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Bulk viscosity in superfluid neutron star cores

I. Direct Urca processes in $npe\mu$ matter

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Abstract. The bulk viscosity of the neutron star matter due to the direct Urca processes involving nucleons, electrons and muons is studied taking into account possible superfluidity of nucleons in the neutron star cores. The cases of singlet-state pairing or triplet-state pairing (without and with nodes of the superfluid gap at the Fermi surface) of nucleons are considered. It is shown that the superfluidity may strongly reduce the bulk viscosity. The practical expressions for the superfluid reduction factors are obtained. For illustration, the bulk viscosity is calculated for two models of dense matter composed of neutrons, protons, electrons and muons. The presence of muons affects the bulk viscosity due to the direct Urca reactions involving electrons and produces additional comparable contribution due to the direct Urca reactions involving muons. The results can be useful for studying damping of vibrations of neutron stars with superfluid cores.

Key words: stars: neutron – dense matter

1. Introduction

The dissipative processes in neutron stars play an important role for some dynamical properties of these unique objects. Shear viscosity damps differential rotation of neutron stars, leading to their uniform rigid-body rotation. Quite generally, the viscosity of neutron star matter implies damping of pulsations of neutron stars. Such pulsations could be excited during the process of neutron star formation. They could also be triggered by instabilities appearing during neutron star evolution, or by some external perturbations. The corresponding damping timescales involve the shear and bulk viscosities of neutron star interior. Both viscosities depend on density, temperature and composition of dense matter. Calculations of damping timescales of pulsations for various models of neutron star interiors have been done by Cutler et al. (1990). Viscous damping of pulsations of newly born hot neutron stars turns out to be due to the bulk viscosity.

Another role of the viscosity of neutron star matter is that it can damp gravitational radiation driven instabilities in rotating neutron stars and, therefore, could be important for determination of the maximum rotation frequency of neutron stars. In the absence of viscosity all rotating neutron stars would be driven unstable by the emission of gravitational waves. Viscous damping timescales enter explicitly the criteria for the appearance of these instabilities. Similarly, as in pulsating non-rotating neutron stars, viscous damping of gravitational radiation driven instabilities in rapidly rotating newly born neutron stars is dominated by bulk viscosity of neutron star interiors (e.g., Lindblom 1995, Zdunik 1996, Lindblom et al. 1998).

In this paper, we focus on the viscosity of matter in the neutron star cores (which extend from the layers of density $\rho \simeq 1.5 \times 10^{14}$ g cm^{-3} to the stellar centers). It is well known that the cores consist of baryons (neutrons n, protons p and possibly hyperons) and leptons (electrons e and muons μ). All constituents of matter are strongly degenerate fermions. The electrons and muons form almost ideal Fermi gases. The electrons are ultrarelativistic while the muons may be non-relativistic or relativistic depending on density. The nucleons are, to a good approximation, non-relativistic and constitute strongly interacting Fermi liquid. At the densities close to the normal nuclear density (baryon number density $n_0 = 0.16$ fm^{-3} which corresponds to the mass density $\rho_0 = 2.8 \times 10^{14}$ g cm^{-3}), neutron star matter is composed of n, p, e, and μ. At higher densities [$\rho \gtrsim$ (3–4) ρ_0] some models of dense matter predict appearance of hyperons. At still higher densities, some calculations indicate possible presence of exotic phases (pion condensate, kaon condensate, deconfined quark matter). We will not consider the hyperonic or exotic phases but restrict ourselves to the study of the $npe\mu$ matter.

Our analysis is additionally complicated by possible superfluidity of nucleons in the neutron star cores. The superfluidity is thought to be produced by Cooper pairing of nucleons due to attractive parts of nucleon-nucleon interaction. The superfluidity of nucleons in the neutron star cores has been the subject of numerous papers (as reviewed, for instance, by Yakovlev et al. 1999). Various microscopic theories predict very different superfluid gaps (critical temperatures T_{cn} and T_{cp}) of neutrons and protons depending on specific model of strong interaction employed and specific many-body theory used to account for medium effects. However, all these results have important com-

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mon features. In particular, the proton pairing occurs mainly in the $^1S_0$-state since the pp interaction is attractive in this state everywhere in the neutron star core due to not too high number density of protons. As for the neutrons, their interaction in the $^1S_0$ state turns from attraction to repulsion at densities $\rho \gtrsim \rho_0$ but the interaction in the $^3P_2$ state may be attractive and may lead to Cooper pairing. The critical temperatures T_{cn} and T_{cp} in the neutron star cores predicted by different microscopic theories depend on density and scatter in a wide range from about 10^8 to 10^{10} K. Under these conditions we will not rely on any specific microscopic theory of nucleon superfluidity, but will treat T_{cn} and T_{cp} as free parameters.

The viscosity, we are interested in, is well known to consist of the shear viscosity and bulk viscosity. The standard source of the shear viscosity of the neutron star matter is scattering between its constituents. Classical calculations of shear viscosity for the npe model of non-superfluid matter were done by Flowers & Itoh (1979). Their results were used in the studies of damping of neutron star pulsations by Cutler et al. (1990). In the superfluid core of a rotating neutron star, there is an additional viscous mechanism, called mutual friction, resulting from the scattering of electrons off the magnetic field trapped in the cores of superfluid neutron vortices (Lindblom & Mendell 1995).

The bulk viscosity may partly be determined by particle scattering. However, this component of bulk viscosity is usually much smaller than the shear viscosity (e.g., Baym & Pethick 1991). The main contribution into the bulk viscosity of sufficiently hot $npe\mu$ matter comes from the neutrino processes of Urca type associated with electron and muon emission and capture by nucleons. We will focus on such bulk viscosity. Generally, the neutrino processes in question are divided into the *direct Urca* and the *modified Urca* processes. A direct Urca process is a sequence of two reactions (direct and inverse one) and can be written as

$$n \to p + l + \bar{\nu}_l, \quad p + l \to n + \nu_l, \tag{1}$$

where l is either electron or muon, and ν_l is an associated neutrino. The most important is the process (Lattimer et al. 1991) involving electrons ($l = e$); it consist of the beta decay of neutron and subsequent beta capture. It should be emphasized that the both direct Urca processes are forbidden by momentum conservation of reacting particles for the simplest model of dense matter as a gas of noninteracting Fermi particles (e.g., Shapiro & Teukolsky 1983) at any density ρ in the neutron star cores. Nevertheless they are allowed (Lattimer et al. 1991) for some realistic equations of state at densities higher than some threshold densities (of several ρ_0). Thus, the direct Urca processes may be open in the inner cores of rather massive neutron stars. The threshold density for the muon process is always higher than for the electron one.

If allowed, the direct Urca processes produce the most powerful neutrino emission from the neutron star cores (Lattimer et al. 1991). Corresponding neutrino emissivities were calculated by Lattimer et al. (1991) and used in numerous simulations of the neutron star cooling as reviewed, for instance, by Yakovlev et al. (1999). In the absence of nucleon superfluidity, the direct Urca processes lead to the *fast* cooling of neutron stars. If allowed, the direct Urca processes produce the main contribution into the bulk viscosity we are interested in.

However, for many equations of state the direct Urca processes are forbidden by momentum conservation at any density in the neutron star cores. Moreover, they are prohibited at $\rho \lesssim 3\,\rho_0$ for the majority of equations of state. In such cases, they do not operate in the low and medium-mass neutron stars and in the outer cores of all neutron star models constructed using these equations of state. If so, the bulk viscosity is determined by the the reactions of the modified Urca processes

$$n + N \to p + N + l + \bar{\nu}_l, \quad p + N + l \to n + N + \nu_l, \tag{2}$$

where N is an additional nucleon required to conserve momentum of the reacting particles. For instance, in npe matter one has two modified Urca processes corresponding to $N = n$ and $N = p$, respectively, which can be referred to as the neutron and proton branches of the modified Urca process (e.g., Friman & Maxwell 1979, Yakovlev & Levenfish 1995). The rates of the modified Urca processes are typically 3–5 orders of magnitude lower than the rates of the direct Urca processes. The modified Urca processes either have no density threshold (as the neutron branch in npe matter) or have much lower density thresholds than the direct Urca processes. Thus they operate in the entire neutron star core. If the direct Urca processes are forbidden and matter is non-superfluid, the modified Urca processes produce the main neutrino emission from the neutron star cores leading to *slow* (*standard*) cooling of neutron stars. Their role in the neutron star cooling theory has been studied in many papers (see, e.g., Yakovlev et al. 1999, for review).

Thus, the problem of calculating the bulk viscosity due to neutrino processes is quite complicated: there are several neutrino processes involved influenced by possible nucleon superfluidity. So far the bulk viscosity has been studied only for non-superfluid npe matter. The viscosity due to the neutron branch of the modified Urca process was analyzed by Sawyer (1989) while the viscosity produced by the nucleon direct Urca process was considered by Haensel & Schaeffer (1992). The effects of superfluidity have not been analyzed for the problem of bulk viscosity but studied thoroughly for the neutrino emissivity produced in different reactions (e.g., Yakovlev et al. 1999, and references therein).

The relative importance of the bulk viscosity produced by neutrino reactions with respect to the shear viscosity produced by collisions can be estimated by comparing the results by Sawyer (1989) and Haensel & Schaeffer (1992) with the values of the shear viscosity calculated by Flowers & Itoh (1979). The comparison shows that the neutrino bulk viscosity dominates in the neutron star cores for temperatures $T \gtrsim 10^8$ K if the direct Urca processes are switched on and for $T \gtrsim 10^9$ K if the direct Urca processes are forbidden. In superfluid matter, the bulk viscosity can be even more important.

In this paper, we consider the bulk viscosity produced by the direct Urca processes in $npe\mu$ matter of the neutron star cores. In analogy with the effects of superfluidity on the neutrino emissivity, we will analyze the effects of superfluidity of nucleons on the bulk viscosity.

2. Bulk viscosity in non-superfluid matter

2.1. Bulk viscosity in $npe\mu$ matter

Consider the bulk viscosity produced by the direct Urca process (involving muons and electrons) in non-superfluid $npe\mu$ matter.

Due to very frequent collisions between particles, dense stellar matter should very quickly (instantaneously on macroscopic time scales) achieve a quasi-equilibrium state with certain temperature T and chemical potentials μ_i of different particle species $i = n, p, e, \mu$. Typically, all particle species are strongly degenerate. We assume that the matter is transparent for neutrinos, which therefore do not contribute to the thermodynamical quantities.

A quasi-equilibrium state described above does not mean full thermodynamic equilibrium. The latter assumes additionally the equilibrium with respect to the beta and muon decay and capture processes. We will call it the *chemical equilibrium*. Relaxation to the chemical equilibrium depends drastically on a given equation of state and local density of matter ρ. It is realized either through direct Urca or modified Urca processes (Sect. 1). Consideration of this subsection is valid for all Urca processes although the practical expressions (Sects. 2.3–2.4) will be obtained for the direct Urca processes.

The Urca processes of both types, direct and modified, are rather slow. Although the chemical relaxation rate depends strongly on temperature, in any case it takes much more time (from tens of seconds to much longer time intervals) than the rapid relaxation to a quasi-equilibrium state described above. Therefore, a neutron star can be in a quasi-equilibrium, but not in the chemical equilibrium, for a long time.

If the chemical equilibrium is achieved, then the chemical potentials satisfy the equalities $\mu_n = \mu_p + \mu_e$ and $\mu_n = \mu_p + \mu_\mu$, which imply $\mu_e = \mu_\mu$. Under these conditions, the rates of the direct and inverse reactions, Γ_l and $\bar{\Gamma}_l$ ($l = e$ or μ), of any Urca process are equal.

Let us assume that the neutron star undergoes radial pulsations of frequency ω. Associated temporal variations of the local baryon number density will be taken in the form $n_b = n_{b0} + n_{b1} \cos \omega t$, where n_{b1} is the pulsation amplitude and n_{b0} is the non-perturbed baryon number density ($|n_{b1}| \ll n_{b0}$). We assume further that n_{b0} corresponds to the chemical equilibrium. This chemical equilibrium is violated slightly in pulsating matter. If the pulsation frequency ω were much smaller than the chemical relaxation rates, the composition of matter would follow instantaneous values of n_b, realizing the chemical equilibrium every moment of time.

In reality, the typical frequencies of the fundamental mode of the radial pulsations $\omega \sim 10^3$–10^4 s^{-1}, are much higher than the chemical relaxation rates. As a result, the partial fractions $X_i = n_i/n_b$ of all the constituents of dense matter are almost unaffected by pulsations (i.e., almost constant). Owing to the slowness of the Urca reactions, these fractions lag behind their instantaneous equilibrium values, producing non-zero differences of instantaneous μ_i:

$$\eta_e = \mu_n - \mu_p - \mu_e, \quad \eta_\mu = \mu_n - \mu_p - \mu_\mu. \tag{3}$$

This causes an asymmetry of the direct and inverse direct Urca reactions, and, hence, slight deviations from the chemical equilibrium. The asymmetry, calculated in the *linear approximation* with respect to η_l, is given by

$$\Gamma_l - \bar{\Gamma}_l = -\lambda_l \, \eta_l \,, \tag{4}$$

where λ_l are the coefficients specified in Sect. 2.4 for the direct Urca reactions. Microscopic calculation (Sect. 2.4) yields $\lambda_e = \lambda_\mu$. In this paper, we will restrict ourselves to the case $|\eta_l| \ll T$ ($l = e, \mu$). Our definition of λ_l is the same as was used by Sawyer (1989) for the case of npe matter. Notice that λ_l defined in this way is negative.

The non-equilibrium Urca reactions provide the energy dissipation which causes damping of stellar pulsations. Accordingly, they contribute to the bulk viscosity of matter, ζ. Using the standard definition of the bulk viscosity, the energy dissipation rate per unit volume averaged over the pulsation period $\mathcal{P} = 2\pi/\omega$ can be written as

$$\left\langle \dot{\mathcal{E}}_{\rm kin} \right\rangle = -\frac{\zeta}{\mathcal{P}} \int_0^{\mathcal{P}} {\rm d}t \, ({\rm div}\, \boldsymbol{v})^2 = -\frac{\zeta\, \omega^2}{2} \left(\frac{n_{b1}}{n_{b0}} \right)^2 , \tag{5}$$

where $\boldsymbol{v}$ is the hydrodynamic velocity associated with the pulsations. The latter equality is obtained from continuity equation for baryons, $\dot{n}_b + n_{b0}\, {\rm div} \boldsymbol{v} = 0$ (pulsations do not change their total number), which yields ${\rm div}\, \boldsymbol{v} = -\dot{n}_b/n_{b0} = \omega\, (n_{b1}/n_{b0}) \, \sin \omega t$.

The hydrodynamic matter flow implied by the stellar pulsations is accompanied by the time variations of the local pressure, $P(t)$. The dissipation of the energy of the hydrodynamic motion is due to irreversibility of the periodic compression-decompression process. Averaged over the pulsation period, this dissipation rate in the unit volume is

$$\left\langle \dot{\mathcal{E}}_{\rm diss} \right\rangle = -\frac{n_b}{\mathcal{P}} \int_0^{\mathcal{P}} {\rm d}t \; P \, \dot{V}. \tag{6}$$

For a strictly reversible process, $\left\langle \dot{\mathcal{E}}_{\rm diss} \right\rangle = 0$. However, in our case the quantities P and V follow variations of n_b in different ways. The specific volume $V = 1/n_b$ varies instantaneously as n_b varies, i.e., the oscillations of V and n_b are in phase but the pressure varies with certain phase shift. In $npe\mu$ matter at quasi-equilibrium the pressure can be regarded as a function of four variables: n_b, X_e, X_μ, and T. Variations of T are insignificant, for our problem, and may be disregarded. Thus, it is sufficient to assume that $P = P(n_b, X_e, X_\mu)$. Variations of the pressure contain the terms oscillating with shifted phases due to the lags of X_e and X_μ.

Let us evaluate the integral (6). We have $\dot{V} = -\dot{n}_b/n_{b0}^2 = \omega\, (n_{b1}/n_{b0}^2) \sin \omega t$. Thus the only terms in P contributing into the energy dissipation are those which are proportional to $\sin \omega t$. At this stage it is convenient to use the formalism of complex variables and write $P = P_0 + \mathrm{Re}\{P_1 \exp(i\omega t)\}$, $X_l = X_{l0} + \mathrm{Re}\{X_{l1} \exp(i\omega t)\}$, where P_0 and X_{l0} are the equilibrium quantities while P_1 and X_{l1} are small complex amplitudes to be determined. We have

$$P_1 = \left(\frac{\partial P}{\partial n_b}\right) n_{b1} + \left(\frac{\partial P}{\partial X_e}\right) X_{e1} + \left(\frac{\partial P}{\partial X_\mu}\right) X_{\mu 1}, \tag{7}$$

where all the derivatives are taken at equilibrium. The real part of P contains the terms with $\sin\omega t$ provided the amplitudes X_{l1} have imaginary part.

The change of the lepton fraction $\dot{X}_l$ is determined by the difference of the direct and inverse reaction rates given by Eq. (4). The quantity η_l in the latter equation varies near its equilibrium value $\eta_{l0} = 0$ as $\eta_l = \eta_{l0} + \mathrm{Re}\{\eta_{l1} \exp(i\omega t)\}$, where

$$\eta_{l1} = \left(\frac{\partial \eta_l}{\partial n_b}\right) n_{b1} + \left(\frac{\partial \eta_l}{\partial X_e}\right) X_{e1} + \left(\frac{\partial \eta_l}{\partial X_\mu}\right) X_{\mu 1}, \tag{8}$$

and all the derivatives are again taken at equilibrium. Combining the expression $n_{b0}\dot{X}_l = \Gamma_l - \bar{\Gamma}_l$ with Eq. (4) and using the formalism of complex variables we obtain the two equations $X_{l1} = -\lambda_l\, \eta_{l1}/(i\omega n_b)$ (for $l = e$ and μ). These two equations supplemented by Eq. (8) constitute a system of equations which solution is

$$X_{l1} = -\frac{n_{b1}}{n_{b0}}\, \frac{C_l\,(B_{l'l'} + i\,\alpha_{l'}) - C_{l'}\, B_{ll'}}{(B_{ll} + i\,\alpha_l)(B_{l'l'} + i\,\alpha_{l'}) - B_{l'l}\, B_{ll'}}, \tag{9}$$

where $l' \neq l$ and $\alpha_l = \omega\, n_{b0}/\lambda_l$. In analogy with Sawyer (1989) we have introduced the notations:

$$B_{ll'} = \frac{\partial \eta_l}{\partial X_{l'}}, \quad C_l = n_{b0}\, \frac{\partial \eta_l}{\partial n_b}. \tag{10}$$

Note that all the derivatives are taken at equilibrium. In the absence of muons from Eq. (9) we have $X_{\mu 1} \equiv 0$ and $X_{e1} = -(n_{b1}/n_{b0})\, C_e/(B_{ee} + i\alpha_e)$. This is the well known limit considered by Sawyer (1989) and Haensel & Schaeffer (1992).

Generally, Eq. (9) is quite complicated. However, in practical applications stellar oscillations are always much more frequent than the beta and muon reaction rates ($|\partial\eta_l/\partial X_l| \ll \omega n_{b0}/|\lambda_l|$) and it is sufficient to use the asymptotic form of the solution in the *high-frequency limit*. In this limit the imaginary part of X_{l1} is related to the amplitude n_{b1} as

$$\mathrm{Im}\,\{X_{l1}\} = \frac{n_{b1}}{n_{b0}}\, \frac{\lambda_l}{\omega}\, \frac{\partial \eta_l}{\partial n_b}\,. \tag{11}$$

Combining this equation with that for $\dot{V}$ (see above) and inserting into Eq. (6) we get the dissipation rate of mechanical energy

$$\left\langle \dot{\mathcal{E}}_{\rm diss} \right\rangle = \frac{\omega^2}{2} \left(\frac{n_{b1}}{n_{b0}}\right)^2 \sum_l \frac{\lambda_l}{\omega^2}\, \frac{\partial P}{\partial X_l}\, \frac{\partial \eta_l}{\partial n_b}. \tag{12}$$

Finally, bearing in mind that $\langle \dot{\mathcal{E}}_{\rm kin} \rangle = -\langle \dot{\mathcal{E}}_{\rm diss} \rangle$, from Eqs. (5) and (12) we have the bulk viscosity

$$\zeta = \zeta_e + \zeta_\mu\,, \quad \zeta_l = \frac{|\lambda_l|}{\omega^2} \left| \frac{\partial P}{\partial X_l} \right| \frac{\partial \eta_l}{\partial n_b}. \tag{13}$$

Here we have taken into account that λ_l and $\partial P/\partial X_l$ are negative and presented the viscosity in the form which clearly shows that ζ_l is positive. The expression for ζ_e was obtained by Sawyer (1989). Let us emphasize that the viscosity we deal with has meaning of a coefficient in the equation that determines the damping rate of stellar pulsations averaged over pulsation period (and it cannot generally be used in exact hydrodynamical equations of fluid motion).

Therefore, in the high frequency limit, which is the most important in practice, the bulk viscosity ζ is a sum of the partial viscosities ζ_e and ζ_μ produced by the electron and muon Urca processes, respectively. This additivity rule greatly simplifies evaluation of ζ. The values of $\partial P/\partial X_l$ and $\partial\eta_l/\partial n_b$ are determined by an equation of state as described in Sects. 2.2 and 2.3. The factors λ_l are studied in Sect. 2.4 for the direct Urca reactions. The results of analogous consideration for the modified Urca reactions will be published elsewhere.

2.2. Partial bulk viscosity

Let us discuss briefly how to calculate the partial bulk viscosity ζ_l of $npe\mu$ matter for a given equation of state. All the quantities in this section and below are essentially (quasi)equilibrium values. Thus we will omit the index 0, for brevity.

Since the electrons and muons constitute almost ideal gases, the matter energy per baryon can be generally written as

$$E = E_N(n_b, X_p) + X_e E_e(n_e) + X_\mu E_\mu(n_\mu), \tag{14}$$

where $E_N(n_b, X_p)$ is the nucleon energy per baryon, $X_p = n_p/n_b$ is the proton fraction, and $E_l(n_l)$ is a lepton energy per one lepton (e or μ). The latter energy is determined by the lepton number density, n_l. Owing to charge neutrality, we have $X_p = X_e + X_\mu$.

The neutron and proton chemical potentials are given by $\mu_n = \partial(n_b E_N)/\partial n_n$ and $\mu_p = \partial(n_b E_N)/\partial n_p$. The derivatives should be taken using $n_p = X_p\, n_b$ and $n_n = (1 - X_p)\, n_b$. This gives $\mu_n - \mu_p = -\partial E_N/\partial X_p$. The chemical potentials of electrons or muons are $\mu_l = (m_l^2 c^4 + p_{{\rm F}l}^2 c^2)^{1/2}$, where $p_{{\rm F}l} = \hbar\left(3\pi^2\, n_l\right)^{1/3}$ is the Fermi momentum. Therefore, the difference of chemical potentials is

$$\eta_l = -\frac{\partial\, E_N(n_b, X_p)}{\partial X_p} - \mu_l. \tag{15}$$

Now let us calculate C_l from Eq. (10). The derivative of μ_l with respect to n_b is evaluated using $n_l = X_l n_b$. The result is

$$C_l = -n_b\, \frac{\partial^2 E_N(n_b, X_p)}{\partial n_b\, \partial X_p} - \frac{c^2 p_{{\rm F}l}^2}{3\,\mu_l}. \tag{16}$$

Using Eq. (14) and the standard thermodynamic relations we obtain the pressure $P = P_N + P_e + P_\mu$, where $P_N = n_b^2\, \partial E_N/\partial n_b$ is the nucleon pressure, while P_e and P_μ are the well known partial pressures of free gases of e and μ, respectively. Direct calculations yields $\partial P/\partial X_l = -n_b\, C_l$. Inserting this derivative into Eq. (13) and using the definition of C_l we come to a very simple equation

$$\zeta_l = \frac{|\lambda_l|}{\omega^2}\, C_l^2. \tag{17}$$

Thus, a partial bulk viscosity ζ_l is expressed through the two factors, C_l and λ_l. Calculation of C_l is discussed in Sect. 2.3, while λ_l is analyzed in Sect. 2.4.

2.3. Illustrative model of $npe\mu$ matter

For illustration, we use a phenomenological equation of state proposed by Prakash et al. (1988). According to these authors the nucleon energy is presented in the familiar form (neglecting small neutron-proton mass difference)

$$E_N(n_b, X_p) = E_{N0}(n_b) + S(n_b)\,(1-2X_p)^2, \tag{18}$$

where $E_{N0}(n_b) = E_N(n_b, X_p = 1/2)$ is the energy of the symmetric nuclear matter and $S(n_b)$ is the symmetry energy. From Eq. (15) at equilibrium ($\eta = 0$) we immediately obtain $\mu_l = 4(1-2X_p)S(n_b)$, and from Eq. (16) we have

$$\begin{aligned} C_l \;&=\; 4\,(1-2X_p)\,n_b\,\frac{\mathrm{d}S}{\mathrm{d}n_b} - \frac{c^2 p_{\mathrm{F}l}^2}{3\,\mu_l} \\ &=\; (1-2X_p)\,n_b\,\frac{\mathrm{d}}{\mathrm{d}n_b}\left(\frac{\mu_l}{1-2X_p}\right) - \frac{c^2 p_{\mathrm{F}l}^2}{3\,\mu_l}. \end{aligned} \tag{19}$$

This is the practical expression for evaluating C_l. The factor C_l is not affected by a possible nucleon superfluidity which has a negligible effect on the equation of state. The relative effect of the superfluidity on the energy per nucleon is $\sim (\Delta/\mu_N)^2 \sim 10^{-4}$–$10^{-3}$, where Δ is the superfluid energy gap, and μ_N is the nucleon Fermi energy.

In accordance with Eqs. (19) and (17), the bulk viscosity is determined by the symmetry energy $S(n_b)$. At the saturation density $n_0 = 0.16$ fm^{-3} the symmetry energy $S_0 = S(n_0)$ is measured rather reliably in laboratory (e.g., Moeller et al. 1988) but at higher n_b it is still unknown. Prakash et al. (1988) presented $S(n_b)$ in the form:

$$S(n_b) = 13\,\mathrm{MeV}\left[u^{2/3} - F(u)\right] + S_0 F(u), \tag{20}$$

where $u = n_b/n_0$, $S_0 = 30$ MeV, and $F(u)$ satisfies the condition $F(1) = 1$. They proposed three theoretical models (I, II and III) for $F(u)$:

$$F_{\mathrm{I}}(u) = u, \quad F_{\mathrm{II}}(u) = \frac{2\,u^2}{u+1}, \quad F_{\mathrm{III}}(u) = \sqrt{u}, \tag{21}$$

and three models for $E_{N0}(n_b)$ in Eq. (18). We do not discuss the latter models here because they are not required to calculate the particle fractions and the bulk viscosity as a function of n_b. Following Sawyer (1989) and Haensel & Schaeffer (1992) we will use models I and II, for illustration. Model I gives lower symmetry energy and accordingly lower excess of neutrons over protons. In contrast to the above authors we will allow for appearance of muons.

The three models for $E_{N0}(n_b)$ correspond to three different values of the compression modulus of symmetric nuclear matter at saturation, K_0=120, 180 and 240 MeV. If, for instance, we take models I and II of $S(n_b)$ and the model of $E_{N0}(n_b)$ with $K_0 = 180$ MeV, we have two model equations of state of matter (models I and II) in the cores of neutron stars. The effective masses of nucleons, renormalized by the medium effects, will be set equal to 0.7 of their bare masses (the same values will be adopted in all numerical examples below). The equations of state I and II obtained in this way are moderately stiff. The maximum neutron star masses for models I and II are $M_{\max} = 1.72\,M_\odot$ and $1.74\,M_\odot$, respectively.

The equilibrium fractions of muons and electrons, X_μ and X_e, can be obtained as numerical solutions of the set of the chemical equilibrium equations at given n_b:

$$\begin{aligned} & X_\mu = \frac{1}{2} - \mathcal{A}\,X_e^{1/3} - X_e, \\ & X_e^{2/3} - X_\mu^{2/3} - \frac{m_\mu^2\,c^2}{\hbar^2\,(3\pi^2 n_b)^{2/3}} = 0, \end{aligned} \tag{22}$$

where $\mathcal{A} = \hbar\,c\,(3\pi^2 n_b)^{1/3}/(8\,S)$. If the muons are absent ($X_\mu = 0$), the second equation should be disregarded, while the the first one determines the equilibrium composition of matter

$$X_e^{1/3} = \left(\sqrt{D} + \frac{1}{4}\right)^{1/3} - \left(\sqrt{D} - \frac{1}{4}\right)^{1/3}, \tag{23}$$

where $D = (\mathcal{A}^3/27) + (1/16)$. At given n_b the equilibrium fraction of electrons in $npe\mu$-matter is always smaller than it would be in npe matter, while the equilibrium fraction of protons is always higher. The threshold of muon appearance is determined by the condition $\mu_e = m_\mu c^2$. For models I and II, the muons appear at the baryon number density 0.150 fm^{-3} and 0.152 fm^{-3}, respectively.

In Fig. 1 we plot the factors $C_e(n_b)$ (solid lines) and $C_\mu(n_b)$ (dot-and-dash lines) which determine the bulk viscosity for models I and II. The dotted lines show $C_e(n_b)$ for the simplified models I and II in which appearance of muons is artificially forbidden. These results coincide with those obtained by Sawyer (1989) and Haensel & Schaeffer (1992). They coincide also with the solid lines at densities below the thresholds of muon appearance but go above the solid lines at higher densities (the presence of muons affects fractions of electrons and protons). As for the factor $C_\mu(n_b)$, it appears in a jump-like manner at the muon threshold, initially exceeds $C_e(n_b)$ and then tends to $C_e(n_b)$ with increasing density.

2.4. Practical expressions for the bulk viscosity of $npe\mu$ matter

Now let us calculate the factor λ_l, which enters the bulk viscosity (17) and determines the asymmetry (4) of the rates of direct and inverse reactions of the direct Urca processes in $npe\mu$ matter. In the absence of superfluidity, the rate of a direct reaction producing a lepton l is given by ($\hbar = c = k_\mathrm{B} = 1$)

$$\begin{aligned} \Gamma_l \;=\; & \int \left[\prod_{j=1}^{2} \frac{\mathrm{d}^3 p_j}{(2\pi)^3}\right] \frac{\mathrm{d}^3 p_l}{2\varepsilon_l (2\pi)^4}\,\frac{\mathrm{d}^3 p_\nu}{2\varepsilon_\nu (2\pi)^3}\, f_1 (1-f_2) \\ & \times\; (1-f_e)(2\pi)^3 \delta(E_f - E_i)\delta(\boldsymbol{P}_f - \boldsymbol{P}_i) \sum_{\mathrm{spins}} |M|^2, \end{aligned} \tag{24}$$

where $\boldsymbol{p}_j$ is a nucleon momentum ($j = 1$ or 2), $\boldsymbol{p}_l$ and ε_l are, respectively, the lepton momentum and energy, $\boldsymbol{p}_\nu$ and ε_ν are the

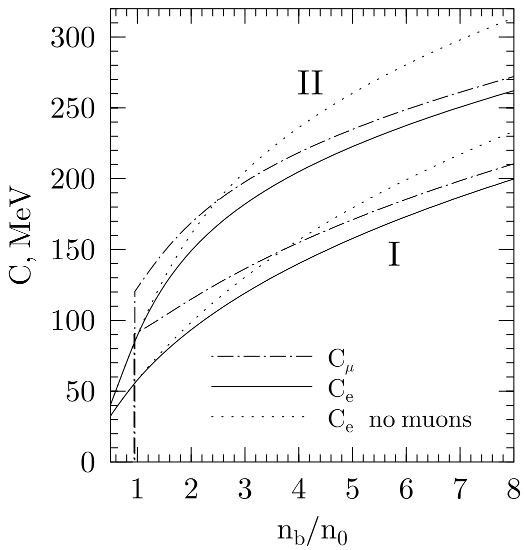

Fig. 1. Linear response of chemical potential difference to baryon number density perturbation, $C_l = n_b\, \partial\eta_l/\partial n_b$, versus the baryon number density n_b for two model equations of state proposed by Prakash et al. (1988) and discussed in the text. Solid curves correspond to $l = e$, while dot-and-dash curves to $l = \mu$. Dotted lines present C_e for the simplified models of matter without muons.

neutrino momentum and energy, $\delta(E_f - E_i)$ and $\delta(\boldsymbol{P}_f - \boldsymbol{P}_i)$ are the delta functions, which conserve energy E and momentum $\boldsymbol{P}$ of the particles in initial (i) and final (f) states, $|M|^2$ is the squared matrix element of the reaction, and f_i is an appropriate Fermi-Dirac function, $f_i = \{1 + \exp\left[(\varepsilon_i - \mu_i)/T\right]\}^{-1}$. Eq. (24) includes the instantaneous chemical potentials μ_i ($i = n, p, l$) and does not generally require the chemical equilibrium.

For further analysis we introduce the dimensionless quantities:

$$x_i = \frac{\varepsilon_i - \mu_i}{T}\,, \quad x_\nu = \frac{\varepsilon_\nu}{T}\,, \quad \xi = \frac{\eta_l}{T}\,, \tag{25}$$

where the chemical potential difference η_l is determined by Eq. (3). Thus, the delta function in Eq. (24) takes the form $\delta(E_f - E_i) = T^{-1}\,\delta(x_n - x_p - x_e - x_\nu + \xi)$, where $\xi = 0$ for chemical equilibrium.

Multidimensional integrals in Eq. (24) are standard (see, e.g., Shapiro & Teukolsky 1983). Eq. (24) is simplified taking into account that nucleons and leptons l (e and μ) are strongly degenerate. The main contribution into the integral comes from the narrow vicinities of momentum space near the Fermi surfaces of these particles. The momenta of nucleons and leptons (e or μ) can be set equal to their Fermi momenta in all smooth functions. The squared matrix element summed over the spin states and averaged over orientations of the neutrino momenta is

$$\sum_{\rm spins} |M|^2 = G^2\,(f_V^2 + 3\,g_A^2). \tag{26}$$

Here, $G = G_{\rm F}\,\cos\theta_{\rm C}$, $G_{\rm F} = 1.436 \times 10^{-49}$ erg cm^3 is the Fermi weak coupling constant, $f_V \approx 1$ is the vector normalization constant, $g_A = 1.23$ is the axial vector normalization constant, and $\theta_{\rm C}$ is the Cabibbo angle ($\sin\theta_{\rm C} = 0.231$). Hence the squared matrix element is constant and can be taken out of the integral. Further procedure consists in the standard energy-momentum decomposition of the integration in Eq. (24). It yields:

$$\Gamma_l = \Gamma_0\, I\,; \quad I \;=\; \int {\rm d}x\, x_\nu^2 \int {\rm d}x_n\, {\rm d}x_p\, {\rm d}x_l\, f(x_n) f(x_p) f(x_l) \times \delta\,(x_n + x_p + x_l - x_\nu + \xi)\,. \tag{27}$$

We have transformed all the blocking factors $(1 - f(x))$ into the Fermi-Dirac functions $f(x)$ by replacing $x \to -x$. The prefactor Γ_0 is given by (returning to the standard physical units)

$$\Gamma_0 \;=\; \frac{G^2\,(1+3g_A^2)}{4\pi^5\,\hbar^{10}\,c^3}\; m_n^*\, m_p^*\, m_e^*\, (k_{\rm B}T)^5\, \Theta_{npl} = 1.667 \times 10^{32}\, \frac{m_n^*}{m_n}\, \frac{m_p^*}{m_p} \left(\frac{n_e}{n_0}\right)^{1/3} T_9^5\, \Theta_{npl}\, {\rm cm}^{-3}\,{\rm s}^{-1}. \tag{28}$$

Here, T_9 is temperature in units of 10^9 K; m_n^* and m_p^* are, respectively, the effective masses of neutrons and protons in dense matter (which differ from the bare nucleon masses due to the in-medium effects). Moreover, we have defined $m_e^* \equiv \mu_e/c^2 \approx p_{{\rm F}e}/c$ and $m_\mu^* \equiv \mu_\mu/c^2 \approx p_{{\rm F}e}/c$, for leptons. The step function Θ_{npl} equals 1 if the direct Urca process is switched on and equals 0 otherwise (Sect. 1). The direct Urca process of study is switched on if the Fermi momenta of the reacting particles satisfy the inequality $p_{{\rm F}n} < (p_{{\rm F}p} + p_{{\rm F}l})$ (Lattimer et al. 1991).

It easy to show that the rate $\bar{\Gamma}_l$ of the inverse reaction of the direct Urca process (lepton capture) differs from the rate of the direct reaction, given by Eq. (24), only by the argument of the delta function in the expression for I (one should replace $\xi \to -\xi$ there). Therefore, the difference of the lepton production and capture rates (4) can be written as

$$\Gamma_l - \bar{\Gamma}_l \;=\; \Gamma_0\, \Delta I, \tag{29}$$

$$\Delta I \;=\; \int_0^\infty {\rm d}x_\nu\, x_\nu^2\, \left[J(x_\nu - \xi) - J(x_\nu + \xi)\right], \tag{30}$$

where

$$J(x) \;=\; \int {\rm d}x_n\, {\rm d}x_p\, {\rm d}x_e\;\; f(x_n) f(x_p) f(x_e) \times \delta\,(x_n + x_p + x_e - x)\,. \tag{31}$$

In a non-superfluid matter, which we consider in this section, the function $J(x)$ is calculated analytically

$$J(x) = \frac{\pi^2 + x^2}{2\,(1 + {\rm e}^x)}\,. \tag{32}$$

We see that the difference (29) of the non-equilibrium rates of the direct Urca reactions in normal matter is determined solely by the parameter $\xi = \eta/T$. Moreover, the integral (30) is taken analytically for any ξ:

$$\Delta I = \frac{17\pi^4}{60}\, \xi\, \mathcal{F}(\xi)\,, \quad \mathcal{F}(\xi) = 1 + \frac{10}{17\pi^2}\,\xi^2 + \frac{1}{17\pi^4}\,\xi^4\,. \tag{33}$$

This relation, with account for Eqs. (4) and (30), gives the factor λ:

$$|\lambda| = \frac{\Gamma_0}{T}\, \frac{\Delta I}{\xi}. \tag{34}$$

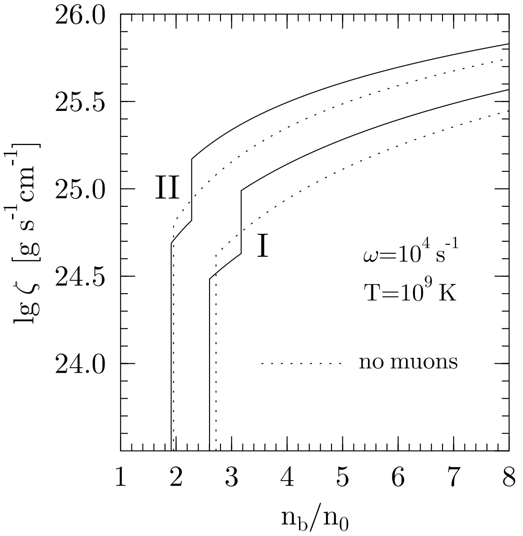

Fig. 2. The bulk viscosity ζ_0 for models I and II of non-superfluid $npe\mu$ matter (solid lines) induced by the non-equilibrium direct Urca processes involving electrons and muons versus the baryon number density n_b, for $T = 10^9$ K and $\omega = 10^4\ \mathrm{s}^{-1}$. Dotted lines show the same bulk viscosity but for the simplified models of matter without muons.

In this paper, we do not consider large deviations from the chemical equilibrium. We restrict ourselves to the deviations $|\eta| \ll T$ for which $\mathcal{F}(\xi) \approx 1$.

Finally, combining Eqs. (17) and (34), we obtain the partial bulk viscosity of $npe\mu$ matter, $\zeta_l = \zeta_{l0}$ (subscript 0 refers to non-superfluid matter), induced by a non-equilibrium direct Urca process for $|\eta| \ll T$:

$$\begin{aligned}\zeta_{l0} \;&=\; \frac{17\,G^2(1+3g_A^2)C_l^2}{240\pi\hbar^{10}c^3\,\omega^2}\; m_n^* m_p^* m_e^*\,(k_\mathrm{B}T)^4\;\Theta_{npl} \\ &=\; 8.553\times 10^{24}\,\frac{m_n^*}{m_n}\,\frac{m_p^*}{m_p}\,\left(\frac{n_e}{n_0}\right)^{1/3} \\ &\quad\times T_9^4\,\frac{1}{\omega_4^2}\,\left(\frac{C_l}{100\ \mathrm{MeV}}\right)^2\Theta_{npl}\ \ \mathrm{g\,cm^{-1}\,s^{-1}}, \end{aligned} \tag{35}$$

where $\omega_4 = \omega/(10^4\ \mathrm{s}^{-1})$. Fig. 2 shows the total bulk viscosity $\zeta_0 = \zeta_{e0} + \zeta_{\mu 0}$ of non-superfluid matter versus nucleon density n_b. We have used models I and II of $npe\mu$ matter described in Sect. 2.3. The dotted lines show the bulk viscosity for the simplified models in which the muons are absent (cf. with Fig. 1). The latter results coincide with those reported by Haensel & Schaeffer (1992).

The results for models I and II are similar. The bulk viscosity due to the direct Urca processes is switched on in a jump-like manner at the threshold density at which the electron direct Urca process becomes operative (this happens at n_b= 0.414 fm^{-3} and 0.302 fm^{-3}, respectively). The presence of muons lowers the threshold density (mainly due to increasing the number density and Fermi momenta of protons). On the other hand, the muons lower the bulk viscosity produced by the electron direct Urca process (by decreasing n_e). At larger n_b, the total bulk viscosity suffers the second jump (at $n_b = 0.503$ fm^{-3} and 0.358 fm^{-3} for models I and II, respectively). This time it is associated with switching on the muon direct Urca process, where muons participate by themselves. The contribution of the muon direct Urca into the bulk viscosity is even larger than the contribution of the electron direct Urca. The total bulk viscosity exceeds the bulk viscosity in npe matter. This is natural: the muons introduce additional non-equilibrium Urca process which makes stellar matter more viscous.

Finally, let us discuss practical calculation of the bulk viscosity. A user possesses usually number densities of different particles for any given equation of state of $npe\mu$ matter. This information is sufficient to calculate factors C_l numerically from the second equality in Eq. (19). Other density dependent quantities which enter the expressions for the bulk viscosity are also expressed through the number densities. Thus, the evaluation of the bulk viscosity for any equation of state is not a problem.

For our illustrative equations of state the functions $q_l(n_b)$=$(n_l/n_0)^{1/3}\ (C_l(n_b)/100\,\mathrm{MeV})^2$, which enter Eq. (35), can be fitted by simple expressions

$$q_l = a_0 + a_1\,u + a_2\,u^2 + a_3\,u^3, \tag{36}$$

where $u = n_b/n_0 \le 15$, $n_0 = 0.16\ \mathrm{fm}^{-3}$, and the maximum error $\lesssim 0.5$%. The fit parameters are: $a_0 = -0.3849$, $a_1 = 0.3338$, $a_2 = 0.0375$, $a_3 = -0.00098$ for $l = e$ and model I; $a_0 = -0.3396$, $a_1 = 0.4334$, $a_2 = 0.0293$, $a_3 = -0.0007$ for $l = \mu$ and model I; $a_0 = -1.0491$, $a_1 = 1.1788$, $a_2 = -0.013$, $a_3 = 0.00044$ for $l = e$ and model II; $a_0 = -0.844$, $a_1 = 1.2715$, $a_2 = -0.02023$, $a_3 = 0.000675$ for $l = \mu$ and model II.

On the other hand, the baryon number number density as a function of mass density (for the model equations of state with the compression modulus $K_0 = 180$ MeV) can be fitted as

$$n_b = n_0\,b_1\rho/(\rho_0 + b_2\rho), \tag{37}$$

where $b_1 = 1.057$ and $b_2 = 0.0322$ for model I; $b_1 = 1.059$ and $b_2 = 0.0343$ for model II. The maximum fit error is $\lesssim 0.9$%, and $\rho_0 = 2.8\times 10^{14}$ g cm^{-3}. These equations allow one to calculate the bulk viscosity as a function of mass density as required in practical applications.

3. Bulk viscosity of superfluid matter

3.1. Superfluid gaps

Superfluidity of nucleons in a neutron star core may strongly affect the bulk viscosity. Neutrons are believed to form Cooper pairs due to their interaction in the triplet state, while protons suffer singlet-state pairing (Sect. 1). While studying the triplet-state neutron pairing one should distinguish the cases of different projections m_J of nn-pair moment onto a quantization axis z (see, e.g., Amundsen and Østgaard 1985): $|m_J| = 0,\,1,\,2$. The actual (energetically most favorable) state of nn-pairs is not known being extremely sensitive to the (still unknown) details of nn interaction. One cannot exclude that this state varies with density and is a superposition of states with different m_J. We will consider the $^3\mathrm{P}_2$-state neutron superfluidity either with

Table 1. Studied type of superfluidity

Type	Pairing state	$F(\vartheta)$	$k_{\rm B}T_c/\Delta(0)$
A	$^1{\rm S}_0$	1	0.5669
B	$^3{\rm P}_2\ (m_J=0)$	$(1+3\cos^2\vartheta)$	0.8416
C	$^3{\rm P}_2\ (\lvert m_J\rvert=2)$	$\sin^2\vartheta$	0.4926

$m_J=0$ or with $|m_J|=2$. In these two cases the effect of superfluidity on the bulk viscosity is qualitatively different. Consideration of the superfluidity based on mixed m_J states is much more complicated and goes beyond the scope of the present paper.

Thus we will study three different superfluidity types: $^1{\rm S}_0$, $^3{\rm P}_2$ $(m_J=0)$ and $^3{\rm P}_2$ $(|m_J|=2)$ denoted as A, B and C, respectively (Table 1). The superfluidity of type A may be attributed to any protons, while superfluidity of types B and C may be attributed to neutrons.

Microscopically, superfluidity introduces an energy gap δ in momentum dependence of the nucleon energy, $\varepsilon(\boldsymbol{p})$. Near the Fermi level ($|p-p_{\rm F}|\ll p_{\rm F}$), this dependence can be written as

$$\begin{aligned}\varepsilon &= \mu-\sqrt{\delta^2+v_{\rm F}^2(p-p_{\rm F})^2}\ \ \text{at}\ \ p<p_{\rm F}\,,\\ \varepsilon &= \mu+\sqrt{\delta^2+v_{\rm F}^2(p-p_{\rm F})^2}\ \ \text{at}\ \ p\ge p_{\rm F}\,,\end{aligned} \tag{38}$$

where $p_{\rm F}$ and $v_{\rm F}$ are the Fermi momentum and Fermi velocity of the nucleon, respectively, and μ is the nucleon chemical potential. In the cases of study one has $\delta^2=\Delta^2(T)F(\vartheta)$, where $\Delta(T)$ is the amplitude which describes temperature dependence of the gap; $F(\vartheta)$ specifies dependence of the gap on the angle ϑ between the particle momentum and the z axis (Table 1). In case A the gap is isotropic, and $\delta=\Delta(T)$. In cases B and C the gap depends on ϑ. Note that in case C the gap vanishes at the poles of the Fermi sphere at any temperature: $F_{\rm C}(0)=F_{\rm C}(\pi)=0$.

The gap amplitude $\Delta(T)$ is derived from the standard equation of the BCS theory (see, e.g., Yakovlev et al. 1999). The value of $\Delta(0)$ determines the critical temperature T_c. The values of $k_{\rm B}T_c/\Delta(0)$ for cases A, B and C are given in Table 1.

For further analysis it is convenient to introduce the dimensionless quantities:

$$v=\frac{\Delta(T)}{T},\quad \tau=\frac{T}{T_c},\quad y=\frac{\delta}{T}. \tag{39}$$

The dimensionless gap y can be presented in the form:

$$y_{\rm A}=v_{\rm A},\quad y_{\rm B}=v_{\rm B}\sqrt{1+3\cos^2\vartheta},\quad y_{\rm C}=v_{\rm C}\,\sin\vartheta. \tag{40}$$

The dimensionless gap amplitude v depends only on τ. In case A the quantity v coincides with the isotropic dimensionless gap, while in cases B and C it represents, respectively, the minimum and maximum gap (as a function of ϑ) on the nucleon Fermi surface. The dependence of v on τ can be fitted as (Levenfish & Yakovlev 1994):

$$\begin{aligned}v_{\rm A} &= \sqrt{1-\tau}\left(1.456-\frac{0.157}{\sqrt{\tau}}+\frac{1.764}{\tau}\right),\\ v_{\rm B} &= \sqrt{1-\tau}\left(0.7893+\frac{1.188}{\tau}\right),\\ v_{\rm C} &= \frac{\sqrt{1-\tau^4}}{\tau}\left(2.030-0.4903\tau^4+0.1727\tau^8\right).\end{aligned} \tag{41}$$

The mean errors of these fits are $\lesssim 1\%$ for all $\tau\le 1$.

3.2. Superfluid reduction factors

Now let us consider the effects of nucleon superfluidity on the bulk viscosity. The dynamics of superfluid is generally much more complicated than the dynamics of ordinary fluids. Even the motion of matter which consists of particles of one species is described by the equations of two-fluid hydrodynamics (normal and superfluid components), and viscous dissipation of the normal component is determined by three coefficients of the second (bulk) viscosity (Landau & Lifshitz 1987). Our main assumption is that stellar pulsations represent fluid motion of the first-sound type (particularly, temperature variations are neglected) in which all constituents of matter move with the same hydrodynamical velocity. In this case the hydrodynamical equations reduce to the equation of one-fluid hydrodynamics with one coefficient of the bulk viscosity ($\zeta=\zeta_2$ in the notation of Landau & Lifshitz, 1987).

We will see that superfluidity *reduces* the bulk viscosity due to the appearance of energy gaps in the nucleon dispersion relation, Eq. (38). Quite generally, the bulk viscosity can be presented in the form

$$\zeta=\sum_l \zeta_{l0}\,R_l, \tag{42}$$

where ζ_{l0} is a partial bulk viscosity of non-superfluid matter, Eq. (35), and R_l is a factor which describes reduction of the partial bulk viscosity by superfluidity of nucleons 1 and 2 involving into a corresponding direct Urca process. If both nucleons, 1 and 2, belong to non-superfluid component of matter, we have $R_l=1$ and reproduce the results of Sect. 2.

Thus the problem consists in calculating the reduction factors R_l. Each factor depends generally on two parameters, v_1 and v_2, which are dimensionless gap amplitudes of nucleons 1 and 2 (and on the type of superfluidity of these nucleons). Let us study the effect of superfluidity on the partial bulk viscosity. For this purpose let us reconsider derivation of the bulk viscosity (Sect. 2.1). If all constituents of matter have the same macroscopic velocity, the superfluidity affects noticeably only the factor λ_l in the expression for the bulk viscosity, Eq. (35). As seen from Eq. (34), the main factor affected by the superfluidity in λ_l is the integral ΔI, Eq. (30), which describes the asymmetry of the lepton production and capture rates in the direct and inverse reactions of the direct Urca process. At $\xi\ll 1$ the integrand of this equation is $J(x_\nu-\xi)-J(x_\nu+\xi)\approx -2\xi\,\partial J(x_\nu)/\partial x_\nu$, where $J(x_\nu)$ is given by Eq. (31). Thus, at small deviations from the equilibrium one can transform Eq. (30) to:

$$\Delta I_0 \;=\; 4\xi\int_0^{+\infty}{\rm d}x_\nu\,x_\nu\int_{-\infty}^{+\infty}{\rm d}x_1\,f(x_1)\int_{-\infty}^{+\infty}{\rm d}x_2\,f(x_2)$$

$$\times \int_{-\infty}^{+\infty} \mathrm{d}x_e\, f(x_e)\, \delta\,(x_1 + x_2 + x_e - x_\nu)\,. \quad (43)$$

Here, the index "0" refers to the non-superfluid case, in which we have obtained $\Delta I_0 = 17\,\pi^4 \xi/60$.

Generalization of λ_l to the superfluid case can be achieved by introducing the neutron and proton energy gaps into Eq. (43). For convenience, let us define the dimensionless quantities

$$x = \frac{v_\mathrm{F}(p - p_\mathrm{F})}{T}, \quad z = \frac{\varepsilon - \mu}{T} = \mathrm{sign}(x)\sqrt{x^2 + y^2}\,, \quad (44)$$

where y is given by Eq. (39). In the absence of superfluidity, we have $y = 0$ and $z = x$.

Let the index $i = 1$ correspond to a nucleon which can suffer superfluidity of type A while $i = 2$ correspond to a nucleon which can suffer any superfluidity, A, B or C. In order to account for superfluidity in Eq. (43) it is sufficient to replace $x_i \to z_i$ for $i = 1$ and 2 [in $f(x_i)$ and in the delta function] and introduce averaging over orientations of $\boldsymbol{p}_2$ (analogous procedure is considered in detail by Levenfish & Yakovlev 1994 for the problem of superfluid reduction of the neutrino emissivity). Then the factor λ_l can be written as

$$\lambda_l \;=\; \lambda_{l0}\, R, \quad R(v_1, v_2) = \int_0^{\pi/2} \frac{\mathrm{d}\Omega}{4\pi}\; \mathcal{J}(y_1, y_2)$$
$$= \int_0^{\pi/2} \mathrm{d}\vartheta\; \sin\vartheta\; \mathcal{J}(y_1, y_2), \quad (45)$$

where λ_{l0} refers to the non-superfluid case, $R = R_l$ is the reduction factor in question, and

$$\mathcal{J}(y_1, y_2) \;=\; \gamma \int_0^{+\infty} \mathrm{d}x_\nu\, x_\nu \int_{-\infty}^{+\infty} \mathrm{d}x_l\, f(x_l) \int_{-\infty}^{+\infty} \mathrm{d}x_1\, f(z_l)$$
$$\times \int_{-\infty}^{+\infty} \mathrm{d}x_2\, f(z_2)\, \delta(z_1 + z_2 + x_l - x_\nu), \quad (46)$$

with $\gamma = 240/(17\pi^4)$. Here, $\mathrm{d}\Omega$ is the solid angle element in the direction of $\boldsymbol{p}_2$.

Thus, we have derived explicit Eqs. (45) and (46) for calculating the reduction factor R. Calculation is quite similar (and in fact, simpler) to that done for the factor which describes superfluid reduction of the neutrino emissivity in the direct Urca process (Levenfish & Yakovlev 1994, Yakovlev et al. 1999). The effect of superfluidity on the bulk viscosity has also much in common with the effect on the emissivity. Thus we omit technical details and present only the results and their brief discussion.

3.3. Superfluidity of neutrons or protons

Consider the superfluidity of nucleon of one species, for instance, of species 2. In this case R depends on the only parameter v_2, and we can set $z_1 = x_1$ in Eqs. (45) and (46). Integration over x_l and x_1 in Eq. (46) reduces to well-known integrals of the theory of Fermi liquids and yields:

$$R \;=\; \gamma \int_0^{\pi/2} \mathrm{d}\vartheta\; \sin\vartheta \int_0^{+\infty} \mathrm{d}x_\nu\, x_\nu \int_0^{+\infty} \mathrm{d}x_2$$
$$\times\; [\, f(z_2) B(x_\nu - z_2) + f(-z_2) B(x_\nu + z_2)\,]\,, \quad (47)$$

where $B(x) = x/(\mathrm{e}^x - 1)$. For $\tau = \tau_2 = T/T_{c2} \geq 1$, one has $R = 1$. If superfluidity is strong ($\tau \ll 1$, $v_2 \gg 1$), the direct Urca process is drastically suppressed by large superfluid gap in the nucleon spectrum and reduces the bulk viscosity. The asymptotic expressions of R for $\tau \ll 1$ can be obtained from Eq. (47):

$$R_\mathrm{A} \;=\; \frac{20\sqrt{2}}{17\pi^{3.5}}\, v^{3.5} \exp(-v) = \frac{0.221}{\tau^{3.5}}\; \exp\left(-\frac{1.764}{\tau}\right), \quad (48)$$

$$R_\mathrm{B} \;=\; \frac{20\sqrt{3}}{51\pi^3}\, v^3 \exp(-v) = \frac{0.0367}{\tau^3}\; \exp\left(-\frac{1.188}{\tau}\right), \quad (49)$$

$$R_\mathrm{C} \;=\; \frac{764\pi^2}{1071}\, \frac{1}{v^2} = 1.7085\, \tau^2. \quad (50)$$

Note that the factors R_A and R_B are suppressed exponentially with decreasing temperature, whereas R_C varies as T^2. The latter fact is associated with the presence of gap nodes at the Fermi surface (Levenfish & Yakovlev 1994).

In addition, we have calculated the reduction factors R numerically in a wide range of v and propose the expressions which fit the numerical results (with a mean error of $\lesssim 1\%$) and reproduce the asymptotes (48)–(50):

$$R_\mathrm{A} \;=\; \left[0.2787 + \sqrt{(0.7213)^2 + (0.1564\, v)^2}\right]^{3.5}$$
$$\times\; \exp\left(2.9965 - \sqrt{(2.9965)^2 + v^2}\right), \quad (51)$$

$$R_\mathrm{B} \;=\; \left[0.2854 + \sqrt{(0.7146)^2 + (0.1418\, v)^2}\right]^{3}$$
$$\times \exp\left(2.0350 - \sqrt{(2.0350)^2 + v^2}\right), \quad (52)$$

$$R_\mathrm{C} \;=\; \frac{0.5 + (0.1086\, v)^2}{1 + (0.2347\, v)^2 + (0.2023\, v)^4}$$
$$+\, 0.5\, \exp\left(1 - \sqrt{1 + (0.5v)^2}\right). \quad (53)$$

Here, $v = v_\mathrm{A}$, $v = v_\mathrm{B}$ and $v = v_\mathrm{C}$ in the factors R_A, R_B and R_C, respectively. Using Eqs. (41) and (51)–(53), one can easily calculate the reduction factors R for any τ. These factors are shown in Fig. 3 versus τ. We see that the reduction can be quite substantial. The strongest reduction is provided by superfluidity A and the weakest by superfluidity C. For instance, at $T = 0.1\, T_c$ we obtain $R_\mathrm{A} \approx 2 \times 10^{-5}$, $R_\mathrm{B} \approx 4 \times 10^{-4}$ and $R_\mathrm{C} \approx 2 \times 10^{-2}$.

3.4. Superfluidity of neutrons and protons

Let both nucleons, 1 and 2, be superfluid at once, and let the superfluidity of nucleon 1 be of type A. In this case R can be calculated from Eqs. (45) and (46). Using the delta function, we remove the integration over x_ν and obtain

$$\mathcal{J}(y_1, y_2) = \gamma \int_{-\infty}^{+\infty} \mathrm{d}x_1\, f(z_1) \int_{-\infty}^{+\infty} \mathrm{d}x_2\, f(z_2)\, H(z_1 + z_2), \quad (54)$$

$$H(z) = \int_{-z}^{+\infty} \mathrm{d}x\; \frac{z + x}{1 + \mathrm{e}^x}. \quad (55)$$

Notice that $H(z) \approx z^2/2$ as $z \to \infty$, and $H(z) \approx \mathrm{e}^z$ as $z \to -\infty$.

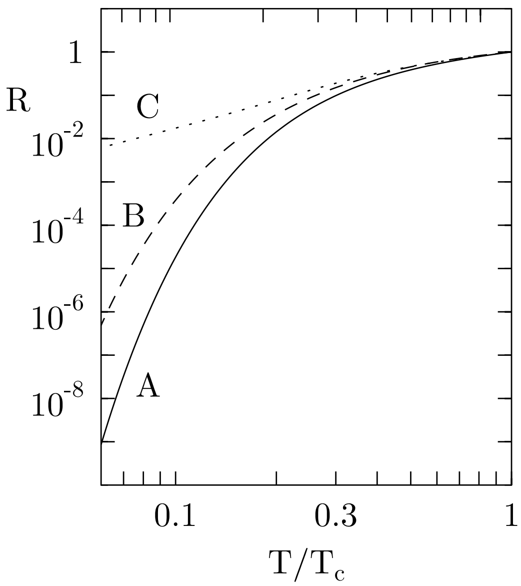

Fig. 3. Reduction factors R of the bulk viscosity versus $\tau = T/T_c$ for three superfluidity types A, B and C (Table 1).

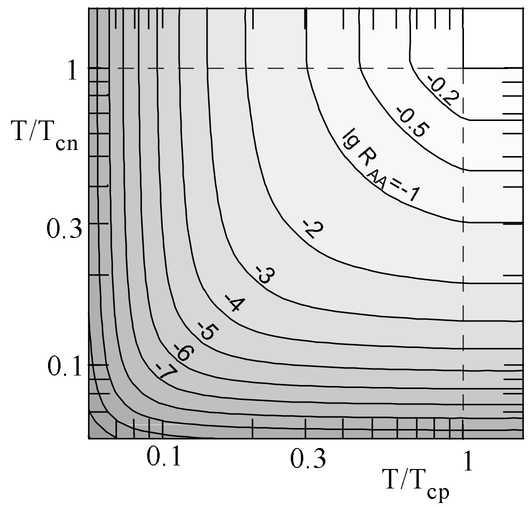

Fig. 4. Isolevels of the reduction factor R of the bulk viscosity by nucleon superfluidity of type AA versus $\tau_1 = T/T_{c1}$ and $\tau_2 = T/T_{c2}$. In the domain $\tau_1 \geq 1$, $\tau_2 \geq 1$ nucleons 1 and 2 are normal and $R_{\rm AA} = 1$. In the domain $\tau_1 < 1$, $\tau_2 \geq 1$ nucleons 1 are superfluid and nucleons 2 normal, while in the domain $\tau_1 \geq 1$, $\tau_2 < 1$ nucleons 1 are normal and 2 superfluid; in these domains R depends on one parameter (τ_1 or τ_2). In the domain $\tau_1 < 1$, $\tau_2 < 1$ both nucleons 1 and 2 are superfluid at once.

First consider the case in which the superfluidities of nucleons 1 and 2 are of type A. Using Eqs. (40) and (45) we get

$$R_{\rm AA}(v_1, v_2) = \mathcal{J}(v_1, v_2) = \mathcal{J}(v_2, v_1), \tag{56}$$

where $v_1 = y_1$ and $v_2 = y_2$. It is evident that $R_{\rm AA}(0, 0) = 1$. We have also derived the asymptote of $R_{\rm AA}$ in the limit of strong superfluidity. Furthermore, we have calculated the factor $R_{\rm AA}$ and derived the fit expression which reproduces the numerical results and the asymptotes. Both, the asymptotes and fits, are given by the complicated expressions presented in the Appendix. In Fig. 4 we show the curves $R_{\rm AA} =$ const as a function of $\tau_1 = T/T_{c1}$ and $\tau_2 = T/T_{c2}$. This visualizes the reduction the bulk viscosity for any T, T_{c1} and T_{c2}.

One can observe (Fig. 4) one important property of the reduction factor R. If both superfluidities are strong, $\tau_1^2 + \tau_2^2 \ll 1$, the factor R is mainly determined by the larger of the two gaps (by the strongest superfluidity):

$$R_{12} \sim \min\left\{R_1,\, R_2\right\}. \tag{57}$$

Here, R_1 and R_2 are the reduction factors for the superfluidity of nucleons of one species. The weaker superfluidity (with smaller energy gap) produces some additional reduction of the viscosity which is relatively small; this is confirmed by the asymptote $R_{\rm AA}$ given in the Appendix. The same effect takes place for the reduction of the neutrino emissivity (e.g., Yakovlev et al. 1999).

Eq. (45) can be used also to evaluate R for the case in which the nucleons of species 1 suffer pairing of type A, while the nucleons of species 2 suffer pairing of types B or C. In the Appendix we present the asymptotes of the factors $R_{\rm AB}$ and $R_{\rm AC}$ in the limit of strong superfluidity of both nucleon species ($\tau_1 \ll 1$, $\tau_2 \ll 1$).

The factors $R_{\rm AB}$ and $R_{\rm AC}$ can be calculated easily in a wide range of τ_1 and τ_2. The calculation reduces to the one-dimensional integration in Eq. (45) of the function $\mathcal{J}(y_1, y_2)$ fitted by Eq. (A3). The results are exhibited in Figs. 5 and 6. One can see that the dependence of the factors $R_{\rm AB}$ and $R_{\rm AC}$ on τ_1 and τ_2 has much in common with the dependence of $R_{\rm AA}$ but $R_{\rm AB}(\tau_1, \tau_2) \neq R_{\rm AB}(\tau_2, \tau_1)$ and $R_{\rm AC}(\tau_1, \tau_2) \neq R_{\rm AC}(\tau_2, \tau_1)$. The simple estimate (57) turns out to be valid in cases AB and AC as well. However since the superfluidity of type C reduces the factor R in a much weaker way than the superfluidities of types A or B, the transition from one dominating superfluidity to the other takes place in a rather wide region of v_1 and v_2 at $v_1 \sim \ln v_2$. Accordingly, for $v_2 \gtrsim v_1 \gg 1$, the reduction factor $R_{\rm AC}$ exceeds greatly $R_{\rm AA}$ and $R_{\rm AB}$.

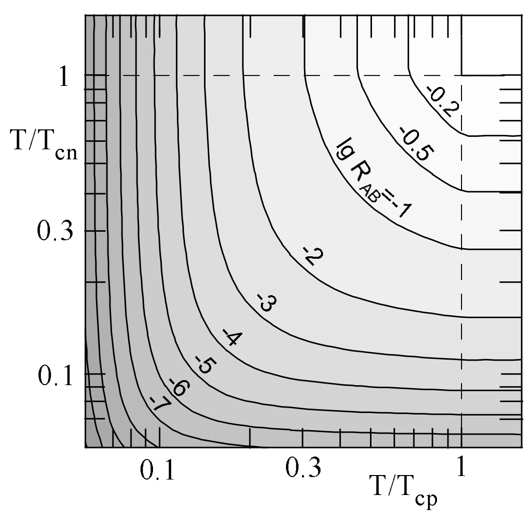

Fig. 5. Same as in Fig. 4 but for the case in which the superfluidity of nucleons 1 is of type A while the superfluidity of nucleons 2 is of type B.

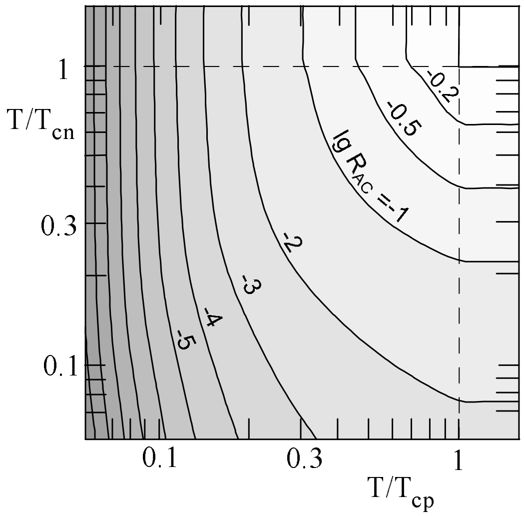

Fig. 6. Same as in Fig. 5 but for the case in which superfluidity of nucleons 2 is of type C.

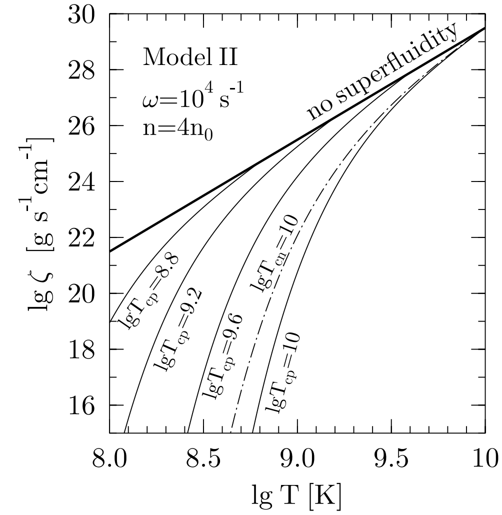

Fig. 7. Bulk viscosity ζ of superfluid $npe\mu$ matter (model II) produced by the electron and muon direct Urca processes at the baryon number density $n_b = 4\,n_0$ and $\omega = 10^4\,\mathrm{s}^{-1}$ as a function of temperature for non-superfluid matter (thick solid line), for matter with superfluid protons (solid curves, $T_{cp} = 10^{10}$, $10^{9.6}$, $10^{9.2}$ and $10^{8.8}$) and normal neutrons, and for matter with superfluid neutrons (dash-and-dotted curve, $T_{cn} = 10^{10}$ K) and normal protons.

For practical calculations of the bulk viscosity in superfluid matter, one needs to know how to evaluate $R_{\rm AA}$, $R_{\rm AB}$ and $R_{\rm AC}$. Corresponding expressions for superfluidity of one nucleon species are given in Sect. 3.3. If nucleons 1 and 2 are superfluid at once, the reduction factor $R_{\rm AA}$ can be determined easily from the fit Eq. (A3). As for the reduction factors $R_{\rm AB}$ and $R_{\rm AC}$, we have generated their extensive tables. These tables and numerical code which generates them are freely distributed.

Finally, Fig. 7 illustrates reduction of the bulk viscosity of $npe\mu$ matter with decreasing temperature by superfluidity of neutrons of type B or protons of type A for $n_b = 4\,n_0$ and $\omega = 10^4\,\mathrm{s}^{-1}$. Thick solid line shows the viscosity of non-superfluid matter (cf. with Fig. 2). Thin solid lines exhibit the bulk viscosity suppressed by the proton superfluidity at several selected critical temperatures T_{cp} indicated near the curves. The dot-and-dashed line shows the effect of neutron superfluidity ($T_{cn} = 10^{10}$ K) for normal protons. We see that the superfluid reduction of the bulk viscosity depends drastically on temperature, superfluidity type, and critical temperatures T_{cn} and T_{cp}. One can hardly expect T_{cn} and T_{cp} higher than 10^{10} K for n_b as large as $4\,n_0$ (e.g., Yakovlev et al. 1999). If so, the superfluid reduction cannot be very large, say, for $T \gtrsim 3 \times 10^9$ K, but it can reach five orders of magnitude in the case of superfluid protons (or six orders of magnitude if n and p are superfluid at once, see Fig. 5) for $T = 10^9$ K at $T_{cn} = T_{cp} = 10^{10}$ K. It grows exponentially with further decrease of T.

4. Conclusions

We have derived practical expressions for the bulk viscosity of matter in the cores of neutron stars under conditions in which the bulk viscosity is determined by the direct Urca processes. We have paid special attention to the case in which dense matter consists of neutrons, protons, electrons and muons (Sects. 2.1–2.4). In addition, we have studied the reduction of the bulk viscosity by superfluidity of neutrons and protons (Sect. 3). We have analyzed the cases of singlet-state superfluidity of protons, and triplet-state superfluidity of neutrons (without and with the nodes of superfluid gaps on the nucleon Fermi surface). These cases are most interesting for applications (Sect. 1). We have obtained the practical expressions for the bulk viscosity of superfluid $npe\mu$ matter. The results can be used for studying the damping of neutron star pulsations (Sect. 1) and in the studies of the gravitational radiation driven instabilities in rotating neutron stars. We will analyze the bulk viscosity induced by the weaker modified Urca processes (2) in a separate paper.

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Appendix

Consider the case in which both nucleon species are superfluid at once. The asymptotic behaviour of the reduction factor R given by Eq. (54) in the limit of strong superfluidity depends on the gap amplitudes v_1 and v_2 and on the type of superfluidities.

Case AA

Let both nucleon superfluidities be of type A. If the superfluidities are strong ($v_1, v_2 \gg 1$) and $(v_1 - v_2) \gg \sqrt{v_1}$ we obtain:

$$R_{\rm AA} \;=\; \gamma \sqrt{\frac{\pi v_1}{2}}\; \mathrm{e}^{-v_1}\, K, \tag{A1}$$

$$K \;=\; \frac{s}{6}\,(v_1^2 + 2v_2^2) - \frac{1}{2} v_1 v_2^2\, \ln\left(\frac{v_1 + s}{v_2}\right), \tag{A2}$$

where $s=\sqrt{|v_1^2-v_2^2|}$. In the limiting case $v_2 \ll v_1$ Eq. (A2) yields $K = v_1^3/60$ and reproduces the asymptote given by Eq. (48). In the opposite limit $\sqrt{v_1} \ll (v_1-v_2) \ll v_1$ one has $K=s^5/(15v_1^2)$. In the intermediate region $|v_2-v_1| \lesssim \sqrt{v_2}$ the asymptote (A2) becomes invalid. One can show that $K \sim \sqrt{v_1}$ for $v_1=v_2$.

We have calculated $R_{\rm AA}$ in a wide range of arguments v_1 and v_2 and proposed the fit expression valid for $\sqrt{v_1^2+v_2^2} \lesssim 50$:

$$R_{\rm AA} = \frac{u}{u+13.528}\, S \,+\, D. \tag{A3}$$

Here,

$$\begin{aligned}
S &= \gamma\,\left(K_0+2.3190\,K_1+0.016689\,K_2\right)\,\left(\frac{\pi}{2}\right)^{1/2} p_s^{1/4} \\
&\quad \times \exp\left(-\sqrt{p_e}\right), \\
K_0 &= \frac{\sqrt{p-q}}{6}\,(p+2q) - \sqrt{p}\,\frac{q}{2}\,\ln\left(\frac{\sqrt{p}+\sqrt{p-q}}{\sqrt{q}}\right), \\
K_1 &= \frac{\pi^2}{6}\sqrt{p-q}, \\
K_2 &= \frac{p}{\sqrt{p-q}}\,\left(1+\frac{\pi^2}{6}\right), \\
2p &= u+10.397+\sqrt{w^2+5.4188\,u+7.2987}, \\
2q &= u+10.397-\sqrt{w^2+5.4188\,u+7.2987}, \\
2p_s &= u+\sqrt{w^2+3748.6\,u+39.797}, \\
2p_e &= u+0.39417+\sqrt{w^2+7.8849\,u+1.8132}, \\
D &= 0.0080487\,(q_1\,q_2)^{3/2}\,\exp(-q_1-q_2+9.9798), \\
q_1 &= 2.79205+\sqrt{v_1^2+(2.19785)^2}, \\
q_2 &= 2.79205+\sqrt{v_2^2+(2.19785)^2},
\end{aligned} \tag{A4}$$

with $u=v_1^2+v_2^2$ and $w=v_1^2-v_2^2$.

Case AB

Let nucleons of species 1 suffer pairing of type A and nucleons of species 2 suffer pairing of type B. In this case $y_{\rm B}=y_2=v_2\sqrt{1+3\cos\vartheta^2}$, see Eq. (40). There are three domains in the (v_1,v_2)-plane, in which the asymptotes of the reduction factor $R_{\rm AB}(v_1,v_2)$ in the limit of strong superfluidity are different.

The first domain corresponds to $v_2>v_1$, i.e., to $y_2>y_1$ for all ϑ. If the both superfluidities are strong ($v_1 \gg 1$ and $v_2 \gg 1$) and $(v_2-v_1) \gg \sqrt{v_2}$, the asymptote of $R_{\rm AB}$ can be obtained by averaging Eq. (A1) over ϑ after making a formal replacement $v_1 \to y_2$ and $v_2 \to v_1$. Since $v_2 \gg 1$ the main contribution into the integral comes from the region in which $|\cos\vartheta| \ll 1$. This allows us to put $y_2 \approx v_2$ in all smooth functions under the integral. In this way we obtain

$$\begin{aligned}
R_{\rm AB} \;&=\; \gamma\,\frac{\pi}{2\sqrt{3}}\,\exp(-v_2) \\
&\quad \times\,\left[\frac{s}{6}\,(v_1^2+2v_2^2)-\frac{1}{2}v_1v_2^2\,\ln\left(\frac{v_1+s}{v_2}\right)\right].
\end{aligned} \tag{A5}$$

The second domain corresponds to $v_1/2>v_2$. We have $y_1>y_2$ for all ϑ in this domain. Then the asymptote of $R_{\rm AB}$ is derived by direct averaging over ϑ of the asymptote $R_{\rm AA}$ given by Eq. (A1), after replacing formally $v_2 \to y_2$. We have:

$$\begin{aligned}
R_{\rm AB} \;&=\; \gamma\sqrt{\frac{\pi}{2}\,v_1}\,\exp(-v_1)\,\left[\frac{t}{24}\,\left(3v_1^2+11v_2^2\right)\right. \\
&\quad +\frac{2\,v_1v_2^2}{6\sqrt{3}}\arcsin\left(\frac{\sqrt{3}\,v_1}{2s}\right)-v_1v_2^2\,\ln\left(\frac{v_1+t}{2v_2}\right) \\
&\quad \left.+\,\frac{v_1^4-6v_1^2v_2^2-3v_2^4}{24\sqrt{3}\,v_2}\,\arcsin\left(\frac{\sqrt{3}\,v_2}{s}\right)\right],
\end{aligned} \tag{A6}$$

where $t=\sqrt{v_1^2-4v_2^2}$.

The third domain corresponds to $v_1/2<v_2<v_1$. While averaging over ϑ we can split the integral into two terms. The first term represents integration over the region $\cos\vartheta_0<|\cos\vartheta|<1$, in which $y_1<y_2$, while the second term contains integration over the region $0 \le |\cos\vartheta| \le \cos\vartheta_0$, in which $y_1>y_2$, with $\cos\vartheta_0^2=(v_1^2-v_2^2)/(3v_2^2)$. The latter term can be taken as an asymptote of the factor $R_{\rm AB}$, since the main contribution into the integral comes, again, from $|\cos\vartheta| \ll 1$. We have

$$R_{\rm AB} = \gamma\sqrt{\frac{\pi\,v_1}{2}}\,\exp(-v_1)\,\frac{\pi\,(v_1-v_2)^3(v_1+3v_2)}{48\sqrt{3}\,v_2}\,. \tag{A7}$$

Note that the asymptotes given by Eqs. (A5) and (A7) are invalid at $|v_2-v_1| \lesssim \sqrt{v_1}$. If $v_2 \ll v_1$, Eq. (A5) reproduces Eq. (48) for $R_{\rm A}$. For $v_1 \ll v_2$, Eq. (A5) transforms into Eq. (A6). Eqs. (A6) and (A7) coincide at $v_1=2v_2$. If $v_1=v_2$, the reduction factor can be estimated as an $R_{\rm AB} \sim v^2{\rm e}^{-v}$.

Case AC

Let superfluidity of nucleons 1 be of type A, as before, while superfluidity of nucleons 2 be of type C. In this case $y_{\rm C}=y_2=v_2\sin\vartheta$. For deriving $R_{\rm AC}$ in the limit of strong superfluidity we will use the asymptote of the function H defined by Eq. (55). If $v_1>v_2 \gg 1$, the main contribution into the integral (54) comes from the region, in which $z_1>z_2$. In this case the asymptote is

$$\begin{aligned}
R_{\rm AC} \;&=\; \gamma\sqrt{\frac{\pi v_1}{2}}\,\exp(-v_1)\,\left[\frac{v_1}{8}(v_1^2+v_2^2)-\,\frac{v_1v_2^2}{3}\,\ln\left(\frac{s}{v_2}\right)\right. \\
&\quad \left.+\,\frac{v_1^4-6v_1^2v_2^2-3v_2^4}{24v_2}\,\ln\left(\frac{v_1+v_2}{s}\right)\right].
\end{aligned} \tag{A8}$$

If $v_2 \ll v_1$, this asymptote reproduces the asymptote (48) for $R_{\rm A}$. For $v_1=v_2$, we have $R_{\rm AC} \approx \gamma\sqrt{\pi v/2}\,{\rm e}^{-v}(3-4\ln 2)\,v^3/12$. Finally, at $v_2>v_1 \gg 1$ the asymptote of $R_{\rm AC}$ is

$$R_{\rm AC} = \gamma\sqrt{\frac{\pi v_1}{2}}\,\exp(-v_1)\,\frac{v_1^5}{5v_2^2}\,. \tag{A9}$$

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