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The influence of turbulence on the solar p-mode oscillations

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Abstract. To investigate the physical nature of solar turbulent convection, we study the influence of turbulent pressure on the solar p-mode oscillations. It is assumed that only the important effects of the Reynolds stress are included in the equations, while all other effects are neglected. The shifts of oscillation frequencies from their adiabatic values due to influence of Reynolds stress are calculated with the help of a time-dependent improved mixing-length theory which takes into account the contribution of the full spectrum of the turbulent convective eddies.

The calculations reveal that although the spatial and temporal component of turbulent spectrum may influence the oscillation frequencies, the frequency shifts depend on sensitively on the details of turbulent energy spectrum $E(k)$. The comparison of numerical results shows that the frequency shift due to anisotropic turbulence is smaller than the frequency shift due to isotropic turbulence toward higher frequencies. It means that turbulent viscosity exerts a non-negligible influence on the solar p-mode oscillations.

Key words: Sun: oscillations – turbulence

1. Introduction

The solar p-mode oscillations have been investigated extensively since the oscillation modes were discovered. Nowadays, it is well known that several million resonant acoustic normal modes are excited stochastically in the subsurface turbulent layer, and their oscillation frequencies are measured with a relative accuracy as high as 10^{-5} (Libbrecht et al. 1990). The frequencies of the modes have been successfully modeled at the one percent level as adiabatic oscillations of a standard solar model (Deubner & Gough 1984; Christensen-Dalsgaard et al. 1985; Libbrecht 1988). However, the discrepancy between the observed and the calculated frequencies for the solar oscillations is too great to be the result of observational or numerical errors, and it requires us to investigate the standard solar models and the non-adiabatic character of the oscillation modes at the solar surface in details. It is obvious that the uncertainties between observation and theory may result mainly from our poor information on the physical properties of turbulence in the outer part of the solar convection zone and non-adiabaticity of oscillation due to radiation and convection.

In most of the convection zone the temperature gradient is nearly adiabatic because of the high efficiency of energy transfer by low-speed convective motions. However, the gradient becomes superadiabatic and convective velocities substantially increase near the surface because of turbulent heat transfer. The stratification is consequently practically adiabatic. This behavior is one candidate for improvement in the mixing-length theory of convection and for shifting oscillation frequencies from their adiabatic values. Therefore, turbulence plays a crucial role on the solar p-mode oscillations (Balmforth 1992; Kosovichev 1995; Gabriel 1995; Böhmer & Rüdiger 1998; Rosenthal et al. 1999).

In the present paper, we calculate the frequency shifts by neglecting the effect of radiation on pulsation and only considering the effect of turbulent convection on pulsation. The purpose of this paper is to treat improvements in the frequencies of oscillation and the theory of turbulent convection. To do so, we use a more realistic model for the turbulence, an incorporation of physically meaningful description of the spatial and temporal spectrum of the turbulent convection (Musielak et al. 1994; Bi & Li 1998).

The paper is organized as follows. The basic physical formulations that describe the influence of turbulent convection on the equilibrium structure and oscillation frequencies are given in Sect. 2 and Sect. 3. The convective model which takes into account the contribution of the full spectrum of the turbulent convective eddies is discussed in Sect. 4. In Sect. 5, we calculate the frequency shifts due to Reynolds stress with the different turbulent energy spectrum and discuss the numerical results. It is found that the frequency shifts depend sensitively on the details of turbulent energy spectrum $E(k)$, and the turbulent viscosity is an important factor for the solar p-mode oscillations.

2. Basic physical formulation

2.1. Dynamic equations of the mean motion of turbulent convection

The equations describing the conservation laws for mass and momentum, and temperature of a incompressible self-

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gravitating fluid of total density $\widetilde{\rho}$, pressure $\widetilde{P}$, velocity vector $\widetilde{\mathbf{v}}$, molecular viscosity υ and thermal conductivity $K \equiv c_p \rho \chi$ are written below ($\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \widetilde{\mathbf{v}} \cdot \nabla$) (Canuto 1993):

$$\frac{d\widetilde{\rho}}{dt} + \widetilde{\rho} \frac{\partial}{\partial x_i} v_i = 0, \tag{1}$$

$$\widetilde{\rho} \frac{d}{dt} v_i + \frac{\partial}{\partial x_i} \left(\widetilde{P}_g + \widetilde{P}_R \right) + \widetilde{\rho} g_i = \upsilon \widetilde{\rho} \frac{\partial^2}{\partial x_j^2} v_i, \tag{2}$$

$$\widetilde{\rho} c_p \frac{d\widetilde{T}}{dt} - \frac{d}{dt} \left(\widetilde{P}_g + \widetilde{P}_R \right) - K \nabla^2 \widetilde{T} = \upsilon \widetilde{\rho} \left[\frac{\partial^2}{\partial x_i \partial x_j} v_i v_j + \left(\frac{\partial v_i}{\partial v_j} \right)^2 \right] + \widetilde{\rho} c_p Q, \tag{3}$$

where the total pressure $\widetilde{P}$ has been written as the combination of gas pressure $\widetilde{P}_g$ and radiation pressure $\widetilde{P}_R$, $\mathbf{g}$ is the gravitational acceleration, and $\widetilde{\rho} c_p Q$ is the gradient of an external flux.

When turbulent convection appears, density, velocity, pressure and temperature are split into their average and fluctuating parts:

$$\widetilde{\rho} = \rho + \rho', \widetilde{\mathbf{v}} = \mathbf{U} + \mathbf{u}, \widetilde{P} = P + p, \widetilde{T} = T + \theta \tag{4}$$

here P, ρ, and $\mathbf{U}$ represent the average fields. The mean is defined as the average weighted by the fluid density $\widetilde{\rho}$,

$$\overline{\rho X} = \overline{\widetilde{\rho} X} \tag{5}$$

where X denotes any variable mentioned above. The fluctuating components have zero average

$$\overline{\rho'} = \overline{p} = \overline{u_i} = \overline{\theta} = 0 \tag{6}$$

Once the turbulence is present, the molecular viscosity term is much smaller than the term of turbulent eddy viscosity for the average motion, and therefore the former can be neglected. In the standard Boussinesq approximation, the term with the fluctuating pressure p/P is absent. Substituting Eqs. (3)-(4)into Eqs. (1)-(3), after an averaging procedure, we obtain the following dynamical equations for the mean values $\left(\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + U_i \frac{\partial}{\partial x_i} \right)$:

$$\frac{D\rho}{Dt} + \rho \frac{\partial U_i}{\partial x_i} = 0, \tag{7}$$

$$\rho \frac{DU_i}{Dt} = -\frac{\partial P}{\partial x_i} - \frac{\partial}{\partial x_j} \rho R_{ij} - \rho g_i, \tag{8}$$

$$\rho \frac{DT}{Dt} = \chi \frac{\partial^2 T}{\partial x_j^2} + \frac{1}{c_p} U_j \frac{\partial P}{\partial x_j} - \frac{\partial}{\partial x_j} \overline{u_i \theta} + Q + \frac{1}{\rho c_p T} \overline{u_j \theta}, \tag{9}$$

here for simplicity we have neglected the terms of pressure-velocity and pressure-temperature correlation. The term of $\overline{u_i \theta}$ is convective flux. It is important to realize that the significant effects of the turbulent dynamical stress is the averaged Reynolds stress

$$R_{ij} \equiv \overline{u_i(\mathbf{x}, t) \, u_j(\mathbf{x}, t)}, \tag{10}$$

contributing to the hydrostatic support and behavior of the solar p-mode oscillations. Convection affects the solar structure and pulsation through this second-order correlation tensor.

2.2. Hydrostatic equilibrium equation

Using Eqs. (8) in the stationary limit, we obtain the new hydrostatic equilibrium equation

$$\frac{1}{\rho} \frac{d}{dr} (P + P_t) = -\frac{Gm}{r^2}. \tag{11}$$

In the meantime, the superadiabatic temperature gradient is different from the gradient without dynamical correction, and it becomes

$$\beta = -\frac{\partial T}{\partial r} - \frac{g}{c_p} \frac{P + P_t}{P} \left(1 + \frac{1}{g} \frac{\partial P_t}{\partial r} \right), \tag{12}$$

where turbulent pressure is defined as

$$P_t \equiv \rho g_{ij} R_{ij}, \tag{13}$$

here g_{ij} is the metric tensor. This result that turbulent pressure modifies the hydrostatic equilibrium of solar structure, which follows naturally from the Reynolds stress approach, contrasts with previous empirical suggestions to include P_t (Baker & Gough 1979).

Therefore the kinematic part of turbulence contributes a turbulent pressure P_t to the hydrostatic equilibrium of solar structure. The turbulent pressure has two effects on the solar structure. It contributes to the total pressure, thus reducing the gas pressure and leading to lower density and sound speed. Turbulence pressure also modifies the superadiabatic gradient and it will results in changes in the properties of convection. The effects of turbulent pressure are often ignored in contributing to the hydrostatic support, mainly owing to the lack of a turbulence model to provide the turbulent spectrum.

3. The effects of oscillation frequencies

So far we have considered linear, adiabatic solar p-mode oscillations with no turbulent pressure correction. However, the turbulent pressure can be as much as 10% of the total pressure (Canuto & Mazzitelli 1991), and therefore, the influence of turbulence on oscillations has to be taken into account.

3.1. Inhomogeneous wave equation

The Eqs. (7) and (8) describing the motion of mean flow are non-linear equations. It is difficult to solve directly these equations to obtain the various properties of the solar oscillation. The usual method adopted is to linearize the non-linear equations. For simplicity, we employ Euler perturbation to establish the linearized oscillation equations. Under the perturbed state, the various physical quantities, such as density ρ, pressure P and velocity $\mathbf{U}$ can be treated as the sum of equilibrium value and a perturbation

$$\rho = \rho_0 + \rho^1, \quad \mathbf{U} = \mathbf{U}_0 + \mathbf{U}^1, \quad P = P_0 + P^1, \tag{14}$$

where the subscript "0" refers to equilibrium values and the superscript "1" refers to perturbation values.

Substituting Eq. (14) into the nonlinear Eqs. (7) and (8), neglecting terms with higher orders of perturbation values, and noting that the equilibrium values fulfill these equations, we can obtain the following equations in first order of the oscillation:

$$\frac{\partial \rho^1}{\partial t} + \nabla \cdot (\rho_0 \mathbf{v}) = 0, \tag{15}$$

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla \left(P^1 + P_t^1\right) - \rho^1 \mathbf{g} - \nabla \cdot \left(\rho^1 \mathbf{R}\right), \tag{16}$$

where the pulsating velocity is defined as $\mathbf{v} \equiv \frac{\partial \xi}{\partial t}$, ξ is the pulsating displacement vector. When turbulent pressure is taken into account, the appropriate effects of turbulent pressure on pulsation should include two parts: the first accounts for the influence of the oscillation on physical quantities, $\rho^1 R_{ij}$, while the second accounts for the influence of the oscillation on turbulent quantities, $\nabla \cdot \left(\rho_0 \mathbf{R}^1\right) \equiv \nabla P_t^1$. In this paper, we neglect the influence of the oscillation on the turbulent quantities.

Since Eqs. (15) and (16) are linear homogeneous equations, whose coefficients are independent of t, we can assume the variables to be proportional to $\exp(-i\sigma t)$, Eqs. (15) and (16) become

$$\rho^1 + \nabla \cdot (\rho_0 \xi) = 0, \tag{17}$$

$$\nabla P^1 + \rho^1 \mathbf{g} - \sigma^2 \rho_0 \xi = -\nabla \cdot \left(\rho^1 \mathbf{R}\right). \tag{18}$$

In quasi-adiabatic approximation, we have a simple relation

$$\frac{\delta P}{P_0} = \Gamma_1 \frac{\delta \rho}{\rho_0}, \tag{19}$$

where $\Gamma_1 = \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_s$.

For a quantity f, the Lagrangian perturbation is related to the Eulerian perturbation by

$$\delta f = f^1 + \xi \cdot \nabla f_0, \tag{20}$$

where the Lagrangian perturbation and Eulerian of physical variables are denoted by δ and the symbol "1".

Since near the surface, the ratio of the horizontal and radial displacements is given by

$$\frac{\xi_h}{\xi_r} \approx \frac{\ell g}{\sigma^2 R} \approx 10^{-3} \ell \left(\frac{3mHz}{\nu}\right)^2, \tag{21}$$

where $\sigma = 2\pi\nu$. If $\ell < 10^3$, then the oscillations of the typical frequency 3mHz are almost radial near the surface. Therefore, for p-modes of non-radial low ℓ, we can restrict ourselves in this paper to radial oscillations, and ξ has only a radial component.

The next step is to obtain some basic properties of the p-mode eigenfunction. Using Eqs. (19) and (20), we reduces the equations for radial oscillations to

$$\frac{1}{\rho_0}\frac{dP^1}{dr} + \frac{g}{\rho_0 c^2} P^1 + \left(N^2 - \sigma^2\right)\xi = -\frac{1}{r^2}\frac{d}{dr}\left[\frac{d}{dr}\left(r^2 \rho_0 \xi\right) R_{rr}\right], \tag{22}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \xi\right) + \frac{1}{\Gamma_1}\frac{d\ln P}{dr}\xi + \frac{P^1}{\rho_0 c^2} = 0, \tag{23}$$

with c^2 the adiabatic sound speed, and the Brunt-Väisälä frequency

$$N^2 = g\left(\frac{1}{\Gamma_1}\frac{d\ln p_0}{dr} - \frac{d\ln \rho_0}{dr}\right). \tag{24}$$

Eliminating P^1 from Eqs. (22) and (23), we easily get the inhomogeneous oscillation wave equation

$$\pounds(\xi) + \sigma^2 \rho_0 r^3 \xi = \rho_0 r^3 D(\xi), \tag{25}$$

where the scale operators $\pounds(\xi)$ and $D(\xi)$ are given in the following

$$\pounds(\xi) = \frac{d}{dr}\left[\Gamma_1 P_0 r^4 \frac{d}{dr}\left(\frac{\xi}{r}\right)\right] + r^3\left\{\frac{d}{dr}\left[(3\Gamma_1 - 4) P_0\right]\right\}\left(\frac{\xi}{r}\right), \tag{26}$$

and

$$D(\xi) = -\frac{1}{r^2}\frac{d}{dr}\left[\frac{d}{dr}\left(r^2 \rho_0 \xi\right) R_{rr}\right]. \tag{27}$$

The effect of the Reynolds stress on the oscillations appears in the scale operator $D(\xi)$.

3.2. Frequency shift with the Reynolds stress

As seen in above, the Reynolds stress affects both the equilibrium structure and the oscillations. For simplicity, we only consider the Reynolds stress correction to the oscillation in the wave equation. If the effect of Reynolds stress on the oscillations is small, the inhomogeneous wave equation of the solar p-mode oscillations can be solved by the perturbation method (Unno et al. 1989).

The eigenvalue σ and eigenfunction ξ are expanded as

$$\sigma_n = \sigma_n^{(0)} + \sigma_n^{(1)} + ..., \tag{28}$$

$$\xi_n = \xi_n^{(0)} + \xi_n^{(1)} + ..., \tag{29}$$

where the first terms in the right-hand sides of Eqs. (28) and (29) represent the n-th eigenmode for the case of no turbulent pressure. Substituting Eqs. (28) and (29) into Eq. (25) and neglecting the terms of the second and higher orders, we obtain

$$\pounds\left(\xi_n^{(0)}\right) + \rho_0 r^3\left[\sigma_n^{(0)2}\xi_n^{(0)} + 2\sigma_n^{(0)}\sigma_n^{(1)}\xi_n^{(0)} + \sigma_n^{(0)2}\xi_n^{(1)}\right] = \rho_0 r^3 D\left(\xi_n^{(0)}\right). \tag{30}$$

Since the zero-th order eigensolution, $\sigma^{(0)}$ and $\xi_n^{(0)}$, satisfies

$$\pounds\left(\xi_n^{(0)}\right) + \rho_0 r^3 \sigma^{(0)2}\xi_n^{(0)} = 0, \tag{31}$$

Eq. (30) becomes

$$2\sigma_n^{(0)}\sigma_n^{(1)}\xi_n^{(0)} + \sigma_n^{(0)2}\xi_n^{(1)} = D\left(\xi_n^{(0)}\right). \tag{32}$$

To solve Eq. (32) with respect to $\sigma_n^{(1)}$ and $\xi_n^{(1)}$, we expand the displacement $\xi_n^{(1)}$ by eigenfunction $\xi_n^{(0)}$

$$\xi_n^{(1)} = \sum_k a_k \xi_k^{(0)}. \tag{33}$$

According to the complete orthogonality of the set of eigenfunctions $\left\{\xi_k^{(0)}\right\}$, the coefficient a_k is given by

$$a_k = \int_0^{M_\odot} \xi_n^{(1)} \cdot \xi_k^{(0)*} dm.$$

Substituting Eq. (33) into Eq. (32), multiplying Eq. (32) by $\xi_k^{(0)*}$, and integrating over the mass, we have

$$\sigma_n^{(1)} = \frac{\int_0^{M_\odot} D\left(\xi_k^{(0)}\right) \xi_k^{(0)*} dm}{8\pi\sigma_n^{(0)} I}. \tag{34}$$

where

$$I = \frac{1}{4\pi} \int_0^{M_\odot} \left|\xi_k^{(0)}\right|^2 dm$$

is defined as the inertia of the mode. In the first order correction approximation, the frequency shift from the adiabatic frequencies ν_a due to Reynolds stress is

$$\Delta\nu \equiv \frac{\sigma_n^{(1)}}{2\pi} \tag{35}$$

4. The treatment of the turbulence

4.1. Mixing-length theory

For a long time, local mixing-length theory (MLT) has been used to calculate the transport of heat of turbulent convection in models of stellar structure. The standard description of the mean stratification of convective layers is based on mixing-length models. In the mixing-length approach, the complicated situation in the actual convection zone is replaced by a group of identical convective elements, all of which travel a distance equal to one mixing length, l, and then thermalize with the ambient medium. Further, the MLT requires that the mean velocity of convective elements go to zero at the boundaries of the unstable region.

From a phenomenological point of view, turbulence in a fluid is often described in terms of eddies. The structure of eddies of the convective zone is very close to being isotropic because the plasma density is high enough that convective flux of low speed are sufficient to transport the convective energy. The convective flux is the excess energy in a convecting fluid parcel times its velocity. In the MLT, the convective flux is given by (Canuto 1996)

$$F_c \sim c_p\rho \left(\frac{\beta}{g\alpha}\right)^{1/2} \int E(k)\, dk, \tag{36}$$

where $\alpha = T^{-1}$ is the volume expansion coefficient, and the energy spectrum of turbulent convection is represented by a δ-function

$$E(k) = E_0 \delta(k/k_0 - 1), \tag{37}$$

here the wavenumber k is roughly related to the inverse of the "size of the eddy". The integral of $E(k)$ yields the total turbulent kinetic energy

$$K = \frac{1}{2} v_t^2 = \int E(k) dk.$$

The mixing-length model treats turbulence associated the spectrum of turbulent eddies as if it is dominated by a single large eddy with size comparable to the local pressure scale height. This is a reasonable approximation for viscous fluids, which are characterized by a rather narrow eddy spectrum. However, the MLT approximation is doubtful about its reliability as a faithful representation of stellar turbulent convection.

4.2. Improved description of turbulence

If L and l_d represent the largest and smallest sizes of the eddy, we have

$$\frac{L}{l_d} = R_e^{3/4},$$

where dissipation scale l_d characterizes the scales where dissipation occurs. In the solar interior, Reynolds number $\sim 10^8$, the spectral function $E(k)$ is usually very wide and can hardly be represented by a δ-function. Therefore the MLT, one eddy model, underestimates the convective flux. The major challenge of a turbulent model is the derivation of the spectral function $E(k)$ which takes into account the contribution of the full turbulent convective eddies.

Because of the shearing motion and the work done by the buoyant force, turbulent eddies receive kinetic energy from the mean field. This process happens at the low wavenumber end (large-scale turbulent eddies) of the turbulent spectrum, and the turbulence in the region of low wave numbers is highly anisotropic. Some newly gained turbulent kinetic energy flows out from the region, while the remaining will cascade into higher and higher wavenumber zones. The turbulent kinetic energy will be transformed into thermal energy through molecular viscous dissipation at the high-wavenumber tail, where the viscous dissipation process is dominant. In the solar interior, the molecule viscosity is very small compared to the turbulent viscosity, implying that the turbulent spectrum spans many decades in wavenumber space. We consider the possibility the cascade process has two cases: one is so-called cascade process of isotropic, homogeneous and incompressible turbulence; the other is the so-called cascade process of anisotropic, inhomogeneous and compressible turbulence (Bi & Xu 1999).

Although the turbulence is phenomena in physics without characteristic length scales, the turbulence occurs in eddies with different size, and the eddies of different sizes are statistically independent, therefore specify different forms for the turbulent energy spectrum in different eddy wavenumber regions. We define three characteristic eddy length scales as below

- k_ℓ characteristic of the largest eddies in the system,
- k_0 characteristic of the energy-containing eddies,
- k_ν characteristic of the eddies at the scales at which viscous effects become important.

Following Stein (1967) and Musielak et al. (1994), we developed a general method for dealing with the turbulence. The turbulent energy spectrum $E(k, \sigma)$ can be factored into a spatial and temporal part

$$E(k,\sigma) = E(k)\,\Delta\left(\frac{\sigma}{ku_k}\right), \tag{38}$$

where u_k is the average velocity of eddy with wavenumber k and is given by

$$u_k = \int_k^{2k}\left[E\left(k^{'}\right)dk^{'}\right]^{1/2}. \tag{39}$$

For isotropic turbulence, the turbulence should be fully developed and a universal inertial range $(k_0 < k < k_v)$ in the turbulence should be present. The spatial component of the turbulent energy spectrum $E(k)$ should be well described by the Kolmogorov spectrum. Assuming that wave numbers near $k_0 \approx 2\pi/H_p$ represent the energy-containing eddies, where the pressure scale height H_p is the dominant local length scale in the convection zone, and considering the small wavenumber contributions, we can use a linear extension of the Kolmogorov energy spectrum as isotropic turbulent spectrum in different eddy wavenumber regions

$$E(k) = \begin{cases} 0 & 0 < k < 0.2k_0, \\ a\frac{u_0^2}{k_0}\frac{k}{k_0} & 0.2k_0 \le k < k_0, \\ a\frac{u_0^2}{k_0}\left(\frac{k}{k_0}\right)^{-5/3} & k_0 \le k < k_v \end{cases} \tag{40}$$

For time-dependent convection theory, the assumption of quasi-isotropism excludes turbulent viscosity caused by anisotropic properties in the real situation. Within the context of the turbulent theory, the effects of turbulent viscosity play a very important role on pulsational behavior. The anisotropic turbulent energy spectrum also has similar energy spectral forms in different eddy wavenumber regions (Frisch 1990)

$$E(k) = \begin{cases} 0 & 0 < k < 0.2k_0, \\ b\frac{u_0^2}{k_0}\left(\frac{k}{k_0}\right)^{-\frac{3-D}{3}} & 0.2k_0 \le k < k_0, \\ b\frac{u_0^2}{k_0}\left(\frac{k}{k_0}\right)^{-\frac{5}{3}-\frac{3-D}{3}} & k_0 \le k < k_\nu \end{cases} \tag{41}$$

where parameter D is fractal of turbulence, $2 < D \le 3$, and it describes anisotropization of eddies. If $D = 3$, it means that the anisotropic and inhomogeneous turbulence becomes the isotropic and homogeneous. We consider parameter D to vary with depth and time, but here it is treated as a constant for simplicity, $D = 2.7$.

The factors a=0.758 and b=0.701 are determined by the normalization condition. u_0 is the turbulence velocity scale. The spectra are normalized by the requirement for turbulent convection that

$$\int_0^\infty E(k)\,dk = \frac{3}{2}u_0^2.$$

With our time-dependent improved mixing length theory, we recognize that not all eddies of wavenumber k have exactly the same lifetime, which there is a distribution of eddy lifetimes. This lifetime distribution then translates into the finite-width temporal component of turbulent energy spectrum. In order to examine the effects of different temporal components of the turbulent energy spectrum on solar p-mode oscillations, we adopt two the temporal component in our present work. The so-called Gaussian frequency factor and shifted Gaussian frequency factor are given as

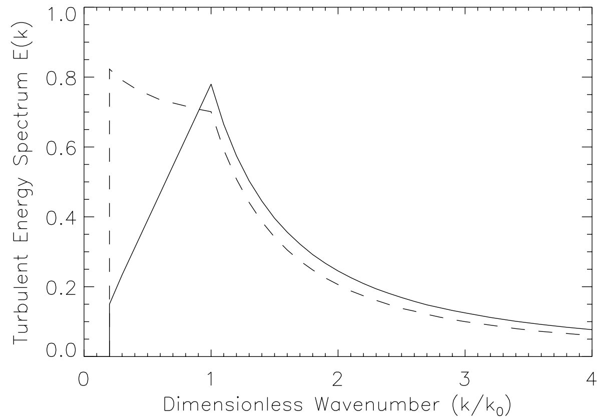

Fig. 1. The spatial component of turbulent energy spectrum $E(k)$ in units of u_0^2/k_0 as a function k/k_0. The forms of the spectrum are shown in two cases. dashed line: improved MLT for anisotropic turbulence; solid line: improved MLT for isotropic turbulence

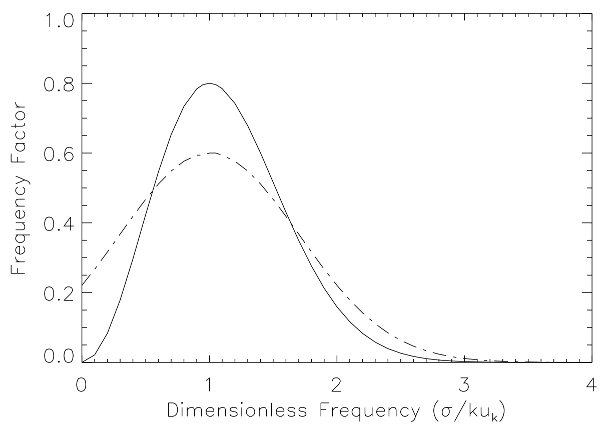

Fig. 2. Turbulent frequency factor $\Delta\left(\frac{\sigma}{ku_k}\right)$ as a function k/k_0. The cases plotted for the modified Gaussian form: solid line; the shifted Gaussian form: dash dotted line

$$\triangle(\sigma/ku_k) = \frac{4}{\sqrt{\pi}}\frac{\sigma^2}{|ku_k|^3}\exp\left[-\left(\frac{\sigma}{ku_k}\right)^2\right], \tag{42}$$

$$\triangle(\sigma/ku_k) = \frac{c}{\sqrt{\pi}\,|ku_k|}\exp\left[-\left(\frac{|\sigma| - |ku_k|}{|ku_k|}\right)^2\right]. \tag{43}$$

with the normalization factor c=1.78 for isotropic turbulence. The three spatial and two temporal component of turbulent energy spectrum used by this paper are shown in Fig. 1 and Fig. 2, respectively.

4.3. Reynolds stress

We introduce the Fourier transform

$$\phi_{ij}(\mathbf{k},\sigma) \equiv \frac{1}{(2\pi)^4}\int d^3r \int dt R_{ij}(\mathbf{r},t)\,e^{-i(\mathbf{k}\cdot\mathbf{r}-\sigma t)}, \tag{44}$$

and its inverse transform

$$R_{ij}\left(\mathbf{r},t\right)=\int d^3k\int d\sigma\phi_{ij}\left(\mathbf{k},\sigma\right)e^{-i(\mathbf{k}\cdot\mathbf{r}-\sigma t)}. \quad (45)$$

For quasi-isotropic approximation, ϕ_{ij} can be expressed in terms of the turbulent energy spectrum $E\left(k,\sigma\right)$, by (Batchelor 1960)

$$\phi_{ij}\left(k,\sigma\right)=\frac{E\left(k,\sigma\right)}{4\pi k^2}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right). \quad (46)$$

Substituting Eq. (45) into (44), taking $d^3k = k^2\sin\theta dkd\theta d\varphi$, and performing analytically the angle integration over θ and φ, we have

$$\begin{aligned}R_{ij}\left(r,t\right)=&\int_0^\infty d\sigma\cos\sigma t\int_0^\infty dkE\left(k,\sigma\right)[\delta_{ij}(\tfrac{\sin kr}{kr}\\&+\tfrac{\cos kr}{k^2r^2}-\tfrac{\sin kr}{k^3r^3})-\tfrac{r_ir_j}{r^2}(\tfrac{\sin kr}{kr}+3\tfrac{\cos kr}{k^2r^2}-3\tfrac{\sin kr}{k^3r^3})]\end{aligned} \quad (47)$$

As mentioned above, we apply these results to radial solar p-modes, and assume that $\mathbf{r}$ is limited to the z-direction $(r_x=r_y=0)$, then obtain (Bi & Li 1998)

$$\begin{aligned}R_{rr}\left(r\right)=2&\int_{-\infty}^{+\infty}dt\exp\left(i\sigma t\right)\int_0^\infty d\sigma\cos\sigma t\\&\int_0^\infty dkE\left(k,\sigma\right)\left(\tfrac{\sin kr}{k^3r^3}-\tfrac{\cos kr}{k^2r^2}\right).\end{aligned} \quad (48)$$

The function $g\left(k,\sigma\right)$ involves only the frequency factors and defined as

$$g\left(k,\sigma\right)\equiv\int_{-\infty}^{+\infty}dt\exp\left(i\sigma t\right)\int_0^\infty d\sigma\cos\sigma t\,\triangle\left(\sigma/ku_k\right). \quad (49)$$

By using Eqs. (37) and (48), and taking $\alpha = ku_k$, we have integral of the type

$$R_{rr}=2\int_0^\infty E(k)g\left(k,\sigma\right)\left(\frac{\sin kr}{k^3r^3}-\frac{\cos kr}{k^2r^2}\right)dk, \quad (50)$$

for modified Gaussian frequency factor

$$g\left(k,\sigma\right)=8\pi^{1/2}\alpha\sigma^2\exp\left(-\frac{\sigma^2}{\alpha^2}\right), \quad (51)$$

and for shifted Gaussian frequency factor

$$g\left(k,\sigma\right)=2c\pi^{1/2}\alpha\exp\left[-\left(\frac{\sigma-\alpha}{\alpha}\right)^2\right]. \quad (52)$$

5. Results and discussion

In this paper, we only consider the influence of Reynolds stress on the solar oscillations by neglecting the contribution of the turbulent pressure to the hydrostatic support.

To investigate the influence of convection on solar p-mode oscillations, one requires a solar model that provides the input data p_0, ρ_0, etc. We use the equilibrium model data which is the same as in Bi & Li (1998). It is characterized by mixing-length parameter $\alpha=\frac{l_0}{H_p}$, which is the ratio of the characteristic convective mixing length l_0 to the pressure scale height. In this paper, we adopt $\alpha = 1.7$. The eigenfunction of solar p-mode oscillations are obtained by numerically integrating the equations that govern the linear radial oscillation (Unno et al., 1989).

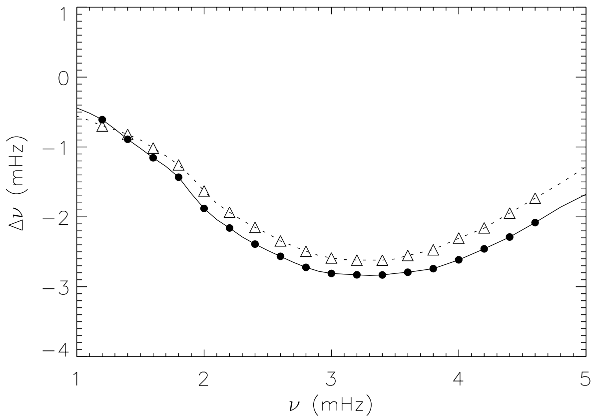

Fig. 3. Frequency shift as function of circular frequency for different turbulent energy spectrum with the modified Gaussian frequency factor. The circles joined by solid line: improved MLT for isotropic turbulence; the triangles joined by dotted line: improved MLT for anisotropic turbulence

The eigenvalues problem is supplemented by the boundary condition for the radial displacement ξ at the center, $\xi\left(0\right)=0$; and for the Lagrangian perturbation of the pressure at the surface, $\delta p = 0$. The adiabatic eigenfrequencies ν_a of corresponding modes of the model whose equilibrium structure is computed without turbulent pressure.

It is interesting to examine whether the effects of Reynolds stress on the oscillation frequencies with improved time-dependent mixing-length convective theory. Our work is divided into two steps. The first is to examine the influence of different choices for the spatial turbulent energy spectrum $E\left(k\right)$ on frequencies; the second is to test the effects of turbulent frequency factor $\Delta\left(\sigma/ku_k\right)$.

According to our time-dependent mixing-length theory, the calculations of two frequency shifts from the adiabatic frequencies ν_a due to Reynolds stress are performed on the same equilibrium model for different choices for the spatial turbulent energy spectrum $E\left(k\right)$, using a modified Gaussian turbulent frequency factor $\Delta\left(\sigma/ku_k\right)$. The Fig. 3. exhibits pronounced oscillatory features, there is a different between our turbulent corrections and adiabatic calculations. The effects of turbulent pressure correction become more important toward higher frequencies in the range below 4mHz.

It can be seen from Fig. 3 that the frequency shifts are different by using different mixing-length formulation, and the shape of frequency shift depends mainly on the choice of turbulent energy spectrum. The frequency shift based on the improved MLT for anisotropic turbulence is smaller than that one based on the improved MLT for isotropic turbulence at higher frequencies when the mixing-length parameter α is to be held constant.

This result is explained by the contribution of turbulent viscosity caused by anisotropic properties in the real situation. The introduction of turbulent eddies viscosity means that there is an additional dynamical viscosity pressure that can oppose smaller increase convective velocity at the outer boundary and is, therefore, equivalent to a lowering of the mixing length comparing

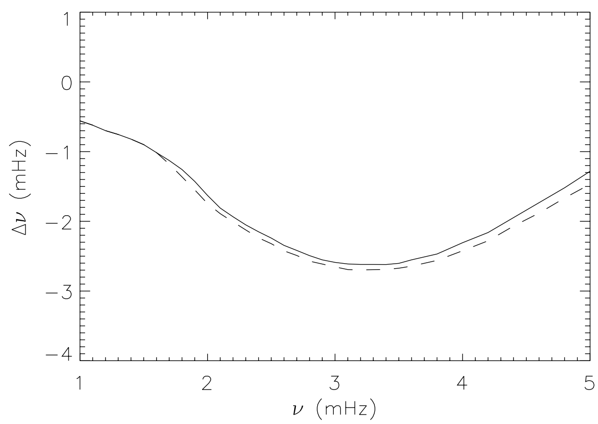

Fig. 4. Frequency shift as function of circular frequency for anisotropic turbulent energy spectrum with different frequency factor. The solid line: the modified Gaussian frequency factor; The dashed line: the shifted Gaussian frequency factor

with isotropic turbulence no eddy viscosity. Consequently, the lower convective velocity would lead to the smaller correction.

Although the comparison shows that the treatment of isotropic turbulence can produce the larger frequency shift closer to the observed value than anisotropic turbulence, considering that isotropic turbulence is less realistic due to its missing eddy viscosity. Since homogeneous and isotropic limits in the theory are determined purely locally, the frequency sensitivity of the response of the convection is artificially enhanced. With the more reliable description of turbulent convection it is likely that the amplitude of the oscillations would be diminished. The frequency shifts we have computed with anisotropic formulation do not appear to be of sufficient magnitude, at the higher frequencies, to reconcile theory with observation. In order to further reduce the difference between observed and computed frequencies, we need to ask for other physical properties which are responsible for large frequency shifts.

Fig. 4 shows examples of computed frequency shifts with the modified Gaussian and shifted Gaussian frequency factors for anisotropic formulation. It is obvious that the functional form of frequency factor may influence the frequency shifts. The result given in Fig. 4 reveals that the magnitude of frequency shift with the shifted Gaussian frequency factor appears to be slightly larger than one with the modified Gaussian frequency factor. This difference is explained by adding contribution for $k < k_0$. The shifted Gaussian frequency factor considers the contribution of the low wavenumber extension.

Comparison of Fig. 3 and Fig. 4 demonstrates that the frequency shifts due to the functional form of frequency factor is smaller than the frequency shifts due to turbulent energy spectrum. Therefore, the frequency shifts depend on sensitively on the details of turbulent energy spectrum $E(k)$.

6. Conclusions

On the basis of the comparison of adiabatic frequencies with corrected frequencies, the main conclusions are given below

1. The introduction of Reynolds stress exerts a non-negligible influence on the oscillation frequencies. Although the spatial and temporal component of turbulent spectrum which takes into account the contribution of the full turbulent convective eddies may influence the oscillation frequencies, the frequency shifts depend on sensitively on the details of turbulent energy spectrum $E(k)$.
2. It is evident that the turbulent viscosity plays a important role for the solar p-mode oscillations because the frequency shifts due to anisotropic turbulence is smaller then the frequency shifts due to isotropic turbulence toward higher frequencies.
3. It would appear that the physical processes due to turbulent pressure and turbulent viscosity contributing to the frequency shifts are insufficient to explain the high frequency shifts suggested by the observations. A reliable theory of turbulence has yet to be developed.

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