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The Parker propagator for spherical solar modulation

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Abstract. An analytical solution for the Green's function of the fundamental transport equation of cosmic rays, i.e. of the Parker equation, is presented. The solution is valid for an arbitrary power-law dependence of the coefficient of spatial diffusion on both the configuration and the momentum space coordinate, and it generalizes earlier more limited solutions. The flexibility with respect to the momentum dependence allows for a study of different turbulence models. Here, three of such models, namely a slab turbulence with Alfvén waves, an isotropic turbulence with fast magnetosonic waves and one consisting of a mixture of slab Alfvén and isotropic fast magnetosonic waves are considered. After the determination of the transport parameters for these turbulence models, the analytical solution for the Parker propagator is applied to the problem of heliospheric modulation of anomalous as well as galactic cosmic rays.

Key words: acceleration of particles – diffusion – turbulence – waves – methods: analytical – ISM: cosmic rays

1. Introduction

An understanding of the solar modulation of cosmic rays (CRs) contributes to the solution of various problems in heliospheric physics and astrophysics. On the one hand, since modulation is tightly connected to the physics of the solar wind expansion and that of the turbulence being present in the wind plasma, it is used as valuable diagnostic of the large- and small-scale heliospheric structure, respectively. On the other hand, a successful and reliable so-called de-modulation of heliospheric CR spectra yields the unmodulated interstellar CR spectra.

The CRs of interest for solar modulation studies can be divided into two populations, namely Galactic and Anomalous Cosmic Rays (GCRs and ACRs). GCRs are accelerated somewhere in the Galaxy and arriving with their interstellar spectra at the heliosphere. According to the present believe based on ideas by Fisk et al. (1974) and Pesses et al. (1981), ACRs are locally accelerated at the solar wind termination shock, for a recent overview see Fisk et al. (1998). Both GCRs upon arrival at the outer boundary of the heliosphere and ACRs after local acceleration are entering the heliospheric region enclosed by the termination shock. Their transport to the inner heliosphere is mainly determined by spatial diffusion, convection with the solar wind background flow and adiabatic cooling.

It was Parker (1965) who first derived the fundamental CR transport equation, now known as the Parker equation, that takes into account all of these processes:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\kappa_{\rm rr}\frac{\partial F}{\partial r}\right) - V\frac{\partial F}{\partial r} + \frac{p}{3r^2}\frac{\partial}{\partial r}\left(r^2 V\right)\frac{\partial F}{\partial p} = -S(r,p) \qquad (1)$$

In this equation, written here for spherical symmetry already, $F(r,p)$ is the quasi-isotropic phase space distribution function of CRs with r and p denoting heliocentric distance and particle momentum, respectively. The solar wind speed is given by V and the coefficient of spatial diffusion by $\kappa_{\rm rr}$. On the right-hand-side, $S(r,p)$ indicates a source function.

Later, Jokipii et al. (1977) recognized the importance of large-scale drifts of CRs in the heliospheric magnetic field and supplemented Eq. (1) by an appropriate term. While being indispensable for a description of multi-dimensional large-scale modulation, it has been demonstrated that drifts effects do not have to be included in all cases to reproduce observations (see, e.g., Reinecke et al. 1993; le Roux & Fichtner 1997).

Already Parker (1965) has given a variety of solutions of Eq. (1) for simplified cases to explore the effects of spatial diffusion, convection and adiabatic energy loss. In the past 34 years many more analytical or semi-analytical solutions were presented in the context of solar modulation (see, e.g., Fisk & Axford 1969; Gleeson & Webb 1974; Cowsik & Lee 1977; Zhang 1999). Although these solutions describe the basic effects of solar modulation, they are not exact in a strict sense as they are employing various approximations and asymptotic expansions or assumptions about source functions.

Exact solutions for the Green's function of Eq. (1), i.e. for the Parker propagator, were presented by several authors in a different context, namely the acceleration of particles in accretion flows assuming a power law dependence of both the flow speed and the coefficient of spatial diffusion on the phase space coordinates:

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$$V(r) \propto r^{\alpha} \quad ; \qquad \kappa_{\rm rr}(r,p) \propto r^{\beta} p^{\gamma} \tag{2}$$

These previously published solutions for the Green's function required $\alpha > -2$ (see Schneider & Bogdan (1989) and Becker (1992), but note that they used a different sign of α) and can be classified using the auxiliary parameter

$$\eta = \frac{1+\alpha-\beta}{2+\alpha} \tag{3}$$

The solution derived by Schneider & Bogdan (1989) was valid for $\eta < 0$, i.e. $\beta - \alpha > 1$ (see their Eq. (2.6)). Noticing this, and generalizing to an arbitrary dependence of $\kappa_{\rm rr}(r,p)$ on momentum, Becker (1992) presented the solution for $\eta < 0$ and $\eta > 1$ (see his Eq. (A7) and his comments following it).

For the case of solar modulation (see the discussion in Sect. 4.1. below), we have indeed $\alpha \geq 0 > -2$, but $0 \lesssim \beta \lesssim 1$, i.e. $0 \lesssim \eta \lesssim 0.5$. This, however, means that the previously published solutions are not applicable to the case of solar modulation.

It is the purpose of this paper to present an exact solution for the Parker propagator of Parker's equation for the case of solar modulation. This analytical representation of the Parker propagator is then used to determine the modulated spectra of both ACRs and GCRs for three turbulence models. These models are formulated from the plasma wave viewpoint and embrace a slab turbulence with Alfvén waves, an isotropic turbulence with fast magnetosonic waves and a mixture of these two cases.

2. Turbulence models and the coefficient of spatial diffusion

Energetic charged particles like ACRs and GCRs can interact resonantly with plasma waves embedded in the solar wind. These waves are of low-frequency and mainly determined by their magnetic field component. Within the framework of quasilinear theory the coefficient of spatial diffusion along an external magnetic field is defined as the integral

$$\kappa_{\|} = \frac{v^2}{8} \int_{-1}^{+1} d\mu \frac{(1-\mu^2)^2}{D_{\mu\mu}} \tag{4}$$

where $\mu = p_{\|}/p$ and v are the pitch-angle and the particle speed, respectively. $D_{\mu\mu}$ denotes a Fokker-Planck coefficient which is determined by the composition and geometry of the plasma wave turbulence.

Here we consider three different turbulence models from the plasma wave viewpoint: slab Alfvén waves (A), isotropic fast magnetosonic waves and a mixture of both (M). Schlickeiser (1989) and Schlickeiser & Miller (1998) have calculated the coefficients of spatial diffusion for these three cases. In their calculations they assumed for the plasma wave spectrum $A(k)$ a Kolmogorov-like power-law dependence above some minimum wavenumber $k_{\min}$, i.e. $A(k) \sim (\delta B)^2\, k^{-q}\, H(k-k_{\min})$ with $q > 1$. Using their results with, for simplicity, (1) considering forward and backward propagating modes with equal intensities, (2) assuming identical spectral shapes, scales and intensities of Alfvén and fast mode waves, i.e. $q_{\rm A} = q_{\rm F} = q$, $k_{\rm min,A} = k_{\rm min,F} = k_{\min}$ and $\delta B_{\rm A}^2 = \delta B_{\rm F}^2 = \delta B^2$, and, furthermore, (3) using the empirical relationship $\kappa_{\perp} = w\kappa_{\|}$ ($w = const. \ll 1$) as well as (4) the approximation $\kappa_{\rm rr} \simeq \kappa_{\perp}$, one finds the following unified representation of the coefficients of spatial diffusion for the three turbulence models:

$$\kappa_{\rm rr}^{(i)} = \kappa_{\rm rr,0}^{(i)} \left(\frac{r}{r_{\rm E}}\right)^{\beta} \left(\frac{p}{p_{\rm R}}\right)^{\gamma^{(i)}} \tag{5}$$

where the superscript (i) refers to the different models labeled (A), (F) and (M) above. The reference value $\kappa_{\rm rr,0}^{(i)}$, which might be different for the three turbulence models, is taken at $r_{\rm E} = 1$AU and $p_{\rm R}$ which denotes an arbitrary reference momentum. The dimensionless exponents $\gamma^{(i)}$ are determined by the composition and geometry of the heliospheric plasma wave turbulence. These exponents follow from the considerations described in Appendix A and are given by $\gamma^{(A)} = 3 - q_{\rm A}$ for slab Alfvén waves, $\gamma^{(F)} = 2$ in the case of isotropic fast mode waves, and $\gamma^{(M)} = 1$ for the mixed turbulence.

3. The Parker propagator

After having established the relevant parameters of transport for the three turbulence models, we now turn to the derivation of the propagator of Parker's Eq. (1).

Using Eq. (5) and the general Ansatz $V(r) = V_0 r^{\alpha}$, Eq. (1) can be manipulated to obtain the form

$$y\frac{\partial^2 F}{\partial y^2} + \Big[b - y\Big]\frac{\partial F}{\partial y} + \frac{2+\alpha}{3\nu}\tau\frac{\partial F}{\partial \tau} = -\frac{r[y,\tau]}{V(r[y,\tau])\nu} S(r[y,\tau],\tau) \tag{6}$$

where we have introduced the new variable (see also Jokipii 1967) $\tau = p$ and the modulation parameter

$$y(r,p) := \frac{\nu}{(1+\alpha-\beta)^2}\frac{rV}{\kappa_{\rm rr}^{(i)}} \quad \text{with} \quad \beta - \alpha \neq 1.$$

Furthermore, we have used the abbreviations $b = 1/\eta = (2+\alpha)/(1+\alpha-\beta)$ and $\nu = (1+\alpha-\beta+\frac{2+\alpha}{3}\gamma)$.

The general solution for the phase space distribution function $F(r,p)$ can be expressed in terms of the Green's function $G(y, y_0, \tau, \tau_0)$, i.e. Parker's propagator:

$$F(y,\tau) = \int dy_0 \int d\tau_0 G(y,y_0,\tau,\tau_0) \times \left[\frac{r[y_0,\tau_0]}{V(r[y_0,\tau_0])\nu} S(r[y_0,\tau_0],\tau_0)\right] \tag{7}$$

The Laplace-transformed Green's function

$$g(y,y_0,s) \equiv \mathcal{L}[G(y,y_0,t=\tau/\tau_0)] = \int_0^{\infty} dt e^{-st} G(y,y_0,t) = \frac{\Gamma(s)}{\Gamma(b)} \begin{cases} M(s,b,y)U(s,b,y_0) & ,y \leq y_0 \\ U(s,b,y)M(s,b,y_0) & ,y \geq y_0 \end{cases} \tag{8}$$

with $\mathcal{L}$ being the Laplace-Transformation, s the corresponding variable, $\Gamma(z)$ the Gamma function, and M and U denoting Kummer's functions, satisfies the ordinary inhomogeneous differential equation

$$y\frac{d^2g}{dy^2} + [b-y]\frac{dg}{dy} - sg = -\frac{3\nu}{(2+\alpha)\tau_0}\delta(y-y_0) \quad (9)$$

resulting from Eq. (6). δ is the delta function. The homogeneous part is the confluent hypergeometric differential equation, also called Kummer's equation.

Following the standard method of solving such equations and constructing the Green's functions one can derive, by applying an inverse Laplace transformation $\mathcal{L}^{-1}$ (see formula 5.20.27 in Erdélyi et al. (1954), the exact solution for the differential intensity (Stawicki 1999):

$$\begin{aligned} j(r,p) &= p^2F(r,p) \\ &= \frac{3}{b}\int dr_0 \int dp_0 \frac{S(r_0,p_0)}{V(r_0)}\frac{p_0y_0}{f}\left(\frac{r_0}{r}\right)^{\frac{1+\beta}{2}} \\ &\times \left(\frac{p_0}{p}\right)^{\frac{3\beta-4\alpha-5}{2(2+\alpha)}} \exp\left(-\frac{y_0(1+h^2)}{f}\right) \\ &\times I_{\frac{1+\beta}{1+\alpha-\beta}}\left(\frac{2y_0h}{f}\right) \end{aligned} \quad (10)$$

Here, r_0 and p_0 are integration variables which result from the construction of the Laplace-transformed Green's function $g(y, y_0, s)$. To simplify the notation we have introduced the functions

$$f = f(p,p_0) = 1-\left(\frac{p}{p_0}\right)^{\frac{3\nu}{2+\alpha}}$$

$$h = h(r,r_0,p,p_0) = \left(\frac{r}{r_0}\right)^{\frac{1+\alpha-\beta}{2}}\left(\frac{p}{p_0}\right)^{\frac{3}{2b}}$$

as well as the modulation parameter $y_0 = y(r_0, p_0)$. The quantity $I_{\rm n}$ is a modified Bessel function of the first kind.

This solution is valid for arbitrary source functions $S(r, p)$ with which super-alfvénic charged particles of momentum p are injected at position r. Notice that, in contrast to the solutions presented previously, no assumptions were made with regard to Kummer's functions or the source function.

For later application we observe two asymptotic representations of the general solution. First, for $r \to 0$ one obtains with the corresponding asymptotic form of the Bessel function:

$$\begin{aligned} j(r\to 0,p) &\simeq \frac{3p^2}{b\Gamma(b)}\int dr_0 \int dp_0 \frac{S(r_0,p_0)}{p_0V(r_0)}\, y_0^b\, f(p,p_0)^{-b} \\ &\times \exp\left\{-\frac{y_0}{f(p,p_0)}\right\} \end{aligned} \quad (11)$$

This limiting value of the differential intensity is finite and shows, that the particle spectrum depends on the source function not only close to the source but also far away from it.

Similarly, one can approximate Eq. (10) for low momenta, i.e. $p \to 0$, and finds

$$\begin{aligned} j(r,p\to 0) &\simeq \frac{3p^2}{b\Gamma(b)}\int dr_0 \int dp_0 \frac{S(r_0,p_0)}{p_0V(r_0)} \\ &\times y_0^b \exp\{-y_0\} \end{aligned} \quad (12)$$

Thus, for the case of sufficiently small p, we get $j = p^2F(r,p) \propto p^2$. In other words, the phase space distribution function $F(r,p)$ becomes constant as p approaches zero.

4. Application to solar modulation

To demonstrate the potential and flexibility provided with the solution Eq. (10) we apply it to the problem of ACR and GCR proton modulation assuming for the turbulence the mixed model (M) with $\gamma^{(M)} = 1$. In order to do so we have to select physical parameters being typical for the heliosphere and specify boundary spectra.

4.1. *Turbulence in the heliosphere*

The theory of the diffusion tensor in the heliosphere is usually formulated in terms of the diffusion along ($\kappa_\parallel$) and perpendicular ($\kappa_\perp$ with respect to the local heliospheric magnetic field $\boldsymbol{B}$. While from quasilinear theory it is found that $\kappa_\parallel \propto (1/B)$ so that $\kappa_\parallel \propto r$ in the outer heliosphere, more recent studies, based on comprehensive magnetohydrodynamic turbulence models including turbulence generation by stream interactions as well as by pick-up ions, resulted in more complicated dependencies of the parallel diffusion coefficient on heliocentric distance (see, e.g., Burger & Hattingh 1998; Zank et al. 1998). Combining these new results with the assumption of long standing that the perpendicular diffusion is of the order of $\kappa_\perp \approx 0.01\kappa_\parallel$ that has been confirmed by recent numerical analyses (e.g., Giacalone & Jokipii 1999) one obtains a radial variation of the spatial diffusion $\kappa_{\rm rr} = \kappa_\perp sin^2\,\psi + \kappa_\parallel cos^2\,\psi \propto r^\beta$ with $\beta < 1$ (ψ denotes the angle between the average magnetic field $\boldsymbol{B}$ and the radial direction, so that $\psi \approx \pi/2$ beyond a few astronomical units from the Sun). Also, recent analyses of ACR and GCR data (Steenberg 1998; Moraal et al. 1999) resulted in the finding that $\beta \in [0, 0.5]$.

In view of this knowledge about the spatial diffusion in the heliosphere, we select $\kappa^{(A)}_{\rm rr,0} = 1.0 \cdot 10^{18}{\rm m}^2\,{\rm s}^{-1}$ and $\beta = 0.2$. Besides being a typical value in the interval suggested by observation and theory, this value has the additional advantage to result in the modified Bessel function $I_{3/2}$ in Eq. (10) when combined with a constant solar wind speed which we assume to be $V = const. = 400\,{\rm km\,s}^{-1}$, i.e. $\alpha = 0$ (Eq. (2) above).

4.2. *The boundary spectra of ACRs and GCRs*

The solution Eq. (10) is formulated in terms of a source function $S(r, p)$. So, the task is to determine source functions for ACRs and GCRs such that the correct boundary spectra $j_{\rm ACR}(r_{\rm sh}, p)$ and $j_{\rm GCR}(r_{\rm sh}, p)$ result.

4.2.1. ACRs

According to the theory of diffusive shock acceleration (for a review see, e.g., Drury 1983), for ACRs one expects the accelerated spectrum at the solar wind termination shock to be related to the scattering centers' compression ratio s (Vainio & Schlickeiser 1999), namely $j_{\rm ACR} \sim p^{2-\mu}$ with $\mu = 3s/(s-1)$. The spectrum cuts off in the range 0.1–$1.0\,GeV$ due to the finite radius of the termination shock.

Such a spectrum will result if the ACR source function is chosen as:

$$S_{\rm ACR}(r,p) \;=\; S_{\rm ACR,n}\,(p/p_n)^{-\mu} \times \exp\left(-p/p_{\rm c}\right)\delta(r-r_{\rm sh}) \qquad (13)$$

i.e. we use an exponential cut-off controlled with the constant momentum $p_{\rm c}$, have defined a normalisation momentum $p_{\rm n}$, and assume the solar wind termination shock to be located at a heliocentric distance $r = r_{\rm sh}$. The factor $S_{\rm ACR,n}$ allows for suitable normalisation.

4.2.2. GCRs

Many studies have been carried out to derive the interstellar proton spectrum (for a recent compilation see, e.g., Fig. 6 in Mori 1997). We selected the one obtained by Webber et al. (1987) who found $j_{\rm GCR} \sim (v/c)(E+0.5\,E_0)^{-2.6}$. E and E_0 denote the kinetic and the rest energy of a proton, and c is the speed of light.

In order to obtain such a spectrum from Eq. (10) the GCR source function has to be taken as

$$S_{\rm GCR}(r,p) \;=\; S_{\rm GCR,n}\,(v/p^2) \times \left[E+0.5\,E_0\right]^{\gamma^{(i)}-2.6}\delta(r-r_{\rm sh}) \qquad (14)$$

Analogously to the ACRs, $S_{\rm GCR,n}$ is a normalisation factor. The delta function indicates that we assume the solar wind termination to define the modulation barrier for GCRs. This barrier might actually be located farther out, however this is unimportant for the illustration below.

For the illustration we assume that the termination shock is located at a heliocentric distance of $r_{\rm sh} = 100\,{\rm AU}$ and has a compression ratio of $s = 2.5$, which appear as reasonable choices according to both observation (Stone et al. 1996) and theory (le Roux & Fichtner 1997).

4.3. Results

The resulting spectra, computed with a numerical integration of Eq. (10), are shown in Fig. 1. Inspection of the figure results, evidently, in the finding that all characteristic features of spherical CR modulation in the heliosphere are clearly visible. At low energies the spectra are dominated by the ACR contribution clearly showing the correct power-law behaviour. At high energies the spectra are dominated by GCRs and the amount of modulation decreases with increasing kinetic energy. The modulated spectra have the expected shape (Christian et al. 1995; le Roux et al. 1996; le Roux & Fichtner 1997). At small heliocentric distances, the spectrum still contains information about the source functions according to the asymptotic limit Eq. (11). Finally, at low kinetic energies the spectra exhibit the expected asymptotic form $j \propto p^2 \propto E$ according to Eq. (12), reflecting the dominance of adiabatic cooling in this energy range.

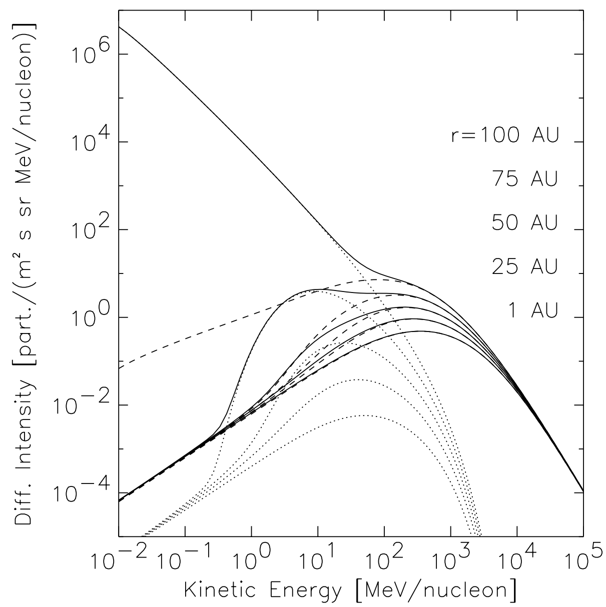

Fig. 1. The modulated spectra of ACRs and GCRs in the heliosphere. The solar wind termination shock marking the position of the sources is located at $r_{\rm sh} = 100\,{\rm AU}$. The solid lines are the combined spectra, the dotted and dashed lines indicate the individual contributions from ACRs and GCRs, respectively.

As a further illustration, we study the dependence of the modulation on the underlaying turbulence model. Fig. 2 displays the ACR spectra at a heliocentric distance of $50\,{\rm AU}$ for the three turbulence models introduced in Sect. 2. The importance of the chosen turbulence model at low energies is obvious. The ordering of the curves is a consequence of the choice $p_{\rm R} = 600\,m_{\rm p}\,V$ (with $m_{\rm p}$ being the proton rest mass). For $p < p_{\rm R}$ one has $(p/p_{\rm R}) < 1$ in Eq. (5), and a greater $\gamma^{(i)}$ yields a lower diffusion coefficient. Thus, the greater $\gamma^{(i)}$, the lower the differential intensities below $p_{\rm R}$. At high energies the flux levels are approaching each other due to the exponential cut-off.

5. Conclusions

In this paper we presented an exact solution for the Parker propagator of the fundamental transport equation of cosmic rays. It generalizes earlier solutions to the paramter range being characteristic for solar modulation. The solution is determined sensitively by the composition and topology of the turbulent wave fields the particle interact with and is valid for arbitrary source functions which are allowed to depend on position as well as momentum.

First illustrative examples for the solar modulation of the spectra of anomalous and galactic cosmic rays, demonstrates the potential and flexibility of the new (semi-) analytical approach.

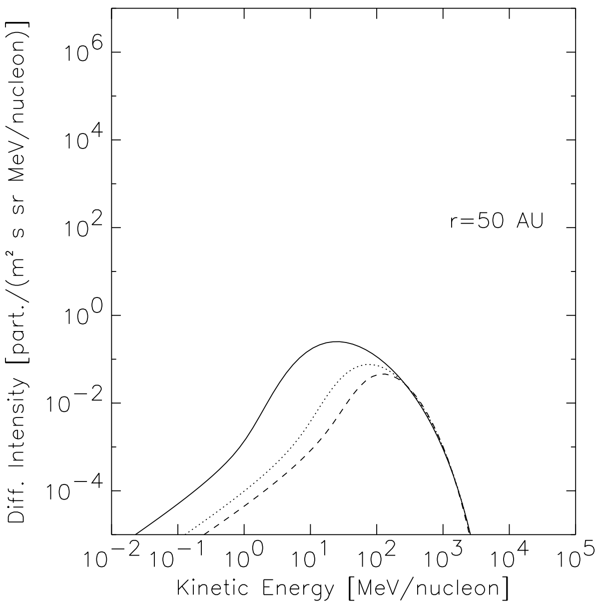

Fig. 2. The modulated spectra of ACRs at a heliocentric distance of $r = 50$ AU for the three turbulence models considered: slab Alfvén waves ($\gamma^{(A)} = 1.5$, dotted line), isotropic fast magnetosonic waves ($\gamma^{(F)} = 2.0$, dashed line) and the mixed case ($\gamma^{(M)} = 1.0$, solid line).

Besides an easy way to compare observations and theory, in particular with respect to the turbulence model determining the spatial diffusion of the cosmic rays, the new solution renders the opportunity to check on modelling results obtained purely numerically with computer codes.

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Appendix A: the spatial diffusion coefficients

The coefficient of parallel diffusion $\kappa_\parallel$ is for a mixture of Alfvén and fast magnetosonic waves given by:

$$\kappa_\parallel^M = \frac{v^2}{8}\int_{-1}^{1} d\mu \frac{(1-\mu^2)^2}{D_{\mu\mu}^A(\mu) + D_{\mu\mu}^F(\mu)} \quad (A.1)$$

Here the Fokker-Planck coefficient

$$D_{\mu\mu}^A(\mu) = \frac{\pi(q_\mathrm{A}-1)\Omega^{2-q_\mathrm{A}}}{4k_{\mathrm{min,A}}^{1-q_\mathrm{A}}}\left(\frac{\delta B_\mathrm{A}}{B_0}\right)^2 (1-\mu^2) \times \sum_{j=\pm1}(1-j\mu\frac{v_\mathrm{A}}{v})^2 |v\mu - jv_\mathrm{A}|^{q_\mathrm{A}-1} \quad (A.2)$$

describes the resonant cyclotron interactions of particles with slab Alfvén waves (see Schlickeiser (1989), Eq. (59a)), where $\Omega = \Omega_0/\gamma$ denotes the relativistic gyrofrequency and q_A, $k_{\mathrm{min,A}}$ as well as δB_A are the Alfvénic spectral index, minimum wavenumber and the plasma wave magnetic field component, respectively.

Schlickeiser & Miller (1998) calculated transport and acceleration parameters for cosmic ray particles interacting resonantly with undamped fast magnetosonic waves. Since fast mode waves are compressional the corresponding Fokker-Planck coefficient (compare Schlickeiser & Miller (1998), Eq. (27))

$$D_{\mu\mu}^F(\mu) = \frac{\pi(q_\mathrm{F}-1)\Omega^2}{4k_{\mathrm{min,F}}^{1-q_\mathrm{F}}}\left(\frac{\delta B_\mathrm{F}}{B_0}\right)^2 (1-\mu^2) \times \sum_{j=\pm1}\sum_{n=-\infty}^{+\infty}\int_{-1}^{+1} d\eta (1+\eta^2)\int_0^\infty dk k^{-q_\mathrm{F}} \times \delta(kv\mu\eta - jv_\mathrm{A}k + n\Omega)\left[J_\mathrm{n}'(\frac{k_\perp v_\perp}{\Omega})\right]^2 \quad (A.3)$$

consist of contributions caused by gyroresonant wave-particle interactions as well as transit-time damping. Here q_F, $k_{\mathrm{min,F}}$ and δB_F are the fast mode spectral index, the corresponding minimum wavenumber and the fast mode magnetic field component. The cosine of the angle between the propagation direction of the fast mode waves and the background magnetic field B_0 is denoted by η. Since $\epsilon = v_\mathrm{A}/v \le 1$ is the condition for particles in order to gain energy by the process of transit-time damping, the integral in Eq. A.1 can be split into two seperate integrals in order to obtain

$$\kappa_\parallel^M = \frac{v^2}{4}\left\{\int_0^\epsilon d\mu \frac{(1-\mu^2)^2}{D_{\mu\mu}^A(\mu) + D_{\mu\mu,\mathrm{G}}^F(\mu)} + \int_\epsilon^1 d\mu \frac{(1-\mu^2)^2}{D_{\mu\mu}^A(\mu) + D_{\mu\mu,\mathrm{T}}^F(\mu) + D_{\mu\mu,\mathrm{G}}^F(\mu)}\right\} \quad (A.4)$$

where we have used symmetry conditions of the Fokker-Planck coefficients with respect to the pitch-angle. $D_{\mu\mu,\mathrm{T}}^F(\mu)$ and $D_{\mu\mu,\mathrm{G}}^F(\mu)$ denote the contributions resulting from transit-time damping ($n = 0$) and gyroresonant interactions ($n \neq 0$).

It is obvious that, considering the limit $\delta B_\mathrm{F} \to 0$, Eq. (A.4) yields the Alfvénic coefficient of spatial diffusion (see Schlickeiser (1989), Eq. (74)). Because $\epsilon \ll 1$, we obtain the following expression:

$$\kappa_\parallel^A = \frac{k_{\mathrm{min,A}}^{1-q_\mathrm{A}}\Omega_0^{q_\mathrm{A}-2}}{\pi(q_\mathrm{A}-1)(2-q_\mathrm{A})(4-q_\mathrm{A})} \times \left(\frac{B_0}{\delta B_\mathrm{A}}\right)^2 \gamma^{2-q_\mathrm{A}} v^{3-q_\mathrm{A}} \quad (A.5)$$

Using $(B_0/\delta B_\mathrm{A})^2 \propto r^\beta$ and considering low-energetic particles, we get, disregarding constants, the dependence on radius r and momentum p

$$\kappa_\parallel^A \propto r^\beta p^{3-q_\mathrm{A}} \quad (A.6)$$

On the other hand, the case of a vanishing intensity of Alfvén waves leads to the spatial diffusion coefficient for particles propagating in a fast magnetosonic turbulence (see also Schlickeiser & Miller (1998), Eq. (100))

$$\kappa_\parallel^F = \frac{2(v_\mathrm{A}k_\mathrm{min,F})^{1-q_\mathrm{F}}\Omega_0^{q_\mathrm{F}-2}}{3\pi(q_\mathrm{F}-1)\zeta(1+q_\mathrm{F})}\left(\frac{B_0}{\delta B_\mathrm{F}}\right)^2\gamma^{2-q_\mathrm{F}}v^2 \tag{A.7}$$

where $\zeta(z)$ is Riemann's zeta function. With the radial dependence of the magnetic field ratio mentioned above, one obtains from Eq. (A.7)

$$\kappa_\parallel^F \propto r^\beta p^2 \tag{A.8}$$

From a mathematical point of view, the treatment of the mixed turbulence, i.e. $\delta B_\mathrm{A} \neq 0$ as well as $\delta B_\mathrm{F} \neq 0$, is more difficult than the pure Alfvénic and magnetosonic case. Following the calculations by Schlickeiser & Miller (1998) one finds, apart from a factor 2, after straightforward algebra the formula

$$\kappa_\parallel^M = \frac{(v_\mathrm{A}k_\mathrm{min,A})^{1-q_\mathrm{A}}\Omega_0^{q_\mathrm{A}-2}}{2\pi(q_\mathrm{A}-1)}\left(\frac{B_0}{\delta B_\mathrm{A}}\right)^2 \times\frac{\gamma^{2-q_\mathrm{F}}v^2}{\frac{v}{v_\mathrm{A}}\gamma^{q_\mathrm{A}-q_\mathrm{F}}+\xi} \tag{A.9}$$

where we have introduced the abbreviation

$$\xi = \frac{3}{4}\zeta(1+q_\mathrm{F})\left(\frac{q_\mathrm{F}-1}{q_\mathrm{A}-1}\right)\frac{k_\mathrm{min,A}^{1-q_\mathrm{A}}}{k_\mathrm{min,F}^{1-q_\mathrm{F}}}\left(\frac{v_\mathrm{A}}{\Omega_0}\right)^{q_\mathrm{F}-q_\mathrm{A}} \times\left(\frac{\delta B_\mathrm{F}}{\delta B_\mathrm{A}}\right)^2 \tag{A.10}$$

Considering, for simplicity, the case that Alfvén and fast magnetosonic waves have equal intensities, i.e. $\delta B_\mathrm{A}^2 = \delta B_\mathrm{F}^2 = \delta B^2$, identical spectral indices and scales, that means $q_\mathrm{A} = q_\mathrm{F} = q$ and $k_\mathrm{min,A} = k_\mathrm{min,F} = k_\mathrm{min}$, one obtains the expression

$$\kappa_\parallel^M = \frac{(v_\mathrm{A}k_\mathrm{min})^{1-q}\Omega_0^{q-2}}{2\pi(q-1)}\left(\frac{B_0}{\delta B}\right)^2\frac{\gamma^{2-q}v^2}{\frac{v}{v_\mathrm{A}}+\xi} \tag{A.11}$$

with the correspondingly simplified abbreviation ξ. Since $v/v_\mathrm{A} \gg \xi$, one derives again dependences in r and p of the following form:

$$\kappa_\parallel^M \propto r^\beta p^1 \tag{A.12}$$

Eqs. (A6), (A8) and (A12) give the values $\gamma^{(A)} = 3 - q_\mathrm{A}$, $\gamma^{(F)} = 2$ and $\gamma^{(M)} = 1$ used in Sect. 2 above.

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