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Dynamical stability for the gravitational evolution of a homogeneous polytrope

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Abstract. The dynamic stability of the spherical gravitational evolution (collapse or expansion) for a homogeneous polytropic gas with any exponent γ, is studied using the lagrangian formalism. We obtain the analytical expression for density perturbations at the first order.

In the case $\gamma = 4/3$, the Jeans' criterion is easily generalized to a self-similar expanding background. The collapsing case is found to be always unstable. The stability of density modes obtained for $\gamma \neq 4/3$ does not introduce any conditions on the wavelength perturbation, but only a criterion on the polytropic index. As a result, stability is obtained for an expanding gas provided $\gamma < 4/3$, and for a collapsing one, for $\gamma > 5/3$.

Key words: stars: formation – hydrodynamics – instabilities – accretion, accretion disks

1. Introduction

Within the framework of high energy laser experiments, the study of dynamic stability for a gas in a microtarget under an external field becomes experimentally possible (Kane et al. 1997a, 1997b, 1999; Remington et al. 1997). The extrapolation of the results to large self-gravitating masses (Ryutov et al. 1999) opens the way to the "laboratory astrophysics". In particular, instabilities in giant molecular hydrogen clouds can be considered as initial seeds to the gravitational collapse and, consequently, to the birth of stars. Due to simple models, it is therefore conceivable to find conditions on protostellar configurations which do, or do not lead to their own gravitational collapse. A first method for dealing with this process is the analysis of non-linear equations by eulerian self-similar techniques (Blottiau et al. 1988; Bouquet et al. 1985a; Shu 1977; Yahil 1983). The lagrangian way, often prefered in numerical studies, has also been used by Blottiau (1989). However, whereas the numerical results seem to agree with theoretical stability obtained from self-similarity analysis, (Blottiau et al. 1988), analytical lagrangian approaches remain in discrepancy (Bonnor 1957; Buff & Gerola 1979).

In this study we use widely and intensively the analytical lagrangian approach to check and to compare our results with the ones previously found by eulerian self-similar ways. The "predilection" model is still the one describing the evolution of a homogeneous polytropic spherical mass. The stability is discussed from the study of the time evolution of density perturbations at the first order (Bonnor 1957; Bouquet 1999). From the simplicity of the assumptions, it is obvious that such treatment cannot describe thoroughly stellar explosions or collapses. However, it can provide relevant conditions and results for the starting processes leading to the dynamic evolution. On the other hand, laboratory experiments will allow us to delimit the domain of validity of such "simple" models, but which are almost the only ones fully computable analytically.

In Sect. 2, similar results of Blottiau et al. (1988) and Bouquet (1999) are refered to and used to generalize the Jeans' criterion in the case of an expanding homogeneous polytropic gas with $\gamma = 4/3$.

Sect. 3 deals with the lagrangian description. The system which consists in the hydrodynamical equations for the density perturbations at first order has been solved analytically. The stability criteria are obtained from the study of the asymptotic behaviour of these solutions for any value of γ. The analytical expression is obtained from an infinite summation over eigenmodes satisfying the appropriate boundary conditions. It turns out that the results confirm and extend those presented in Sect. 2. The conclusion is given in Sect. 4.

2. Eulérian collapse

2.1. Previous results

The study of self-gravitating configurations can be made with the use of scaling transformations (Bouquet et al. 1985a; Blottiau et al. 1988; Chièze et al. 1997; Hanawa & Nakayama 1997; Saigo & Hanawa 1998; Nakamura et al. 1999; Hanawa & Matsumoto 1999). The equations governing the evolution of the gravitational system, written in the new space (rescaled space) of transformed physical quantities (rescaled quantities) are often easier to solve and to understand than in the physical one. In

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particular, the dynamic stability problem may reduce to a static one. The Euler self-similar approach of Blottiau et al. (1988) deals with a homogeneous self-gravitating infinite mass which follows a polytropic equation of state:

$$P = K\rho^{\gamma} \tag{1}$$

with an exponent $\gamma = 4/3$ and where P and ρ are respectively the pressure and the density of the medium. The case $\gamma \neq 4/3$ was also studied but only in a numerical way. In the present paper, we first recall the Euler analytical study for $\gamma = 4/3$, and we recast it into the lagrangian frame. Second, we extend this approach, analytically, to any value of the polytropic exponent γ.

The evolution of the system is governed by the Euler, Poisson and continuity equations which read respectively:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + g \tag{2}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 g\right) = -4\pi\rho G \tag{3}$$

$$\frac{\partial \rho}{\partial t} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\rho v\right) \tag{4}$$

where r, t, $v(r,t)$ and $g(r,t)$ are respectively the radial position, the time, the eulerian velocity field and the value of the gravitational field at the event (r,t). A newtonian self-similar solution for these equations is a parabolic homogeneous collapse, therefore without any velocity at infinity (Blottiau et al. 1988; Henriksen & Wesson 1978):

$$r_o(m,t) = \hat{r_o}(1+\Omega t)^{\frac{2}{3}} \tag{5}$$

$$\rho_o(t) = \hat{\rho_o}(1+\Omega t)^{-2} \tag{6}$$

$$\hat{r_o} = \left(\frac{9Gm}{2\Omega^2}\right)^{\frac{1}{3}} \tag{7}$$

$$\hat{\rho_o} = \frac{\Omega^2}{6\pi G} \tag{8}$$

where $r_o(m,t)$ and $\rho_o(t)$ are, respectively, the position of the shell whose interior mass is m and the uniform density, both of them being taken at time t. The quantities $\hat{r_o}$ and $\hat{\rho_o}$ represent, respectively, the position of the shell (labelled by m) and the uniform density at the initial time $t = 0$. The parameter Ω (Blottiau et al. 1988; Bouquet et al. 1985a) is an integrating constant which reflects the freedom in the choice of the time-origin. It is just proportional to the *initial* Jeans frequency, $\hat{\Omega}_J$, given by:

$$\hat{\Omega}_J = \sqrt{4\pi\hat{\rho_o}G} \tag{9}$$

and the relationship between Ω and $\hat{\Omega}_J$ is just (Eq. (8)):

$$\Omega^2 = \frac{3}{2}\hat{\Omega}_J^2. \tag{10}$$

This parameter may seen redundant with the Jeans frequency. However, we are going to explain how relevant it can be. Usually, one works with the variable t which varies from $-\infty$ up to $+\infty$. But, generally, a singularity arises at $t = 0$ which, in our opinion, is not so easy to be managed. For instance, the spatial extension of the configuration must be zero.

In opposition, the introduction of the parameter Ω allows us to leave $t = 0$ as the initial time in any case. In order to describe expansions, we take the positive solution in Eq. (10), $\Omega = +\sqrt{3/2}\,\hat{\Omega}_J$ and t elapses from 0 up to $+\infty$. In contrast, collapses will be obtained for $\Omega = -\sqrt{3/2}\,\hat{\Omega}_J$ (negative solution in Eq. (10)) and the final gravitational singularity will arise for $1+\Omega t_{sing} = 0$, i.e., at $t_{sing} = -1/\Omega$ (which, of course, is a positive value since Ω is negative). It should be noted that this very simple remark provides, in a very straightforward and easy way, the free fall-time for a homogeneous gravitational system:

$$t_{ff} = \sqrt{\frac{1}{6\pi\hat{\rho_o}G}}.$$

In addition, and for any situation, t remains positive and its initial value is finite and is always $t = 0$. Moreover, since at $t = 0$ no singularity arises, the extension of the configuration can be not (and is not) zero whereas removing the parameter Ω could give rise to expansions beginning at the singularity ($r = 0$ and $t = 0$) which, in our mind, does not make sense. Thanks to the parameter Ω, we may specify any spatial profile (for the density, for the velocity, etc.) at the initial time and study its influence on the further evolution of the system. These properties are very convenient from a physical viewpoint. This parameter is not only useful in astrophysical studies but it can also be used very fruitfully in evolution problems: plasma physics (Bouquet et al. 1985b; Burgan et al. 1978, 1983), nonlinear evolution equations and dynamic systems (Bouquet 1995; Cairó & Feix 1998; Cairó & Feix 1999) and other interesting domains.

We are going to see that by means of scaling transformations, the time dependence of the solutions (Eq. (5) and Eq. (6)) can be removed. The dynamic problem of stability reduces, therefore, to a static one. The new physical quantities in this rescalled space are written with a hat " ˆ " and, according to Blottiau et al. (1988) and Bouquet et al. (1985a), we have:

$$\hat{r} = r(1+\Omega t)^{\gamma-2} \tag{11}$$

$$\hat{t} = \frac{1}{\Omega}\ln(1+\Omega t) \tag{12}$$

$$\hat{\rho}(\hat{r},\hat{t}) = (1+\Omega t)^2\rho(r,t) \tag{13}$$

$$\hat{P}(\hat{r},\hat{t}) = (1+\Omega t)^{2\gamma}P(r,t) \tag{14}$$

$$\hat{g}(\hat{r},\hat{t}) = (1+\Omega t)^{\frac{4}{3}}g(r,t) \tag{15}$$

where we set $\gamma = 4/3$ in the following. From these equations, it is clear that at the initial time $t = \hat{t} = 0$, the quantities with and without " ˆ " coincide (the rescaled space and the physical one are initially identical). Moreover, for $\Omega > 0$ (t and $\hat{t}$ go from 0 to $+\infty$), the transformation describes an expansion, while for $\Omega < 0$, the configuration collapses up to the central singularity in a finite time given by $t = -1/\Omega$. It must be noted that for this case ($\Omega < 0$), the times t and $\hat{t}$ vary respectively, in the ranges $[0, -1/\Omega[$ and $[0, +\infty[$. It can be easily shown (Blottiau et al. 1988; Bouquet et al. 1985a; Bouquet 1999) that the system formed by Eqs. (2) to (4), becomes stationary in the new space without any explicit dependence upon $\hat{t}$. Moreover, assuming that:

$$\hat{\rho}(\hat{r},\hat{t}) \;=\; \hat{\rho_o} + \delta\hat{\rho}(\hat{r},\hat{t}) \qquad (16)$$

$$\delta\hat{\rho}(\hat{r},\hat{t}) \;=\; A(\hat{t})\sin(\hat{k}\hat{r})/(\hat{k}\hat{r}) \qquad (17)$$

where $\hat{k}$ is the wave number in the rescaled space and where:

$$A(\hat{t}) \;=\; A_o \exp(\omega\hat{t}). \qquad (18)$$

The quantities A_o and ω are two constants. The study of the evolution of the perturbations for the various transformed quantities, at the first order, provides a dispersion equation for the density modes. This dispersion relationship is (Blottiau et al. 1988):

$$\omega^2 + \frac{\Omega}{3}\omega + \hat{k}^2\hat{c}^2 - \hat{\Omega}_J^2 = 0. \qquad (19)$$

where $\hat{c}$ is the initial sound velocity given by:

$$\hat{c}^2 = \gamma K \hat{\rho_o}^{\gamma-1} \qquad (20)$$

and where $\hat{\Omega}_J$ is related to Ω from Eq. (10). Their physical values at time t are obtained from the inverse scale transformation (Blottiau et al. 1988):

$$\Omega_J(t) \;=\; \hat{\Omega}_J(1+\Omega t)^{-1} \qquad (21)$$

$$c(t) \;=\; \hat{c}(1+\Omega t)^{-\frac{1}{3}}. \qquad (22)$$

Coming back to Eq. (19), the eigenmodes are obtained by the resolution of the dispersion equation, quadratic in ω with the discriminant:

$$\Delta_{\hat{k}} = \frac{25\hat{\Omega}_J^2 - 24\hat{k}^2\hat{c}^2}{6}. \qquad (23)$$

According to the sign of $\Delta_{\hat{k}}$, we obtain, therefore, the two solutions for ω:

$$\hat{k} < \hat{k}_{trans} \Rightarrow \omega_{r\pm} \;=\; -\frac{\Omega}{6} \pm \frac{\sqrt{\Delta_{\hat{k}}}}{2} \qquad (24)$$

$$\hat{k} > \hat{k}_{trans} \Rightarrow \omega_{i\pm} \;=\; -\frac{\Omega}{6} \pm i\frac{\sqrt{-\Delta_{\hat{k}}}}{2} \qquad (25)$$

with:

$$\hat{k}_{trans} = \sqrt{\frac{25}{24}}\frac{\hat{\Omega}_J}{\hat{c}}. \qquad (26)$$

The imaginary values, $\omega_{i\pm}$, for ω are obtained for $\hat{k} > \hat{k}_{trans}$. These solutions give rise to evanescent modes. This is the stability criterion, in the rescaled space, found by Blottiau et al. (1988). In the next section, we are going to show that it is equivalent to the Jeans' criterion in the physical space.

2.2. Equivalence to the Jeans' criterion

The time dependence of the density perturbations, in the physical space, is deduced from Eq. (12), Eq. (17) and Eq. (18). We obtain:

$$A(t) = A_o(1+\Omega t)^{\frac{\omega}{\Omega}}. \qquad (27)$$

Moreover, since ρ and $\delta\rho$ rescale in the same way, we have, at the first order:

$$\frac{\delta\rho}{\rho} = \frac{\delta\hat{\rho}}{\hat{\rho}} \sim \frac{\delta\hat{\rho}}{\hat{\rho}_o}. \qquad (28)$$

The asymptotic time evolution of $\delta\rho/\rho$ is, therefore, directly given by the real part sign of the exponent ω/Ω.

First, for $\hat{k} < \hat{k}_{trans}$, ω is real but a critical value of $\hat{k}$, $\hat{k}_{crit}$, makes changing the sign of ω_{r+}/Ω. With Eq. (24) it comes:

$$\hat{k} < \hat{k}_{crit} < \hat{k}_{trans} \;\Rightarrow\; \frac{\omega_{r+}}{\Omega} > 0 \qquad (29)$$

$$\hat{k}_{crit} < \hat{k} < \hat{k}_{trans} \;\Rightarrow\; \frac{\omega_{r+}}{\Omega} < 0 \qquad (30)$$

where

$$\hat{k}_{crit} = \frac{\hat{\Omega}_J}{\hat{c}}. \qquad (31)$$

We notice that $\hat{k}_{crit}$ corresponds to the value given by Jeans (1961). In addition, keeping in mind the permanent negative sign of ω_{r-}/Ω, and since the solution is written as the linear superposition of the two modes, the asymptotic behaviour is given by the leading term. We get:

$$\hat{k} < \hat{k}_{crit} < \hat{k}_{trans} \Rightarrow \begin{cases} \forall\Omega > 0 \;\; \lim_{t\to\infty} \frac{\delta\rho}{\rho} = \infty \\ \forall\Omega < 0 \;\; \lim_{t\to-\frac{1}{\Omega}} \frac{\delta\rho}{\rho} = \infty \end{cases} \qquad (32)$$

$$\hat{k}_{crit} < \hat{k} < \hat{k}_{trans} \Rightarrow \begin{cases} \forall\Omega > 0 \;\; \lim_{t\to\infty} \frac{\delta\rho}{\rho} = 0 \\ \forall\Omega < 0 \;\; \lim_{t\to-\frac{1}{\Omega}} \frac{\delta\rho}{\rho} = \infty. \end{cases} \qquad (33)$$

Second, for $\hat{k} > \hat{k}_{trans}$, the imaginary part of ω (given by Eq. (25)) produces an oscillating contribution to $A(t)$. In contrast, the real part gives a time-power evolution with a negative exponent $-1/6$. Consequently, one gets:

$$\hat{k} > \hat{k}_{trans} \Rightarrow \begin{cases} \forall\Omega > 0 \;\; \lim_{t\to\infty} \frac{\delta\rho}{\rho} = 0 \\ \forall\Omega < 0 \;\; \lim_{t\to-\frac{1}{\Omega}} \frac{\delta\rho}{\rho} = \infty. \end{cases} \qquad (34)$$

Eqs. (32) to (34) emphasize that the asymptotic behaviour of the density perturbations depends strongly on the value of the wave number $\hat{k}$, which is connected to the value of k in the physical space by the inverse transformation of Eq. (11) (Blottiau et al. 1988):

$$k = \hat{k}(1+\Omega t)^{-\frac{2}{3}}. \qquad (35)$$

As a result, for explosions ($\Omega > 0$), the density perturbations are unstable as soon as the instantaneous wave number, $k(t)$, satisfies $k(t) < k_{crit}(t)$ with $k_{crit}(t) = \Omega_J(t)/c(t)$ is the instantaneous Jeans wave number. However, since k and k_{crit} have the same time-dependence, if the criteria is satisfied at $t = 0$, it is satisfied for any time. Consequently, the result obtained by Jeans (1961) for a static background is also valid for an expanding one provided $\gamma = 4/3$. This is closely akin to Bonnor's results (1957). On the other hand, in the implosion case ($\Omega < 0$), we always have instabilities: every density perturbation is amplified during the collapse. Finally, note that the value $\hat{k}_{trans}$ is not relevant to stability, but indicates only changes in behaviour with wave number: beyond this value, the perturbation oscillates and increases, and below, it explodes as a time-power dependence.

2.3. Conclusion

The stability criterion for an eulerian self-similar evolution does not agree with the one given by Buff & Gerola (1979). Instead of Eq. (31), they find a Jeans wave number equal to $\sqrt{3/2}\, k_{crit}$. As a matter of fact, their dispersion equation, derived in the physical space, for a fixed mass collapse is:

$$\omega^2 - \frac{2}{3}\Omega_J^2 + k^2 c^2 = 0. \tag{36}$$

Buff & Gerola (1979) have chosen a density perturbation, at first order, under the form $A(t)\sin(kr)/(kr)$ with $A(t) = A_o \exp(\omega t)$. In our opinion, this ansatz is not possible. The reason for this is that, in opposition to our approach in which we obtain a second order automous differential equation for the density perturbations, they get a linearized equation with time-varying coefficients. But, in that case, it is well known that the exponential solution, $\exp(\omega t)$, is no longer valid. Consequently, the meaning of Eq. (36) is not clear and one would have to assume that ω be an explicit function of time. However, under this assumption, additional terms proportional to $d\omega/dt$ should appear and Eq. (36) would be modified. In the next section, an analytical lagrangian calculation is performed. It is shown that obtaining a dispersion relation is not necessary and we are going to recover and to extend the results found by Blottiau et al. (1988) and by Bonnor (1957).

3. Lagrangian collapse

Let M be the mass of a spherical homogeneous configuration with initial radius $\hat{R}$ submitted to its own gravitational field and initially at rest. In the following, the physical quantities will be expressed, either as a function of the lagrangian variable m (where m is the internal mass of a shell), or in terms of $\hat{r_o}$ (with $\hat{r_o}$ being the initial radius of the shell labelled by m), plus the time, t, in both cases. The stability is again studied via the time-evolution of density perturbations at the first order, $\delta\rho$. All parameters with the subscript "o" are associated with the non-perturbated solution. Finally, it must be pointed out that this study is performed analytically for any arbitrary value of the polytropic exponent.

3.1. Equation of evolution

The evolution of the non-perturbed system obeys the hydrodynamical equations (1) to (4) with the solution given by (5) and (6). The perturbation is then written in the form:

$$\rho(m,t) \;=\; \rho_o(t) + \delta\rho(m,t) \tag{37}$$

$$r(m,t) \;=\; r_o(m,t) + \delta r(m,t). \tag{38}$$

The solution will no longer be homogeneous and we have to keep the pressure gradient term in the Euler equation (2). This gradient is expressed as a function of the density according to the polytropic equation of state (1). After elimination of the zero order terms from Eqs. (5) and (6), the Euler equation reads:

$$\frac{\partial^2 \delta r}{\partial t^2} = -4\pi r_o^2 c^2 \frac{\partial \delta\rho}{\partial m} + \frac{8\pi}{3} G\rho_o \delta r. \tag{39}$$

The time-dependent sound velocity, c, is written at the zero order:

$$c^2(t) \;\simeq\; c_o^2(t) = \frac{\gamma P}{\rho} = \hat{c_o}^2 (1+\Omega t)^{2(1-\gamma)} \tag{40}$$

where

$$\hat{c_o}^2 \;=\; \gamma K \hat{\rho_o}^{\gamma-1}. \tag{41}$$

In addition, the conservation of mass from the non-perturbed to the perturbed configuration provides the second equation:

$$dm = 4\pi r^2 \rho dr = 4\pi r_o^2 \rho_o dr_o \tag{42}$$

which becomes, after some straight calculation:

$$(3m)^{\frac{2}{3}} \frac{\partial \delta r}{\partial m} + 2(3m)^{-\frac{1}{3}} \delta r = -\frac{\delta\rho}{(4\pi\rho_o^4)^{\frac{1}{3}}}. \tag{43}$$

The differential system formed by (39) and (43) can be solved by direct integration with the physical assumption that there is no perturbation at the center of the configuration ($\delta r << r_o$ at $r_o = 0$ gives $\delta r|_{r_o=0} = 0$), which is a zero mass point. The solution of Eq. (43) is, therefore:

$$\delta r(m,t) = -\frac{1}{(36\pi\rho_o^4 m^2)^{\frac{1}{3}}} \int_0^m \delta\rho(\mu,t) d\mu. \tag{44}$$

Plugging this solution into Eq. (39), the evolution equation for the density perturbation writes:

$$m^{\frac{4}{3}} \frac{\partial^2 \delta\rho}{\partial m^2} + \frac{4}{3} m^{\frac{1}{3}} \frac{\partial \delta\rho}{\partial m} = \\ = \frac{1}{(36\pi\rho_o)^{\frac{2}{3}} c_o^2} \left[\frac{\partial^2 \delta\rho}{\partial t^2} + 8\frac{|\Omega|}{\Omega} \sqrt{\frac{8\pi G\rho_o}{3}} \frac{\partial \delta\rho}{\partial t} + 24\pi G\rho_o \delta\rho \right]. \tag{45}$$

As expected, Eq. (45) is linear but with a partial differentiation with respect to the independant variables m and t. The eigenmodes may be found by the technique of separation of variables. Then, the general solution will be the superposition of all modes with the constraint that the boundary conditions must be satisfied.

3.2. Density eigenmodes

Introducing the separation of variables for $\delta\rho(m,t)$ under the form:

$$\delta\rho = \delta T(t) \delta R(m) \tag{46}$$

the equation for the mass dependence becomes:

$$\frac{d^2 \delta R}{dl^2} + \frac{2}{l}\frac{d\delta R}{dl} - 9\varepsilon\eta_k^2 \delta R = 0 \tag{47}$$

where the independant variable, l, is given by:

$$l = m^{1/3}. \tag{48}$$

In Eq. (47), we have decided to write the separation constant as $\varepsilon\eta_k^2$ with $\eta_k \geq 0$ which has the dimension $[M]^{-1/3}$. The parameter $\varepsilon = \pm 1$ has been introduced for choosing the sign. From η_k, let us introduce, now, the wave number, k, labelling each density eigenmode, and a dimensionless number, lets say N_k, which will help to separate the various stability regimes:

$$k = (36\pi)^{\frac{1}{3}} \hat{\rho_o}^{\frac{1}{3}} \eta_k \tag{49}$$
$$N_k = \frac{k\hat{c_o}}{|\Omega|}. \tag{50}$$

The quantity η_k is equivalent to a "massic pulsation" since we have:

$$3\eta_k m^{\frac{1}{3}} = k\hat{r_o}. \tag{51}$$

On the other hand, it comes from Eq. (45) that the time differential equation, for $\gamma \neq 4/3$, is:

$$z^2 \frac{d^2\delta S}{dz^2} + z\frac{d\delta S}{dz} - (\varepsilon z^2 + n^2)\delta S = 0 \tag{52}$$

where the new variable, z, and function, $\delta S(z)$, are given by:

$$z = \frac{N_k(1+\Omega t)^\mu}{|\mu|} \tag{53}$$
$$\delta T(t) = N_k^{-\frac{13}{6\mu}} (1+\Omega t)^{-\frac{13}{6}} \delta S(z) \tag{54}$$
$$\mu = \frac{4}{3} - \gamma \tag{55}$$
$$n = \frac{5}{6|\mu|}. \tag{56}$$

The special case $\gamma = 4/3$ gives, from Eq. (45), the following second order differential equation:

$$\frac{d^2\delta T}{dy^2} + \frac{13}{3}\frac{d\delta T}{dy} + (4 - \varepsilon N_k^2)\delta T = 0 \tag{57}$$

with the new independant variable:

$$y = \ln(1+\Omega t). \tag{58}$$

It turns out that Eqs. (47) and (52) are the so-called classical and modified Bessel equations according to the value of ε. On the other hand, for $\gamma = 4/3$, Eq. (57) is a linear homogeneous differential equation with constant coefficients. It is therefore readily integrable in terms of the exponential functions. This separation naturally leads us to distinguish between the eigenmodes for $\gamma = 4/3$ from the ones for $\gamma \neq 4/3$ (see Sect. 3.2.2).

3.2.1. Mass dependence of the solution

The requirement of a finite value for the density perturbation at the center of the configuration restricts the solutions of Eq. (47) to be:

$$\delta R^k_{\varepsilon=-1}(m) \propto \frac{\sin(3\eta_k m^{\frac{1}{3}})}{3\eta_k m^{\frac{1}{3}}} \tag{59}$$
$$\delta R^k_{\varepsilon=1}(m) \propto \frac{\sinh(3\eta_k m^{\frac{1}{3}})}{3\eta_k m^{\frac{1}{3}}}. \tag{60}$$

Note that the hyperbolic sine appears in the case $\varepsilon = 1$.

3.2.2. Time dependence of the solution

Case $\gamma = 4/3$.

The roots, $s^\varepsilon_\pm(k)$, of the characteristic equation associated with Eq. (57) are:

$$s^\varepsilon_\pm(k) = -\frac{13}{6} \pm \frac{\sqrt{\Delta^\varepsilon_k}}{2} \tag{61}$$

where the discriminant is:

$$\Delta^\varepsilon_k = \frac{25 + 36\varepsilon N_k^2}{9} = \frac{25\hat{\Omega}_J^2 + 24\varepsilon k^2 \hat{c_o}^2}{9\hat{\Omega}_J^2}. \tag{62}$$

Then, for the hyperbolic modes ($\varepsilon = 1$), it becomes:

$$\delta T^k_+(t) = \beta_k (1+\Omega t)^{s^+_+(k)} + \gamma_k (1+\Omega t)^{s^+_-(k)} \tag{63}$$

where β_k and γ_k are two arbitrary real constants and where the superscript sign in the exponents is just the sign of the parameter ε.

The trigonometric modes (with $\varepsilon = -1$) introduce roots with imaginary part provided $N_k < 5/6$, i.e., $k < \sqrt{25/24}\,\hat{\Omega}_J/\hat{c_o}$. We have, therefore, two kinds of solution:

$$k < \sqrt{\frac{25}{24}} \frac{\hat{\Omega}_J}{\hat{c}_o} \Rightarrow \delta T^k_-(t) = \beta_k (1+\Omega t)^{s^-_+(k)} + \gamma_k (1+\Omega t)^{s^-_-(k)} \tag{64}$$

$$k > \sqrt{\frac{25}{24}} \frac{\hat{\Omega}_J}{\hat{c}_o} \Rightarrow \delta T^k_-(t) = \alpha_k (1+\Omega t)^{-\frac{13}{6}} \times \cos\frac{\sqrt{|\Delta^-_k|}\ln(1+\Omega t)}{2} \tag{65}$$

where α_k is an arbitrary real constant and where the unimportant phase in the cosine have been dropped. Let us notice the transition at the same value of the wave number than in the eulerian self-similar case, k_{trans}, which according to Eq. (26), is given by:

$$k_{trans} = \sqrt{\frac{25}{24}} \frac{\hat{\Omega}_J}{\hat{c}_o}. \tag{66}$$

It must be noted that, now, the transition wave number is no longer defined in a rescaled space, but it applies directly in the physical one.

Case $\gamma \neq 4/3$.

From the inversion of the independent variables (z and t in Eq. (53)) and the dependent ones (δS and δT in Eq. (54)), it turns out that for all k and for $\varepsilon = 1$, we have:

$$\delta T^k_+(t) = (1+\Omega t)^{-\frac{13}{6}} \times \left[\alpha_k I_n\left(\frac{N_k(1+\Omega t)^\mu}{|\mu|}\right) + \beta_k K_n\left(\frac{N_k(1+\Omega t)^\mu}{|\mu|}\right)\right] \tag{67}$$

where I_n and K_n are respectively, the modified Bessel functions of first and second kind of order n.

The other case is $\varepsilon = -1$, and we get:

$$\delta T_-^k(t) = (1+\Omega t)^{-\frac{13}{6}} \times \times \left[\alpha_k J_n\left(\frac{N_k(1+\Omega t)^\mu}{|\mu|}\right) + \beta_k Y_n\left(\frac{N_k(1+\Omega t)^\mu}{|\mu|}\right)\right]. \quad (68)$$

The functions J_n and Y_n are respectively the first and second, classical Bessel functions.

3.3. Stability of eigenmodes

From the analytical expressions of the eigenmodes, it is possible to derive their asymptotic behaviour. In the case where $\Omega > 0$ (expanding background), the time elapses from $t = 0$ to $t \to +\infty$. On the other hand, in a collapsing background ($\Omega < 0$), the initial time is again $t = 0$, while the final one is defined when the singularity at $r_o = 0$ arises, i.e., $t \to -1/\Omega$.

3.3.1. Stability of the eigenmodes for $\gamma = 4/3$

According to Eqs. (63)–(65), the asymptotic time behaviour of the perturbation is given by the value of the limit of $(1+\Omega t)^q$ where the exponent q is either $s_\pm^\varepsilon$ or $-13/6$, according to the studied case. This value depends upon the sign of q and Ω.

Now, the relevant quantity is the density contrast $\delta\rho/\rho_o$. Keeping in mind the square contained into Eq. (6), the asymptotic variations of the hyperbolic modes ($\varepsilon = 1$) are readily obtained. Since from Eq. (61), we have the inegality $s_+^+(k) > s_-^+(k)$, it is clear that in Eq. (63) the asymptotic leading behaviour for a collapse (resp. for an expansion) is given by the variation of the second term (resp. the first term) of the right hand side of Eq. (63). A trivial calculation provides:

$$\Omega < 0 \Rightarrow \frac{\delta T_+^k}{\rho_o} \propto \lim_{t\to -\frac{1}{\Omega}} (1+\Omega t)^{s_-^+(k)+2} = \infty \quad (69)$$

$$\Omega > 0 \Rightarrow \frac{\delta T_+^k}{\rho_o} \propto \lim_{t\to \infty} (\Omega t)^{s_+^+(k)+2} = \infty. \quad (70)$$

As a consequence, all hyperbolic modes are unstable for any value of the wave number.

The behaviour of trigonometric modes ($\varepsilon = -1$) introduces the Jeans wave number through the exponant sign, like in the eulerian derivation. A transition beetween oscillating and non-oscillating modes is also obtained for the “pivot” value, k_{trans}, given by Eq. (66). For the implosions ($\Omega < 0$), it becomes:

$$k < k_{trans} \Rightarrow \frac{\delta T_-^k}{\rho_o} \propto \lim_{t\to -\frac{1}{\Omega}} (1+\Omega t)^{s_-^-(k)+2} = \infty \quad (71)$$

$$k > k_{trans} \Rightarrow \frac{\delta T_-^k}{\rho_o} \propto \lim_{t\to -\frac{1}{\Omega}} \frac{\cos\left(\ln(1+\Omega t)\right)}{(1+\Omega t)^{\frac{1}{6}}} = \infty \quad (72)$$

and for the explosion ($\Omega > 0$):

$$0 < k < k_J \Rightarrow \frac{\delta T_-^k}{\rho_o} \propto \lim_{t\to\infty} (\Omega t)^{s_+^-(k)+2} = \infty \quad (73)$$

$$k_J < k < k_{trans} \Rightarrow \frac{\delta T_-^k}{\rho_o} \propto \lim_{t\to\infty} (\Omega t)^{s_+^-(k)+2} = 0 \quad (74)$$

$$k > k_{trans} \Rightarrow \frac{\delta T_-^k}{\rho_o} \propto \lim_{t\to\infty} \frac{\cos\left(\ln(\Omega t)\right)}{(\Omega t)^{\frac{1}{6}}} = 0 \quad (75)$$

where k_J is the Jeans wave number given by:

$$k_J = \frac{\hat{\Omega}_J}{\hat{c}_o}. \quad (76)$$

Similarly to k_{trans}, the wave number k_J is now significant in the physical space. Here, again, we have kept the leading order term in $\delta T_-^k(t)$. The trigonometric modes are found to be unstable only for $k < k_J$ in a expanding background and for all k in a collapsing one. This wave number is identical to k_{crit} defined in Sect. 2.2. We fit closely, therefore, with the self-similar approach. However, an oscillating phase occurs before the final divergence for $k > k_{trans}$ for collapses. For expansions, an oscillating phase arises for the stable case too (see Eq. (75)).

3.3.2. Stability of eigenmodes with $\gamma \neq 4/3$

We use the same method as in the previous section. However the eigenmodes are now expressed in terms of the Bessel functions. It is, therefore, necessary to know their asymptotic form when their argument goes to zero or to infinity. We have (Abramowitz & Stegun 1972):

$$x \to 0 \Rightarrow \begin{cases} J_n(x) & \sim \frac{1}{\Gamma(n+1)}\left(\frac{x}{2}\right)^n \\ Y_n(x) & \sim -\frac{\Gamma(n)}{\pi}\left(\frac{2}{x}\right)^n \\ I_n(x) & \sim \frac{1}{\Gamma(n+1)}\left(\frac{x}{2}\right)^n \\ K_n(x) & \sim \frac{\Gamma(n)}{2}\left(\frac{2}{x}\right)^n \end{cases} \quad (77)$$

and:

$$x \to \infty \Rightarrow \begin{cases} J_n(x) & \sim \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \\ Y_n(x) & \sim \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \\ I_n(x) & \sim \sqrt{\frac{1}{2\pi x}}\exp(x) \\ K_n(x) & \sim \sqrt{\frac{\pi}{2x}}\exp(-x). \end{cases} \quad (78)$$

Moreover, in Eqs. (67) and (68), the exponent μ, i.e., the quantity $(\gamma - 4/3)$ appears and, consequently, the asymptotic behaviour will be dependent on whether $\gamma < 4/3$ or $\gamma > 4/3$.

Hyperbolic modes ($\varepsilon = 1$). From Eqs. (67), (77) and (78), the explosion case ($\Omega > 0$) behaves according to:

$$\gamma < \frac{4}{3} \Rightarrow \frac{\delta T_+^k}{\rho_o} \propto \lim_{t\to\infty} \frac{\exp(\Omega t)^{\frac{4}{3}-\gamma}}{(\Omega t)^{\frac{5/3-\gamma}{2}}} = \infty \quad (79)$$

$$\gamma > \frac{4}{3} \Rightarrow \frac{\delta T_+^k}{\rho_o} \propto \lim_{t\to\infty} (\Omega t)^{\frac{2}{3}} = \infty. \quad (80)$$

and for $\Omega < 0$ (implosions), we have:

$$\gamma < \frac{4}{3} \Rightarrow \frac{\delta T_+^k}{\rho_o} \propto \lim_{t\to -\frac{1}{\Omega}} (1+\Omega t)^{-1} = \infty \quad (81)$$

$$\gamma > \frac{4}{3} \Rightarrow \frac{\delta T_+^k}{\rho_o} \propto \lim_{t\to -\frac{1}{\Omega}} \frac{\exp(1+\Omega t)^{\frac{4}{3}-\gamma}}{(1+\Omega t)^{\frac{5/3-\gamma}{2}}} = \infty. \quad (82)$$

We conclude that all hyperbolic modes are unstable for any values of both the wave number k and the polytropic exponent γ. It is important to notice the exponential rise of the perturbation in Eqs. (79) and (82). Moreover, it is quite surprising to see that expansions and collapses behave exactly in the same way for $\gamma < 4/3$ and $\gamma > 4/3$, respectively. The dependence upon the value of γ is very sensitive and it is interesting that, in addition to the "critical value" $\gamma = 4/3$, the difference $\gamma - 5/3$ arises quite naturally. This was not expected from the beginning of the study. On the other hand, we see that for $\gamma > 4/3$ (resp. $\gamma < 4/3$), the leading time evolution of the instability for explosions (resp. implosions) does not depend any more upon the value of γ.

Trigonometric modes ($\varepsilon = -1$). In the same way, from the asymptotic form of the classical Bessel functions (Abramovitz & Stegun 1972), we have in the explosive case ($\Omega > 0$):

$$\gamma < \frac{4}{3} \;\Rightarrow\; \frac{\delta T_-^k}{\rho_o} \propto \lim_{t\to\infty} (\Omega t)^{\frac{\gamma-5/3}{2}} \cos\left[(\Omega t)^{\frac{4}{3}-\gamma}\right] = 0 \qquad (83)$$

$$\gamma > \frac{4}{3} \;\Rightarrow\; \frac{\delta T_-^k}{\rho_o} \propto \lim_{t\to\infty} (\Omega t)^{\frac{2}{3}} = \infty. \qquad (84)$$

and for implosions ($\Omega < 0$):

$$\gamma < \frac{4}{3} \;\Rightarrow\; \frac{\delta T_-^k}{\rho_o} \propto \lim_{t\to -\frac{1}{\Omega}} (1+\Omega t)^{-1} = \infty \qquad (85)$$

$$\frac{4}{3} < \gamma < \frac{5}{3} \;\Rightarrow\; \frac{\delta T_-^k}{\rho_o} \propto \lim_{t\to -\frac{1}{\Omega}} \frac{\cos\left[(1+\Omega t)^{\frac{4}{3}-\gamma}\right]}{(1+\Omega t)^{\frac{5/3-\gamma}{2}}} = \infty \qquad (86)$$

$$\gamma > \frac{5}{3} \;\Rightarrow\; \frac{\delta T_-^k}{\rho_o} \propto \lim_{t\to -\frac{1}{\Omega}} \frac{\cos\left[(1+\Omega t)^{\frac{4}{3}-\gamma}\right]}{(1+\Omega t)^{\frac{5/3-\gamma}{2}}} = 0. \qquad (87)$$

It turns out that, for an expanding background, all modes with polytropic exponent $\gamma < 4/3$ are stable. This property is valid for any value of the wave number k. For the collapsing case, only the modes with $\gamma > 5/3$ vanish. In particular, and in the frame of this simple model, the core of a supernova, which can be described by a polytrope with $\gamma \simeq 2$, is stable during the implosion regarding the evolution of density perturbations.

On the other hand, from Eq. (77) and Eq. (78), the bessel function of the second kind, Y_n, behaves near the origin, like a power divergent function, whereas it is oscillating for arguments greater than the first zero. It is, therefore, possible that the initial value of the argument z be greater than the first zero of Y_n. In addition, if z decreases with time, we may have transient oscillating modes.

Let z_n^o be the first zero of Y_n. From Eq. (53), the argument $z(0)$ of Y_n at $t = 0$ is:

$$z(0) = \frac{N_k}{|\mu|} = \sqrt{6}\frac{k\hat{c_o}}{\hat{\Omega}_J|4-3\gamma|}. \qquad (88)$$

Consequently, for a given γ, the value of this argument can be greater than z_n^o if the wave number is large enough, and satisfies:

$$k > \frac{|4-3\gamma|}{\sqrt{6}} z_n^o k_J = k_{J,\gamma}. \qquad (89)$$

In this derivation, we have used Eq. (76). At time t, the argument of Y_n is written from Eq. (53):

$$z(t) = z(0)(1+\Omega t)^{\frac{4}{3}-\gamma} \qquad (90)$$

and provided the condition $\Omega(\gamma - 4/3) > 0$ is satisfied, the variable z will decrease to zero as the time will elapse. This is the proof of our claim that, if $z(0) > z_n^o$, the first zero will be crossed over in that case. The consequence for the evolution is that the expanding (resp. collapsing) configuration will oscillate for $\gamma > 4/3$ (resp. $\gamma < 4/3$) before the final divergence at $t \to \infty$ (resp. $t \to -1/\Omega$). The amplitude of such oscillating modes increases with time and as soon as $z < z_n^o$, the mode grows as a power of time according to Eq. (84) and Eq. (85). Note that this behaviour is observed only for eigenmodes with $k > k_{J,\gamma}$. The other ones grow immediatly as a power of time. This is also a result found by Bouquet (1999) and he calls it the "dynamic Jeans criterion". In fact, this is just a change in the behaviour, but it might lead to a true criterion in an improved model. Thus, we have:

$$k < k_{J,\gamma} \;\Rightarrow\; \frac{\delta T_-^k}{\rho_o} \to \infty \quad \text{no oscillation} \qquad (91)$$

$$k > k_{J,\gamma} \;\Rightarrow\; \frac{\delta T_-^k}{\rho_o} \to \infty \quad \text{transient oscillations} \qquad (92)$$

3.4. *Perturbations in a finite medium and influence of boundary conditions*

Each eigenmode, of wave number k, can be written as (46):

$$\delta\rho_\varepsilon^k(m,t) = \delta R_\varepsilon^k(m)\delta T_\varepsilon^k(t). \qquad (93)$$

Moreover, we have to distinguish between the two cases $\gamma = 4/3$ and $\gamma \neq 4/3$. The evolution of the radius, R, of a configuration with total mass, M, is given by Eq. (5) and the special form of the density, Eq. (6), means that the mass is preserved during the evolution. Considering that the configuration is embedded into the interstellar medium, we consider that the pressure remains constant at the surface $r = R(t)$. The equation of state (1) and the continuity of the pressure through the surface make the density perturbation zero at $r = R(t)$. Thus, we must have $\delta\rho(M,t) = 0$ at all times. Since each eigenmode has its own time variation, this condition should be applied to each of them. Eqs. (59) and (60) provide respectively, for all k:

$$\delta\rho_{\varepsilon=-1}^k(M,t) = 0 \;\Rightarrow\; \frac{\sin(k\hat{R})}{k\hat{R}} = 0 \qquad (94)$$

$$\delta\rho_{\varepsilon=1}^k(M,t) = 0 \;\Rightarrow\; \frac{\sinh(k\hat{R})}{k\hat{R}} = 0 \qquad (95)$$

where $\hat{R}$ is the initial radius of the configuration. The second condition leads to $k = 0$, and, thus, all hyperbolic modes are zero. The first one (trigonometric modes, $\varepsilon = -1$) gives a quantification for the values of the wave number k:

$$k_q = \frac{\pi q}{\hat{R}} = \left(\frac{\hat{\Omega}_J^2}{3G}\right)^{\frac{1}{3}} \frac{\pi q}{M^{\frac{1}{3}}} \qquad \text{with} \quad q \in \mathbf{N}^*. \qquad (96)$$

As we can see in this equation, the lagrangian representation with $(\hat{R}, t)$ is more useful than the (M, t) one. However, both of them are strictly equivalent and the connection between the $(\hat{r_o}, t)$ and the (m, t) coordinates is obtained from the conservation of mass:

$$\frac{\partial}{\partial m} = \frac{1}{4\pi r_o^3 \rho_o}\frac{\partial}{\partial r_o} = \frac{1}{4\pi \hat{r_o}^3 \hat{\rho_o}}\frac{\partial}{\partial \hat{r_o}}. \quad (97)$$

The most general expression for the density perturbation is a discrete sum over the eigenmodes satisfaying Eq. (96):

$$\delta\rho(\hat{r_o}, t) = \sum_{q=1}^{\infty} \delta\rho^{k_q}_{\varepsilon=-1}(\hat{r_o}, t). \quad (98)$$

3.4.1. Case $\gamma \neq 4/3$

Plugging Eq. (96) into Eq. (68) and using Eq. (46) the density perturbation is, therefore:

$$\delta\rho(\hat{r_o}, t) = (1+\Omega t)^{-\frac{13}{6}} \sum_{q=1}^{\infty} \frac{\sin(k_q \hat{r_o})}{k_q \hat{r_o}} \times \times \left[\alpha_q J_n\left(\frac{N_q(1+\Omega t)^\mu}{|\mu|}\right) + \beta_q Y_n\left(\frac{N_q(1+\Omega t)^\mu}{|\mu|}\right)\right] \quad (99)$$

where constant factors and integration constants have been absorbed in the coefficients α_q and β_q. The orthonormalization of trigonometric functions allow us to find their expressions (Abramovitz & Stegun 1972). For the sake of simplicity, let us introduce the quantities:

$$I^p_\rho = \int_0^{2\hat{R}} \delta\rho(\hat{r_o}, 0)\hat{r_o}\sin(k_p \hat{r_o}) d\hat{r_o} \quad (100)$$

$$\dot{I}^p_\rho = \int_0^{2\hat{R}} \frac{\partial \delta\rho}{\partial t}(\hat{r_o}, 0)\hat{r_o}\sin(k_p \hat{r_o}) d\hat{r_o} \quad (101)$$

$$J^p_n = J_n\left(\frac{k_p \hat{c_o}}{|\mu\Omega|}\right), \quad J^{p'}_n = \left.\frac{dJ_n(x)}{dx}\right|_{x=\frac{k_p \hat{c_o}}{|\mu\Omega|}} \quad (102)$$

$$Y^p_n = Y_n\left(\frac{k_p \hat{c_o}}{|\mu\Omega|}\right), \quad Y^{p'}_n = \left.\frac{dY_n(x)}{dx}\right|_{x=\frac{k_p \hat{c_o}}{|\mu\Omega|}}. \quad (103)$$

After easy but rather long calculations, we obtain:

$$\alpha_p = \frac{\frac{|\mu\Omega|}{\mu\Omega}\frac{Y^p_n}{\hat{c_o}}\dot{I}^p_\rho - \left(k_p Y^{p'}_n - \frac{13}{6}\frac{|\mu\Omega|}{\mu\hat{c_o}}\right) I^p_\rho}{\hat{R}(J^p_{n-1}Y^p_n - J^p_n Y^p_{n-1})} \quad (104)$$

$$\beta_p = \frac{\frac{|\mu\Omega|}{\mu\Omega}\frac{J^p_n}{\hat{c_o}}\dot{I}^p_\rho - \left(k_p J^{p'}_n - \frac{13}{6}\frac{|\mu\Omega|}{\mu\hat{c_o}}\right) I^p_\rho}{\hat{R}(J^p_n Y^p_{n-1} - J^p_{n-1} Y^p_n)}. \quad (105)$$

3.4.2. Case $\gamma = 4/3$

The discretization of wave numbers obeys Eq. (96). However, because of the critical value, k_{trans} (see Eq. (64) to Eq. (66)), of the wave number, k, we are obliged to separate the sum in two parts. With $p_{trans} \in \mathbf{N}$ defined as:

$$p_{trans} = Int\left(\frac{k_{trans}\hat{R}}{\pi}\right) = Int\left(\frac{5}{6}\frac{|\Omega|}{\pi \hat{c_o}}\hat{R}\right) \quad (106)$$

the perturbation is expressed as:

$$\delta\rho(\hat{r_o}, t) = \sum_{q=1}^{p_{trans}-1} \frac{\sin(k_q \hat{r_o})}{k_q \hat{r_o}} \left[\beta_q (1+\Omega t)^{s^-_+(k_q)} + \right. \left. + \gamma_q (1+\Omega t)^{s^-_-(k_q)}\right] + \sum_{q=p_{trans}}^{\infty} \frac{\sin(k_q \hat{r_o})}{k_q \hat{r_o}} (1+\Omega t)^{-\frac{13}{6}} \times \times \alpha_q \cos \frac{\sqrt{-\Delta^-_{k_q}} \ln(1+\Omega t)}{2}. \quad (107)$$

As in the case $\gamma \neq 4/3$, the initial conditions, $\delta\rho(\hat{r_o}, 0)$ and $\dot{\delta\rho}(\hat{r_o}, 0)$, completely define the parameters α_q, β_q and γ_q. From the orthonormalization conditions, it comes:

$$\alpha_p = \frac{\pi p}{\hat{R}^2} I^p_\rho \quad (108)$$

$$\beta_p = \frac{\hat{\Omega}_J k_p}{\sqrt{\Delta^-_{k_p}}}\left(\dot{I}^p_\rho - s^-_-(k_p) I^p_\rho\right) \quad (109)$$

$$\gamma_p = \frac{\hat{\Omega}_J k_p}{\sqrt{\Delta^-_{k_p}}}\left(-\dot{I}^p_\rho - s^-_+(k_p) I^p_\rho\right). \quad (110)$$

4. Conclusion

In this paper, we have studied the dynamic stability of a homogeneous collapsing or expanding spherical polytropic configuration. In opposition to the usual studies performed up to now, we have used the lagrangian formalism instead of the eulerian one. It turns out that the polytrope must be split in the two cases $\gamma = 4/3$ and $\gamma \neq 4/3$. This is not really surprising because the $\gamma = 4/3$–polytrope is highly self-similar: $\partial v/\partial t \propto v\partial v/\partial r \propto (1/\rho)\partial p/\partial r \propto g \propto (1+\Omega t)^{-4/3}$ (Blottiau et al. 1988). However the langrangian approach makes the study more difficult than the eulerian one because of the lack of dispersion relation. Nevertheless, we have been able to come to a conclusion about the gravitational stability and, unexpectedly, it comes out that the polytrope $\gamma = 5/3$ also plays a special role.

Let us come back to the particular $\gamma = 4/3$–polytrope in more detail. In spite of the decelerated (or accelerated) motion of the expanding background, part of the stability criterion is still given by the Jeans' result derived for a static configuration (Jeans 1961). We recover the classical threshold for the wave number, $k_J = \hat{\Omega}_J/\hat{c_o}$, but, in addition, a second pivot value, $k_{trans} = \sqrt{25/24}\, k_J$, separates oscillating solutions ($k > k_{trans}$) from monotonic ones ($k < k_{trans}$), and both of them are stable provided $k > k_J$. It is really amazing that the macroscopic expanding motion of the background does not alter the Jeans' criterion. In our opinion, this is due to the beautiful property of "sharp" self-similarity. Collapses behave in a quite different way: although, the pivot value, k_{trans}, plays exactly the same role as in expansions, we find that any disturbance is instable.

Now, let us examine the case $\gamma \neq 4/3$. As written above, the lagrangian treatment does not lead to a dispersion relation. The condition derived by Buff & Gerola (1979) does not agree with our results. According to us, the time variation of the coefficients arising in their linearized dispersion equation has not been taken into account. In opposition to the case $\gamma = 4/3$, we find that the stability does not depend any longer on the value of the wave number (excepted for the apparition of transcient oscillating phases). The critical parameter for the stability is just the value of the polytropic exponent γ. For expansions, $\gamma = 4/3$ is a threshold and stability (resp. unstability) is obtained for $\gamma < 4/3$ (resp. $\gamma > 4/3$). Collapses are more complicated since two critical values arise, i.e., $\gamma = 4/3$ and $\gamma = 5/3$. For $\gamma < 5/3$, unstable collapse occurs with monotonic (resp. oscillating) behaviour for $\gamma < 4/3$ (resp. for $\gamma > 4/3$). On the other hand, for $\gamma > 5/3$, collapses are always stable. To our knowledge, this is the first time that the $\gamma = 5/3$ has been derived as a threshold for gravitational stability. The case $\gamma = 4/3$ is not surprising, it corresponds to a perfect gas of photons plus matter and it is very relevant in astrophysics (Chandrasekhar 1967). The value $\gamma = 5/3$ corresponds to the monoatomic perfect gas but, up to now, we have not been able to associate this value with a specially important phenomena in astrophysics.

Finally, it would be very interesting to check numerically these theoritical predictions. This will be the next step in further studies.

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