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Consistent determination of quasi force-free magnetic fields from observations in solar active regions

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Abstract. The classical concept of extrapolating the photospheric magnetic field from line-of-sight measurements B_ℓ assuming a pure force-free field is revisited and a new procedure is proposed which is based on:

1) non-vanishing magnetic forces,
2) known horizontal forces from direct measurement of pressure or a related quantity (continuum brightness),
3) a small perturbation to the potential or linear force-free field which matches the B_ℓ distribution.

Orders of magnitude are estimated from observational data and they show that magnetic forces should be actually as large as expected from dimensional analysis.

Key words: Sun: activity – Sun: magnetic fields

1. Introduction

The extrapolation of solar magnetic fields from measurements of the line-of-sight component B_ℓ using the force-free assumption

$$\boldsymbol{J} \wedge \boldsymbol{B} = \boldsymbol{0} \tag{1a}$$
$$\nabla \cdot \boldsymbol{B} = 0, \tag{1b}$$

or equivalently

$$\nabla \wedge \boldsymbol{B} = \alpha \boldsymbol{B} \tag{2a}$$
$$\nabla \cdot \boldsymbol{B} = 0, \tag{2b}$$

holds fairly well in the corona, where the plasma $\beta(= \mu_0 p/B^2)$ is much smaller than unity. Note that modern magnetographs (THEMIS: Mein & Rayrole 1985, Rayrole 1992; LEST: Stenflo 1985) will provide all of the three components of $\boldsymbol{B}$, at least at one level in the photosphere, and therefore from the set of Eqs. (2) written as

$$\frac{\partial B_x}{\partial z} = \alpha B_y + \frac{\partial B_z}{\partial x} \tag{3a}$$
$$\frac{\partial B_y}{\partial z} = -\alpha B_x + \frac{\partial B_z}{\partial y} \tag{3b}$$
$$\frac{\partial B_z}{\partial z} = -\frac{\partial B_x}{\partial x} - \frac{\partial B_y}{\partial y} \tag{3c}$$
$$\alpha = \frac{\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}}{B_z} = \frac{\mu_0 J_z}{B_z} \tag{3d}$$

will enable us to determine α only from the vertical component J_z of the density current vector $\boldsymbol{J}$ (3d) **if (1) holds**, but not from (3a) or (3b) since the vertical derivatives $\partial B_x/\partial z$ and $\partial B_y/\partial z$ cannot be easily estimated from observations. On the opposite, system (3) can be used to extrapolate $\boldsymbol{B}$ to the lower corona (Wu et al. 1985, 1990; Cuperman et al. 1989, 1990), but lots of difficulties arise because of the ill-posedness of the problem (Amari et al. 1997). Other methods, existing or presently under development, are based on iterations or MHD codes (Amari & Démoulin 1992; Amari et al. 1997; Démoulin et al. 1997; Mc Clymont et al. 1997) and they are expected to produce stable solutions.

Nevertheless, at the photosphere $\beta \approx 1$ holds, and the force-free balance (1a) must be replaced by the magnetostatic balance (4a) on the horizontal

$$\boldsymbol{F} = \boldsymbol{J} \wedge \boldsymbol{B} = \nabla_h p \tag{4a}$$
$$\nabla \cdot \boldsymbol{B} = 0, \tag{4b}$$

as far as inertial terms of acceleration ($|\rho \partial \boldsymbol{V}/\partial t| \approx 10^{-4}$ J·m^{-3}) and advection ($|\rho \boldsymbol{V} \cdot \nabla \boldsymbol{V}| \approx 10^{-5}$ J·m^{-3}) are negligible with respect to the pressure force ($|\nabla_h p| \approx 10^{-3}$ J·m^{-3}) and the magnetic force ($|\boldsymbol{J} \wedge \boldsymbol{B}| \approx 10^{-3}$ J·m^{-3}), estimated on the basis of typical values ($V \approx 10^3$ m·s^{-1}; $B \approx 10^{-1}$ T; $L \approx 10^7$ m; $\tau \approx 10^3$ s; $p \approx 10^4$ Pa; $\rho \approx 10^{-4}$ kg·m^{-3}). This is a starting assumption, since we know that velocity and magnetic field are well correlated in sunspots (Berton 1986). Moreover, there is serious observational evidence that magnetic forces are important up to 400 km above the photosphere and become negligible beyond (Metcalf et al. 1995).

If horizontal pressure gradients are by some means measured, one may think to solve Eqs. (4) with standard boundary conditions (B_ℓ known at the photosphere). In a first step brightness fluctuations I currently observed across sunspots may be related to horizontal temperature gradients through the black-body assumption of radiation transfer

$$I \approx A e^{-h\nu/kT} \tag{5}$$

where h and k denote respectively Planck and Boltzmann constants, and the absolute temperature T is connected to the thermal pressure p by the perfect gas law

$$p = \rho\, r\, T \tag{6}$$

where the density ρ is assumed to vary only with height ($r \approx 6.4 \times 10^3$ m^2·s^{-2}·K^{-1} in the photosphere). Actually, the relative brightness ϕ is measured at the photospheric level

$$\phi = \frac{I - I_0}{I_0} \tag{7}$$

with respect to the continuum value I_0 far from the line centre.

With this background, the relationship between I, p and T will be discussed in Sect. 2, the calculation algorithms will be explained with demonstrations of the well-posedness in Sect. 3, and prospects will be proposed in the concluding Sect. 4.

2. Radiative model

Quite generally, the radiative model should provide a relationship between the brightness intensity I and the temperature

$$I = f(T). \tag{8}$$

In the photosphere, the Local Thermodynamic Equilibrium (LTE) is not too bad an assumption (Bray et al. 1984; Mihalas et al. 1984), so that the intensity emerging from the photosphere at frequency ν can be expressed as the blackbody source function, or Planck's function

$$I \approx \frac{A}{e^{h\nu/kT} - 1} \tag{9}$$

which can be approximated at the optical wavelengths under concern $(h\nu/kT \gg 1)$ by Wien's law

$$I \approx Ae^{-T_1/T}, \tag{10}$$

with the temperature constant (at the wavelength $\lambda = 525$ nm)

$$T_1 = \frac{h\nu}{k} = \frac{hc}{k\lambda} = 27.5 \times 10^3\,K. \tag{11}$$

Expression (10) can be cast into the form

$$\phi \approx e^{-\Delta\Theta} - 1 \tag{12}$$

with (7) and

$$\Delta\Theta = T_1 \left(\frac{1}{T} - \frac{1}{T_0} \right), \tag{13}$$

which clearly accounts for the contrast between the "cool" umbra at some 4000 K and the surrounding "hot" photosphere at 6000 K. Inversion of relation (12) yields

$$\Delta\Theta = -Log(\phi + 1). \tag{14}$$

Let us here insist upon the fact that relation (14) provides a hint about the variations of temperature over the active region, which will be checked in Sect. 4 to be not so far from reality. Of course, the direct measurement of the pressure field remains the best way to provide reliable information (e.g. from the pressure broadening of spectral lines).

3. Perturbation method

3.1. Initial potential field

Let us write the total magnetic field $\boldsymbol{B}$ in the half-space $\mathcal{D}$ above the photosphere as the sum of the potential field $\boldsymbol{B}_0$ satisfying the observed component B_ℓ at the photosphere $\mathcal{S}$, and a small deviation $\boldsymbol{b}$

$$\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{b} \qquad \text{in } \mathcal{D} \tag{15a}$$

$$\boldsymbol{B}_0 \cdot \boldsymbol{\ell} = B_\ell \qquad \text{on } \mathcal{S} \tag{15b}$$

where $\boldsymbol{\ell}$ denotes the unit vector on the line of sight. The total current density splits into

$$\boldsymbol{J} = \boldsymbol{J}_0 + \boldsymbol{j} \tag{16}$$

with

$$\begin{cases} \mu_0 \boldsymbol{J}_0 = \nabla \wedge \boldsymbol{B}_0 \\ \nabla \cdot \boldsymbol{B}_0 = 0 \end{cases} \tag{17a,b}$$

$$\begin{cases} \mu_0 \boldsymbol{j} = \nabla \wedge \boldsymbol{b} \\ \nabla \cdot \boldsymbol{b} = 0 \end{cases} \tag{17c,d}$$

and $\boldsymbol{B}_0$ is assumed to be potential

$$\boldsymbol{J}_0 = \boldsymbol{0} \quad \text{or equivalently} \quad \nabla \wedge \boldsymbol{B}_0 = \boldsymbol{0}. \tag{18}$$

Now, we are interested in currents $\boldsymbol{j}$ yielding magnetic forces counterbalanced by pressure gradients, like in (4)

$$\boldsymbol{F} = \boldsymbol{J} \wedge \boldsymbol{B}. \tag{19}$$

Therefore, writing $\boldsymbol{b}$ and $\boldsymbol{j}$ as the sum of their components along $\boldsymbol{B}$ and perpendicular to it

$$\boldsymbol{b} \ = \ \boldsymbol{b}_\| + \boldsymbol{b}_\perp \tag{20a}$$

$$\boldsymbol{j} \ = \ \boldsymbol{j}_\| + \boldsymbol{j}_\perp \tag{20b}$$

and inserting (15a)(16) and (20) into (19) with use of (18), one gets the Lorentz force

$$\boldsymbol{F} \approx \boldsymbol{j}_\| \wedge \boldsymbol{B}_0 + \boldsymbol{j}_\perp \wedge \boldsymbol{B}_0 \tag{21}$$

after neglect of the quantity $\boldsymbol{j} \wedge \boldsymbol{b}$ assumed one order smaller than $\boldsymbol{j} \wedge \boldsymbol{B}_0$. Likewise, from (15a) and (20), the expressions $\boldsymbol{j}_\| \wedge \boldsymbol{B}$ and $\boldsymbol{j}_\perp \cdot \boldsymbol{B}$ can be approximated as

$$\begin{cases} \boldsymbol{j}_\| \wedge \boldsymbol{B} \approx \boldsymbol{j}_\| \wedge \boldsymbol{B}_0 \\ \boldsymbol{j}_\perp \cdot \boldsymbol{B} \approx \boldsymbol{j}_\perp \cdot \boldsymbol{B}_0 \end{cases} \tag{22a,b}$$

after neglect of the quantities $\boldsymbol{j}_\| \wedge \boldsymbol{b}$ and $\boldsymbol{j}_\perp \cdot \boldsymbol{b}$ assumed one order smaller than $\boldsymbol{j}_\| \wedge \boldsymbol{B}_0$ and $\boldsymbol{j}_\perp \cdot \boldsymbol{B}_0$ respectively. Since by definition, $\boldsymbol{j}_\| \wedge \boldsymbol{B} = \boldsymbol{0}$ and $\boldsymbol{j}_\perp \cdot \boldsymbol{B} = 0$, we eventually get the equalities

$$\begin{cases} \boldsymbol{j}_\| \wedge \boldsymbol{B}_0 \approx \boldsymbol{0} \\ \boldsymbol{j}_\perp \cdot \boldsymbol{B}_0 \approx 0 \end{cases} \tag{23a,b}$$

which provide the other two equations we shall use. Actually, relation (23a) means that the first term in (21) disappears, and there simply remains the system

$$\begin{cases} \boldsymbol{F} \approx \boldsymbol{j}_\perp \wedge \boldsymbol{B}_0 \\ \boldsymbol{j}_\perp \cdot \boldsymbol{B}_0 \approx 0 \end{cases} \tag{24a,b}$$

This system can be now solved directly for $\boldsymbol{j}_\perp$, without any iteration

$$\boldsymbol{j}_\perp = \frac{\boldsymbol{B}_0 \wedge \boldsymbol{F}_h}{B_0^2} \tag{25}$$

with

$$B_0^2 = B_x^2 + B_y^2 + B_z^2 \tag{26a}$$
$$\boldsymbol{F}_h = \nabla_h p, \tag{26b}$$

wherever $B_0 \neq 0$. Then by Biot-Savart's law, the magnetic field vector $\boldsymbol{b}$ is derived (h denotes here the thickness of the photosphere)

$$\boldsymbol{b} = \frac{\mu_0}{4\pi}\int\limits_{S'} \frac{\boldsymbol{j}_\parallel \wedge \boldsymbol{r}'}{r'^3} h dS' + \frac{\mu_0}{4\pi}\int\limits_{S'} \frac{\boldsymbol{j}_\perp \wedge \boldsymbol{r}'}{r'^3} h dS' \tag{27}$$

and the contribution due to $\boldsymbol{j}_\perp$ (second integral) can be computed. Note that the contribution due to $\boldsymbol{j}_\parallel$ (first integral) is arbitrary and can be dropped since it does not produce any magnetic force

$$\boldsymbol{b} = \frac{\mu_0}{4\pi}\int\limits_{S'} \frac{\boldsymbol{j}_\perp \wedge \boldsymbol{r}'}{r'^3} h dS'. \tag{28}$$

Moreover, it should be kept in mind that, in general, there is no simple relation between the quantities $\boldsymbol{j}_\parallel$, $\nabla \wedge \boldsymbol{b}_\perp$, $\boldsymbol{j}_\perp$ and $\nabla \wedge \boldsymbol{b}_\parallel$ (see Appendix). For any vector $\boldsymbol{v}$ (in particular $\boldsymbol{b}$ or $\boldsymbol{j}$), its components $\boldsymbol{v}_\parallel$ and $\boldsymbol{v}_\perp$ along $\boldsymbol{B}_0$ and perpendicular to it must be calculated by means of the formulas

$$\boldsymbol{v}_\parallel = \frac{\boldsymbol{v} \cdot \boldsymbol{B}_0}{B_0^2}\boldsymbol{B}_0 \tag{29a}$$
$$\boldsymbol{v}_\perp = \boldsymbol{v} - \boldsymbol{v}_\parallel = \frac{\boldsymbol{B}_0 \wedge (\boldsymbol{v} \wedge \boldsymbol{B}_0)}{B_0^2} \tag{29b}$$

wherever $B_0 \neq 0$.

Let us now demonstrate the well-posedness of this boundary value problem restricted to the $\boldsymbol{j}_\perp$-contribution. It is defined by the three properties of existence, unicity of the solution and its continuous dependence on the boundary conditions (Amari et al. 1997). In our problem, the existence and unicity are guaranteed by the existence and unicity of the potential field $\boldsymbol{B}_0$, which is known to be solution of a well-posed boundary value problem, and by the fact that the explicit relations (25) and (28) determine uniquely the current and induction corrections to $\boldsymbol{B}_0$. Moreover, since the potential field $\boldsymbol{B}_0$ is solution of a well-posed boundary value problem, it depends continuously on the boundary condition B_ℓ. Relation (25) shows that $\boldsymbol{j}_\perp$ depends continuously on $\boldsymbol{F}_h$ and $\boldsymbol{B}_0$ provided $B_0 \neq 0$, and relation (28) shows that $\boldsymbol{b}$ depends continuously on $\boldsymbol{j}_\perp$. It results that $\boldsymbol{b}$ depends continuously on $\boldsymbol{F}_h$ and B_ℓ.

3.2. *Initial force-free field*

The basic method described above can be extended to the situation where the initial non-perturbed field $\boldsymbol{B}_0$, instead of being potential, is linear force-free. Then (18) is replaced by

$$\boldsymbol{J}_0 \wedge \boldsymbol{B}_0 = \boldsymbol{0} \quad \text{or equivalently}$$
$$\nabla \wedge \boldsymbol{B}_0 = \alpha \boldsymbol{B}_0 \quad \text{i.e.} \quad \boldsymbol{J}_0 = \frac{\alpha}{\mu_0}\boldsymbol{B}_0 \tag{30}$$

with α constant in space. Therefore the magnetic force becomes, after inserting (15a)(16) and (20) into (19) with use of (30)

$$\boldsymbol{F} \approx \boldsymbol{J}_0 \wedge \boldsymbol{b}_\parallel + \boldsymbol{J}_0 \wedge \boldsymbol{b}_\perp + \boldsymbol{j}_\parallel \wedge \boldsymbol{B}_0 + \boldsymbol{j}_\perp \wedge \boldsymbol{B}_0 \tag{31}$$

and neglect of the quantity $\boldsymbol{j} \wedge \boldsymbol{b}$, like in (21). Inserting the expression of $\boldsymbol{J}_0$ (30b) into (31) yields the magnetic force

$$\boldsymbol{F} = \left(\boldsymbol{j}_\parallel - \frac{\alpha}{\mu_0}\boldsymbol{b}_\parallel\right) \wedge \boldsymbol{B}_0 + \left(\boldsymbol{j}_\perp - \frac{\alpha}{\mu_0}\boldsymbol{b}_\perp\right) \wedge \boldsymbol{B}_0. \tag{32}$$

Using the same arguments as previously, one derives (23) which holds here again, and moreover the expressions $\boldsymbol{b}_\parallel \wedge \boldsymbol{B}$ and $\boldsymbol{b}_\perp \cdot \boldsymbol{B}$ can be approximated as

$$\begin{cases} \boldsymbol{b}_\parallel \wedge \boldsymbol{B} \approx \boldsymbol{b}_\parallel \wedge \boldsymbol{B}_0 \\ \boldsymbol{b}_\perp \cdot \boldsymbol{B} \approx \boldsymbol{b}_\perp \cdot \boldsymbol{B}_0 \end{cases} \tag{33a,b}$$

after neglect of the quantities $\boldsymbol{b}_\parallel \wedge \boldsymbol{b}$ and $\boldsymbol{b}_\perp \cdot \boldsymbol{b}$ assumed one order smaller than $\boldsymbol{b}_\parallel \wedge \boldsymbol{B}_0$ and $\boldsymbol{b}_\perp \cdot \boldsymbol{B}_0$ respectively. Since by definition, $\boldsymbol{b}_\parallel \wedge \boldsymbol{B} = \boldsymbol{0}$ and $\boldsymbol{b}_\perp \cdot \boldsymbol{B} = 0$, we eventually get the equalities

$$\begin{cases} \boldsymbol{b}_\parallel \wedge \boldsymbol{B}_0 \approx \boldsymbol{0} \\ \boldsymbol{b}_\perp \cdot \boldsymbol{B}_0 \approx 0 \end{cases} \tag{34a,b}$$

The relations (23a) and (34a) imply that the first term in (32) disappears, and there simply results the system

$$\begin{cases} \boldsymbol{F} \approx \left(\boldsymbol{j}_\perp - \frac{\alpha}{\mu_0}\boldsymbol{b}_\perp\right) \wedge \boldsymbol{B}_0 \\ \boldsymbol{j}_\perp \cdot \boldsymbol{B}_0 \approx 0 \\ \boldsymbol{b}_\perp \cdot \boldsymbol{B}_0 \approx 0 \end{cases} \tag{35a,b,c}$$

Now this system can be solved directly for the following linear combination of $\boldsymbol{j}_\perp$ and $\boldsymbol{b}_\perp$

$$\boldsymbol{j}_\perp = \frac{\alpha}{\mu_0}\boldsymbol{b}_\perp = \frac{\boldsymbol{B}_0 \wedge \boldsymbol{F}_h}{B_0^2} \tag{36}$$

corresponding to (25), with B_0 and $\boldsymbol{F}_h$ given by (26), wherever $B_0 \neq 0$. Then, as previously, the magnetic field vector $\boldsymbol{b}$ produced by the $\boldsymbol{j}_\perp$-contribution is derived by Biot-Savart's law, Eq. (28).

Unlike the potential case yielding an explicit solution (25), the present situation needs solving an integro-differential system (36) and (28). In a way similar to that one used for force-free fields (Grad & Rubin 1958; Bineau 1972; Aly 1989), we may imagine an iterative solving scheme, based on the sequence

$$\begin{cases} \boldsymbol{j}_\perp^{(n+1)} = \frac{\alpha}{\mu_0}\boldsymbol{b}_\perp^{(n)} + \frac{\boldsymbol{B}_0 \wedge \boldsymbol{F}_h}{B_0^2} \\ \boldsymbol{b}^{(n)} = \frac{\mu_0}{4\pi}\int\limits_{S'} \frac{\boldsymbol{j}_\perp^{(n)} \wedge \boldsymbol{r}'}{r'^3} h dS' \end{cases} \tag{37a,b}$$

for any integer n, with the initial seed

$$\boldsymbol{b}^{(0)} = \boldsymbol{0}. \tag{38}$$

The well-posedness of this problem can be demonstrated by means of Bineau's arguments (Bineau 1972). From (37a), using the same norm as Bineau, we derive the sequence

$$\|\boldsymbol{j}_\perp^{(n+2)} - \boldsymbol{j}_\perp^{(n+1)}\| = \frac{\alpha}{\mu_0}\|\boldsymbol{b}_\perp^{(n+1)} - \boldsymbol{b}_\perp^{(n)}\| \tag{39}$$

and after Bineau's second lemma, we know there exists a scalar L_0 such that

$$\|\boldsymbol{j}_\perp^{(n+2)} - \boldsymbol{j}_\perp^{(n+1)}\| \leq \frac{\alpha L_0}{\mu_0}\|\boldsymbol{j}_\perp^{(n+1)} - \boldsymbol{j}_\perp^{(n)}\|. \tag{40}$$

Therefore, provided α is not too large $(\alpha L_0 < \mu_0)$, the difference $\boldsymbol{b}_\perp^{(n+1)} - \boldsymbol{b}_\perp^{(n)}$ converges to zero, and eventually $\boldsymbol{b}_\perp$ and $\boldsymbol{j}_\perp$ converge toward a limit. This proves the existence of $\boldsymbol{j}_\perp$ and $\boldsymbol{b}_\perp$ solutions of (36) and (28).

Next, in order to establish the unicity of solutions, let $(\boldsymbol{j}_{\perp 1}, \boldsymbol{b}_{\perp 1})$ and $(\boldsymbol{j}_{\perp 2}, \boldsymbol{b}_{\perp 2})$ be two different solutions of Eq. (36), i.e. such that

$$\begin{cases} \boldsymbol{j}_{\perp 1} - \frac{\alpha}{\mu_0}\boldsymbol{b}_{\perp 1} = \frac{\boldsymbol{B}_0 \wedge \boldsymbol{F}_h}{B_0^2} \\ \boldsymbol{j}_{\perp 2} - \frac{\alpha}{\mu_0}\boldsymbol{b}_{\perp 2} = \frac{\boldsymbol{B}_0 \wedge \boldsymbol{F}_h}{B_0^2} \end{cases} \tag{41a,b}$$

If we subtract Eqs. (41) to each other, and take the norm of both sides, there results the equality

$$\|\boldsymbol{j}_{\perp 1} - \boldsymbol{j}_{\perp 2}\| = \frac{\alpha}{\mu_0}\|\boldsymbol{b}_{\perp 1} - \boldsymbol{b}_{\perp 2}\| \tag{42}$$

and again using Bineau's second lemma, we know there exists a scalar L_{12} such that

$$\|\boldsymbol{j}_{\perp 1} - \boldsymbol{j}_{\perp 2}\| \leq \frac{\alpha L_{12}}{\mu_0}\|\boldsymbol{j}_{\perp 1} - \boldsymbol{j}_{\perp 2}\| \tag{43}$$

or

$$\left(1 - \frac{\alpha L_{12}}{\mu_0}\right)\|\boldsymbol{j}_{\perp 1} - \boldsymbol{j}_{\perp 2}\| \leq 0. \tag{44}$$

Provided α is not too large $(\alpha L_{12} < \mu_0)$, the inequality (44) leads to $\boldsymbol{j}_{\perp 1} = \boldsymbol{j}_{\perp 2}$, which means unicity of the $\boldsymbol{j}_\perp$-solution and therefore of the $\boldsymbol{b}_\perp$-solution from (36).

As for the third necessary condition of well-posedness, we know that the linear force-free field $\boldsymbol{B}_0$ is solution of a well-posed boundary value problem (Bineau 1972), which implies that it depends continuously on the boundary condition B_ℓ. Then, inverting (35a) for $\boldsymbol{B}_0$ yields

$$\boldsymbol{B}_0 = -\frac{\left(\boldsymbol{j}_\perp - \frac{\alpha}{\mu_0}\boldsymbol{b}_\perp\right) \wedge \boldsymbol{F}_h}{\left|\boldsymbol{j}_\perp - \frac{\alpha}{\mu_0}\boldsymbol{b}_\perp\right|^2} \tag{45}$$

wherever $\boldsymbol{j}_\perp \neq \frac{\alpha}{\mu_0}\boldsymbol{b}_\perp$ or equivalently $\boldsymbol{F}_h \neq \boldsymbol{0}$. This latter relation (45) shows the continuous dependence of the field $\boldsymbol{B}_0$ upon $\boldsymbol{b}_\perp$ and $\boldsymbol{j}_\perp$. We can say that if $\boldsymbol{b}$ did not depend continuously on B_ℓ and $\boldsymbol{F}_h$, then $\boldsymbol{B}_0$ would not as well, and this would contradict the assumptions. Finally, by contraposition of this argument we derive the desired property.

4. Discussion

In order to get some insight into the validity of the procedure described in the previous sections, let us examine real data (Fig. 1). It is a part of an active region observed on 1974 September 12 at 12:04 for which two-dimensional measurements of longitudinal magnetic field (B_ℓ: Fig. 1a), relative brightness (ϕ: Fig. 1b) and longitudinal velocity field (V_ℓ: Fig. 1c) are available. From (6) and (14), the pressure writes

$$\frac{\rho}{p} = \frac{\rho_0}{p_0} - \frac{Log(1+\phi)}{rT_1} \tag{46}$$

where p_0 and ρ_0 are typical pressure and density in the quiet photosphere, and the constant T_1 has been calculated at the wavelength $\lambda = 525\,\text{nm}$ with expression (11). The pressure force follows from (46)

$$|\nabla p| = \left|\frac{p^2}{\rho r T_1}\frac{\nabla\phi}{1+\phi} + p\frac{\nabla\rho}{\rho}\right| = |\nabla p_1 + \nabla p_2|\,, \tag{47}$$

and it is composed of two contributions: the first one (∇p_1) due to temperature variations, which can be estimated via brightness gradients and the second one (∇p_2) due to density variations, which are not directly measured. Nevertheless it is important to notice that **both are directed from the umbra outwards**.

With $|\nabla\phi| \approx 0.04\,\text{arcsec}^{-1}$ or $0.04/(7\times 10^5)\,\text{m}^{-1}$ found in the umbra of the leading spot (north polarity) of our Fig. 1, and typical values of $p \approx 10^4$ Pa, $\rho \approx 3\times 10^{-4}\,\text{kg}\cdot\text{m}^{-3}$, $B_0 \approx 0.1$ T, we get the following estimates of the first pressure gradient contribution

$$|\nabla p_1| = \frac{p^2}{\rho r T_1}\frac{|\nabla\phi|}{1+\phi} \approx 10^{-4}\,\text{J}\cdot\text{m}^{-3}, \tag{48}$$

and of the resulting current

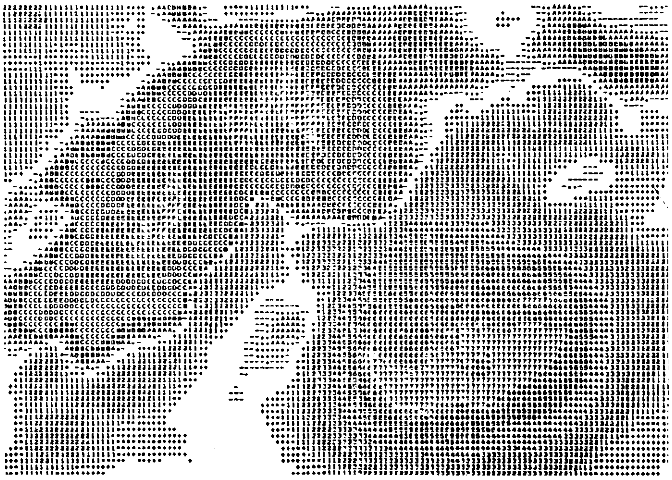
$$|\boldsymbol{j}_{\perp 1}| = \frac{|\nabla p_1|}{B_0} \approx 10^{-3}\,\text{A}\cdot\text{m}^{-2}, \tag{49}$$

which, with the values $\mu_0 = 4\pi \times 10^{-7}\,\text{H}\cdot\text{m}^{-1}$, $L \approx 10^7$ m, produces a magnetic field perturbation

$$|\boldsymbol{b}_1| = \frac{\mu_0}{4\pi}\,|\boldsymbol{j}_{\perp 1}|\,L \approx 10^{-3}\,T, \tag{50}$$

or approximately 10 gauss. The corresponding relative deviation amounts $|\boldsymbol{b}_1|/B_0 \approx 1\%$, which is **at least equal to** the accuracy on measurements (about 1%), and could be detected. Let us notice that brightness gradients of such orders of magnitude or slightly larger are actually found by other authors (Sobotka et al. 1992) from the observed contrast of umbral dots, to be on the average $|\nabla\phi| \approx 0.09\,\text{arcsec}^{-1}$ for a mean dot size of 0.8 arcsec. The electric current perturbation estimated above is also consistent with the maximum values of the full current generally found in observations (Gary & Démoulin 1995).

Note that the umbra is assumed here to be homogeneous at the working resolution of one arcsecond, though the fluctuations we put into evidence from our observations are compatible with the two-component model (Sobotka et al. 1992) cited above. We know that the heterogeneous structure of the umbra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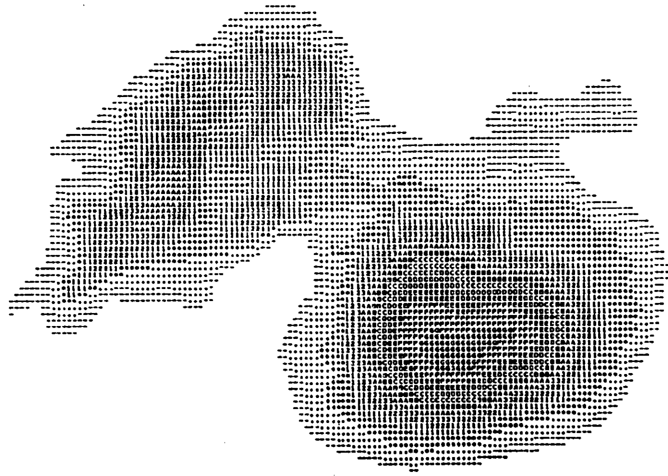
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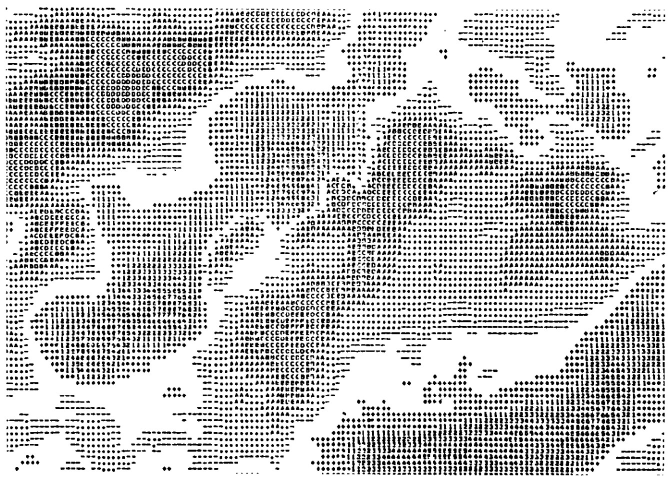
c

Fig. 1a–c. Observation of 1974, Sept. 12. **a** Magnetic field B_ℓ (values in gauss encoded according to the following scale for module with the symbols in parentheses $(B_\parallel < 0/B_\parallel > 0)$: **0** (blank) **20** $(-/+)$ **40** $(\bullet/*)$ **60** (A/1) **120** $(B/2)$ **250** (C/3) **500** (D/4) **800** (E/5) **1200** (F/6) **1600** (G/7) **2000** (H/8) **2500** (I/9) **3000** (J/0)); **b** Relative brightness ϕ (dimensionless values encoded with a step of 0.04); **c** Velocity field V_ℓ (values in m/s encoded according to the following scale for module with the symbols in parentheses $(V_\parallel < 0/V_\parallel > 0)$: **0** (blank) **50** $(-/+)$ **100** $(\bullet/*)$ **200** (A/1) **300** (B/2) **400** (C/3) **500** (D/4) **600** (E/5) **700** (F/6) **800** (G/7) **900** (H/8) **1000** (I/9) **1100** (J/0))

can undergo magnetodynamic changes at small scales as well as global scales, as we put it into evidence a few years ago by detecting velocity-coupled torsional oscillations of umbral magnetic fields (Berton & Rayrole 1985).

As for the second contribution due to density variations, we can derive estimates via models (Michard 1953). With the same typical values of p, ρ, B_0 as above we get the following estimates of the second pressure gradient contribution

$$|\nabla p_2| = p\frac{|\nabla\rho|}{\rho} \approx 10^{-4}\,\mathrm{J}\cdot\mathrm{m}^{-3}, \tag{51}$$

and the resulting current

$$|\boldsymbol{j}_{\perp 2}| = \frac{|\nabla p_2|}{B_0} \approx 10^{-3}\,\mathrm{A}\cdot\mathrm{m}^{-2}, \tag{52}$$

which, with the same numerical constants μ_0 and L as above, produces a magnetic field perturbation

$$|\boldsymbol{b}_2| = \frac{\mu_0}{4\pi}\,|\boldsymbol{j}_{\perp 2}|\,L \approx 10^{-3}\,\mathrm{T}, \tag{53}$$

or again approximately 10 gauss. The corresponding relative deviation amounts $|\boldsymbol{b}_2|/B_0 \approx 1\%$, which again is at least equal to the accuracy on measurements (about 1%), and could be detected. So, both contributions are of same order of magnitude, and since they are directed outwards, they are not likely to cancel, but rather to add together.

Therefore, the method proposed in this paper gives a reliable estimate of the magnetic correction due to pressure forces. Moreover, it should be emphasized that our procedure provides a **consistent** way of extrapolating $\boldsymbol{B}$ from B_ℓ which does not contradict the force-free procedure, but on the contrary adds information. The estimated forces are one order smaller than previously calculated in the introduction, but they seem to be of same order as the acceleration term $\rho\partial\boldsymbol{V}/\partial t$. Actually, the term $\rho\partial\boldsymbol{V}/\partial t$ may not be negligible in a next step, and account for the observed evolution of velocity fields in typical time scales of 10^3 s (≈ 20 min). This effect has been already considered in a former paper (Berton 1987) and could be coupled with the pressure contributions dealt with here.

Appendix A: components of b and j with respect to $\boldsymbol{B}_0$

Let us prove that there is generally no simple relation between the components of $\boldsymbol{b}$ and $\boldsymbol{j}$ along $\boldsymbol{B}_0$ and perpendicular to it. In the following calculations for the vector $\boldsymbol{b}$

$$\begin{aligned}\left(\nabla \wedge \boldsymbol{b}_{\|}\right) \wedge \boldsymbol{B} &\approx \left(\nabla \wedge \boldsymbol{b}_{\|}\right) \wedge \boldsymbol{B}_0 \\ &= \left(\boldsymbol{B}_0 \cdot \nabla\right) \boldsymbol{b}_{\|} + \left(\boldsymbol{b}_{\|} \cdot \nabla\right) \boldsymbol{B}_0 \\ &\quad -\nabla\left(\boldsymbol{B}_0 \cdot \boldsymbol{b}_{\|}\right) - \mu_0 \boldsymbol{J}_0 \wedge \boldsymbol{b}_{\|}\end{aligned} \tag{A.1}$$

$$\begin{aligned}\left(\nabla \wedge \boldsymbol{b}_{\perp}\right) \wedge \boldsymbol{B} &\approx \left(\nabla \wedge \boldsymbol{b}_{\perp}\right) \wedge \boldsymbol{B}_0 \\ &= \left(\boldsymbol{B}_0 \cdot \nabla\right) \boldsymbol{b}_{\perp} + \left(\boldsymbol{b}_{\perp} \cdot \nabla\right) \boldsymbol{B}_0 \\ &\quad -\nabla\left(\boldsymbol{B}_0 \cdot \boldsymbol{b}_{\perp}\right) - \mu_0 \boldsymbol{J}_0 \wedge \boldsymbol{b}_{\perp}\end{aligned} \tag{A.2}$$

$$\begin{aligned}\left(\nabla \wedge \boldsymbol{b}_{\|}\right) \cdot \boldsymbol{B} &\approx \left(\nabla \wedge \boldsymbol{b}_{\|}\right) \cdot \boldsymbol{B}_0 \\ &= -\nabla \cdot \left(\boldsymbol{B}_0 \wedge \boldsymbol{b}_{\|}\right) + \mu_0 \boldsymbol{J}_0 \cdot \boldsymbol{b}_{\|} \\ &= \mu_0 \boldsymbol{J}_0 \cdot \boldsymbol{b}_{\|}\end{aligned} \tag{A.3}$$

$$\begin{aligned}\left(\nabla \wedge \boldsymbol{b}_{\perp}\right) \cdot \boldsymbol{B} &\approx \left(\nabla \wedge \boldsymbol{b}_{\perp}\right) \cdot \boldsymbol{B}_0 \\ &= -\nabla \cdot \left(\boldsymbol{B}_0 \wedge \boldsymbol{b}_{\perp}\right) + \mu_0 \boldsymbol{J}_0 \cdot \boldsymbol{b}_{\perp} \\ &= -\nabla \cdot \left(\boldsymbol{B}_0 \wedge \boldsymbol{b}_{\perp}\right)\end{aligned} \tag{A.4}$$

the terms $\boldsymbol{b}_{\|} \wedge \boldsymbol{B}_0$, $\boldsymbol{b}_{\|} \wedge \boldsymbol{J}_0$, $\boldsymbol{b}_{\perp} \cdot \boldsymbol{B}_0$ and $\boldsymbol{b}_{\perp} \cdot \boldsymbol{J}_0$ vanish because of the collinearity of $\boldsymbol{J}_0$ and $\boldsymbol{B}_0$ for a potential or force-free field $\boldsymbol{B}_0$, and the properties (34a,b). This yields the expressions

$$\begin{aligned}\left(\nabla \wedge \boldsymbol{b}_{\|}\right) \wedge \boldsymbol{B} &\approx \left(\boldsymbol{B}_0 \cdot \nabla\right) \boldsymbol{b}_{\|} + \left(\boldsymbol{b}_{\|} \cdot \nabla\right) \boldsymbol{B}_0 \\ &\quad -\nabla\left(\boldsymbol{B}_0 \cdot \boldsymbol{b}_{\|}\right)\end{aligned} \tag{A.5}$$

$$\begin{aligned}\left(\nabla \wedge \boldsymbol{b}_{\perp}\right) \wedge \boldsymbol{B} &\approx \left(\boldsymbol{B}_0 \cdot \nabla\right) \boldsymbol{b}_{\perp} + \left(\boldsymbol{b}_{\perp} \cdot \nabla\right) \boldsymbol{B}_0 \\ &\quad -\mu_0 \boldsymbol{J}_0 \wedge \boldsymbol{b}_{\perp}\end{aligned} \tag{A.6}$$

$$\left(\nabla \wedge \boldsymbol{b}_{\|}\right) \cdot \boldsymbol{B} \approx \mu_0 \boldsymbol{J}_0 \cdot \boldsymbol{b}_{\|} \tag{A.7}$$

$$\left(\nabla \wedge \boldsymbol{b}_{\perp}\right) \cdot \boldsymbol{B} \approx -\nabla\left(\boldsymbol{B}_0 \wedge \boldsymbol{b}_{\perp}\right) \tag{A.8}$$

which do not vanish in general. Nevertheless, $\boldsymbol{J}_0 \cdot \boldsymbol{b}_{\|}$ vanishes in the particular case of a potential field $\boldsymbol{B}_0$, so that relation (A.7) becomes

$$\left(\nabla \wedge \boldsymbol{b}_{\|}\right) \cdot \boldsymbol{B} \approx 0. \tag{A.9}$$

This proves that $\nabla \wedge \boldsymbol{b}_{\|}$ and $\boldsymbol{j}_{\perp}$ are both perpendicular to $\boldsymbol{B}$ (or $\boldsymbol{B}_0$), but we cannot conclude they are collinear. Similar relations to (A5)–(A8) hold for $\boldsymbol{j}$ instead of $\boldsymbol{b}$.

Let us notice that Eqs. (17c,d) expressing $\boldsymbol{b}$ and $\boldsymbol{j}$ are solenoidal

$$\nabla \cdot \boldsymbol{b} = 0 \tag{A.10a}$$

$$\nabla \cdot \boldsymbol{j} = 0 \tag{A.10b}$$

imply after decomposition the relations

$$\nabla \cdot \boldsymbol{b}_{\|} = -\nabla \cdot \boldsymbol{b}_{\perp} \tag{A.11a}$$

$$\nabla \cdot \boldsymbol{j}_{\|} = -\nabla \cdot \boldsymbol{j}_{\perp} \tag{A.11b}$$

which actually do not inter-link the components of $\boldsymbol{b}$ and $\boldsymbol{j}$.

In conclusion, $\boldsymbol{b}$ and $\boldsymbol{j}$ should be decomposed **geometrically** as follows

$$\boldsymbol{b}_{\|} = \frac{\boldsymbol{b} \cdot \boldsymbol{B}_0}{B_0^2} \boldsymbol{B}_0 \tag{A.12a}$$

$$\boldsymbol{j}_{\|} = \frac{\boldsymbol{j} \cdot \boldsymbol{B}_0}{B_0^2} \boldsymbol{B}_0 \tag{A.12b}$$

and

$$\begin{aligned}\boldsymbol{b}_{\perp} &= \boldsymbol{b} - \boldsymbol{b}_{\|} \\ &= \boldsymbol{b} - \frac{\boldsymbol{b} \cdot \boldsymbol{B}_0}{B_0^2} \boldsymbol{B}_0 \\ &= \frac{\boldsymbol{B}_0 \wedge \left(\boldsymbol{b} \wedge \boldsymbol{B}_0\right)}{B_0^2}\end{aligned} \tag{A.13a}$$

$$\begin{aligned}\boldsymbol{j}_{\perp} &= \boldsymbol{j} - \boldsymbol{j}_{\|} \\ &= \boldsymbol{j} - \frac{\boldsymbol{j} \cdot \boldsymbol{B}_0}{B_0^2} \boldsymbol{B}_0 \\ &= \frac{\boldsymbol{B}_0 \wedge \left(\boldsymbol{j} \wedge \boldsymbol{B}_0\right)}{B_0^2}\end{aligned} \tag{A.13b}$$

wherever $B_0 \neq 0$.

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