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ASTRONOMY AND ASTROPHYSICS

Studies on stellar rotation

II. Gravity-darkening: the effects of the input physics and differential rotation. New results for very low mass stars

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Abstract. In the second paper of this series we present the calculations of the gravity-darkening exponent for low mass stars. Such calculations are based on a modified version of the method developed by us and published in an earlier paper. The corresponding apsidal motion constants for VLMS are also presented. The mass range for which both parameters are available is now extended from 0.08 up to 40 $M_\odot$.

The derived β_1 is compared with the results of the radiation hydrodynamics simulations. Although a systematic difference with decreasing effective temperature is detected, the interagreement can be considered as good (maximum discrepancy around 10%). The local maximum for log $T_{eff} \approx 3.7$ is not detected in the RHD formalism, although calculations for lower effective temperatures are needed to draw a more definitive conclusion.

It is pointed out that the gravity-darkening phenomenon is related not only with atmospheric parameters but also with the internal stellar structure and with details of the rotation law. The influence of changing the input physics on the gravity-darkening exponent is investigated and it is found that they depend slightly on the chemical composition mainly in the zone of the radiative/convective phase transition. For deep convective envelopes, it is found no significant differences in β_1's computed for different mixing-length parameter.

It is emphasized that at the actual level of light curve quality of eclipsing binaries it is not possible to discriminate effects of third order in the gravity-darkening exponents such as chemical composition, theory of convection, irradiation, etc.

On the other hand, the role of the differential rotation is discussed in the light of its possible relation with moderately anomalous gravity-darkening exponents for some particular cases. Concerning the high values of the gravity-darkening exponent reported by some authors, this is not supported by very recent simultaneous $uvby$ observations, at least in the case of VV Uma.

Key words: stars: binaries: eclipsing – stars: evolution – stars: fundamental parameters – stars: interiors – stars: rotation

1. Introduction

The light we observe from the components of a binary system depends not only on the geometrical configuration but also on how the specific intensities are distributed along the stellar disk. Often, such components are distorted by rotation and tides and the flux distribution is not uniform over their surfaces. In a pioneering paper von Zeipel 1924 shows that the flux distribution over the surface of a pseudo-barotrope depends on the effective gravity, that is, $T^4_{eff} \propto g^{\beta_1}$, where β_1=1. This means that in the distorted stars the polar zones are hotter than the equatorial ones. This phenomenon is known as gravity-darkening though some authors prefer to call it gravity-brightening. While the limb-darkening is symmetric with respect to longitude, gravity-darkening is symmetric with respect to the equator. The result by von Zeipel is valid for configurations in radiative equilibrium. Some decades later Kippenhahn 1977 and Maeder 1999 derived a generalization of such a relationship. Kopal 1959 computed the gravity-darkening coefficient - do not confuse with β_1 - as a function of the effective temperature and wavelength. Barman 1991 extended these results up to fourth order of accuracy.

For stars presenting convective envelopes Lucy (1967), through a numerical procedure, demonstrated that the gravity-darkening exponent - GDE - is smaller than 1. His method is based on the calculation of the adiabatic constant for a series of stellar envelopes. As an average value, he found β_1=0.32. Webbink 1976, assuming a simple opacity law, pointed out that if the entropy is constant in an envelope the values of β_1 are compatible with those derived by Lucy. On the other hand, Anderson & Shu 1977 found that there is no relation between the effective temperature and effective gravity. However, the method they used predicted null albedo, which is in disagreement with the observational data (Rafert & Twigg 1980).

Applying a perturbation theory Smith & Worley 1974 obtained β_1 = 0.5 for conservative rotation laws and radiative envelopes. For non-conservative rotation the corresponding relationship between flux and effective gravity was found to be a more complex function. Using the same technique, Smith 1975 analyzed the case of magnetic stars and found a strong relationship between the mentioned parameters. Sarna 1989 confirmed the results obtained by Lucy although a slight variation with mass was also detected.

Under the observational point of view, only a few papers were dedicated to infer the GDE from the light curves analysis. Eaton et al. 1980 analyzed the light curves of three systems, namely, CC Com, RT Lac and W Uma and recommended a weighted value of 0.22. A more comprehensive sample was studied by Rafert & Twigg 1980. These authors analyzed systems with radiative and convective envelopes. The corresponding mean values of β_1 were 0.96 and 0.31, in good agreement with theoretical predictions. Kitamura & Nakamura 1987 derived very anomalous GDE's for the secondaries of semi-detached systems. Their sample was also constituted by stars with convective and radiative envelopes. They have found a high dependence of the surface brightness with the gravity. The inferred values are between 2.3 and 9.4 which is in disagreement with the classical theoretical values. These results were interpreted later by Unno et al. 1994 as an enthalpy transport linked with the mass exchange process in semi-detached systems.

The application of the GDE is not limited to the binary stars investigations. In order to take into account the effects of rotation on the emergent flux of isolated stars, the knowledge of β_1 is of capital importance. Several works were dedicated to the subject (Collins 1963, 1965, 1966; Hardorp & Strittmatter 1968, Maeder & Peytremann 1970, Collins & Smith 1985). The range of the applicability of such calculations is unfortunately limited to a few models. The work by Pérez Hernández et al. (1999) changes the situation since they compute the effects of rotation on isochrones considering modern interior and atmospheric models and adequate values of gravity-darkening exponent as computed by Claret 1998a (Paper I). The results were presented in terms of the rotating and non-rotating counterparts. An extension of this work is under preparation (Pérez Hernández & Claret 2000).

Returning to theoretical computation of the GDE's itself, Claret 1998a, 1999b introduced a new procedure based on the triangles strategy usually employed in stellar evolution calculations. Such a technique allows for more extensive calculations than the previous ones since β_1 is computed for each point of a given evolutionary track no matter if the envelope is in radiative or convective equilibrium. In other words, the GDE's are given, for the first time, as a function of time, mass and effective temperature. These calculations cover the mass range from 1 up to 40 $M_\odot$.

Very recent investigations on radiation hydrodynamical calculations (Ludwig et al. 1999) indicate that the values of β_1 for convective envelopes is around 0.28–0.40 confirming, within uncertainties, the results based on the mixing-length theory of convection for late-type stars.

In the first paper of this series it was investigated the role of rotation on the apsidal motion constants (Claret 1999a). In the present work, we basically explore three points: a) the influence of the physics input on the GDE's b) the calculation of the gravity-darkening exponent and apsidal motion constants for models with masses down to 0.08 $M_\odot$ c) an analysis of the influence of the differential rotation on the GDE's an its possible relation with moderately anomalous β_1's. The new calculations of the apsidal motion constants and the corresponding moment of inertia for the VLMS are very important in order to test the tidal evolution theory in this mass range. Furthermore, the new computations of the corresponding GDE's can be used in order to analyze the light curves of eclipsing binaries presenting VLMS components as well as to take into account the effects of rotation in brown dwarfs.

2. An analytical approximation for β_1 in convective envelopes

Using a graphical approach we have shown in Paper I that a rough estimation of the GDE can be directly obtained from the shape of the theoretical HR diagram. Fitting the tracks for 1 and 10 $M_\odot$ by straight lines, we derived β_1=0.24 and 1.0 in good agreement with the old usual values for stars with convective and radiative envelopes respectively (Lucy 1967, von Zeipel 1924; see also Fig. 3 in Paper I). In this section we shall present an analytical approximation for β_1 for stars which present a significant amount of convective flux in their outermost layers. In order to do that, let us write the average convective flux as

$$F_c = \psi \rho v_s c_P T \tag{1}$$

where

$$\psi = \frac{1}{2}\frac{\overline{v}\Delta T}{v_s T} \tag{2}$$

ρ is the density, T the temperature, v_s the local velocity of sound, $\overline{v}$ the mean convective velocity and ΔT is the excess of temperature of a convective cell with respect to its neighboring and c_P is the specific heat at constant pressure. At a given fractional radius b the total flux is given by $b^2\sigma T^4_{eff}$. Defining the convective efficiency f as the ratio of the convective to total flux (Cox & Giuli 1968), one gets

$$f \equiv \frac{\psi c_P \rho T v_s}{b^2 \sigma T^4_{eff}} \tag{3}$$

The convective efficiency is hard to handle since the functions which appear in Eq. (3) are not so easy to treat analytically if the structure of the envelope changes in space and time. However, it is useful to analyze where and when convection is important in the transfer of energy and consequently, how the GDE depends on it. Taking typical values for c_P, ψ, μ (mean molecular weight) and Γ_1 (the ratio of specific heats),

$$f = \frac{A\psi P T^{1/2}}{b^2 T^4_{eff}} \tag{4}$$

where A is a constant characterizing a given layer. It can be shown that the pressure $P(T)$ is proportional to $g^{1/2}T^{-9/2}_{eff}$ and considering the loci of constant convective efficiency in the HR diagram, we have

$$T_{eff} \propto g^{1/17} \tag{5}$$

which would give a mean value of β_1 of 0.24, in good agreement with the estimation made by Lucy 1967 and Claret 1998a for stars with deep convective envelopes. This value depends, for example, on the relationship between the opacity, the pressure and temperature.

3. The connection gravity-darkening with the internal stellar structure

Before Paper I, the computational method used in the papers dedicated to evaluate the GDE and, in particular for convective envelopes, is the same as that developed by Lucy (1967) who introduced atmospheric models. However, the distortions caused by rotation, which are the real responsible ones for the non uniform flux distribution, depend on the internal structure of the stars (Kopal 1959). Starting with the total potential of a rotating star and assuming that the angle between the normal and the radius vector is small - only the radial component of the derivative is considered to compute the local gravity - the corresponding equation which describes how much a configuration is distorted by rotation can be written as

$$\frac{g - g_o}{g_o} = \sum_j \left(1 - \frac{5}{\Delta_j}\right)\left(\frac{r}{a} - 1\right) \tag{6}$$

where g_o is the local gravity taken as reference, $\Delta_j = 1 + 2k_j$, k_j being the apsidal motion constant of order j. If we take only the terms for j = 2, the radius of an equipotential r can be written as

$$r = a\,(1 - f_2 P_2(\theta, \phi)) \tag{7}$$

with

$$f_2 = \frac{5\omega^2 a^3}{3GM_\psi(2 + \eta_2)} \tag{8}$$

where ω is the angular velocity, $P_2(\theta, \phi)$ is the second surface harmonic, a the mean radius of the level surface, η_2 is the logarithmic derivative of the spherical harmonic defined through Radau's equation and M_ψ is the mass enclosed by an equipotential.

This equation indicates that gravity-darkening is not only an atmospheric phenomenon since in order to compute it it is necessary to know what is the shape of the distorted stellar configuration, which depends on its internal structure. Moreover, the calculations of the GDE's also depend on the details of the rotation law (see Sect. 6). Anticipating the results which will be presented in Sect. 5, it is interesting to note in Fig. 1 the similarity between the behavior of β_1 and k_2. The effective temperature of the transition zone (convective/radiative) coincides with that for which k_2 is also minimum. The corresponding log T_{eff} is around 3.86 and m≈1.5 $M_\odot$ for homogeneous models. In this range there is a change of the predominant source of thermonuclear energy from proton-proton chain to CNO cycle. This change is accompanied by a readjustment of the mass concentration, this fact having clear repercussion on how a star would react under distortions and consequently on the parameter β_1 (remember that the distortion depends on k_2). But this zone is not only important due to this fact. Besides the mentioned

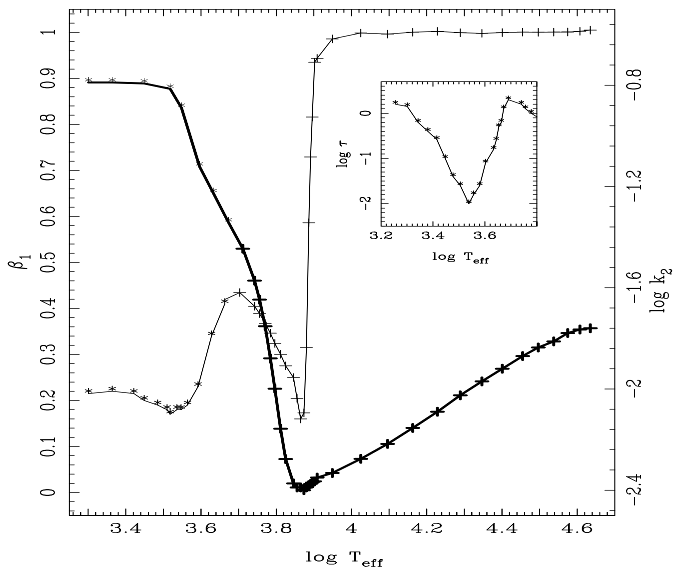

Fig. 1. Gravity-darkening exponent (thin line) and apsidal motion constant (thick line) for homogeneous models. The asterisks indicate computations performed in this paper while crosses denote calculations performed in Paper I. The picture in the upper right corner shows the onset of convection as a function of the effective temperature.

change in internal structure, convection begins to become important for effective temperatures smaller than that of the critical point. In this way, this phase transition does not only depend on the internal constitution but also on the structure of the outermost layers. Both mechanisms may be acting simultaneously and the individual contribution depends on the effective temperature range. For very low mass stars the internal structure has minor importance and the convective efficiency drives the behavior of β_1 (Sect. 5).

The first term between parenthesis of the right hand side in Eq. (6) can be computed integrating the Radau equation for a given configuration. As mentioned, Eq. (6) describes how the harmonic distortion is related with the relative variation of the surface gravity but it can be used to estimate and to check limits on the numerical values of $\Delta log\, g$ used in the method introduced by Lucy 1967 and its variants and the new one proposed in Paper I. Let us evaluate the parameters which appear in that equation: a typical value of the apsidal motion k_2 is 0.01, $|P_2(\cos\theta)| \approx 1$ and $f_2 \approx \lambda/2$, where $\lambda = \frac{2}{3}\omega^2 R^3/(GM)$. The maximum value of λ is 0.3 which corresponds to the break up velocity. In this case, we have $\Delta log\, g \approx 0.6$. As indicated by Lucy, the basic equation used in his numerical method is

$$\beta_1 = -\frac{\Delta logK}{\Delta log\, g} / \frac{\Delta logK}{\Delta logT_{eff}} \tag{9}$$

where $K \equiv K(g, T_{eff})$ represents the equation of an adiabatic. Remember that Eq. (9) was derived *assuming that the variations of the local gravity are small*. This is an important aspect of the problem. Some authors who used the Lucy method (Alencar & Vaz 1997, for example) used values of $\Delta log\, g$ of the order of 1.5 which is around 3 times larger than the maximum value obtained in the case of *break up velocity*. If one takes the models displayed in their Fig. 1 it can be concluded that equatorial radius is around 8 times larger than the polar one. That extreme situation obviously does not meet the requirements mentioned above and those calculations were carried out inconsistently. In Paper I and in the present work we have adopted $|\Delta log\, g|_{max}$=0.08.

4. The influence of the physics input on the gravity-darkening exponent

Before directly explorating the influence of the input physics on β_1, let us summarize the main aspects of the numerical method to compute GDE developed in Paper I. This method, on the contrary to the other ones, makes use of a stellar evolution code. If the external boundary conditions are unchanged at the fitting point as a model evolves, the corresponding outer layer integrations will be the same as the previous ones. Three envelope computations are performed corresponding to three points in the HR diagram and while a model evolves and remains within this triangle the boundary conditions are unchanged. This is valid only if the triangle is sufficiently small, mainly for convective envelopes. This is the triangle strategy introduced by Kippenhahn et al. (1967) to save computational time. To compute the GDE one is interested in a distorted model with different effective temperature distributions over the surface. To represent such a model a modified version of the triangle strategy was used (increasing the number of points) in such way that several envelopes with different temperature distribution are computed but with the same physical conditions at an interior point where the ionization of hydrogen and helium is complete. Finally, by differentiating numerically the neighboring envelopes, we obtain β_1. A sketch of the method is shown in Fig. 2 where we have plotted only three points for sake of clarity. Figs. 3 and 4 summarize the numerical results and illustrate the time evolution of GDE with mass and effective temperature for models with masses from 1 up tp 40 M$_\odot$. Note the transition zone in the gravity-darkening exponent for masses between 1.58 and 1.78 M$_\odot$ and for log $T_{eff} \approx$ 3.86. The position, form and depth of this phase transition depends on the chemical composition through the opacities as well as on computational details. We expect to investigate such aspects in future papers.

This new method presents several advantages with respect to the old ones:

1. it is self consistent since the interior models are used to derive β_1 as well as to predicted limb-darkening coefficients, radius, effective temperature, apsidal motion constants, circularization and synchronization times
2. the method can be applied for convective or radiative envelopes
3. one can go as deep as necessary into the interior of the model to impose the boundary conditions without loss of generality.
4. the GDE's are given, for the first time, as a function of the mass, chemical composition and time. This permits an self-consistent light curves analysis (see item 1)
5. changes in the chemical profile of the envelope as, for example, those due to mixing, are taken into account

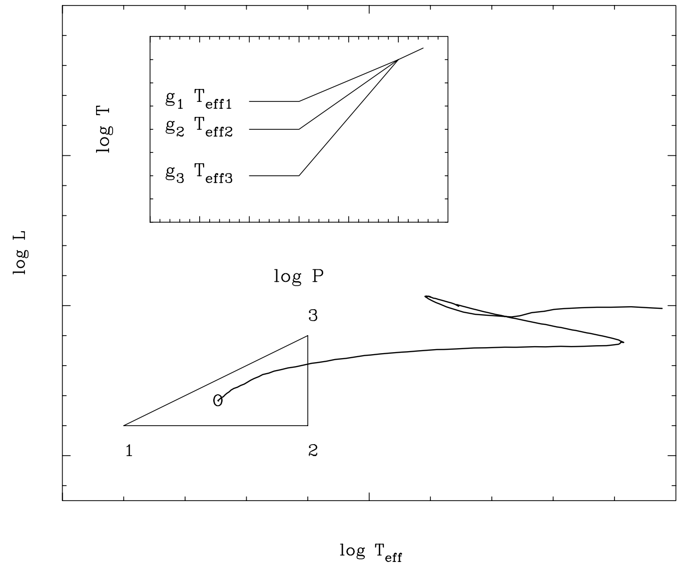

Fig. 2. The thick line denotes a generic evolutionary model. The point marked with the symbol O has the same P×T relationship at a given depth as the points labeled by 1, 2 and 3. Such points represent the distorted models (see the upper left corner).

6. it does not present possible effects of sphericity given the spherical symmetry of the structure equations
7. it can be easily adapted to stellar evolution codes
8. if needed, more realistic atmosphere models can be introduced as external boundary condition, as we have done for brown-dwarfs stars. This possibility embraces the method introduced by Lucy and its variants.

In order to investigate the effects of changing of chemical composition, we have computed β_1 for homogeneous models for different initial chemical compositions - same metallicity - using the method summarized above. Although the hydrogen content was changed by 0.1 with respect to the chemical composition (X, Z)=(0.70, 0.02) the values of the respective β_1 are not so much sensitive to these changes except near the transition zone. Concerning evolved models the situation is a little more conspicuous. Let us take as standard the 2 M$_\odot$ model with (X, Z)=(0.70, 0.02) in order to study how the changes in the hydrogen content affects β_1. This model was selected due to that during its evolution the envelope is first predominantly radiative, then convective, again radiative and finally convective in the giant phase. In Fig. 5 we compare the resulting GDE's for this model during its evolution with those for a 2 M$_\odot$ with (X, Z)=(0.60, 0.02) and (0.80, 0.02). For a higher hydrogen content, the corresponding mean molecular weight is smaller and the effective temperature is also somewhat smaller. In this way, the X60Z02 models are brighter and hotter than the models X70Z02 and X80Z02. In consequence, X60Z02 models present envelopes in radiative equilibrium even in the red-blue hook stages. The values of the GDE's for these points are almost constant and the peak in β_1 around log T$_{eff}$=3.86 is not detected.

For X80Z02 models the scenario is opposite. Even for slightly evolved models the effective temperatures are smaller

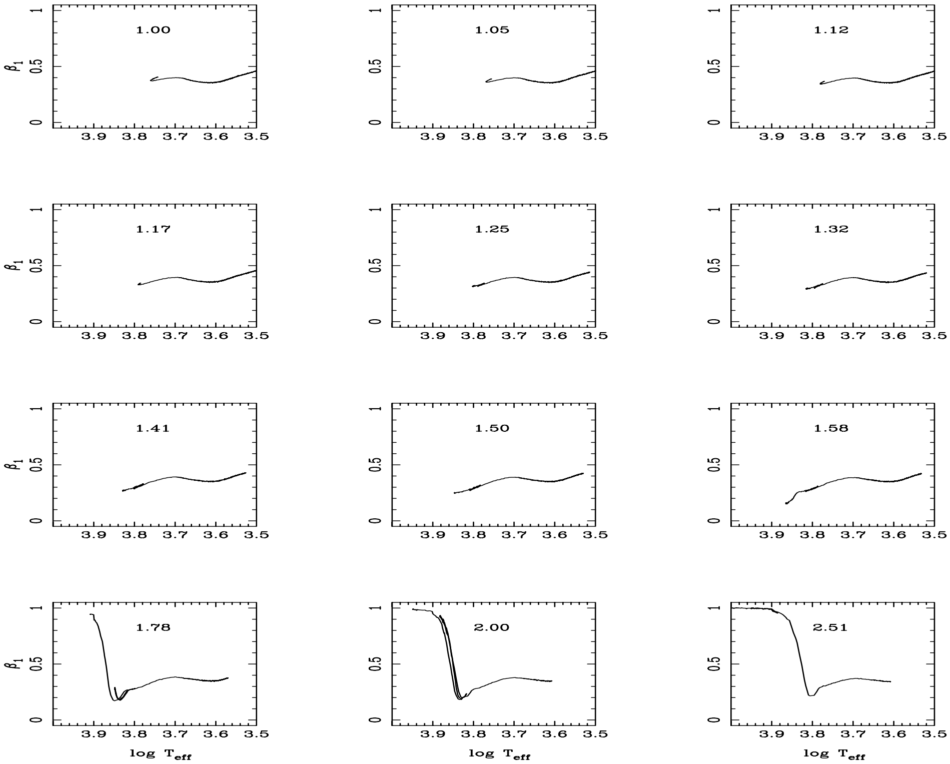

Fig. 3. Gravity-darkening exponent time evolution as a function of the mass and effective temperature for models with masses between 1 and 2.51 $M_\odot$

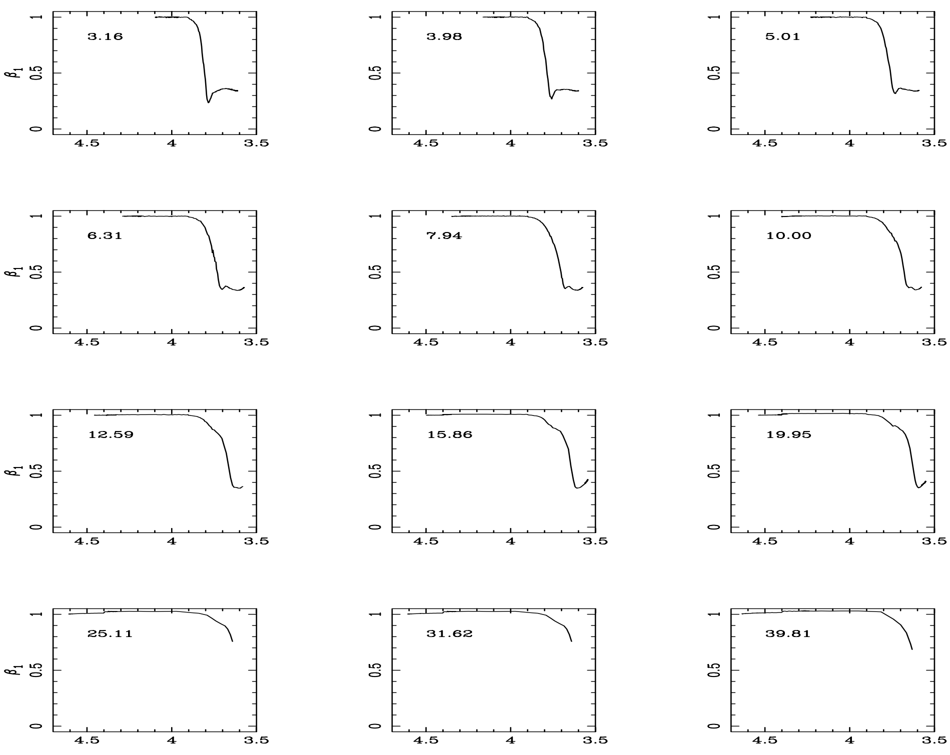

Fig. 4. The same as in Fig. 3 for models with masses between 3.16 and 40 $M_\odot$

than those of the standard model and there is some contribution of convection to the total flux. Convection begins to predominate even in the red-blue hook points and consequently these models always show GDE's smaller than 1.0. As models evolve out the Main Sequence, the convective envelope becomes deeper and the GDE's tend asymptotically to a common value. Note that the minimum value of β_1 in red-blue hook for the three models depends on the hydrogen content.

Now we shall examine how the metallicity affects β_1 for a given mass and hydrogen content. Fig. 6 is somewhat similar

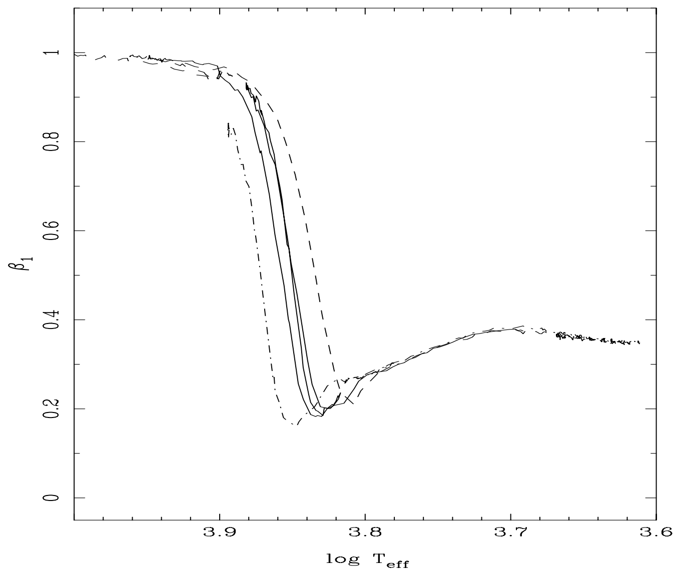

Fig. 5. Gravity-darkening exponent time evolution for a 2 $M_\odot$ model. Continuous line denotes (X, Y) = (0.70, 0.02), dashed one indicates (0.60, 0.02) while dotted-dashed line represents (0.80, 0.02) models.

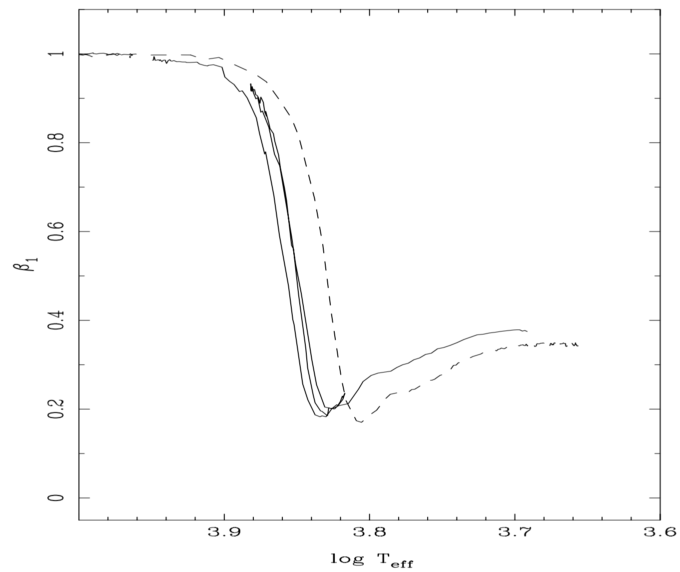

Fig. 6. Gravity-darkening exponent time evolution for a 2 $M_\odot$ model. Continuous line denotes (X, Y) = (0.70, 0.02), and dashed one indicates (0.70, 0.004) models.

to Fig. 5 in the sense that the hotter models (less metallic ones) "mimic" the properties of X60Z02 case. However, there is a remarkable difference: for deep convective envelopes the GDE's for X70Z004 models are systematically smaller than those computed for X70Z02 models. The influence of the physical conditions in the envelopes during this phase, and in particular, the influence of the opacity, is determinant to explain such a difference. The position of the transition phase also depends on the metallicity as it can be seen inspecting Fig. 6.

The influence of the mixing-length parameter α on the GDE's is displayed in Fig. 7. Again, we have used the 2 $M_\odot$ model with (X, Z) = (0.70, 0.02), $\alpha = 1.52$ as standard. The evolutionary tracks are similar from the ZAMS up to the beginning of the hydrogen-shell burning (log $T_{eff} \approx 3.8$) for the three models since convection has no larger influence in this range. The points where the phase transition is detected depends on α but for deep convective envelopes there is no significant difference among the β_1's.

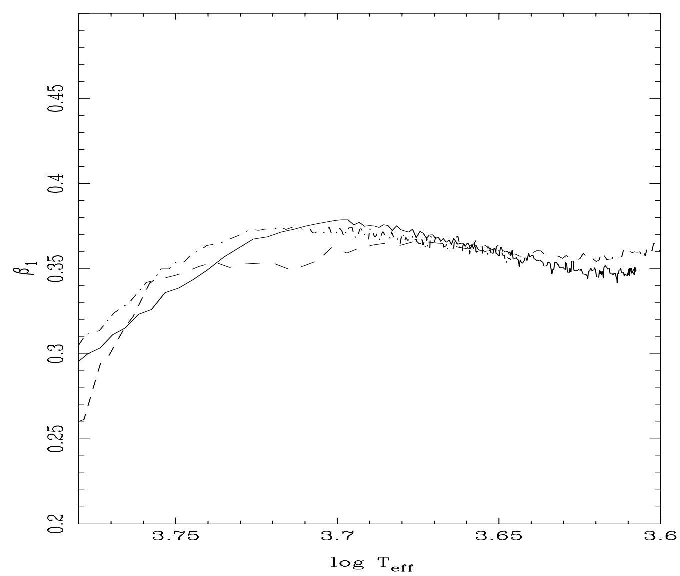

Fig. 7. Influence of the mixing-length parameter α in the gravity-darkening exponent time evolution for a 2 $M_\odot$ model with (X, Y) = (0.70, 0.02). Continuous line represents models with α=1.52, dashed one models with α=1.0 while dotted-dashed denotes models with α=2.0. Only models with convective envelopes are shown.

5. The interior-atmosphere coupling for very low mass stars: apsidal motion constants and GDE's

In Paper I, we limit the calculations of β_1 down to mass equal to 1.0 $M_\odot$. The main reason for such a restriction is based on the fact that very low mass stars (VLMS) present cool and sometimes very dense configurations. Under these conditions, the stellar models must be computed using more complex input physics. The implementations one should introduce in the codes of stellar evolution to obtain more realistic models of VLMS are (see Baraffe 1999a):

1. a more elaborate equation of state (Christensen-Dalsgaard & Däppen 1992, Saumon et al. 1995) since the ideal equation of state is not appropriate to describe the internal structure of VLMS (Chabrier & Baraffe 1997).
2. coupling of atmospheres models with the interior as a boundary condition. In the range of effective temperatures, which characterizes the VLMS, the presence of several stable molecules, provides an additional and important source of opacity. Within this perspective, the grey approximation as exterior boundary condition produces hotter models, for a given mass and chemical composition, than in the case of

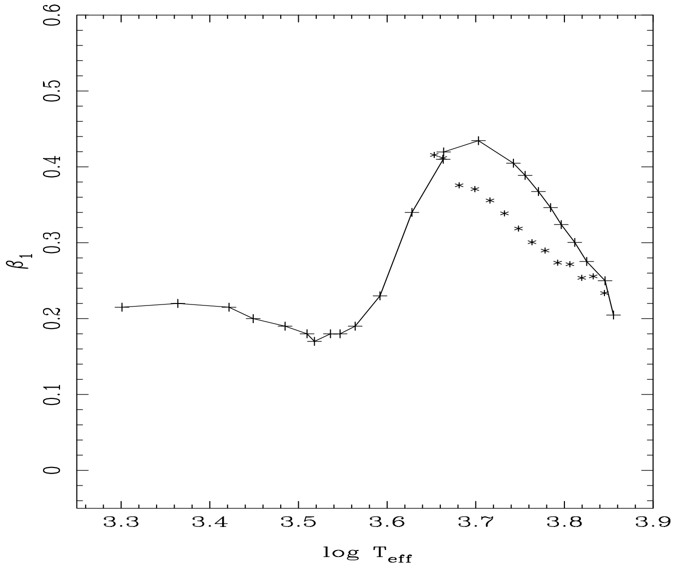

Fig. 8. The gravity-darkening exponent using our method adopting the mixing-length theory of convection (continuous line, crosses) and those derived from the radiation hydrodynamics simulations (asterisks).

more realistic atmospheric models in the range of 0.2 up to 0.9 $M_\odot$.

3. the presence of convection in the optically thin layers since the molecular hydrogen recombination reduces the adiabatic gradient favoring the convective flux.

Important improvements were implemented in the codes of stellar atmospheres and in particular, in the PHOENIX code (see Allard et al. 1997 for a review). The influence of the input physics in the atmosphere models also affects other areas of interest for binary stars researchers as, for example, the limb-darkening calculations. Claret 1998b detected a small but systematic difference between the coefficients of limb-darkening computed using the models of atmosphere generated by the ATLAS code (Kurucz 1993) with the NextGen PHOENIX calculations. These effects are particularly conspicuous for models with log $T_{eff} \leq$ 3.7. Such discrepancies may be explained due to the different adopted equation of state and opacity tables in both codes.

In order to extend our calculations to VLMS we shall adopt the interior models by Baraffe et al. 1998 which are consistent with the atmospheric models generated by the code PHOENIX. These models incorporate the necessary improvements pointed out previously. The corresponding atmosphere models were kindly provided by Hauschildt (1998, 1999). The grid of atmosphere models covers the range of effective temperatures 2000 K $\geq T_{eff} \leq$ 9800 K with log g varying between 3.5 and 5.5. The dust opacities were not included.

Some of these VLMS models were used to extend the mass range of applicability of the apsidal motion and tidal interactions investigation. This parameter is presented for the first time for realistic VLMS models. The calculations were performed for homogeneous models with mass 0.08, 0.1, 0.2, and 0.4 $M_\odot$ which were kindly provided by Baraffe (1999b). As we have done in our previous papers on theoretical apsidal motion, we used the Runge-Kutta method of fourth order to integrate the respective differential equation.

In Fig. 1 we plot the log k_2 versus effective temperatures for such models. The new computations are shown as asterisks while the computations for more massive models are those based on the models by Claret 1995. From that figure we can note that the k_2's for VLMS are only slightly dependent on mass and they are almost constant for log m $\leq$ -0.4 or log $T_{eff} \leq$ 3.5. For this mass range the value of k_2 is about 0.15 which would correspond approximately to one configuration with a polytropic index n=1.5, as expected.

We have also included in Fig. 1 the calculations of β_1 for those VLMS models. We restrict our analysis on the homogeneous models for sake of clarity. As we have seen in Sect. 2, the internal structure and the configuration of the envelopes drive β_1. But when going to lower effective temperature the outermost layers influence begins to predominate. In the upper right corner we have plotted the optical depth of the convection onset as a function of log T_{eff}. The hydrogen recombination is the main responsible for this feature since convection is also present in thin layers. There is a local maximum for log $T_{eff} \approx$ 3.7. For effective temperatures lower than this point there is a decreasing of the onset optical depth indicating an increasing of the convective efficiency. As a consequence β_1 decreases until log $T_{eff} \approx$ 3.54. For log $T_{eff} <$ 3.54 the situation is inverted and β_1 increases a little but remains within the range 0.20–0.25.

For the range 3.7 $<$ log $T_{eff} <$ 3.86, convection loses importance progressively and the corresponding β_1 also decreases. This is due to that for these envelopes the entropy jumps are increasing and therefore the convective efficiency is decreasing. Another way to interpret this decreasing of β_1 is link it with RHD - radiation hydrodynamics simulations - calculations. Translating such results to the mixing-length theory language, α is minimum for log $T_{eff} \approx$ 3.86 which corresponds to the effective temperature of the transition zone (see Fig. 5 in the paper by Ludwig et al. 1999). For effective temperatures larger than that of the phase transition, convection is superadiabatic and the von Zeipel value is progressively restored.

Let us compare these values of GDE's with those derived from the 2D numerical RHD (Ludwig et al. 1999). These authors obtain very interesting results concerning convective efficiency and gravity-darkening exponents. In fact, the derived exponent is independent of the mixing-length theory. We have used the fitting formulae given in their appendix to calculate the entropy in the envelopes and therefore to compute the respective numerical derivatives to obtain β_1.

The comparison is displayed in Fig. 8. The variation of β_1 for RHD simulations is approximately linear with log T_{eff}. The values coincide for both methods for envelopes with low convective efficiency. However, there is a systematic deviation when going to lower effective temperatures: RHD always presenting smaller values. Although the discrepancies as whole are not so large it is worth to try to elucidate why they appear. The physics involved in both formulations is different since RHD uses varying mixing-length parameter α. Of course, the structure of a convective envelope computed under varying α is different from that computed considering a constant mixing-length parameter and this may be in part responsible for the differences shown in Fig. 8.

The local maximum for log T_{eff} around 3.7 following our method is not detected if one compares with RHD results. However, as the RHD calculations are available only for log $T_{eff} \geq 3.63$ it is difficult to draw any definitive conclusion concerning this local maximum since it may be present for log $T_{eff} < 3.63$. Only when RHD calculations would be extended to lower effective temperatures it will be possible to clarify the situation. In spite of the mentioned discrepancies (around 10% at maximum) we can consider that the predicted gravity-darkening exponent using our method and those derived from the radiation hydrodynamical simulations agree well within the uncertainties.

6. Differential rotation: anomalous GDE?

The actual calculations of GDE (using interior models and/or atmosphere models) did not take into account explicitly cause-effect. In this section we begin to explore such an interaction investigating simple models with radiative envelopes. Until now we did not specify what kind of rotation law we are dealing with. Often, it was assumed that the angular velocity depends only on the distance from the axis of rotation, that is, $\omega = \omega(r)$. However, in a more general case one expects that stars rotates differentially and that there is also a dependence on the colatitude angle θ. In this way, $\omega = \omega(r, \theta)$. Let us assume that the dependence of the angular velocity with the distance and the colatitude is given by:

$$\omega^2 = h(r) \sin^N \theta \qquad (10)$$

Making use of a perturbation theory, it can be shown that the integrating factor for the potential equation is

$$\Lambda = 1 + \frac{1}{N+2}\chi(N - 2\alpha)\sin^2\theta. \qquad (11)$$

with $\alpha = \frac{\partial ln\omega}{\partial lnr}$. Note that the integration factor was introduced in such a way that $\Lambda \boldsymbol{g}$ could be derived from a potential. Let us consider the case of ideal gas and homogeneous envelope. Moreover, let us also assume that the density and temperature are functions only of the potential

$$\rho = d(\Psi)\Lambda$$
$$T = \frac{t(\Psi)}{\Lambda} \qquad (12)$$

After some algebra, we can write the relative difference between the flux at the equator and at the pole as

$$\frac{F_e - F_p}{F_p} \approx -\frac{\chi}{N+2}\left[N + 4 + (\kappa_\rho - \kappa_T + 4)(N - 2\alpha)\right] \qquad (13)$$

where

$$\kappa_\rho \equiv \frac{\partial ln\kappa}{\partial ln\rho}$$
$$\kappa_T \equiv \frac{\partial ln\kappa}{\partial lnT}$$
$$\chi = \frac{r^3\omega^2}{Gm} \qquad (14)$$

where κ is the opacity. In order to simplify, let us assume that the configuration of the stars is the same as that given by the Roche model. Considering the variation of the gravity at the equator and pole we have

$$\frac{g_e - g_p}{g_p} = -\frac{4+N}{2+N}\chi. \qquad (15)$$

Finally, approximating the relative differences of the flux and log g by the respective logarithmic derivatives, we write

$$\beta_1 = 1 + \frac{[(4 + \kappa_\rho - \kappa_T)(N - 2\alpha)]}{4+N} \qquad (16)$$

From Eq. (16) one can see that when ω does not depend on the colatitude angle, for no opacity dependence and for constant or null angular velocity we obtain the value derived by von Zeipel, β_1 =1. As already known, this means that the von Zeipel theorem only holds for uniform rotation where the isobars and equipotential coincide. Of course, there are also the trivial solutions $\kappa_\rho = \kappa_T - 4$ and $N = 2\alpha$. For the remaining cases, β_1 will depend on the angular velocity distribution, on the opacity law and on the star‘s differential rotation. Kippenhahn 1977 analyzed the case N=0 and concluded that for Kramer's opacity, and depending on angular velocity distribution, the flux in the pole is larger (or smaller) than at the equator. In other words, it may turn out to be a rotational darkening or brightening.

There are interesting real configurations which we can relate with Eq. (16). Concerning the radial distribution of the angular velocity, one would expect that for evolved stars the interior rotates faster than the outer layers, i.e, $\alpha < 0$. But configurations with $\alpha > 0$ are also possible: the systems presenting an accretion disk. In these systems the external layers rotate faster than the interior ones due to the increase of the angular momentum. Moreover, the dependence on the colatitude (Eq. (10)) is also important and both mechanisms may be acting in systems like Algols with a mass-gaining early-type component or in other systems showing an accretion disk.

As it was stressed in Paper I, to compare the values of theoretical GDE's with those obtained from the light curve analysis is not a simple and accurate task. Several correlations among the different parameters involved in the light curve solution and the fact that the gravity-darkening is a second order effect limit severely a more accurate analysis, as for example, one can do for the relative radii, effective temperatures, etc. Keeping in mind these remarks, we have done such a confrontation in Fig. 7 of Paper I for all available data published by Rafert & Twigg 1980 achieving a general good agreement. Let us concentrate here on more massive components of the sample since in this section we are dealing essentially with envelopes in radiative equilibrium. The large scattering of β_1 for systems with effective temperature larger than, say, 9000 K may be due to the fact that both mechanisms - dependence of ω with θ and/or with the distance - may be present in the observational sample. However, this guess should be taken with care since the quality of the observational data is not good enough to test it with confidence.

Very anomalous GDE's were reported by Kitamura & Nakamura 1987 for some systems with secondaries filling the Roche lobe. The derived GDE's are between 3 and 10. They consider secondaries with convective and radiative envelopes. Unno et al. 1994 interpreted these results as a consequence of the enthalpy transport associated with a mass-loss rate from the secondary of the order of 10^{-6} $M_\odot$ per year. Values of GDE's as high as 3 or 10 are also possible if Eq. (16) is used. But this possibility must be taken with caution. Indeed, recently Lázaro et al. 2000 have observed one of the systems of the Uno et al. list, namely, VV Uma using simultaneous four-color photometry $uvby$ with the 90 cm telescope at Sierra Nevada, Granada, Spain. The light curves analysis does not indicate any anomaly concerning β_1 nor for the primary ($T_{eff} \approx 9000$ K) neither for the secondary ($T_{eff} \approx 5000$ K). On the contrary, the classical values of 1.0 and 0.3 as computed following our method seem to be very adequate. The only detected "anomalous" characteristic is an indication of pulsation of the primary. A critical revision of the light curves of the systems of the Unno et al. list seems to be necessary before concluding that they really present very anomalous gravity-darkening.

7. Conclusions and further perspectives

We have carried out the calculation of the gravity-darkening exponents and apsidal motion constants for low mass stars. These parameters are presented for the first time as a function of the mass, chemical composition and time. In order to compute the GDE's for VLMS we have introduced a variation of our method described in the Paper I: atmospheric models were coupled with the interior structure in order to handle more realistic models. The present calculations extend the range of mass investigated from 0.08 up to 40 $M_\odot$.

It was emphasized that the gravity-darkening phenomenon is connected not only with the envelope physics but also with the internal structure of a configuration, given that the distortions caused by rotation depend on it. A more accurate method to calculate GDE's should take into account the exact shape of the star as well as the details of the rotation law. The later was shown making use of a simple model in radiative equilibrium.

On the other hand, the impact of the input physics on the GDE's is also analyzed. The differences found depend on the chemical composition but in our opinion, at the actual level of noise of observed light curves, it is not possible to detect such an influence nor other effects of third order. Even the confrontation between the theoretical β_1 with the inferred values by Rafert & Twigg (1980) is not a critical test though some knowledge may be achieved. Perhaps only with the next generation of telescopes, mainly from the space, would it be possible to compare β_1 with theoretical predictions and discriminate with confidence the effects of third order as theory of convection, chemical composition, irradiation, etc.

Hydrodynamical simulations indicate a monotonic increasing of GDE's for late-type stars while the predictions based on the mixing-length theory indicate a local maximum around log $T_{eff} \approx 3.7$. Although we can consider the global interagreement as a good one, such a discrepancy may lead us to explore new ways to approach the problem of flux distribution in distorted convective envelopes.

A possible explanation of some anomalies (not so high as those reported by Unno et al. 1994) in the GDE's is discussed. On the basis of the work of Kippenhahn 1977 it was guessed that moderately anomalous GDE's may be found in some particular systems but we remark that it is hard to compare such predictions with observations given the actual light curve quality. Concerning the very anomalous gravity-darkening reported by Unno et al. 1994, recent $uvby$ observations performed in January 2000 of one of the systems quoted by these authors - VV Uma - indicate that there are no anomalies in the gravity-darkening exponent.

We are planning for future works to implement in our evolution code the possibility of varying the mixing-length parameter as well as other RHD simulation results in order to compute the GDE's and compare them with those obtained using the classical mixing-length theory. Another point of interest is to consider the motions driven by the pressure gradient in the equations as early suggested by Lucy 1967.

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