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Approximations of the self-similar solution for a blastwave in a medium with power-law density variation

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Abstract. Approximations of the Sedov self-similar solution for a strong point explosion in a medium with the power-law density distribution $\rho^o \propto r^{-m}$ are reviewed and their accuracies are analyzed. The Taylor approximation is extended to cases $m \neq 0$ and spherical, cylindrical and plane geometry. Two approximations of the solution are presented in the Lagrangian coordinates for all types of geometry. These approximations may be used for the investigation of the ionization structure of the adiabatic flow, e.g., inside adiabatic supernova remnants.

Key words: hydrodynamics – ISM: supernova remnants

1. Introduction

Self-similar (Sedov 1946, 1959) solutions for a strong point explosion in a uniform medium $\tilde{\rho}^o = \mathrm{const}$ or in a medium with power-law density distribution

$$\tilde{\rho}^o(\tilde{r}) = \tilde{\rho}^o(0) r^{-m}, \tag{1}$$

where $\tilde{r}$ is the distance from the center of explosion, are widely used for modelling adiabatic supernova remnants, solar flares and processes in active galactic nuclei.

Sedov (1946, 1959) has obtained the exact solution for the system of hydrodynamic differential equation based on dimensional methods. Independently, Taylor (1950) solved the same task in the case of the uniform medium numerically and in analytical form approximately. Taylor's main idea was to approximate the fluid velocity variation behind the shock front.

Kahn (1975) has proposed an approximation to the Sedov solution in a uniform medium. His technique consists of approximating the mass distribution inside the shocked region. Using Kahn's methodology, Cox & Franco (1981) approximated the exact solution in a power-law medium (1) for $m < 2$. With the same technique, Cox & Anderson (1982) presented the approximation for description of the shocked region and blastwave motion in a uniform medium of finite pressure.

Ostriker & McKee (1988) basing on the virial theorem have given a number of approximations for the fluid characteristic variation as one- or two-power polinomia's.

Hnatyk (1987) proposed to approximate the relation between the Eulerian and Lagrangian coordinates of the flow elements.

In the present work, the Taylor approximation is written for a medium with power-law density variation (1) and different types of geometry. Using Hnatyk's approach, we develop two approximations for the Sedov solution for power-law media with $m \leq 2$ in Lagrangian coordinates useful for investigations of the nonequilibrium ionization processes in a shocked plasma, e.g., inside adiabatic supernova remnants. One of the approximations presented here is based on the approximate hydrodynamic method for description of the nonspherical strong point explosion in the medium with arbitrary large-scale nonuniformity developed by Hnatyk & Petruk (1999). Therefore, it may also be considered as additional test of this method.

2. Sedov solution and its approximations

2.1. Sedov solution

If a strong ($P_{\mathrm{s}}/P^o_{\mathrm{s}} \to \infty$) point ($R_o/R \to 0$) explosion with finite energy E_o originates at the point with coordinate $\tilde{r} = 0$ in time $t = 0$, the blastwave forms and propagates with velocity D in the ambient medium with density $\tilde{\rho}^o(\tilde{r})$ (P_{s} and P^o_{s} are pressure of the shocked gas and the gas of the ambient medium at the shock front position, R_o is the size of the body exploded, R is the radius of the blastwave). It is also assumed that injected mass is small and no energy is lost from the shocked region during the motion.

The evolution is described by a system of hydrodynamic differential equations. Sedov (1959) gives the analytical self-similar solution for description of the motion of shock front and the distribution of fluid parameters inside the shocked region for a strong point explosion in a uniform ambient medium and at the center of symmetry of a radially stratified medium (1).

This solution shows that a strong blastwave in a medium with the power-law density distribution (1) moves with deceleration if $m < N+1$ and accelerates if $m > N+1$. If $m \geq N+1$, both the mass inside any sphere, which contains the center of the symmetry, and kinetic energy equal infinity. We will consider $m < N+1$ cases only.

The radius R and velocity D of the strong blastwave in the medium (1) with $m < N+1$ are (Sedov 1959):

$$R = \left(\frac{E_o}{\alpha_A\ \tilde{\rho}^o(0)}\right)^{1/(N+3-m)} t^{\,2/(N+3-m)}\ , \tag{2}$$

$$D(R) = \frac{2}{N+3-m}\left(\frac{E_o}{\alpha_A\ \tilde{\rho}^o(0)}\right)^{1/2} R^{-(N+1-m)/2}\ , \tag{3}$$

where $N = 0, 1, 2$ for plane, cylindrical and spherical wave, respectively, α_A is a self-similar constant.

Distributions of the fluid characteristics behind the shock front are self-similar, i.e., for any time t the density $\tilde{\rho}$, pressure $\tilde{P}$, fluid velocity $\tilde{u}$ variations and coordinate $\tilde{a}$ are

$$\tilde{\rho}(\tilde{r}, t) = \tilde{\rho}_{\rm s}(t) \cdot \rho(r), \tag{4}$$

$$\tilde{P}(\tilde{r}, t) = \tilde{P}_{\rm s}(t) \cdot P(r), \tag{5}$$

$$\tilde{u}(\tilde{r}, t) = \tilde{u}_{\rm s}(t) \cdot u(r), \tag{6}$$

$$\tilde{a}(\tilde{r}) = R(t) \cdot a(r), \tag{7}$$

where $r = \tilde{r}/R(t)$, $\tilde{a}$ is the original position of the fluid mass element and superscript "s" corresponds to values of the parameters at the shock front (Fig. 1).

Gas occupies the whole shocked region ($0 \le \tilde{r} \le R$) when $m \le m_1$,

$$m_1 = \frac{1 + 3N + (1-N)\gamma}{\gamma + 1}\ . \tag{8}$$

When $m \to m_1$ the central pressure $P(0) \to 0$. Shock waves in media with steep density gradients ($m > m_1$) develop a cavity around the center of explosion. Such a cavity forms in the uniform medium ($m = 0$) when $\gamma > \gamma_1 = (1+3N)/(N-1)$. Sedov has also presented a solution for hollow blastwaves. A review of approximations for these cases is given by Ostriker & McKee (1988). We do not consider $m > m_1$ in this paper.

For $m = m_1$ (or $\gamma = \gamma_1$ in the uniform medium) the solution has very simple form:

$$\rho(r) = r^{N-1}\ , \qquad P(r) = r^{N+1}\ ,$$
$$u(r) = r\ , \qquad a(r) = r^{(\gamma+1)/(\gamma-1)}\ . \tag{9}$$

Singularities in the solution also appear with $m_2 = (N+1)(2-\gamma)$ and $m_3 = (2\gamma + N - 1)/\gamma$, when some exponents in the solution equal infinity. Similarity solutions for these cases are deduced by Korobejnikov & Rjazanov (1959). For $N = 2$ and $\gamma = 5/3$ $m_1 = 2$, $m_2 = 1$, $m_3 = 13/5$.

The self-similar constant $\alpha_A = \alpha_A(N, \gamma, m)$ in Eqs. (2)-(3) for R and D may be found from the energy balance equation with variations of density $\tilde{\rho}$, pressure $\tilde{P}$ and mass velocity $\tilde{u}$ inside the shocked region

$$\frac{E_o}{\sigma} = \int\limits_0^R \frac{\tilde{\rho}(r,t)\tilde{u}(r,t)^2}{2} r^N dr + \int\limits_0^R \frac{\tilde{P}(r,t)}{\gamma - 1} r^N dr\ , \tag{10}$$

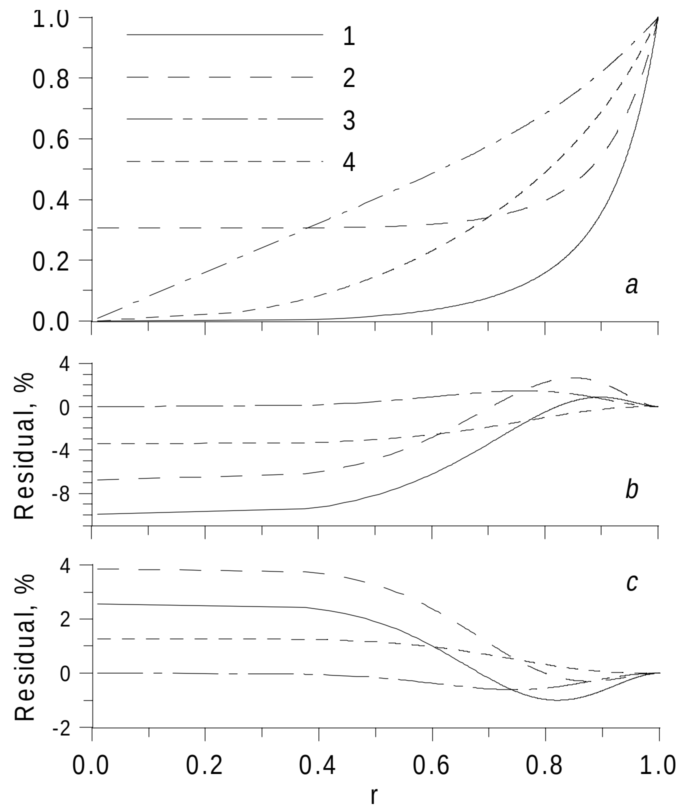

Fig. 1a–c. Sedov solution and the accuracy of Taylor and Kahn approximations of the solution in the uniform medium: **a** exact Sedov solution, **b** relative differences of Taylor approximation, **c** relative differences of Kahn approximation. Lines: 1 – $\rho(r)$, 2 – $P(r)$, 3 – $u(r)$, 4 – $a(r)$. $\gamma = 5/3$.

where $\sigma = 4\pi$ for $N = 2$, $\sigma = 2\pi$ for $N = 1$ and $\sigma = 2$ for $N = 0$ or, generally, $\sigma = 2\pi N + (N-1)(N-2)$. If we proceed to normalized parameters using (4)-(6) and general shock front conditions

$$\tilde{\rho}_{\rm s} = \frac{\gamma+1}{\gamma-1}\tilde{\rho}^o_{\rm s},\ \tilde{P}_{\rm s} = \frac{2}{\gamma+1}\tilde{\rho}^o_{\rm s}D^2,\ \tilde{u}_{\rm s} = \frac{2}{\gamma+1}D \tag{11}$$

we will find that $E_o = \beta_A \cdot MD^2/2$ with mass

$$M = \sigma\tilde{\rho}^o(0)R^{N+1-m}/(N+1-m), \tag{12}$$

constant shape-factor

$$\beta_A = \frac{4(N+1-m)}{\gamma^2 - 1} \cdot (I_{\rm K} + I_{\rm T}) \tag{13}$$

and constant integrals

$$I_{\rm K} = \int\limits_0^1 \rho(r)u(r)^2 r^N dr\ , \qquad I_{\rm T} = \int\limits_0^1 P(r) r^N dr\ . \tag{14}$$

Also we will have a self-similar constant

$$\alpha_A = \frac{2\sigma}{(N+1-m)(N+3-m)^2} \cdot \beta_A\ . \tag{15}$$

Simple formula gives $\alpha_A(N, \gamma, m_1)$:

$$\alpha_A = \frac{2\sigma(\gamma+1)}{(N+1)(\gamma-1)\big((N+1)\gamma - N + 1\big)^2}\ . \tag{16}$$

The distributions (4)-(7) in the exact solution are parametric functions of an internal parameter. The expressions for the functions are complicated. These factors stimulate developing the approximations of the self-similar solution.

2.2. Taylor approximation

Basing on his numerical results, Taylor (1950) proposed to approximate the velocity variation $u(r)$ behind a spherical ($N=2$) shock front moving into the uniform medium ($m=0$) as

$$\frac{\tilde{u}(r,t)}{D}=\frac{r}{\gamma}+\alpha r^{n}, \tag{17}$$

where α and n are found to give exact values of $\tilde{u}_{\rm s}$, $\tilde{u}(0)=0$ and first derivative $\tilde{u}_r^{\rm s}$ in respect to r. Substituting this approximation into the continuity equation and into the equation of state for perfect gas, the approximate distributions of the density and pressure obtain. They exactly yield $\tilde{\rho}_{\rm s}$, $\tilde{P}_{\rm s}$ and $\tilde{\rho}_r^{\rm s}$, $\tilde{P}_r^{\rm s}$. Taylor did not give the dependence $a(r)$, but it may be taken from the adiabaticity condition $P(a)\rho(a)^{-\gamma}=P(r)\rho(r)^{-\gamma}$ and (71)-(72):

$$a^{\gamma m-(N+1)}=P(r)\rho(r)^{-\gamma}, \tag{18}$$

with approximations for $P(r)$ and $\rho(r)$.

So, Taylor's approximation for the variations of density ρ, pressure P, fluid velocity u and coordinate a are:

$$\rho(r)=r^{\,3/(\gamma-1)}\left(\frac{\gamma+1}{\gamma}-\frac{r^{n-1}}{\gamma}\right)^{-p}, \tag{19}$$

$$P(r)=\left(\frac{\gamma+1}{\gamma}-\frac{r^{n-1}}{\gamma}\right)^{-q}, \tag{20}$$

$$u(r)=\frac{\gamma+1}{2}\left(\frac{r}{\gamma}+\frac{\gamma-1}{\gamma+1}\frac{r^{n}}{\gamma}\right), \tag{21}$$

$$a(r)=r^{\,\gamma/(\gamma-1)}\left(\frac{\gamma+1}{\gamma}-\frac{r^{n-1}}{\gamma}\right)^{-s}, \tag{22}$$

where $n=(7\gamma-1)/(\gamma^2-1), p=2(\gamma+5)/(7-\gamma), q=(2\gamma^2+7\gamma-3)/(7-\gamma), s=(\gamma+1)/(7-\gamma)$. The self-similar constant $\alpha_A=\alpha_A(2,\gamma,0)$ goes with (15) and approximated profiles of ρ, P and u. Fig. 1 and Table 5 demonstrate the accuracy of Taylor approximation in comparison with the exact solution.

This approximation is extended to cases $m\neq 0$ and $N=0,1,2$ in Sect. 3.

2.3. Kahn approximation

Kahn (1975) applied his methodology to a strong spherical blastwave ($N=2$) in a uniform medium ($m=0$) with $\gamma=5/3$. It is proposed to approximate first the mass distribution

$$\mu(r)=\frac{M(r,t)}{M(R,t)}=3\int\limits_0^r\rho(r)r^2dr. \tag{23}$$

Sedov solution shows that $P_r(r)=0$ near the centre. This implies that $\mu_r/\mu=15/(2r)$ at $r=0$. Based of the equation of motion, $\mu_r/\mu=12$, $\mu_{rr}=168$ and $(\mu_r/\mu)_r=24$ at $r=1$. Therefore ratio μ_r/μ is proposed to be approximated as

$$\frac{\mu_r}{\mu}=\frac{15}{2r}+\frac{9}{2}r^7. \tag{24}$$

This formula satisfies all prescribed boundary conditions at both ends.

The mass distribution finds as integral from (24):

$$\mu(r)=r^{15/2}\ \exp\left(\frac{9}{16}\left(r^8-1\right)\right). \tag{25}$$

The density distribution follows from (23) and (25):

$$\rho(r)=\mu_r/3r^2. \tag{26}$$

The adiabaticity condition gives pressure variation

$$P(r)=\left(\frac{2}{3}\right)^{5/3}\frac{1}{32}\ \frac{\mu_r^{5/3}}{\mu\, r^{10/3}}\ . \tag{27}$$

Velocity is deduced from the mass conservation equation

$$u(r)=\frac{4}{3}r-\frac{4\mu}{\mu_r}\ . \tag{28}$$

If the present location of mass element a is r, then $a(r)$ may be found from the condition of mass conservation $\mu(a)=\mu(r)$ and relation (74) $\mu(a)=a^3$:

$$a(r)=\mu(r)^{1/3}\ . \tag{29}$$

The expressions for Kahn's approximation are the same as (30)-(34) with $m=0$. The accuracy of this approximation is shown on the Fig. 1.

2.4. Approximation of Cox & Franco

Applying Kahn's approximation technique, Cox & Franco (1981) obtain the approximation of the self-similar solution for an ambient medium with a power-law density distribution (1) with $m<2$ for $\gamma=5/3$ and $N=2$. The approximations of Cox & Franco are:

$$\rho(r)=\left(\frac{5}{8}+\frac{3}{8}r^{\,8-4m}\right)\cdot r^{\,(9-5m)/2}\times\exp\left(\frac{3}{8}\frac{(3-m)}{(2-m)}\left(r^{\,8-4m}-1\right)\right)\ , \tag{30}$$

$$P(r)=\left(\frac{5}{8}+\frac{3}{8}r^{\,8-4m}\right)^{5/3}\times\exp\left(\frac{3}{4(2-m)}\left(r^{\,8-4m}-1\right)\right)\ , \tag{31}$$

$$u(r)=4\,r\ \frac{1+r^{\,8-4m}}{5+3\,r^{\,8-4m}}\ , \tag{32}$$

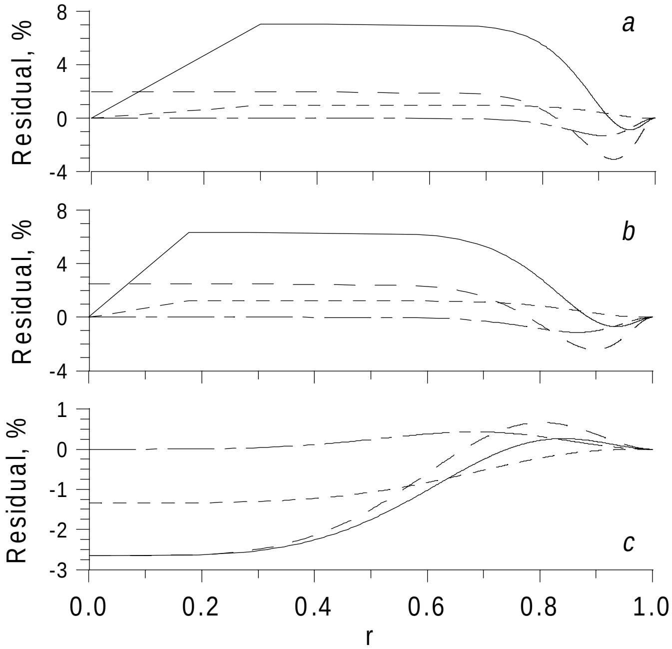

Fig. 2a–c. Accuracy of Cox & Franco approximation of the self-similar solution in the power-law medium (1): **a** relative differences of the approximation for $m = -4$, **b** relative differences for $m = -2$, **c** relative differences for $m = 1$. Lines are the same as in Fig. 1. The breaks in the curves for $\rho(r)$ and $a(r)$ are due to very strong dependence of the relevant Sedov distributions on the internal parameter, which change in these wide intervals of r by 10^{-10} only.

$$a(r) = r^{5/2}\ \exp\left(\frac{3}{8(2-m)}\left(r^{\,8-4m} - 1\right)\right)\ , \qquad (33)$$

$$\mu(r) = r^{\,5(3-m)/2}\ \exp\left(\frac{3}{8}\frac{(3-m)}{(2-m)}\left(r^{\,8-4m} - 1\right)\right)\ . \qquad (34)$$

The author's approximation for β_A is

$$\beta_A = 1.125 \cdot (0.22 + 0.52 \cdot (3-m)/3). \qquad (35)$$

The accuracy of Cox & Franco approximation is shown in Fig. 2 and Table 5.

2.5. *Approximations of Ostriker & McKee*

Ostriker & McKee (1988) in the framework of the virial theorem approach applied to a spherical blastwave (N=2) in a power-law ambient medium (1) and time-dependent energy injection $E_o(t) \propto t^s$, present a number of approximations for the self-similar solution. We consider further $s = 0$.

The authors introduce the dimensionless moments of coordinate r and velocity u:

$$K_{ij} = l_\mu \int\limits_0^1 r^i u(r)^j \rho(r) r^2 dr\ , \qquad (36)$$

where $l_\mu = (\gamma+1)(3-m)/(\gamma-1)$, and consider three types of approximations for $u(r)$ and $\rho(r)$: linear velocity approximation (LVA)

$$u(r) = r, \qquad \rho(r) = r^{(6-(\gamma+1)m)/(\gamma-1)}, \qquad (37)$$

one-power aproximation (OPA)

$$u(r) = r^{l_u}, \qquad \rho(r) = r^{l_\rho}, \qquad (38)$$

and two-power aproximation (TPA)

$$u(r) = a_u r^{l_{u,1}} + (1-a_u) r^{l_{u,2}}, \qquad (39)$$

$$\rho(r) = a_\rho r^{l_{\rho,1}} + (1-a_\rho) r^{l_{\rho,2}}. \qquad (40)$$

In such an approach the self-similar constant α_A as well as exponents l_u and l_ρ may be expressed in terms of moments K_{02} and K_{11}. Namely, under self-similarity $\alpha_A = 2\pi\eta^2\beta_A/(3-m)$, where $\eta = 2/(5-m)$ and factor β_A equals

$$\beta_A = \frac{2}{3} \cdot \frac{2K_{02}(3\gamma-5) + (5-m)(\gamma+1)K_{11}}{(\gamma^2-1)(\gamma+1)}\ . \qquad (41)$$

Exponents in OPA are

$$l_u = \frac{2K_{20} - K_{11}(1+K_{20})}{(1-K_{20})K_{11}}\ , \quad l_\rho = \frac{5K_{20}-3}{1-K_{20}}\ . \qquad (42)$$

Derivatives at the shock front are used to obtain the moments. So,

$$K_{ij} = \frac{1}{1 + s_{ij}/l_\mu}\ , \qquad (43)$$

where $s_{ij} = i + j$ in LVA and $s_{ij} = i + j + j(m_1 - m)/2$ in OPA. Using (43) α_A may be written in a simple form in LVA:

$$\alpha_A = \frac{16\pi}{3(5-m)^2} \cdot \frac{11\gamma - 5 - m(\gamma+1)}{(\gamma^2-1)\big(5\gamma+1-m(\gamma+1)\big)}\ . \qquad (44)$$

Moments have a more complicated form in TPA. In this approach the expression for $u(r)$ coincides with the approximation (21) of Taylor with $n = 1 + \gamma(m_1 - m)/(\gamma-1)$ that equals Taylor's n at $m = 0$. So, TPA is extension of Taylor's approximation of $u(r)$ to $m \neq 0$. Contrary to Taylor's approach to find $\rho(r)$ and $P(r)$ from the hydrodynamic equations, Ostriker & McKee find the density variation independently as TPA (40) with

$$a_\rho = \frac{\gamma(m_1 - m)}{10 - \gamma - (\gamma+2)m}\ ,$$
$$l_{\rho,1} = \frac{3-\gamma m}{\gamma-1}\ , \quad l_{\rho,2} = \frac{6 + (\gamma+1)(m_1 - 2m)}{\gamma-1}\ , \qquad (45)$$

where $\gamma > 1$. For $m = 0$ variation $\rho(r)$ in TPA coincides with the result of Gaffet (1978) for case of a uniform medium (Ostriker & McKee 1988).

The pressure distribution is also restored independently. It may be found in OPA as a linear pressure approximation (LPA) and for TPA in the frame of pressure-gradient approximation (PGA).

Most of mass is concentrated near the shock front and distribution $u(r)$ is close to a linear function of r. Therefore, as noted by Gaffet (1978), the right side of Euler equation

$$\frac{\partial \tilde{P}(\tilde{r},t)}{\partial M(\tilde{r},t)} = -\frac{1}{4\pi}\frac{1}{\tilde{r}^2}\frac{d\tilde{u}(\tilde{r},t)}{dt} \qquad (46)$$

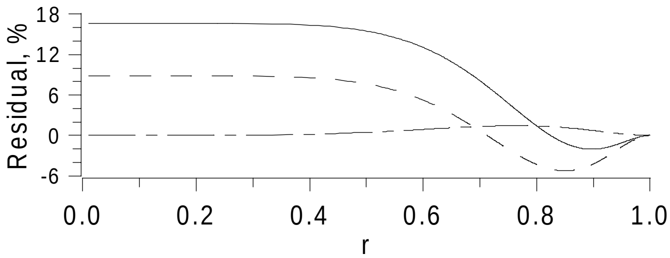

Fig. 3. Relative differences of Ostriker & McKee two-power approximation of the self-similar solution for the uniform medium. $\gamma = 5/3$. Lines are the same as in Fig. 1.

is nearly a constant. LPA (Gaffet 1978; Ostriker & McKee 1988) use this feature assuming the pressure to be a linear function of the mass fraction $\mu(r)$

$$P(r) = P(0) + (P_{\rm s}^{*}/l_{\mu})\ \mu(r). \quad (47)$$

The logarithmic derivative of pressure at the shock front is $P_{\rm s}^{*} = (d\ln P/d\ln r)_{\rm s} = (2\gamma^2+7\gamma-3-\gamma m(\gamma+1))/(\gamma^2-1)$. Mass in OPA is $\mu(r) = 3l_{\mu}^{-1}r^{l_{\mu}}$. P(0) in LPA is (Gaffet 1978)

$$P(0) = 1 + \frac{\overline{u_t^{\rm s}}}{\omega(3-m)} \quad (48)$$

where $\overline{u_t^{\rm s}} = \tilde{u}_t^{\rm s} R/D^2 = \omega\big(4\omega^2 - 3\omega - 4 + m(2-\omega)\big)/2$ (Hnatyk 1987), $\omega = 2/(\gamma+1)$.

Such an approach (substitution with $\tilde{r}_{\rm s}^{-2}\tilde{u}_t^{\rm s}$ instead of $\tilde{r}^{-2}\tilde{u}_t$ in (46)) was also used by Laumbach & Probstein (1969) to develop the sector approximation.

In PGA a power-law form for the pressure gradient

$$\frac{dP(r)}{dr} = P_{\rm s}^{*} r^{l_{p,2}-1} \quad (49)$$

is used to give the two-power approximation for the pressure

$$P(r) = P(0) + a_p r^{l_{p,2}}\ , \quad (50)$$

where $a_p = P_{\rm s}^{*}/l_{p,2}$ and

$$P(0) = \frac{(\gamma+1)^2(m_1-m)}{3\gamma^2+20\gamma+1-(\gamma+1)(3\gamma+1)m}\ ,$$
$$l_{p,2} = \frac{3\gamma^2+20\gamma+1-(\gamma+1)(3\gamma+1)m}{2(\gamma^2-1)}\ . \quad (51)$$

The accuracy in determination of α_A and $P(0)$ in the approximations of Ostriker & McKee is shown in Table 5 and in the flow parameters in Fig. 3.

2.6. Cavaliere & Messina approximation of α_A

Cavaliere & Messina (1976) with a simple technique approximate the equations for the radius and velocity of shock in the power-law medium (1) and $E_o(t) \propto t^s$. For $s = 0$ this approximation gives

$$\beta_A = \frac{4}{\gamma^2-1}\left(\frac{\gamma-1}{\gamma+1} + \frac{1}{2}\frac{N+1-m}{N+1}\right). \quad (52)$$

2.7. Approximate methods for an explosion in medium with arbitrary large-scale nonuniformity

In this subsection we point out a number of approximate methods for description of a point explosion in an arbitrary nonuniform medium. These methods may also be applicable for a medium with power-law density variation. Bisnovatyi-Kogan & Silich (1995) and Hnatyk (1987) have given the reviews of these methods, their applications and accuracy.

2.7.1. Thin-layer approximation

The thin-layer approximation was first used by Kompaneets (1960) and other authors to find analytical solutions for evolution of the shock front in a number of types of nonuniform media. It is assumed in this approach that all the swept-up mass is concentrated in an infinitely thin layer just after shock front and the motion is driven by the hot gas inside the shocked region with uniform pressure distribution $P(r) = 0.5$ (excepting $P_{\rm s} = 1$). The layer of gas moves with velocity $u_{\rm s}$. This method was developed to calculate only the shock front dynamics and therefore does not reveal the distribution of the fluid parameters behind the shock front.

The thin-layer approximation gives for a spherical blastwave in the uniform medium (Andriankin et al. 1962)

$$\alpha_A = \frac{16\pi(3\gamma-1)}{75(\gamma-1)(\gamma+1)^2}\ . \quad (53)$$

2.7.2. Sector approximation

In the sector approximation, the characteristics of a flow in each one-dimentional sector of two- or tree-dimentional disturbed region are found by decomposition into series at the shock front.

Laumbach & Probstein (1969) have proposed the sector approximation applying it to spherical blastwaves in a plane-stratified exponential medium. The authors use Lagrangian coordinate a and propose to approximate the pressure variation in the form equivalent to $P(a) = 1 + P_a^{\rm s}(a-1)$ (Hnatyk 1987). Density variation is given by the adiabaticity condition and the relation $r = r(a)$ by continuity equation. The fluid velocity field is not determined. For shock radius and its velocity, Laumbach & Probstein approximation yields in the uniform medium limit

$$\alpha_A = \frac{32\pi(4\gamma^2-\gamma+3)}{225(\gamma-1)(\gamma+1)^3}\ . \quad (54)$$

Gaffet (1978, 1981) uses Lagrangian mass coordinates μ and finds pressure variation as a linear pressure approximation $P(\mu) = 1 + P_{\mu}^{\rm s}(\mu-1)$. Gaffet (1978, 1981) also proposes to improve the accuracy of the approximation, taking into account the second order coefficients in the series. The author calculates such coefficients in terms of Lagrangian mass coordinate μ. Hnatyk (1987), considering different modifications of the sector approximation, presents the coefficients up to the second order in terms of a.

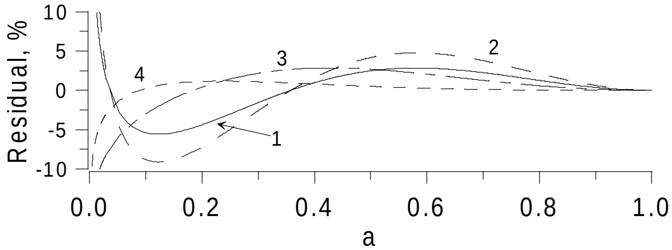

Fig. 4. Accuracy of Hnatyk approximation of Sedov solution for the uniform medium. Lines: 1 – $\rho(a)$, 2 – $P(a)$, 3 – $u(a)$, 4 – $r(a)$. $\gamma = 5/3$.

2.7.3. Hnatyk approximation

Hnatyk (1987) introduces also the idea to aproximate first the relation $\tilde{r} = \tilde{r}(a, t)$ between the Lagrangian a and Eulerian r coordinates of the gas element in each sector of shocked region. Density ρ, pressure P and velocity u variations behind the shock front are exactly deduced from this relation. Really, the continuity equation

$$\tilde{\rho}^o(\tilde{a})\tilde{a}^N d\tilde{a} = \tilde{\rho}(\tilde{r})\tilde{r}^N d\tilde{r} \tag{55}$$

gives us the density distribution

$$\rho(a) = \frac{\tilde{\rho}(a,t)}{\tilde{\rho}^{\rm s}(t)} = \frac{\tilde{\rho}^o(\tilde{a})}{\tilde{\rho}(R,t)} \left(\frac{\tilde{a}}{\tilde{r}(\tilde{a},t)}\right)^N \left(\frac{\partial \tilde{r}(\tilde{a},t)}{\partial \tilde{a}}\right)^{-1}, \tag{56}$$

the equation of adiabaticity

$$\tilde{P}(\tilde{a},t) = K\tilde{\rho}(\tilde{a},t)^\gamma \tag{57}$$

yields the distribution of pressure

$$P(a) = \frac{\tilde{P}(\tilde{a},t)}{\tilde{P}^{\rm s}(t)} = \left(\frac{\tilde{\rho}^o(\tilde{a})}{\tilde{\rho}^o(R)}\right)^{1-\gamma} \left(\frac{D(\tilde{a})}{D(R)}\right)^2 \left(\frac{\tilde{\rho}(\tilde{a},t)}{\tilde{\rho}(R,t)}\right)^\gamma \tag{58}$$

and relation $\tilde{r} = \tilde{r}(\tilde{a}, t)$ gives velocity

$$u(a) = \frac{\tilde{u}(\tilde{a},t)}{\tilde{u}^{\rm s}(t)} = \frac{\gamma+1}{2}\frac{1}{D(R)}\frac{d\tilde{r}(\tilde{a},t)}{dt}\ . \tag{59}$$

The author proposes to approximate $r(a)$ as

$$r(a) = a^\alpha \exp\big(\beta(a-1)\big) \tag{60}$$

with

$$\alpha = (r^{\rm s}_a)^2 - r^{\rm s}_{aa} \quad \text{and} \quad \beta = r^{\rm s}_{aa} + r^{\rm s}_a - (r^{\rm s}_a)^2\ . \tag{61}$$

Such an expression ensures the edge condition $r(0) = 0, r_{\rm s} = 1$ and values of the derivatives

$$r^{\rm s}_a = 1 - \omega, \tag{62}$$

$$r^{\rm s}_{aa} = \omega(1-\omega)\big[3B + N(2-\omega) - m\big] \tag{63}$$

where $B = R\ddot{R}/\dot{R}^2$, $\dot{R} = dR/dt$ is the shock velocity, $m = -d\ln\rho^o(R)/d\ln R$ and subscript "a" denotes a partial derivative with respect to a.

This approximation is accurate near the shock front, but around the explosion site (for $a < 0.1$ or $r < 0.4$) characteristics do not restore correctly (Fig. 4). This approximation does not take into consideration any derivatives of $r(a)$ near the center and the distributions of $\rho(a)$, $P(a)$, $u(a)$ do not hold there, causing such a situation. This approximation is extended to the central region in Sect. 4.5.

3. Extension of Taylor approximation to $m \neq 0$ and different N

In this section Taylor's technique is applied to the case of a medium with the power-law density distribution (1) with $m < m_1$ and $N = 0, 1, 2$. Substitution with (17) into the equations of continuity and state gives

$$\frac{\rho_r}{\rho} = \frac{N + 1 + \alpha\gamma(n+N)r^{n-1} - m\gamma}{(\gamma-1)r - \alpha\gamma r^n}\ , \tag{64}$$

$$\frac{P_r}{P} = \frac{\alpha\gamma^2(n+N)r^{n-1}}{(\gamma-1)r - \alpha\gamma r^n}\ . \tag{65}$$

Simple relation $n = P^*_{\rm s} - N$ follows from (65).

Boundary conditions $u_{\rm s} = 1$ and

$$u^{\rm s}_r = \big(N(3-\gamma) + 3\gamma - m(\gamma+1)\big)/\big(2(\gamma+1)\big)$$

yield $\alpha = (\gamma-1)/\big(\gamma(\gamma+1)\big)$ and

$$n = \frac{(2-N)\gamma^2 + (3N+1)\gamma - 1 - m\gamma(\gamma+1)}{\gamma^2 - 1}\ . \tag{66}$$

This n coincides with n of Ostriker & McKee if $N = 2$ and with Taylor's one if $N = 2$ and $m = 0$. After integration of (65) and taking into account the condition $P_{\rm s} = 1$, the pressure variation will be expressed with (20) where

$$q = \frac{2\gamma^2 + (3N+1)\gamma - (N+1) - m\gamma(\gamma+1)}{3N + 1 - (N-1)\gamma - m(\gamma+1)}\ . \tag{67}$$

The density follows from (64) and $\rho_{\rm s} = 1$:

$$\rho(r) = r^{\ (N+1-m\gamma)/(\gamma-1)} \left(\frac{\gamma+1}{\gamma} - \frac{r^{n-1}}{\gamma}\right)^{-p}, \tag{68}$$

where

$$p = \frac{2\big(\gamma + 2N + 1 - m(\gamma+1)\big)}{3N + 1 - (N-1)\gamma - m(\gamma+1)}\ . \tag{69}$$

Eq. (22) gives $a(r)$ with

$$s = \frac{\gamma+1}{3N + 1 - (N-1)\gamma - m(\gamma+1)}\ . \tag{70}$$

Exponents $n = 6 - 5m/2$, $q = (16-5m)/\big(3(2-m)\big)$, $p = (5-2m)/(2-m)$ and $s = 1/(2-m)$ for $N = 2$ and $\gamma = 5/3$. This extended Taylor approximation is compared with the exact solution in Fig. 5.

4. Approximations of the Sedov solution in Lagrangian coordinates

In this section we present two analytical approximations of the self-similar solution for a medium with the power-law density distribution expressed in Lagrangian geometric coordinates a.

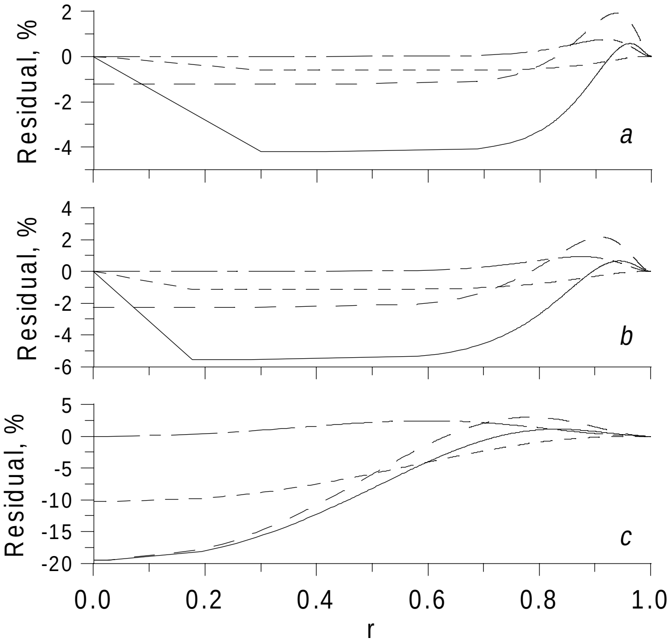

Fig. 5a–c. Accuracy of the extended Taylor approximation of the self-similar solution in the power-law medium (1): **a** relative differences of the approximation for $m = -4$, **b** relative differences for $m = -2$, **c** relative differences for $m = 1$. Lines are the same as in Fig. 1. $\gamma = 5/3$.

4.1. Flow characteristic distributions

Exact expressions for normalized density ρ and pressure P variations behind the shock front moving into the power-law medium (1) follow from (56) and (58):

$$\rho(a) = \frac{\gamma - 1}{\gamma + 1} \cdot a^{N-m} \cdot \left(r(a)^N \cdot r_a(a)\right)^{-1}, \qquad (71)$$

$$P(a) = \left(\frac{\gamma - 1}{\gamma + 1}\right)^{\gamma} \cdot a^{N(\gamma-1)-1} \cdot \left(r(a)^N \cdot r_a(a)\right)^{-\gamma}. \qquad (72)$$

The distribution of the fluid velocity $u(a)$ may be found from (59). Due to $\tilde{r} = rR$ the time derivative $d\tilde{r}/dt = Rr_t + Rr_a a_t + rD$ (subscript "t" denotes a partial derivative with respect to t). We have also that $a_t = -aD/R$ and, in the self-similar case, $r_t = 0$. So,

$$u(a) = \frac{\gamma + 1}{2}\left(r(a) - r_a(a)a\right). \qquad (73)$$

The distribution $\mu(a)$ follows from the definition (23) and (55):

$$\mu(a) = a^{(N+1)-m}. \qquad (74)$$

4.2. Self-similar constant α_A

The self-similar constant $\alpha_A(N, \gamma, m)$ in equations for R and D (2)-(3) obtains from (15) and (13):

$$\alpha_A = \frac{8}{\gamma^2 - 1} \cdot \frac{\sigma}{(3 + N - m)^2} \cdot (I_{\rm K} + I_{\rm T}), \qquad (75)$$

Table 1. $P(0)$ calculated according to the self-similar solution and C for a strong point explosion in the power-law medium.

N	m	$P(0)$ $\gamma = 7/5$	$P(0)$ $\gamma = 5/3$	C $\gamma = 7/5$	C $\gamma = 5/3$
0	0	0.3900	0.3532	1.1429	1.1670
1	0	0.3729	0.3215	1.0863	1.1112
2	0	0.3655	0.3062	1.0618	1.0833
	−4	0.4268	0.3954	1.0233	1.0293
	−3	0.4193	0.3848	1.0276	1.0350
	−2	0.4088	0.3696	1.0339	1.0433
	−1	0.3928	0.3463	1.0438	1.0570
	1	0.3087	0.2217	1.1054	1.1556
	2	0.1273	0.0000	1.3648	1.0000

with

$$I_{\rm K} = \frac{\gamma^2 - 1}{4} \int_0^1 \left(r(a) - r_a(a)a\right)^2 a^{N-m} da, \qquad (76)$$

$$I_{\rm T} = \left(\frac{\gamma - 1}{\gamma + 1}\right)^{\gamma} \int_0^1 \left(r(a)^N r_a(a)\right)^{1-\gamma} a^{N(\gamma-1)-1} da. \qquad (77)$$

4.3. Factor C and exponent x

In the Sedov self-similar solution, if $r \to 0$ then the dependence $r(a)$ is

$$r = C \cdot a^x. \qquad (78)$$

For $m < m_1$, if we substitute (78) into (72) we obtain the connection between the factor C and the normalized central pressure $P(0)$:

$$C = \left(\frac{\gamma}{\gamma + 1} \cdot P(0)^{-1/\gamma}\right)^{1/(N+1)}. \qquad (79)$$

We have to put $x = (\gamma - 1)/\gamma$ during this transformation in order to satisfy the condition $P(0) \neq 0$. In the case $m = m_1$ the exact solution (9) gives x and $C = 1$. The general formula for exponent x is

$$x = \begin{cases} (\gamma - 1)/\gamma & \text{for } m < m_1 \\ (\gamma - 1)/(\gamma + 1) & \text{for } m = m_1 \end{cases}. \qquad (80)$$

Analytical expressions for $P(0)$ from self-similar solution are presented in Appendix 5. Calculated values of $P(0)$ and C for a number of N, γ and m are shown in Table 1.

4.4. Derivatives at the shock front

Expressions for the derivatives $r_a^{\rm s}$, $r_{aa}^{\rm s}$, $r_{aaa}^{\rm s}$ may be obtained with the technique of Gaffet (1978) from the set of hydrodynamic equations for a perfect gas and conditions on the shock

Table 2. Derivatives of the relation between the Lagrangian and Eulerian coordinates at the shock front moving into the power-law medium (1).

Derivative	γ
	$\gamma=7/5$
$r^{\rm s}_a=\dfrac{1}{6}$	
$r^{\rm s}_{aa}=\dfrac{5}{2^3 3^3}\left(-2N+3m-9\right)$	
$r^{\rm s}_{aaa}=\dfrac{5}{2^5 3^5}\left(-2N^2+9Nm+183N-9m^2-270m+675\right)$	
	$\gamma=5/3$
$r^{\rm s}_a=\dfrac{1}{4}$	
$r^{\rm s}_{aa}=\dfrac{3}{2^6}\left(-N+2m-6\right)$	
$r^{\rm s}_{aaa}=\dfrac{3}{2^9}\left(Nm+19N-2m^2-36m+94\right)$	

front (see Hnatyk & Petruk (1999) for details). Derivatives r_a, r_{aa} are given with (62)-(63) and

$$\begin{aligned} r^{\rm s}_{aaa}=\omega(1-\omega)\Big[&3(7-5\omega)B^2+\\ &+\big[(-5\omega^2+4\omega+8)N+(4\omega-11)m\big]B+\\ &+\omega(2\omega^2-7\omega+6)N^2+(\omega^2+\omega-4)Nm-\\ &-\omega(2-\omega)N-(\omega-2)m^2+(2\omega-1)m+\\ &+(2\omega-1)m'+(6\omega-4)Q\Big], \end{aligned} \quad (81)$$

where $Q=R^2R^{(3)}/\dot{R}^3$ and $m'=-dm/d\ln R$.

In the power-law medium (1) $m'=0$. Taking into consideration the equations for the shock radius (2) and shock velocity (3) we may also write

$$B=-\frac{N-m+1}{2},\ \ Q=\frac{(N-m+1)(N-m+2)}{2}. \quad (82)$$

Reduced expressions for the derivatives r^s_a, r^s_{aa}, r^s_{aaa} are shown in Table 2.

4.5. Second order approximation

So, to approximate the self-similar solution, we approximate the relation $r=r(a)$ between Eulerian r and Lagrangian a coordinates of flow elements. Following Hnatyk's approach (60) and the similar relation (33), $r=r(a)$ may be aproximated in the form

$$r(a)=a^x\exp\Big(\alpha(a^\beta-1)\Big) \quad (83)$$

with x given by (80) and

$$\alpha=\frac{(r^{\rm s}_a-x)^2}{r^{\rm s}_{aa}+r^{\rm s}_a-(r^{\rm s}_a)^2}\ , \quad \beta=\frac{r^{\rm s}_{aa}+r^{\rm s}_a-(r^{\rm s}_a)^2}{r^{\rm s}_a-x}\ , \quad (84)$$

or, after substitution with (62)-(63),

$$\alpha=\frac{2(1-\omega-x)^2}{\omega(1-\omega)(N+m-1-2N\omega)}\ , \qquad \beta=\alpha^{-1}(1-\omega-x). \quad (85)$$

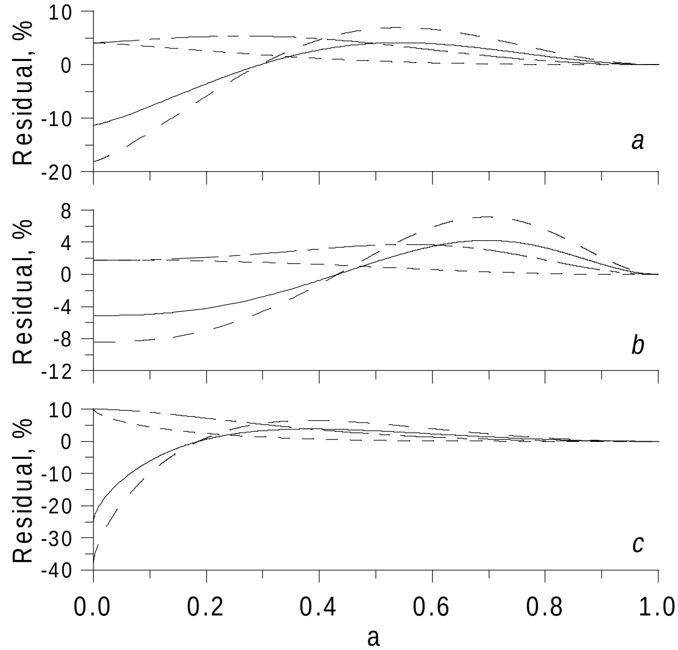

Fig. 6a–c. Accuracy of the second order approximation of the self-similar solution in the power-law medium (1) for $\gamma=5/3$ and $N=2$: **a** relative differences for $m=0$, **b** relative differences for $m=-2$, **c** relative differences for $m=1$. Lines are the same as in Fig. 4.

Such a second order approximation, besides $r(0)=0$, $r_{\rm s}=1$, $r^{\rm s}_a$, $r^{\rm s}_{aa}$, gives $(\partial\ln r/\partial\ln a)^0=x$, and, unlike Hnatyk's approximation, extends the description of a flow to the central region.

Variations of $\rho(a)$, $P(a)$ and $u(a)$ follow from (71)-(73). For the case $N=2$, $\gamma=5/3$ and $m<2$ these relations give $\beta=5(2-m)/8$ and

$$r(a)=a^{2/5}\exp\left(-\frac{6}{25(2-m)}(a^\beta-1)\right), \quad (86)$$

$$\rho(a)=\left(\frac{8}{5}-\frac{3}{5}a^\beta\right)^{-1}\cdot a^{(9-5m)/5} \times\exp\left(\frac{18}{25(2-m)}(a^\beta-1)\right), \quad (87)$$

$$P(a)=\left(\frac{8}{5}-\frac{3}{5}a^\beta\right)^{-5/3}\exp\left(\frac{6}{5(2-m)}(a^\beta-1)\right), \quad (88)$$

$$u(a)=a^{2/5}\left(\frac{4}{5}+\frac{1}{5}a^\beta\right)\exp\left(-\frac{6}{25(2-m)}(a^\beta-1)\right). \quad (89)$$

Approximation (86)-(89) may be considered as an inversion of Cox & Franco approximation (30)-(33). Unfortunately, the accuracy of presented formulae is lower (Fig. 6, Table 5).

4.6. Third order approximation

In order to improve accuracy, we postulate the approximation $r=r(a)$ to give exact values of two additional derivatives:

Table 3. Self-similar constant $\alpha_{\rm A}(N,\gamma,m)$ calculated with the third order approximation of $r(a)$ (90).

N	m	$\alpha_{\rm A}$	
		$\gamma=7/5$	$\gamma=5/3$
0	0	1.0763	0.6018
1	0	0.9841	0.5644
2	0	0.8519	0.4944
	−4	0.2295	0.1270
	−3	0.2960	0.1650
	−2	0.3966	0.2232
	−1	0.5598	0.3192
	1	1.4631	0.8722
	2	3.3537	1.8235

Table 4. Coefficients in approximation (92). $\gamma=5/3$.

N	m	B_0	B_1	B_2	B_3	B_4
0	0	1.1670	0.1333	−0.1127	−0.1074	−0.02833
1	0	1.1112	−0.01510	−0.24655	−0.1530	−0.03276
2	0	1.0833	−0.05189	−0.2130	−0.08708	−0.009294
	−4	1.0293	0.2837	0.9302	0.9766	0.3008
	−3	1.0350	0.1744	0.6152	0.6971	0.2213
	−2	1.0433	0.07170	0.3041	0.4162	0.1406
	−1	1.0570	−0.01334	0.01359	0.1452	0.06128
	1	1.1556	0.08965	−0.1753	−0.1471	−0.03778
	2	1	0	0	0	0

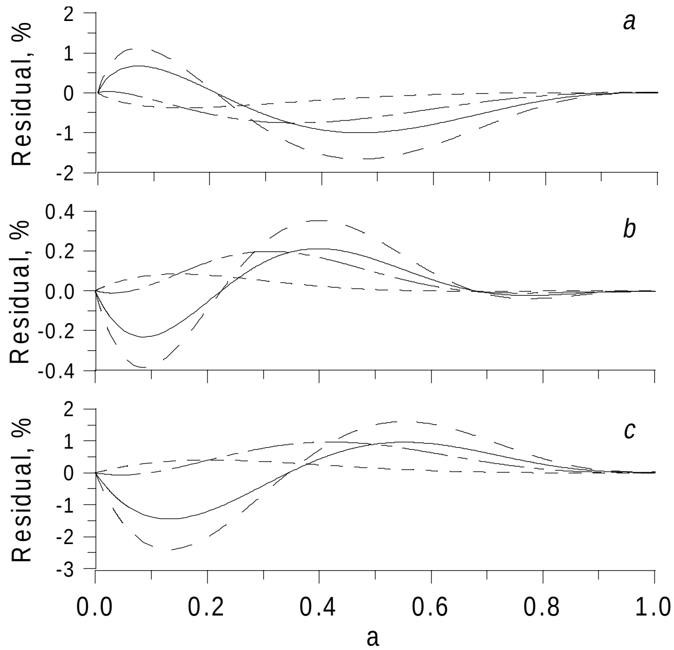

Fig. 7a–c. Accuracy of the third order approximation of the Sedov solution in the uniform medium ($m=0$) for $\gamma=5/3$: **a** relative differences of the approximation for $N=0$, **b** relative differences for $N=1$, **c** relative differences for $N=2$. Lines are the same as in Fig. 4.

third order $r^{\rm s}_{aaa}$ and $(\partial r/\partial(a^x))^0=C$. Consideration of $r^{\rm s}_{aaa}$ is equivalent to consideration of the second order derivatives $\rho^{\rm s}_{aa}$, $P^{\rm s}_{aa}$, $u^{\rm s}_{aa}$ in expansion of relevant characteristics in the series near the shock front. This approximation is the same as used in the approximate hydrodynamical method for modelling an asymmetrical strong point explosion in a medium with a large-scale density nonuniformity (Hnatyk & Petruk 1999). Contrary to the method, we assume here that both the self-similar constant α_A and factor C are different for different m.

Namely, if at time t the shock position is $R(t)$, we approximate a connection $r=r(a)$ as follows

$$r(a)=a^x\cdot(1+\alpha\cdot\xi+\beta\cdot\xi^2+\varsigma\cdot\xi^3+\delta\cdot\xi^4), \qquad (90)$$

where $\xi=1-a$. Coefficients $\alpha,\beta,\varsigma,\delta$ and exponent x are choosen from the condition that the partial derivatives r^s_a, $r^{\rm s}_{aa}$, $r^{\rm s}_{aaa}$ at the shock front ($a=1$) as well as $(\partial\ln r/\partial\ln a)^0=x$ and $(\partial r/\partial(a^x))^0=C$ in the place of explosion ($a=0$) equal their exact values:

$$\begin{aligned}
&\alpha=-r^{\rm s}_a+x,\\
&\beta=\frac{1}{2}\cdot\left(r^{\rm s}_{aa}-2x\cdot r^{\rm s}_a+x(x+1)\right),\\
&\varsigma=\frac{1}{6}\cdot\big(-r^{\rm s}_{aaa}+3x\cdot r^{\rm s}_{aa}-\\
&\qquad -3x(1+x)\cdot r^{\rm s}_a+x(x+1)(x+2)\big),\\
&\delta=C-(1+\alpha+\beta+\varsigma).
\end{aligned} \qquad (91)$$

In terms of a relation (90) and its first derivative are

$$r(a)=a^x(B_0-B_1a+B_2a^2-B_3a^3+B_4a^4), \qquad (92)$$

$$r_a(a)=a^{x-1}(A_0-A_1a+A_2a^2-A_3a^3+A_4a^4), \qquad (93)$$

with
$$\begin{aligned}
&B_0=1+\alpha+\beta+\varsigma+\delta=C, && A_0=xB_0,\\
&B_1=\alpha+2\beta+3\varsigma+4\delta, && A_1=(1+x)B_1,\\
&B_2=\beta+3\varsigma+6\delta, && A_2=(2+x)B_2,\\
&B_3=\varsigma+4\delta, && A_3=(3+x)B_3,\\
&B_4=\delta, && A_4=(4+x)B_4.
\end{aligned}$$

The distributions of $\rho(a)$, $P(a)$ and $u(a)$ obtain from (71)-(73). The self-similar constant $\alpha_{\rm A}$ is given with (75). To simplify the procedure, numerical values of $\alpha_A(N,\gamma,m)$ in this appoximation are presented in Table 3. Table 4 gives ready-calculated values of the coefficients in the approximation (92) for a number of cases.

The accuracy of flow characteristic distributions in this approximation is high for a uniform medium (Fig. 7). The approximation coincides with the exact solution (9) for case $m=m_1$. For other $m\neq 0$, differences increase with increasing $|m|$ but maximal errors occurs in the region with low densities (Fig. 8). We compare also numerical values of α_A and $P(0)$ in this approximation with those from exact Sedov solution in Table 5. α_A in the approximation is close to the exact values and gives accurate shock radius R and velocity D (Table 5).

Table 5. Comparison of the self-similar constant $\alpha_{\rm A}$ and pressure $P(0)$ calculated with: S – Sedov (1946) solution (Kestenboim et al. 1974); T – Taylor (1950) approximation; CF – approximation of Cox & Franco (1981); LVA, OPA and TPA of Ostriker & McKee (1988); CM – approximation of Cavaliere & Messina (1976); TL – thin-layer (53) approximation; LP – approximation (54) of Laumbach & Probstein (1969); SOA – second order (83) and TOA – third order (90) approximations. Uniform medium, $\gamma = 5/3$ and $N = 2$.

	S	T	CF	LVA	OPA/LPA	TPA/PGA	CM	TL	LP	SOA	TOA
α_A	0.4936	0.4957	0.4930	0.5386	0.5027	0.4957	0.5655	0.5655	0.4398	0.4981	0.4944
$P(0)$	0.3062	0.2855	0.3140	–	0.3333	0.3333	–	0.5000	0.3333	0.2507	0.3062

5. Conclusions

In this paper, we review approximations of the self-similar solution for a strong point explosion in the power law medium $\rho^o \propto r^{-m}$ and compare their accuracy with the exact Sedov solution of the problem. Different approaches result in the different basic approximations. Namely, Taylor (1950) and Ostriker & McKee (1988) approximate firstly the fluid velocity variation behind the shock front. Taylor used approximated $u(r)$ substituting it into the hydrodynamic equations to obtain a full description of the flow. Contrary to this, Ostriker & McKee approximate $\rho(r)$ and $P(r)$ independently. Kahn's (1975) technique, used also by Cox & Franco (1981), consists in approximation of the fluid mass variation $\mu(r)$ and further usage of the system of hydrodynamic equations. Gaffet (1978), Laumbach & Probstein (1969), Ostriker & McKee (1988) base their approaches on the approximation of $P(\mu)$ or $P(r)$. The thin layer approximation may also be included into this group. Hnatyk (1987) takes an approximation of the connection between Eulerian and Lagrangian coordinates as the basic relation. So, practically all possible approaches are used to approximate the self-similar solution.

In this paper we apply Taylor's methodology to describe a strong point explosion in a power-law medium, extending his approximation written for uniform medium, and write also two approximations expressed in Lagrangian geometric coordinates, approaching $r(a)$ with different accuracy.

Errors of all approximations are caused only by errors in the basic approximation. When the first approximation has higher accuracy we have more accurate approximation for parameters of the shock and flow.

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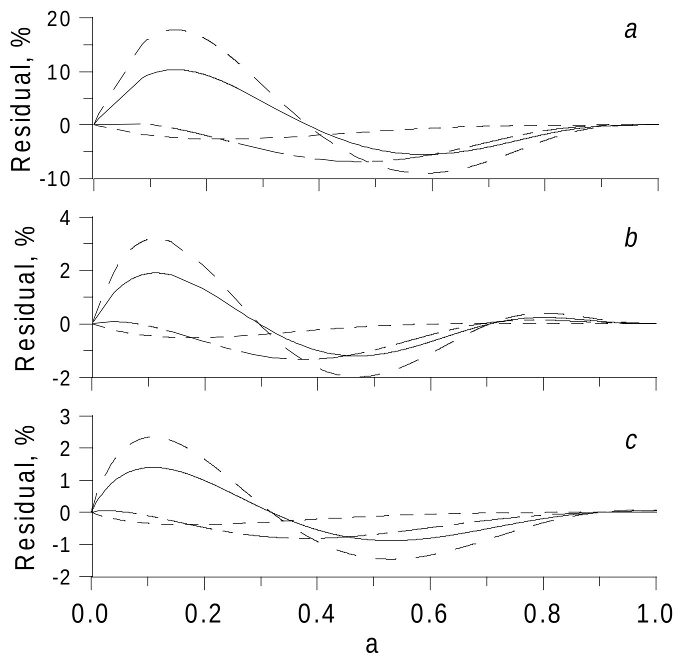

Fig. 8a–c. Accuracy of the third order approximation for power-law medium, $\gamma = 5/3$ and $N = 2$: **a** relative differences for $m = -4$, **b** relative differences for $m = -2$, **c** relative differences for $m = 1$. Lines are the same as in Fig. 4.

Appendix: Central pressure $P(0)$

In this appendix, we give exact expression for $P(0)$ in self-similar solution when $m \le m_1$ (Sedov 1959) and when $m = m_2$ (Korobejnikov & Rjazanov 1959). These relations complete the full set of formulae to build the third order approximation of the Sedov solution for any γ, $m \le \min(N+1, m_1)$ and type of symmetry (plane, cylindrical or spherical blastwave).

$P(0) = 0$ for $m = m_1$.

In the case of $m < m_1$ and $m \ne m_2$

$$P(0) = \left(\frac{1}{2}\right)^{\varepsilon_1} \left(\frac{\gamma+1}{\gamma}\right)^{\varepsilon_2} \left(\frac{m-m_3}{m-m_1}\right)^{\varepsilon_3\varepsilon_4}, \qquad (94)$$

$$\begin{aligned}
\varepsilon_1 &= \frac{2(N+1)}{N+3-m},\\
\varepsilon_2 &= \frac{2(N+1)}{N+3-m} - \frac{\gamma\big(N+1-m\big)}{(N+1)(2-\gamma)-m},\\
\varepsilon_3 &= \frac{(N+1-m)(N+3-m)}{(N+1)(2-\gamma)-m} + m - 2,\\
\varepsilon_4 &= \frac{\gamma+1}{(N+1)(\gamma-1)+2} - \frac{2}{N+3-m}\\
&\quad + \frac{\gamma-1}{\gamma(2-m)+N-1}.
\end{aligned}$$

If $m = m_2$ then

$$P(0) = \left(\frac{1}{2}\right)^{\varepsilon} \left(\frac{\gamma+1}{\gamma}\right)^{\varepsilon\gamma N/(N+1)} \exp\left(-\frac{\gamma}{2}\,\varepsilon\right), \qquad (95)$$

$$\varepsilon = \frac{2(N+1)}{(N+1)(\gamma-1)+2}.$$

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