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Linear adiabatic dynamics of a polytropic convection zone with an isothermal atmosphere

II. Quasi-stationary solutions

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Abstract. For a plane model of the exterior parts of the sun, the behavior of adiabatic waves with complex frequencies is investigated. The equilibrium configuration is a one-layer model with isentropic stratification at great depth and an asymptotically isothermal atmosphere. The wave equation reduces to Whittaker's equation with complex parameters. By the assumption that only outgoing progressive waves are present in the atmosphere, we obtain a discrete spectrum of complex frequencies. The dispersion relation $F(\omega, k) = 0$ is a third-order algebraic equation in ω^2 with real coefficients. There are no connections of the ridges of the eigenmodes with the ridges of the quasi-stationary waves. Instead, there are striking gaps, and the ridges of quasi-stationary waves extend into the region below the acoustic cut-off frequency. The findings indicate that the ridges of the quasi-stationary solutions cannot explain the ridges of the observed pseudo-modes. As the solutions are not quadratic integrable and form no basis, they do not represent eigenmodes. The behavior of the quasi-stationary solutions is related to the behavior of quasi-stationary states of certain quantum mechanical systems. To answer the question whether quasi-stationary waves are limiting cases of instationary waves, we consider a simple one-dimensional two-layer model. For this case, instationary solutions are compared with the corresponding quasi-stationary solutions.

Key words: hydrodynamics – Sun: atmosphere – Sun: chromosphere

1. Introduction

To study basic properties of solar p-modes with $l \gg 1$, the approach of describing the outer regions of the sun by a plane layer with constant gravity is common. In the first paper of the present series (Schmitz & Steffens **?**, I. General features and real modes, called paper I), we have presented a one layer model where an isothermal atmosphere is smoothly matched to a polytropic convection zone. We have reduced the wave equation of such a layer to Whittaker's equation and have calculated the dispersion curves of p-modes and g-modes. In the present paper, we shall investigate the behavior of solutions of the wave equation with complex frequencies. We assume that the convection zone is isentropic as $z \to -\infty$, i.e. $\gamma = 1 + 1/n$, as opposed to paper I, where we have studied also the general case with $\gamma \neq 1 + 1/n$.

We are motivated by the question of the origin of the ridges above the acoustic cut-off frequency first observed by Libbrecht (1988). The ridges of the $k - \omega$-diagram extend far beyond the acoustic cut-off frequency of the chromosphere (Libbrecht 1988, Steffens et al. 1995, Jefferies et al., 1998, Straus et al. 1999, and Antja & Basu 1999). A slope change of the observed ridges at about 6 mHz was noted first by Steffens et al. (1995), and also Straus et al. (1999) and Antja & Basu (1999) have found such an effect.

Meanwhile, there are three potential explanations of the upper parts of the ridges: Two interpretations assume that there are no significant wave reflections at the coronal transition layer. This property is attributed to temporal and spatial variations of the transition layer.

1. Kumar et al. (1990) and Kumar (1991) have shown that constructive interferences of running waves can generate ridges above the cut-off frequency.

2. Also adiabatic waves with complex frequencies can cause ridges above the acoustic cut-off-frequency. Such solutions of the adiabatic wave equation have been studied by Hindman & Zweibel (1994). Also Milford et al. (1993), Kumar et al. (1994), and Nigam et al. (1998) have used such solutions.

3. If, however, the coronal transition layer acts like an ideal reflector, a global p-mode cavity is formed. An extensive discussion of this subject is given by Balmforth & Gough (1990). Hindman & Zweibel (1994) and Steffens & Schmitz (2000) have studied the behavior of p-modes under the influence of a reflecting transition layer.

In the present paper, we deal with adiabatic waves with complex frequencies. We call such waves quasi-stationary waves, as opposed to other authors who use the term modes. The model of a convection zone with an isothermal atmosphere presented in paper I, allows us to study the behavior of waves with complex

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frequencies analytically. As the results are given in terms of simple mathematical expressions, they are very transparent. We will not consider all types of waves. We only deal with causal solutions of the wave equation, namely outgoing progressive waves.

In Sect. 2 we present some basic relations which are used in this paper. In Sect. 3 we select quasi-stationary solutions by conditions for convergence at infinity and present the dispersion relation of these solutions. In Sect. 4 we discuss the selection of outgoing, time damped progressive waves. The representative simplest quasi-stationary wave is studied and properties and ridges of quasi-stationary solutions are displayed. Sect. 5 deals with the mathematical and the physical meaning of quasi-stationary waves. To study relations between quasi-stationary and instationary waves, we take a simple two layer model. Details of the corresponding calculations are given in an appendix.

2. Basic relations

As in paper I, z is the outwards directed vertical coordinate, m the column mass, a the isothermal, c the adiabatic sound speed, and k the horizontal wave number. The temperature profile of the layer is given by

$$a^2(m) = a_0^2 + \epsilon\, m^\lambda \quad \text{with} \quad \lambda = 1/(1+n) \quad \text{and} \quad \epsilon > 0\,. \qquad (1)$$

The layer is isothermal for $m \to 0$ or $z \to +\infty$ and polytropic for $m \to \infty$ or $z \to -\infty$. As a convection zone is nearly in adiabatic equilibrium, we put $\gamma = 1 + 1/n$, as opposed to paper I, where we have studied the general case.

The waves are described by their Lagrangian pressure perturbations $\Delta p(m,t)$. With a dimensionless independent variable x and a new dependent variable η defined by

$$x = \frac{2\,k\,\epsilon}{\lambda\, g}\, m^\lambda \quad \text{and} \quad \Delta p = x^\nu\, \eta \quad \text{with} \quad \nu = \frac{1-\lambda}{2\,\lambda}\,, \qquad (2)$$

the original time-independent wave equation reduces to Whittaker's equation with parameters κ and μ given by

$$\kappa = \frac{1}{2\lambda\gamma g k}\,(\,\omega^2 - 2k^2 c_0^2\,)\,, \qquad (3)$$

$$\mu = \frac{1}{2\lambda}\sqrt{1 - \frac{4c_0^2}{\gamma^2\omega^2 g^2}\left[\omega^4 - k^2 g^2\,(1-\gamma) - \omega^2 k^2 c_0^2\right]}\,. \qquad (4)$$

Here, c_0 is the adiabatic sound speed at $m = 0$ given by $c_0^2 = \gamma\, a_0^2$. The geometrical coordinate z is related to x by

$$z = -\frac{1}{g}\left[\,\frac{a_0^2}{\lambda}\,\ln\, x + \frac{g}{2\,k}\, x\,\right]. \qquad (5)$$

We put $c_0 = 7$ km/s, $g = 274$ m/s^2, and $\gamma = 5/3$.

3. Solutions of the wave equation with complex frequencies and their dispersion relation

In paper I, two independent Lagrangian pressure perturbations are presented. We require that the pressure perturbation vanish for $z \to -\infty$ or $x \to +\infty$. This assumption corresponds to the condition of vanishing non-radial pressure perturbations in the center of a star. Therefore, we have to use the solution

$$\Delta p = x^{1/2+\nu}\, x^\mu\, e^{-x/2}\, U(\frac{1}{2} + \mu - \kappa, 1 + 2\mu, x)\,, \qquad (6)$$

as opposed to paper I, where another solution was selected by a boundary condition at $x = 0$ or $z \to +\infty$. By the asymptotic expansion of the confluent hypergeometric function U (Abramowitz & Stegun, 1965) we obtain

$$\Delta p \to e^{-x/2}\, x^{\nu+\kappa} \to 0 \quad \text{as} \quad x \to \infty\,. \qquad (7)$$

From the standard representation for U (Abramowitz & Stegun 1965) we obtain:

$$\Delta p = x^{1/2\lambda}\, x^\mu\, e^{-x/2} \cdot \Big[\frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\kappa)}\, M(\frac{1}{2}+\mu-\kappa, 1+2\mu, x)$$
$$+ \frac{\Gamma(2\mu)\, x^{-2\mu}}{\Gamma(\frac{1}{2}+\mu-\kappa)}\, M(\frac{1}{2}-\mu-\kappa, 1-2\mu, x)\Big]\,. \qquad (8)$$

As $M(a,b,0) = 1$, for $x \to 0$, this formula passes into

$$\Delta p = x^{1/2\lambda}\Big[\frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\kappa)}\, x^\mu + \frac{\Gamma(+2\mu)}{\Gamma(\frac{1}{2}+\mu-\kappa)}\, x^{-\mu}\Big]. \qquad (9)$$

In the following, we suppose that $Re(\mu) > 0$. The expressions $x^{1/2\lambda}\, x^{\pm\mu} = m^{1/2}\, m^{\pm\lambda\mu}$ represent waves in an isothermal atmosphere. For $x \to 0$ or $z \to \infty$, from Eq. (5), we obtain:

$$x^{1/2\lambda}\, x^{\pm\mu} = \exp\big(-[\frac{1}{2H} \pm \frac{\lambda\,\mu}{H}]\, z\,\big) \quad \text{with} \quad H = \frac{a_0^2}{g}\,. \qquad (10)$$

For real ω and real $\mu > 0$, these expression represent decaying $(+\mu)$ and reflected $(-\mu)$ evanescent waves. For real frequencies ω above the acoustic cut-off frequency of the isothermal atmosphere, where the frequency spectrum is continuous, the exponent μ is imaginary. This case is considered in paper I.

For complex frequencies ω, the exponent μ is complex. As shall be shown below, expression (10) represents a complex outgoing wave for $-\mu$ and a complex incoming wave for $+\mu$. From Eq. (9) we see that we have to put

$$\frac{1}{2} - \mu - \kappa = -\,j\,, \qquad j = 0, 1, 2, 3.... \qquad (11)$$

to exclude incoming waves. Then, $1/\Gamma(\frac{1}{2} - \mu - \kappa) = 0$, so that the first parts of the pressure perturbations (8) and (9) vanish. We obtain

$$\Delta p = x^{1/2\lambda-\mu}\, e^{-x/2}\, M(\frac{1}{2}-\mu-\kappa, 1-2\mu, x). \qquad (12)$$

Its limiting form is

$$\Delta p = x^{1/2\lambda}\, x^{-\mu} \quad \text{as} \quad x \to 0\,. \qquad (13)$$

For $\frac{1}{2} - \mu - \kappa = -\,j$, the function M is a polynomial of degree j, which, when suitably normalized, is a generalized Laguerre-polynomial. Taking the square of the condition

$$-\,\mu = \kappa - j - \frac{1}{2}\,, \qquad (14)$$

and inserting κ and μ, as given by Eq. (3) and Eq. (4) we finally obtain the dispersion relation:

$$y^3 - 2\,\gamma\,\lambda\,(2j+1)\,y^2 +$$

$$[\,\lambda^2\,\gamma^2\,(2j+1)^2 + \lambda\,\frac{k}{k_0}\,(2j+1) - \gamma^2\,]\,y + \lambda\,\frac{k}{k_0} = 0\,, \qquad (15)$$

where, instead of ω^2 and instead of the wave number k, we have introduced the quantity y and the relative wave number k/k_0 with

$$y = \frac{\omega^2}{g\,k} \qquad \text{and} \qquad k_0 = \frac{g}{4\,a_0^2\,\gamma^2}\;. \qquad (16)$$

As this equation is obtained also by squaring of $+\,\mu = \kappa - j - \frac{1}{2}$, it describes also the eigenmodes selected by this condition in paper I. For all horizontal wave numbers $0 < k < \infty$ the dispersion equation has a negative real root y or ω^2 which shall not be considered here.

4. Dispersion curves and properties of quasi-stationary waves

In the following, we select those quasi-stationary solutions which describe outgoing progressive waves. Then, the case $j = 0$ is discussed in detail. Finally we present results for $j = 0, 1, 2, 3, 4$.

4.1. Outgoing progressive waves

Eq. (15) has two complex conjugated roots

$$\omega^2 = \alpha_0 \pm i\,\beta_0 \quad \text{with} \quad \alpha_0 > 0,\ \beta_0 > 0\;. \qquad (17)$$

The corresponding values of μ given by Eq. (4) are

$$\mu = q \mp i\,r \quad \text{with} \quad q > 0,\ r > 0\;. \qquad (18)$$

Eq. (17) yields four frequencies

$$\omega = \pm\,(\alpha + i\,\beta) \text{ and } \omega = \pm\,(\alpha - i\,\beta) \text{ with } \alpha > 0,\ \beta > 0\,. \qquad (19)$$

Putting $\Delta p \propto e^{+\,i\,\omega\,t}$, time-damped waves are represented by

$$\omega = \pm\,\alpha + i\,\beta\;. \qquad (20)$$

As regards the spatial dependence of Δp at infinity, we have to take the part

$$x^{-\,\mu} = x^{-\,q\,\pm\,i\,r}\;, \qquad (21)$$

to obtain outgoing complex waves. Then,

$$e^{\,i\,\omega\,t}\,x^{1/2\lambda}\,x^{-\,\mu} = e^{-\,\beta\,t\,\pm\,i\,\alpha\,t}\;x^{1/2\lambda}\,x^{-\,q\,\pm\,i\,r} =$$

$$\exp[\,-\,\frac{z}{2H} + \frac{\lambda\,q}{H}\,z - \beta\,t\,\pm\,i\,\alpha\,t\,\mp\,i\,\frac{\lambda\,r}{H}\,z\,]\;. \qquad (22)$$

The corresponding real expression

$$\Delta p = \exp[\,-\,\frac{z}{2H}\,]\;\exp[\,\frac{\lambda\,q}{H}\,z - \beta\,t\,]\;\sin(\,\alpha\,t - \frac{\lambda\,r}{H}\,z) \qquad (23)$$

represents an outgoing time-damped wave. Without the reciprocal steepening factor $\exp[-\,z/2H]$, the amplitude $\exp[\lambda q\,z/H]$ increases exponentially.

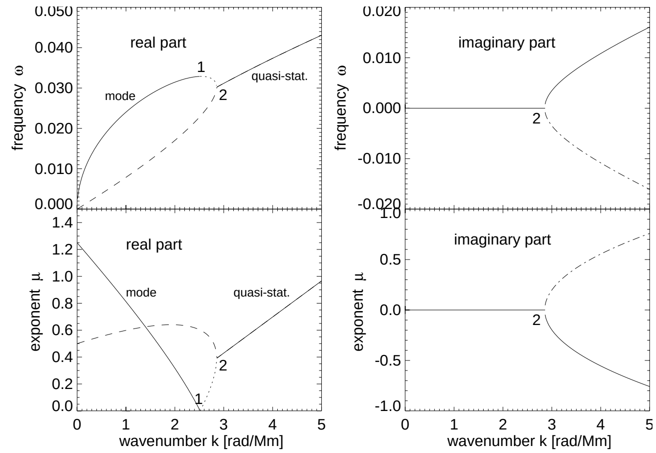

Fig. 1. The behavior of the p_0-mode and the first quasi-stationary wave for $\gamma = 5/3$. The dashed and dotted curves show reflected evanescent waves. The dashed-dotted curves of the imaginary parts represent time-reversed solutions.

4.2. The case $j = 0$

The case $j = 0$ is representative. Here, the real root of the cubic equation (15) is $y = -1$. Separating this solution, we obtain the quadratic equation

$$y^2 - y\,(2\gamma - 1) + \lambda\,\frac{k}{k_0} = 0\;. \qquad (24)$$

The solutions are:

$$\omega^2 = \frac{g\,k}{2}\,[2\gamma - 1 \pm \sqrt{(2\gamma-1)^2 - 16(\gamma-1)\frac{k\,c_0^2}{g}}\;]\;. \qquad (25)$$

Let k_2 be the value of the wave number where the real branches $\omega(k)$ end. We have:

$$k_2 = \frac{g}{16\,c_0^2}\;\frac{(2\gamma-1)^2}{(\gamma-1)}\;. \qquad (26)$$

The case of real ω has been considered in paper I. For $k_2 < k < \infty$ there are two complex conjugate solutions. These solutions represent quasi-stationary waves. Let k_1 be the value where the real function $\omega(k)$ reaches its maximum. For $0 \le k \le k_1$, the solution with the $+$ sign represents the p_0-mode. As discussed in paper I., the other branches represent downward directed waves. Therefore, we have a gap for $k_1 < k < k_2$.

Fig. 1 shows the detailed behavior of the frequencies ω and the corresponding exponents μ. The solid curves represent the ridges of the p-modes and the ridges of the quasi-stationary waves. The dashed and dotted curves of the real parts are solutions which represent incoming waves. The dashed-dotted curves of the imaginary parts represent time-reversed solutions.

Finally, we consider the form of the pressure perturbation of the quasi-stationary wave. As $M = 1$ for $j = 0$, the pressure perturbation $\Delta p(z, t)$ is given by:

$$\Delta p = x^{1/2\lambda - \mu}\,e^{-x/2}\,e^{i\omega t} = x^{1/2\lambda - q \pm ir}\,e^{-x/2}\,e^{-\beta t \pm i\alpha t}. \qquad (27)$$

Its real representation is

$$\Delta p = x^{1/2\lambda}\,e^{-x/2}\,x^{-q}\,e^{-\beta t}\;\sin(\alpha t + r\,\ln x) \qquad (28)$$

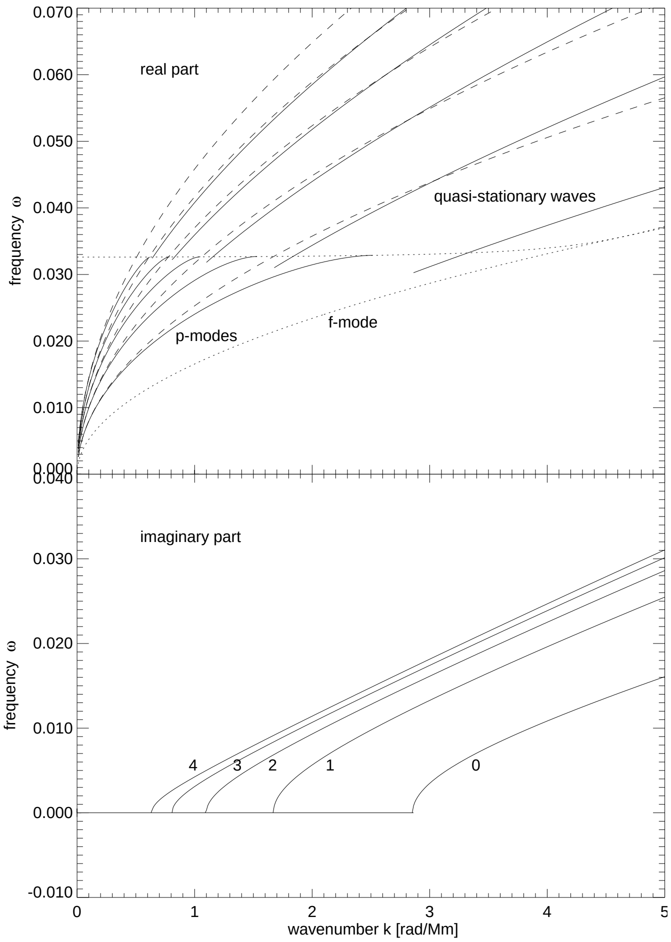

Fig. 2. The frequencies ω of modes and quasi-stationary waves (solid curves). Modes of the simple polytropic convection zone (dashed), the atmospheric f-mode and the critical frequency of the isothermal atmosphere (dotted)

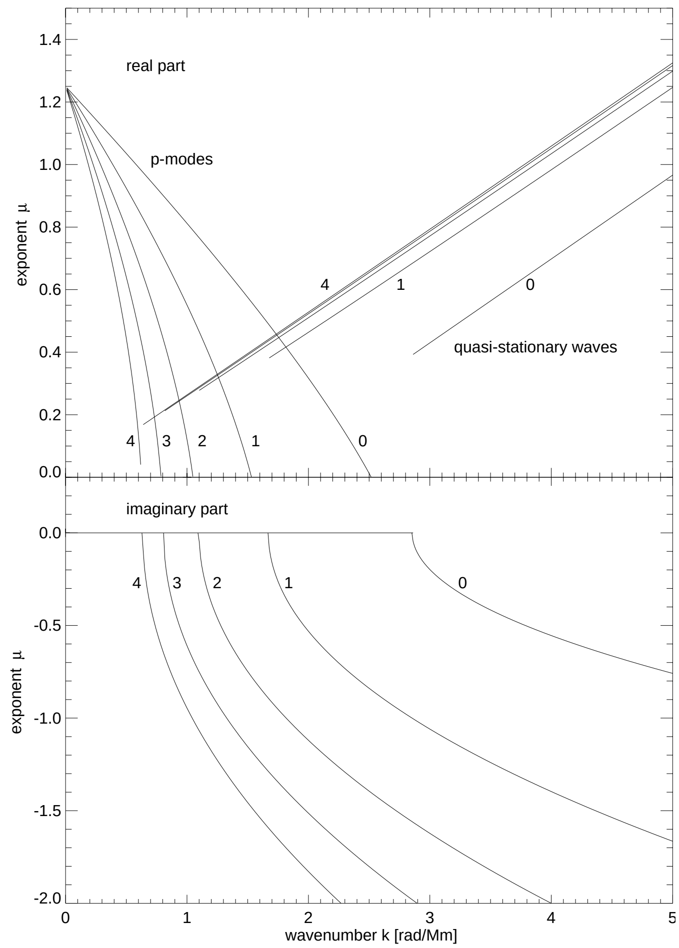

Fig. 3. The behavior of the exponent μ of modes and quasi-stationary waves.

where the dimensionless quantity x is related to the geometrical height z by Eq. (5). For $z \to \infty$, where $\ln x = -g\,\lambda\,z/a_0^2$, the real representation (28) reduces to Eq. (23). For $z \to -\infty$, where $x = -2\,k\,z$, it passes into

$$\Delta p \sim (-z)^{1/2\lambda}\, e^{k\,z}\,(-z)^{-q}\, e^{-\beta t}\,\sin(\alpha t + r\,\ln(-z))\,. \quad (29)$$

This expression represents a wave the amplitude of which decays exponentially when $z \to -\infty$.

4.3. *The general case* $j = 0, 1, 2, 3, 4$.

As the roots of a cubic equation with real coefficients are simple algebraic expressions, we do not present the corresponding formulas. For all wave numbers k, the cubic equation has a negative real root which has no physical meaning. As regards the other two roots, their behavior corresponds to the behavior of the roots of the quadratic equation (24).

Fig. 2 shows real and imaginary parts of the frequencies. Real frequencies represent modes, complex frequencies quasi-stationary waves. Besides, modes of the polytropic convection zone without an overlying atmosphere are shown by dashed lines. Fig. 3 shows the corresponding real and imaginary parts of the exponent μ. In the high frequency limit, the imaginary part β of the frequencies and the real part q of the exponent μ become straight lines. In this case, $\beta = \lambda q/(H\,c)$, so that the time-damping $\exp(-\beta t)$ is compensated by the spatial increase $\exp(\lambda q\,z/H)$ of the amplitude. Each wave of the wave-train propagates with constant amplitude.

The ridges of the real p-modes and the ridges of the quasi-stationary waves are not connected. There are gaps between the ridges. The ridges of the quasi-stationary waves extend into the region below the acoustic cut-off frequency. Both findings have not been noticed in previous papers. Besides, the observed pseudo-mode ridges give another picture. For an isothermal atmosphere, Wang et al. (1995) have shown that waves with complex frequencies propagate at frequencies below the acoustic cut-off frequency.

The f - mode is the atmospheric, pressure-free mode with $\omega^2 = g\,k$. There are no gravity modes although our layer is isentropic only in the limit $z \to -\infty$. This result corresponds to the results of Hindman & Zweibel (1994). Their "chromospheric model", a convection zone matched by a temperature minimum region and a chromosphere has only two g - modes at large

k which obviously stem from the chromospheric structure. As in our model, their convection zone is isentropic only at great depth.

5. The physical and mathematical meaning of the quasi-stationary waves

Quasi-stationary waves are no normal modes. Possibly, they are limiting cases of a certain class of instationary waves. The following considerations deal with this aspect.

5.1. *Quasi-stationary solutions of the ordinary wave equation*

We consider separable solutions of the ordinary wave equation. For $-\infty < z < +\infty$, the frequency spectrum is continuous. We have progressive waves (for simplicity we consider only waves travelling upwards) and standing waves:

$$\sin(\omega\, t - k\, z) \;\text{ and }\; \sin(\omega\, t)\, \cos(k\, z) \quad \text{with} \quad \omega = c\, k. \tag{30}$$

The latter are orthogonal base functions. With complex separation constants we obtain the following solutions of the wave equation:

$$\exp(-\,\alpha\,[\omega t - kz])\, \sin(\omega t - kz) \tag{31}$$

or

$$\exp(-\,\alpha\,[\omega t - kz])\, \sin(\omega t) \cos(kz)\;. \tag{32}$$

These particular solutions are not integrable and do not form a basis for general instationary solutions.

In the case of a semi-infinite medium ($0 \le z < \infty$) with homogeneous initial values and the following boundary condition at $z = 0$:

$$u(t) = \exp(-\,\alpha\omega t)\;\sin(\omega t) \quad \text{for} \quad t > 0, \tag{33}$$

which can be realized by a damped piston, a wave travels upwards. This wave has the form

$$u(z,t) = \exp(-\,\alpha\,[\omega t - kz])\; \sin(\omega\, t - k\, z)\, \Theta(t - \frac{z}{c})\,, \tag{34}$$

where Θ is the step function. The quasi-stationary solution (31) is a limiting case of this instationary solution for $t \to \infty$. The spatial increase of the amplitude is compensated by the time-damping so that each phase $\omega t - kz$ propagates with constant amplitude. The increase of the amplitude is due to the property that pulses emitted later, have smaller amplitudes.

5.2. *The general case*

The Lagrangian displacement $\boldsymbol{\xi}(\boldsymbol{r}, t)$ of linear adiabatic non-radial pulsations of a star is governed by an equation

$$\rho\, \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = -\,\rho\,\omega^2\,\xi = \mathcal{L}\,(\boldsymbol{\xi}) \tag{35}$$

where $\mathcal{L}$ is a linear spatial operator. (see e.g. Saio, 1993). A profound treatment of this subject is given by Lynden-Bell & Ostriker (1967). Such an equation holds also for the plane layer with constant gravity. In the case of a layer that is isothermal when $z \to \infty$, there are discrete eigenfrequencies below the critical frequency with orthogonal eigenfunctions. Above the critical frequency, there is a continuum of real frequencies. In the limit $z \to \infty$, the corresponding eigenfunctions behave as

$$\boldsymbol{\xi}(\boldsymbol{r}, t) = \boldsymbol{\xi}_0\; e^{i\,\omega t}\, e^{z/(2H)}\, e^{i\,\boldsymbol{k}\,\boldsymbol{r}}\,. \tag{36}$$

Only for real ω and real wave vectors $\boldsymbol{k}$, such eigenfunctions are integrable in a generalized sense. As $\rho(z) \sim \exp(-\,z/2H)$, the corresponding integrals

$$\int \rho\; \boldsymbol{\xi}_{\boldsymbol{k}}\, \boldsymbol{\xi}^*_{\boldsymbol{k}'}\, d^3\, \boldsymbol{r} \tag{37}$$

exist in the distribution sense, and the eigenfunctions are orthogonal. This is not the case for outgoing quasi-stationary waves with complex wave vectors $\boldsymbol{k}$. There, the corresponding solutions are not integrable.

This situation occurs also in quantum mechanics, where complex values of the energy are used to describe non-stationary systems. There are examples where the time-damped wave function of an outwards travelling particle increases exponentially with respect to the spatial coordinate. States with complex energies are called quasi-stationary states, and it is pointed out that these non-integrable states differ from the real, instationary states which are integrable. The problem is dealt with in some text-books: Blochinzew (1957), Macke (1959) and Landau & Lifshitz (1959).

For the layer considered in this paper, the question remains how quasi-stationary solutions are related to instationary solutions.

5.3. *A two-layer model*

To study the approach of an instationary solution by a quasi-stationary solution, we consider two homogeneous layers with one-dimensional wave propagation: A hot (interior) region with finite thickness a and an overlying cool (exterior) region with infinite extension. Let

$$\frac{\partial^2\, u}{\partial t^2} = c^2\;\; \frac{\partial^2\, u}{\partial z^2} \tag{38}$$

be the wave equation of a perturbation $u(z, t)$. Let

$$c = c_1 \;\text{ for }\; 0 < z < a \;\text{ and }\; c = c_2 \;\text{ for }\; a < z < \infty$$

with $c_2 < c_1$. At $z = 0$ we put a rigid boundary condition: $u(0, t) = 0$. We study the behavior of quasi-stationary waves and instationary waves. The more general problem, a two layer model with gravity has the shortcomings that the instationary case cannot be solved analytically. In the following, we will discuss quasi-stationary solutions and special instationary solutions, and compare these solutions.

a) *Quasi-stationary waves.* This case has been considered by Schatzman (1956) and by Balmforth & Gough (1990). These authors have studied two atmospheric layers with finite gravity. In this case, the dispersion relation cannot be solved analytically.

We present the results for $g = 0$, a particular case not considered by the above authors.

In the following, ω, $k_1 = \omega/c_1$, $k_2 = \omega/c_2$, and α are real quantities. In region 1, the most general quasi-stationary solution fulfilling the rigid boundary condition $u(0,t) = 0$ is the linear combination

$$u_1(z,t) = \exp(-\alpha\omega t + \alpha k_1 z)\ \sin(\omega t - k_1 z)$$

$$- \exp(-\alpha\omega t - \alpha k_1 z)\ \sin(\omega t + k_1 z)\ . \quad (39)$$

In region 2 only outgoing progressive waves are present. There, the most general quasi-stationary solution is:

$$u_2(z,t) = \exp(-\alpha\omega t + \alpha k_2 z) \cdot$$

$$[D\ \sin(\omega t - k_2 z + k_2 a) + E\ \cos(\omega t - k_2 z + k_2 a)]\ . \quad (40)$$

By the condition for continuity of $u(z,t)$ and its derivative at $z = a$ we obtain the results:

$$\omega = \frac{l\,\pi\,c_1}{a} \quad \text{and} \quad \alpha = \frac{1}{l\,\pi}\ \tanh^{-1}(\frac{c_2}{c_1}) \quad (41)$$

with $l = 1, 2, 3, ...$ and

$$D = (-1)^l\ 2\ e^{-\alpha k_2 a}\ \sinh(\alpha k_1 a)\ ,\ \ E = 0 \quad (42)$$

Finally, the solution in the exterior region 2 reads:

$$u_2(z,t) = 2\ (-1)^l\ \sinh(\alpha k_1 a) \cdot$$

$$\exp(-\alpha\omega t + \alpha k_2\,(z-a))\ \sin(\omega t - k_2\,(z-a))\ . \quad (43)$$

This is an outgoing wave exponentially damped with respect to time, but exponentially increasing with respect to the spatial coordinate z.

b) *Instationary waves.* If $c_2 = 0$, the interior layer is a resonator with the boundary condition $u(a,t) = 0$. The eigenfunctions are $u(z,t) = \sin(kz)$, the eigenfrequencies are $\omega = c_1 k$ with $k = l\pi/a$ for $l = 1, 2, ...$ It is evident that the following initial value problem should yield an outwards propagating instationary wave which might be described by a quasi-stationary wave:

$$u(z,0) = \sin(kz) \text{ for } 0<z<a\,,\ \ u(z,0) = 0 \text{ for } a<z<\infty$$

$$\text{and} \quad \dot{u}(z,0) = 0 \quad \text{for} \quad 0 < z < \infty$$

with the rigid boundary condition $u(0,t)=0$ for $0<t<\infty$.

By the method of the Laplace transform (some details are given in an appendix), we obtain the solution in the interior region $z < a$:

$$u_1(z,t) = \cos(\omega t)\ \sin(kz)\ -\ \frac{c_2}{c_1 + c_2}\ \cdot \quad (44)$$

$$\Big[\sin(\omega[t + z/c_1)])\ \sum_{m=0}^{\infty} R^m \Theta(t + \frac{z - (2m+1)\,a}{c_1})$$

$$-\ \sin(\omega[t - z/c_1)])\ \sum_{m=0}^{\infty} R^m \Theta(t - \frac{z + (2m+1)\,a}{c_1}\Big]$$

and the solution in the exterior region $z > a$:

$$u_2(z,t) = (-1)^{n+1}\ \frac{c_2}{c_1 + c_2}\ \sin(\omega[t - \frac{z-a}{c_2}])\ \cdot \quad (45)$$

$$\Big[\,\Theta(t-\frac{z-a}{c_2}) + \sum_{1}^{\infty}(R^m - R^{m-1})\,\Theta(t - \frac{z-a}{c_2} - \frac{2a\,m}{c_1}\Big)\,]$$

with the Heaviside step function $\Theta(\xi) = 0$ for $\xi < 0$ and $\Theta(\xi) = 1$ for $\xi > 0$, and with

$$R = \frac{c_1 - c_2}{c_1 + c_2}\ .$$

c) *Quasi-stationary solutions as limiting cases of instationary waves.* Now let us study the behavior of the exterior solution in the limit $c_2/c_1 \to 0$. Let M be an integer with

$$M \approx \frac{c_1}{2a}\ (t - \frac{z-a}{c_2}) \gg 1\ . \quad (46)$$

Then, Eq. (46) reduces to:

$$u(z,t) = (-1)^{n+1}\ \frac{c_2}{c_1 + c_2}\ \sin(\omega[t - \frac{(z-a)}{c_2}])\ R^M\ . \quad (47)$$

We have:

$$R^M = \Big(\frac{1 - \frac{c_2}{c_1}}{1 + \frac{c_2}{c_1}}\Big)^M \to \Big(1 - \frac{c_2}{c_1}\Big)^{2M} \quad \text{if} \quad \frac{c_2}{c_1} \to 0\ . \quad (48)$$

Using Eq. (53), for $c_1/c_2 \to \infty$, we obtain

$$R^M = \left[\Big(1 - \frac{c_2}{c_1}\Big)^{c_1}\right]^{\frac{1}{a}\,(t - \frac{z-a}{c_2})} \to \exp\big(-\frac{c_2}{a}\,(t - \frac{z-a}{c_2})\big)\ . \quad (49)$$

For $c_2/c_1 \to 0$ the parameter α given by Eq. (41) reduces to

$$\alpha = \frac{1}{k\,a}\ \frac{c_2}{c_1} = \frac{c_2}{a\,\omega}\ , \quad (50)$$

so that $c_2/a = \alpha\,\omega$. With $k_2 = \omega/c_2$, we finally obtain

$$u_2(z,t) = (-1)^{n+1}\ \frac{c_2}{c_1 + c_2}\ \cdot$$

$$\exp(-\alpha\omega t + \alpha k_2 (z-a))\ \sin(\omega t - k_2 (z-a))\,, \quad (51)$$

for t as given by Eq. (46). This solution corresponds to the outgoing quasi-stationary wave (43) Here, we have considered the limit $c_2/c_1 \to 0$. The picture changes when we consider a finite ratio c_2/c_1. Fig. 4 shows the second harmonics for $c_1/c_2 = 4$. This ratio roughly corresponds to solar conditions. The instationary solution shows a staircase appearance. This feature vanishes only in the limit $c_2/c_1 \to 0$, as there $k_2 \to \infty$. For finite c_2/c_1, however, there are differences between an instationary solution and its corresponding quasi-stationary counterpart. It is not only the front of the wave train which is not described by a quasistationary wave, but also the whole wave train shows differences.

6. Conclusions

Whereas in paper I, we concentrated on waves with real frequencies, the present paper deals with waves with complex frequencies. The derived dispersion relation $F(\omega, k) = 0$ holds for both cases. However, the branches $\omega(k)$ of the modes and the branches $\omega(k)$ of the quasi-stationary waves result from different physical conditions. It is shown, that solutions with time-damped amplitudes represent waves travelling outwards. The amplitude of these waves increase exponentially with height. In a sense, this spatial increase compensates the time-damping.

There are striking gaps between the ridges of the modes and the ridges of the quasi-stationary waves. Further, the ridges of the quasi-stationary waves extend into the region below the acoustic cut-off frequency. Both findings have not been noticed in previous papers. Besides, the observed ridges give another picture.

Solutions with real frequencies above the critical frequency are quadratic integrable in the distribution sense, and, together with the discrete modes, form a set of basis functions. Solutions with complex frequencies, however, are not integrable, and form no fundamental system. With a simple one-dimensional two-layer model of a leaky resonator we have studied relations between quasi-stationary and instationary solutions. An asympotic evaluation shows that both solutions are closely related in the limit of weak leakage of the resonator. However, for conditions corresponding to the structure of the exterior layers of the sun, some differences are evident. Thus, the physical meaning of quasi-stationary waves and their ridges is doubtful.

Our findings give no evidence that the observed ridges of the diagnostic diagram can be attributed to ridges of quasi-stationary solutions.

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Appendix A: Calculation of an instationary solution by the Laplace transformation

We have solved the initial value problem of Sect. 5.3 by the method of the Laplace transform. As we did not find textbooks, hand-books or other works dealing with this problem, we present here some steps of the evaluation without going into details. (As regards heat conduction, corresponding initial value problems are treated in the literature.)

Let $v(z,p) = \mathcal{L}\{u(z,t)\}$ be the Laplace transform of $u(z,t)$ with respect to t. Taking the Laplace transform of the wave equation (38), we get:

$$c_1^2 \, \frac{d^2 v_1}{dz^2} - p^2 \, v_1 = -\, p \, \sin(kz) \quad \text{for} \quad 0 < z < a$$

and

$$c_2^2 \, \frac{d^2 v_2}{dz^2} - p^2 \, v_2 = 0 \qquad \text{for} \quad a < z < \infty \, .$$

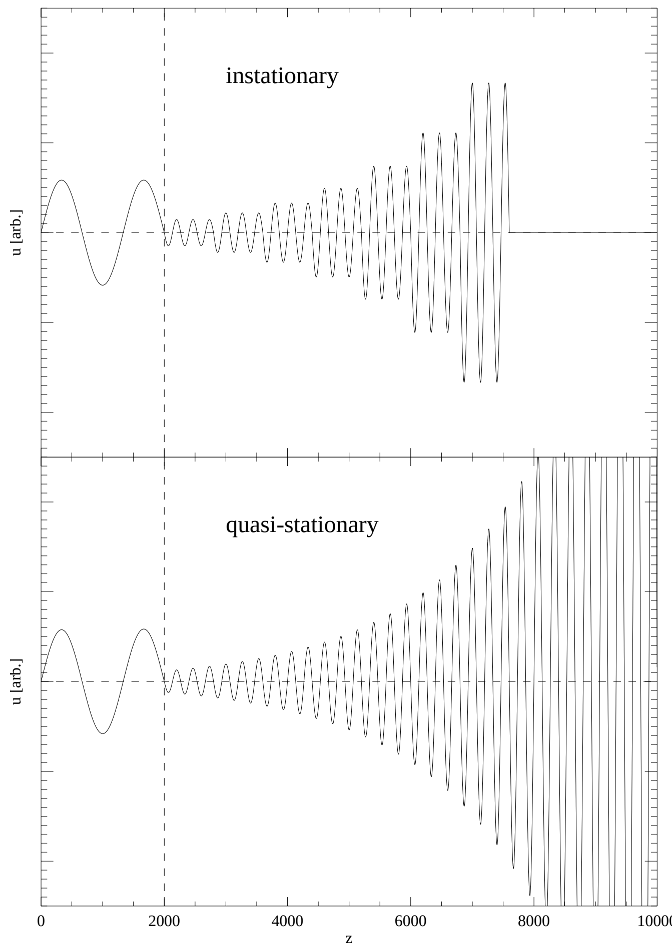

Fig. 4. Instationary and quasi-stationary solution. The case $c_1/c_2 = 4$ and $l = 3$.

The equation of the interior region has the solution:

$$v_1(z,p) = A\left[e^{+pz/c_1} - e^{-pz/c_1}\right] + \frac{p}{p^2+\omega^2}\, \sin(kz)\, ,$$

with $\omega = c_1\, k$. The solution in the exterior region is

$$v_2(z,p) = B e^{-pz/c_2} \, .$$

The constants A and B are determined from the conditions for continuity at $z = a$. We finally obtain:

$$v_1(z,p) = D\, \frac{p}{p^2+\omega^2}\, \frac{1}{p}\, \frac{\left(e^{+p(z-a)/c_1} - e^{-p(z+a)/c_1}\right)}{1 - R\, e^{-2\,pa/c_1}}$$

$$+ \frac{p}{p^2+\omega^2}\, \sin(kz)\, ,$$

$$v_2(z,p) = D\, \frac{p}{p^2+\omega^2}\, \frac{e^{-p(z-a)/c_2}}{p}\, \frac{1 - e^{-2\,pa/c_1}}{1 - R\, e^{-2\,pa/c_1}}\, ,$$

where

$$D = (-1)^{n+1}\, \frac{c_1\, c_2\, k}{c_1 + c_2} \quad \text{and} \quad R = \frac{c_1 - c_2}{c_1 + c_2}\, .$$

By series expansion of the denominators, we get:

$$v_1(z,p) = \frac{p}{p^2+\omega^2}\ \sin(kz) +$$

$$D\Big[\sum_0^\infty R^m\ \frac{p}{p^2+\omega^2}\,\frac{1}{p}\,e^{+p(z-(2m+1)\,a)/c_1}$$

$$-\ \sum_0^\infty R^m\ \frac{p}{p^2+\omega^2}\,\frac{1}{p}\,e^{-p(z+(2m+1)\,a)/c_1}\Big]$$

and

$$v_2(z,p) = D\Big[\frac{p}{p^2+\omega^2}\,\frac{1}{p}\,e^{-p[(z-a)/c_2]}$$

$$+\ \frac{R-1}{R}\sum_1^\infty R^m\ \frac{p}{p^2+\omega^2}\ \frac{1}{p}\ e^{-p[(z-a)/c_2+2am/c_1]}\Big]\,.$$

With the relations

$$\mathcal{L}\left\{\cos(\omega t)\right\} = \frac{p}{p^2+\omega^2}, \quad \mathcal{L}\left\{\Theta(t-\alpha)\right\} = \frac{1}{p}\,e^{-\,\alpha p}\,,$$

and the convolution theorem we obtain the results

$$u_1(z,t) = \cos(\omega t)\ \sin(kz)$$

$$+\,D\Big[\sum_0^\infty R^m\int_0^t\cos(\omega[t-\tau])\,\Theta(\tau+\frac{z-(2m{+}1)a}{c_1})d\tau$$

$$-\sum_0^\infty R^m\int_0^t\cos(\omega[t-\tau])\,\Theta(\tau-\frac{z+(2m{+}1)a}{c_1})d\tau\Big]$$

and

$$u_2(z,t) = D\ \Big[\int_0^t\cos(\omega[t-\tau])\ \Theta(\tau-\frac{z{-}a}{c_2})\ d\tau$$

$$+\,\frac{R{-}1}{R}\sum_1^\infty R^m\int_0^t\cos(\omega[t{-}\tau])\,\Theta(\tau-\frac{z{-}a}{c_2}-\frac{2ma}{c_1})d\tau\Big]\,.$$

Evaluating the integrals, we get the results (44) and (45).

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