right]} \tag{41}$$

where

$$\varepsilon(\vartheta) = \frac{1}{2}\cos 2\vartheta \sum_{i=2}^{n} m_i \frac{\left(\Delta_{i1}^0\right)^2 - \left(\Delta_{i2}^0\right)^2}{\left[\left(\Delta_{i1}^0\right)^2 + \left(\Delta_{i2}^0\right)^2\right]^2} + \\ + \sin 2\vartheta \sum_{i=2}^{n} m_i \frac{\Delta_{i1}^0 \Delta_{i2}^0}{\left[\left(\Delta_{i1}^0\right)^2 + \left(\Delta_{i2}^0\right)^2\right]^2} \tag{42}$$

and $\Delta_i^0 = \left(\sqrt{m_1}\cos\vartheta - x_{i1}; \sqrt{m_1}\sin\vartheta - x_{i2}\right)$.

In many studies of the central caustic of a planetary system, a principle of duality between planets external and internal to the Einstein ring (hereafter, we shall simply refer to them as external and internal planets, respectively) was often claimed on the basis of the observation of the shape of numerical caustics (Griest & Safizadeh 1997; Dominik 1999). This principle can be directly verified on these formulae which are invariant under the transformation

$$\mathbf{x}_i \to \frac{m_1}{|\mathbf{x}_i|^2}\mathbf{x}_i \tag{43}$$

So the conjecture of duality has an effective analytical basis.

The central caustic is generally a self–intersecting curve, except for the case of the single planet, when it has the already encountered diamond shape. Then, its area can be calculated with the same method of the previous sections. The result is:

$$A = \frac{\pi m_2^2}{4m_1}\left(1 - \frac{\left(m_1 + \rho_2^2\right)\left(m_1^2 - 4m_1\rho_2^2 + \rho_2^4\right)}{\left|m_1 - \rho_2^2\right|^3}\right) \tag{44}$$

It positively diverges when $\rho_2 \to \sqrt{m_1}$. This happens because in this limit the perturbative approach is no longer valid and the fusion between the central and the planetary caustic occurs. The area vanishes when $\rho_2 \to 0$ or $\rho_2 \to \infty$ because, in these limits, the Schwarzschild lens is recovered and the central caustic reduces to a point.

5.2. Planetary caustics

The planetary critical curve is always localized in the neighbourhood of the planet and assumes the shape of an elongated ring, when the planet is outside of the star's Einstein ring, or splits into two specular ovals, when the planet is inside.

The zero point of the expansion of these critical curves is then the position $\mathbf{x}_2$ of the planet we are considering. The first non trivial order is $\mathbf{x} - \mathbf{x}_2 \sim \sqrt{m_2}$, since the critical curve of a very far planet is nothing but its Einstein ring with radius $\sqrt{m_2}$.

In (Bozza 1999), the first order planetary critical curves were derived and their relations with the Chang–Refsdal ones was discussed. It is interesting to go farther in the expansion to understand how this relation is broken and to see the effects of other planets.

It is useful to consider polar coordinates around the planet position $\mathbf{x}_2$. Let's then set $\mathbf{x} - \mathbf{x}_2 = (r\cos\vartheta; r\sin\vartheta)$. Now we expand the radial coordinate r:

$$r = \varepsilon_1 + \varepsilon_2 + ... \tag{45}$$

where $\varepsilon_i \sim m_2^{i/2}$.

The first two orders of the equation $\det J = 0$ give:

$$1 - \frac{m_1^2}{\rho_2^4} - \frac{m_2^2}{\varepsilon_1^4} - \frac{2m_1m_2\cos(2\vartheta - 2\varphi_2)}{\varepsilon_1^2\rho_2^2} + \\ + \frac{4m_2^2}{\varepsilon_1^5}\varepsilon_2 + \frac{4m_1m_2\cos(2\vartheta - 2\varphi_2)}{\varepsilon_1^3\rho_2^2}\varepsilon_2 + \\ + \frac{4m_1\left[m_1\varepsilon_1^2\cos(\vartheta - \varphi_2) + m_2\rho_2^2\cos(3\vartheta - 3\varphi_2)\right]}{\varepsilon_1\rho_2^5} = 0 \tag{46}$$

We see that the first row, representing the lowest order Jacobian, is just the Jacobian of the Chang–Refsdal lens m_2 with shear $\gamma = \frac{m_1}{\rho_2^2}$. Then the lowest order critical curve is exactly Chang–Refsdal:

$$\varepsilon_1 = \sqrt{m_2 \frac{\frac{m_1}{\rho_2^2}\cos(2\vartheta - 2\varphi_2) \pm \sqrt{1 - \frac{m_1^2}{\rho_2^4}\sin^2(2\vartheta - 2\varphi_2)}}{1 - \frac{m_1^2}{\rho_2^4}}} \tag{47}$$

According to the double sign, two branches are present. For external planets only the higher is real, while for internal planets both branches are real in two intervals centred on $\vartheta = \pm\frac{\pi}{2}$ (Bozza 1999).

We can notice that if we perform an expansion of Eq. (47) to the first order in the shear, we obviously get the same result of Sect. 3, with the roles of m_2 and m_1 interchanged. However, in this section we are analysing the perturbations in the masses of the planets, so this expansion is not interesting for our purposes, though it gives the connection with the previous calculation.

The full Eq. (46) can now be employed to find the second order perturbation:

$$\varepsilon_2 = -\frac{m_1\varepsilon_1^4\left[m_1\varepsilon_1^2\cos(\vartheta - \varphi_2) + m_2\rho_2^2\cos(3\vartheta - 3\varphi_2)\right]}{m_2\rho_2^3\left[m_1\varepsilon_1^2\cos(2\vartheta - 2\varphi_2) + m_2\rho_2^2\right]} \tag{48}$$

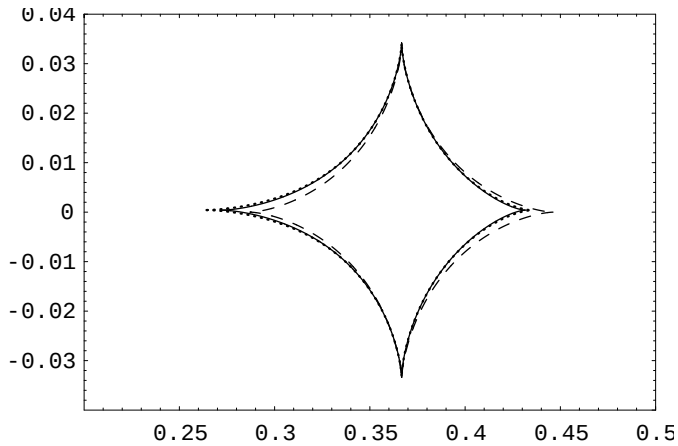

Fig. 6. Planetary caustic of a jovian planet ($m_2 = 10^{-3}$) at position $(1.2; 0)$. The caustic is perturbed by the presence of another jovian planet at $(0; 1.5)$. The dashed curve is the first order approximation, the solid one is the second order approximation, which is almost coincident with the exact one (dotted curve).

As usual, through the lens equation we can write down the formulae for the caustics:

$$\begin{aligned} y_1(\vartheta) \;=\; & \left(\rho_2 - \frac{m_1}{\rho_2}\right)\cos\varphi_2 + \left(\varepsilon_1 - \frac{m_2}{\varepsilon_1}\right)\cos\vartheta + \\ & + \frac{m_1\varepsilon_1\cos(\vartheta - 2\varphi_2)}{\rho_2^2} + \left(1 + \frac{m_2}{\varepsilon_1^2}\right)\varepsilon_2\cos\vartheta + \\ & + m_1\frac{\varepsilon_2\rho_2\cos(\vartheta - 2\varphi_2) - \varepsilon_1^2\cos(2\vartheta - 3\varphi_2)}{\rho_2^3} + \\ & - \sum_{i=3}^{n} m_i \frac{\rho_2\cos\varphi_2 - \rho_i\cos\varphi_i}{\rho_2^2 + \rho_i^2 - 2\rho_2\rho_i\cos(\varphi_2 - \varphi_i)} \end{aligned} \tag{49}$$

$$\begin{aligned} y_2(\vartheta) \;=\; & \left(\rho_2 - \frac{m_1}{\rho_2}\right)\sin\varphi_2 + \left(\varepsilon_1 - \frac{m_2}{\varepsilon_1}\right)\sin\vartheta + \\ & - \frac{m_1\varepsilon_1\sin(\vartheta - 2\varphi_2)}{\rho_2^2} + \left(1 + \frac{m_2}{\varepsilon_1^2}\right)\varepsilon_2\sin\vartheta + \\ & + m_1\frac{\varepsilon_2\rho_2\sin(\vartheta - 2\varphi_2) - \varepsilon_1^2\sin(2\vartheta - 3\varphi_2)}{\rho_2^3} + \\ & - \sum_{i=3}^{n} m_i \frac{\rho_2\sin\varphi_2 - \rho_i\sin\varphi_i}{\rho_2^2 + \rho_i^2 - 2\rho_2\rho_i\cos(\varphi_2 - \varphi_i)} \end{aligned} \tag{50}$$

At the second order a sum containing the effects of the other planets appears. It does not depend on ϑ and then only represents a displacement of the caustic towards the other planets.

Fig. 6 shows a comparison between the perturbative results and the exact curve for an external planet. The loss of symmetry of the figure, not present in the first approximation, is reliably followed by the second order curve. Moreover, it is to notice that the displacement of the caustic, due to other planets, is correctly reproduced.

The critical curves of internal planets are not defined for all values of ϑ. This fact brings in divergences at the border points where the solution switch from imaginary to real. So the second order results are very good everywhere except for a neighbourhood of these border values of the angle.

As for the other caustics, we can easily calculate the area of planetary caustics exploiting their symmetries. The case of external planets is analogue to the previous ones, so we can directly give the result:

$$A = 2m_2\left[\pi - 4E\left(\frac{\pi}{2}; \frac{m_1^2}{\rho_2^4}\right) + 2F\left(\frac{\pi}{2}; \frac{m_1^2}{\rho_2^4}\right)\right] \tag{51}$$

where:

$$F(\varphi; m) \;=\; \int_0^{\varphi} \left[1 - m\sin^2\vartheta\right]^{-1/2} d\vartheta \tag{52}$$

$$E(\varphi; m) \;=\; \int_0^{\varphi} \left[1 - m\sin^2\vartheta\right]^{1/2} d\vartheta \tag{53}$$

are the elliptic integrals of the first and the second kind respectively.

The calculation of the area for the caustics of the internal planets is not very different. We have to integrate on the higher branch and subtract the integral of the lower so as to obtain the area included between the two branches. The formula (16) must then be modified this way:

$$A = \int_{\vartheta_{\min}}^{\vartheta_{\max}} \left(y_2(\vartheta)\frac{\mathrm{d}y_1}{\mathrm{d}\vartheta}\right)_{h.b.} \mathrm{d}\vartheta - \int_{\vartheta_{\min}}^{\vartheta_{\max}} \left(y_2(\vartheta)\frac{\mathrm{d}y_1}{\mathrm{d}\vartheta}\right)_{l.b.} \mathrm{d}\vartheta \tag{54}$$

where $\vartheta_{\min}$ and $\vartheta_{\max}$ are the extremes of the interval where the two branches are real.

The result is:

$$\begin{aligned} A \;=\; & m_2\left(2E\left(2\vartheta_{\min}; \frac{m_1^2}{\rho_2^4}\right) - 2E\left(2\vartheta_{\max}; \frac{m_1^2}{\rho_2^4}\right) + \right. \\ & \left. - F\left(2\vartheta_{\min}; \frac{m_1^2}{\rho_2^4}\right) + F\left(2\vartheta_{\max}; \frac{m_1^2}{\rho_2^4}\right)\right) \end{aligned} \tag{55}$$

Of course, as there are two planetary caustics for internal planets, this number must be doubled.

The area of the planetary caustics is of order m_2. In fact it is well known that they are much greater than the central caustic, which is instead of order m_2^2 (see Eq. (44)) (Dominik 1999).

In Fig. 7 we see a plot of the area of the planetary caustic as a function of the separation of the planet from the star. We see the divergence in correspondence of the "resonant lensing" situation ($\rho_2 = \sqrt{m_1}$) where the perturbative theory cannot be applied.

6. Conclusions

The results presented in this work prove the great flexibility of perturbative methods in the resolution of problems in gravitational lensing. By different choices of perturbative expansions, it is possible to obtain very useful analytical approximations, which can be employed for further applications. In the case of a discrete set of point-like lenses, we gain a deep understanding of the distortion effects with respect to the Schwarzschild lens.

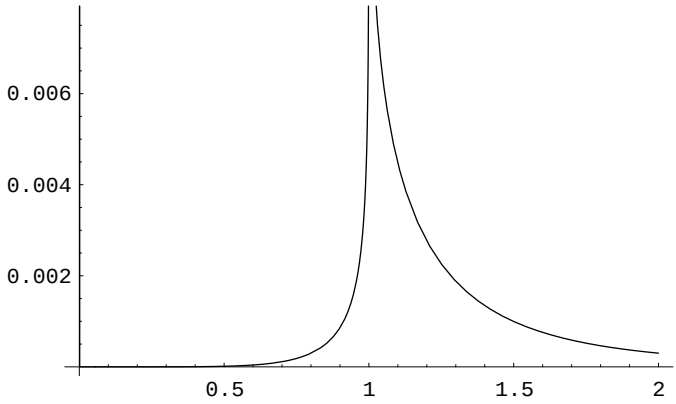

Fig. 7. Area of the planetary caustics as a function of the distance of the planet from the star.

This is the only analytical knowledge we have about the shape of caustics in multiple lenses.

Carrying the expansions to higher orders allows some interesting considerations about the validity of some approximations such as the Chang–Refsdal or the quadrupole lens and the limits of applicability of a superposition principle for the effects coming from different masses.

The calculation of the area of the caustics is another important possibility offered by the achievement of these analytical approximations.

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