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ASTRONOMY AND ASTROPHYSICS

The diffusion of radiation in moving media

I. Basic assumptions and formulae

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Abstract. The 3D radiative transfer equation for differentially moving media is derived upon the assumption that the motions are sufficiently slow. Its solution is then applied to the limiting case of large optical depths, i.e. to the diffusion approximation. Although the effective extinction for static 1D media has been derived by Rosseland already in 1924, it is for the first time in this Paper that for moving 3D media with many spectral lines general expressions for radiative quantities are derived in a rigorous way. Given are simple to use monochromatic as well as wavelength-integrated expressions for the flux and the radiative acceleration, and a generalized version of the Rosseland mean opacity. The essential effects of the motions upon the radiative flux are discussed for the simple case of a single spectral line on a continuum.

Key words: diffusion – radiative transfer – stars: interiors – stars: novae, cataclysmic variables – stars: supernovae: general

1. Introduction

There are many differentially moving, optically thick astronomical objects in which the radiation field is important for the energy and/or momentum balance and therefore the total fluxes and/or radiative accelerations have to be calculated accurately in a modeling. Typical celestial systems of this kind are novae, supernovae, collapsing molecular clouds, and accretion discs. Ideally, one would solve the full appropriate radiative transfer equation which is possible only numerically and is unfortunately extremely CPU time and memory consuming if many spectral lines contribute to the opacity; in fact, in non-stationary models this solution is virtually impossible. However, in many cases one can exploit the large optical thickness and derive a simpler form of the solution of the transfer equation that is valid only in this limit: the radiation diffusion equation.

For static 1D media the problem has been solved already many years ago (Rosseland 1924). Karp et al. (1977) were the first to discuss differentially moving media. They used infinitely narrow lines and showed that the *expansion opacity*, i.e. the modification of the opacity due to the motions that has to be employed to Rosseland's formula for the flux, may be quite large. Subsequently, the problem has been addressed – mainly in the context of supernova explosions, spectra and light-curves – e.g. by Eastman & Kirshner (1989), Höflich (1990, 1995), Höflich et al. (1993), Eastman & Pinto (1993), Blinnikov & Bartunov (1993), Jeffery (1995), Blinnikov (1996a,b), Baron et al. (1996), and recently Pinto & Eastman (2000). These authors make use of the particular conditions in these objects as e.g. the small intrinsic width of the lines and the coincidence of the directions of the velocity and of the temperature gradients.

Here we present the first paper of a series which is devoted to the discussion of the effects of differential motions on the radiative quantities in optically thick absorbing and scattering media of general shape with arbitrary non-relativistic velocities. In particular the directions of the flow and of the temperature gradient may be different which introduces additional components to the flux vector. The expressions are derived in a rigorous way from the comoving-frame transfer equation. Our approach leads to the correct limit of static media, and can handle spectral lines of arbitrary shape and strength as well as edges and continua. In addition, it allows easy physical interpretations of the effects of the motion on the radiative quantities.

In Sect. 2 of *this* paper we first recall the results of the long known "conventional", static diffusion limit and give the definitions of the radiative quantities, the abbreviations etc. used in this series. In Sect. 3 we derive the transfer equation for *slowly* differentially moving 3D media. In the diffusion limit it is sufficient in most cases to consider small velocities v since only velocity differences over one free mean path length in the continuum are relevant. This simplifies the equations significantly as only first order terms in $\beta = v/c$ (c velocity of light) have to be considered. In addition, the aberration/advection terms are so small that they can be neglected, and the characteristics continue to be straight lines. Subsequently, in Sect. 4 we obtain the solution of the transfer equation for the limit of large optical depths, and in Sect. 5 give expressions for the flux and the radiative force in the diffusion limit, and present a generalized version of the Rosseland opacity valid for differentially moving media. The essential effects of the velocity field upon the flux are then

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demonstrated in Sect. 6 on the basis of a numerical evaluation of the simple case that the extinction is due to a continuum and a single line only. Finally, Sect. 7 contains the conclusions and an outlook.

In Paper II of this series we discuss the radiative quantities in the diffusion approximation for moving media for the *limiting* cases of *large* as well as of *small* velocity gradients for deterministic spectral lines of *finite* width. Paper III then will deal with *stochastic* distributions of lines (of finite width) described by a Poisson point process which has been shown by Wehrse et al. (1998) to be flexible and adequate for the treatment of very many spectral lines. In Paper IV *infinitely sharp* lines – which had e.g. been used by Karp et al. (1977) in their discussion of the expansion opacity – will be considered.

2. Static case and nomenclature

The aim of our diffusion calculations is to derive simple expressions for the radiative flux and radiative acceleration at positions far away from the surface in a medium

(i) that is optically very thick,
(ii) in which the extinction coefficient hardly varies over a photon mean free path, and
(iii) in which the variation of the source function can be approximated by a linear function in the neighborhood of the point s_0 where the radiative quantities are to be calculated.

For this purpose we start with the general form of the transfer equation for a *static* 3D medium for the monochromatic intensity $I \equiv I(s, \xi, \mathbf{n})$ at a wavelength λ along a ray described by $\mathbf{s}$ or the unit vector $\mathbf{n}$, respectively,

$$\frac{dI}{d\mathbf{s}} = \mathbf{n} \cdot \nabla I = -\chi(I - S)\,. \tag{1}$$

Here S is the source function and χ the monochromatic extinction coefficient comprising absorption as well as scattering. In cartesian coordinates Eq. (1) reads

$$n_x \frac{\partial I}{\partial x} + n_y \frac{\partial I}{\partial y} + n_z \frac{\partial I}{\partial z} = -\chi(I - S) \tag{2}$$

with $\mathbf{n} = \{n_x, n_y, n_z\} = \{\cos\varphi \sin\vartheta, \sin\varphi \sin\vartheta, \cos\vartheta\}$ (cf. Oxenius 1986).

We introduce *logarithmic* wavelengths

$$\xi = \ln \lambda \tag{3}$$

which are mostly used throughout this paper in place of the usual wavelengths λ in order to obtain Doppler shifts that depend only on spatial coordinates and angles. Then $d\xi = d\lambda/\lambda$, $d\lambda = \exp(\xi) d\xi$.

In the diffusion limit, the mean intensities are very close to the *Planck function* $B(T, \xi)$ of the local temperature $T = T(s)$. This implies that LTE conditions prevail and the source function is identical to the Planck function,

$$S = B(T, \xi)\,. \tag{4}$$

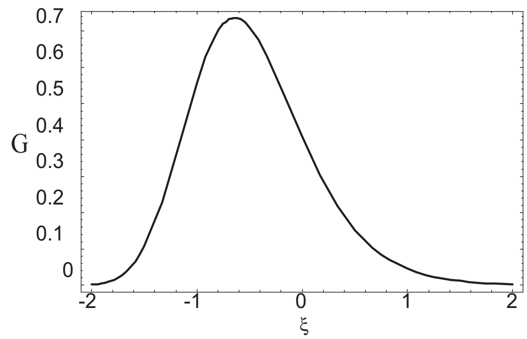

Fig. 1. Weighting function $G(s_0, \xi) = \left(\frac{\partial B(T,\xi)}{\partial T} / \frac{\partial B(T)}{\partial T}\right) \exp(\xi)$ as function of ξ for $h/(kT) = 1$.

In the following we need also the wavelength-integrated Planck function denoted by

$$B = B(T) = \int_{-\infty}^{\infty} B(T, \xi) \mathrm{e}^{\xi} d\xi = \frac{\sigma_{\mathrm{SB}}}{\pi} T^4 \tag{5}$$

with σ_{SB} being the Stefan-Boltzmann constant, the spatial derivatives of $B(T, \xi)$ and $B(T)$, respectively,

$$\frac{\partial B(T, \xi)}{\partial T} \mathbf{n} \cdot \nabla T = \frac{\partial B(T, \xi)}{\partial T} \frac{\partial T}{\partial s} = g(s, \xi, \mathbf{n})\,, \tag{6}$$

$$\begin{aligned} \frac{\partial B(T)}{\partial T} \mathbf{n} \cdot \nabla T &= \frac{\partial B(T)}{\partial T} \frac{\partial T}{\partial s} = \int_{-\infty}^{\infty} g(s, \xi, \mathbf{n})\, \mathrm{e}^{\xi} d\xi \\ &= g(s, \mathbf{n}) \end{aligned} \tag{7}$$

with $\partial B(T)/\partial T = 4\sigma_{\mathrm{SB}} T^3/\pi$, and the weighting function

$$G(s, \xi) = \frac{g(s, \xi, \mathbf{n})}{g(s, \mathbf{n})} \mathrm{e}^{\xi} = \left(\frac{\partial B(T, \xi)}{\partial T} \bigg/ \frac{\partial B(T)}{\partial T} \right) \mathrm{e}^{\xi}\,. \tag{8}$$

This weighting function, which enters the Rosseland mean opacity (Eq. (19)), is normalized according to $\int_{-\infty}^{\infty} G(s, \xi) d\xi = 1$ and is *independent* of $\mathbf{n}$. It decreases exponentially with ξ for very large as well as for very small ξ (Fig. 1).

In terms of these quantities the source function in the neighborhood of s_0 can be written – see assumption (iii) above – as

$$B(s, \xi) = p(s_0, \xi) + q(s_0, \xi, \mathbf{n}) \cdot (s - s_0)\,. \tag{9}$$

Here $p = B(s_0, \xi)$ and

$$\begin{aligned} q(s_0, \xi, \mathbf{n}) &= \left. \frac{\partial B(s, \xi)}{\partial s} \right|_{s_0} \\ &= \left. \frac{\partial B(T, \xi)}{\partial T} \right|_{T(s_0)} \cdot \left. \frac{\partial T}{\partial s} \right|_{s_0} = g(s_0, \xi, \mathbf{n}) \end{aligned} \tag{10}$$

are sufficiently *slowly* varying functions of s_0 over a few mean free photon paths $1/\chi_{\mathrm{c}}$ (in the continuum).

In order to clarify the nomenclature, we present the relevant *static* radiative quantities first (cf. Cox & Giuli 1968, Mihalas 1978, Mihalas & Weibel-Mihalas 1984).

2.1. *Radiative flux*

The *total* flux

$$\mathbf{F}_{\rm tot}(s) = \int_{-\infty}^{\infty} \mathbf{F}(s,\xi)\, {\rm e}^{\xi} d\xi \tag{11}$$

is obtained by integration over all wavelengths from the vector of the *monochromatic* flux,

$$\begin{aligned}
\mathbf{F}(s,\xi) &= \int_{4\pi} I(s,\xi,\mathbf{n})\,\mathbf{n}\,d\omega \\
&= \int_0^{\pi}\int_0^{2\pi} I(s,\xi,\varphi,\vartheta) \begin{pmatrix}\cos\varphi\sin\vartheta\\ \sin\varphi\sin\vartheta\\ \cos\vartheta\end{pmatrix} d\varphi\sin\vartheta d\vartheta \\
&= \int_0^{\pi/2}\int_0^{2\pi} \Big[I(s,\xi,\varphi,\vartheta) - I(s,\xi,\varphi+\pi,\vartheta-\pi/2)\Big] \\
&\qquad \times \begin{pmatrix}\cos\varphi\sin\vartheta\\ \sin\varphi\sin\vartheta\\ \cos\vartheta\end{pmatrix} d\varphi\sin\vartheta d\vartheta\,.
\end{aligned} \tag{12}$$

It is evident from Eq. (12) that the essential physics is contained in the *antisymmetric* average of the specific intensities $[(I(s,\xi,\varphi,\vartheta) - I(s,\xi,\varphi+\pi,\vartheta-\pi/2)]$ which has a flux-like character (cf. Mihalas 1978). In fact, it is the (monochromatic) flux in the two-stream approximation. In the following we therefore will concentrate on this average and – in order to simplify the notation – subsequently write

$$\begin{aligned}
I(s,\xi,\varphi,\vartheta) \;&-\; I(s,\xi,\varphi+\pi,\vartheta-\pi/2) \\
&\equiv \mathcal{I}^{-}(s,\xi) - \mathcal{I}^{+}(s,\xi) \equiv \mathcal{F}(s,\xi)
\end{aligned} \tag{13}$$

and

$$\mathcal{F}_{\rm tot}(s) = \int_{-\infty}^{\infty} \mathcal{F}(s,\xi)\, {\rm e}^{\xi} d\xi\,. \tag{14}$$

2.2. *Radiative acceleration*

The vector of the net radiative acceleration (force per unit volume) has the same direction as that of the flux so that the *total* radiative acceleration is

$$\mathbf{a}_{\rm rad,tot}(s) = \int_{-\infty}^{\infty} \mathbf{a}_{\rm rad}(s,\xi)\, {\rm e}^{\xi} d\xi \tag{15}$$

with

$$\begin{aligned}
\mathbf{a}_{\rm rad}(s,\xi) &= \frac{1}{c}\int_{4\pi} \chi(s,\xi)\, I(s,\xi,\mathbf{n})\,\mathbf{n}\,d\omega \\
&= \frac{1}{c}\int_0^{\pi/2}\int_0^{2\pi} \Big[I(s,\xi,\varphi,\vartheta) \\
&\qquad -I(s,\xi,\varphi+\pi,\vartheta-\pi/2)\Big] \\
&\qquad \times \chi(s,\xi) \begin{pmatrix}\cos\varphi\sin\vartheta\\ \sin\varphi\sin\vartheta\\ \cos\vartheta\end{pmatrix} d\varphi\sin\vartheta d\vartheta
\end{aligned} \tag{16}$$

being the *monochromatic* acceleration. Here the role of the antisymmetric average of the intensities is the same as that occurring in the flux. Analogously we write

$$\frac{1}{c}\chi(s,\xi)\mathcal{F}(s,\xi) \equiv a_{\rm rad}(s,\xi)\,. \tag{17}$$

2.3. *Static radiative quantities in the diffusion limit*

For a static medium we find from the well-known solution of Eq. (1) along a ray at a position s_0 well inside the medium

$$\begin{aligned}
\mathcal{F}(s_0,\xi) &= \lim_{\substack{s_0\to\infty\\(\tilde{s}-s_0)\to\infty}} \Bigg(\int_{s_0}^{\tilde{s}} {\rm e}^{-\chi(\xi)\cdot(\ell-s_0)}\chi(\xi)B(\ell,\xi)d\ell \\
&\qquad - \int_0^{s_0} {\rm e}^{-\chi(\xi)\cdot(s_0-\ell)}\chi(\xi)B(\ell,\xi)d\ell\Bigg) \\
&= \lim_{\substack{s_0\to\infty\\(\tilde{s}-s_0)\to\infty}} \Bigg(\int_{s_0}^{\tilde{s}} {\rm e}^{-\chi(\xi)\cdot(\ell-s_0)}\chi(\xi) \\
&\qquad \times\Big(p(s_0,\xi)+q(s_0,\xi)(\ell-s_0)\Big)d\ell \\
&\qquad - \int_0^{s_0} {\rm e}^{-\chi(\xi)\cdot(s_0-\ell)}\chi(\xi) \\
&\qquad \times\Big(p(s_0,\xi)+q(s_0,\xi)(\ell-s_0)\Big)d\ell\Bigg) \\
&= 2\,\frac{q(s_0,\xi,\mathbf{n})}{\chi(\xi)} = 2\,\frac{g(s_0,\xi,\mathbf{n})}{\chi(\xi)}\,.
\end{aligned} \tag{18}$$

The wavelength-integrated flux can be expressed in terms of the (static) *Rosseland* mean opacity $\bar{\chi}_{\rm R}$, which is defined by

$$\frac{1}{\bar{\chi}_{\rm R}(s)} = \int_{-\infty}^{\infty} \frac{G(s,\xi)}{\chi(s,\xi)}\,d\xi \tag{19}$$

with $G(s,\xi)$ being the weighting function given in Eq. (8). Then we obtain from Eqs. (14) and (18)

$$\mathcal{F}_{\rm tot}(s) = 2\int_{-\infty}^{\infty} \frac{g(s,\xi,\mathbf{n})}{\chi(s,\xi)}\,{\rm e}^{\xi}d\xi = 2\,\frac{g(s,\mathbf{n})}{\bar{\chi}_{\rm R}(s)}\,, \tag{20}$$

and from Eq. (12)

$$\begin{aligned}
\mathbf{F}_{\rm tot}(s) &= \int_{4\pi} \mathcal{F}_{\rm tot}(s)\,\mathbf{n}\,d\omega \\
&= 2\;\frac{1}{\bar{\chi}_{\rm R}(s)}\,\frac{\partial B(T)}{\partial T}\int_{4\pi}(\mathbf{n}\cdot\nabla T)\,\mathbf{n}\,d\omega \\
&= \frac{4\pi}{3}\,\frac{1}{\bar{\chi}_{\rm R}(s)}\,\frac{\partial B(T)}{\partial T}\,\nabla T\,,
\end{aligned} \tag{21}$$

the classical result of Rosseland (1924).

According to Eq. (17) the expressions for the radiative acceleration become

$$a_{\rm rad}(s,\xi) = \frac{2}{c}\,g(s,\xi,\mathbf{n}) = 2\,\frac{\partial B(T,\xi)}{\partial T}\,\mathbf{n}\,\nabla T\,, \tag{22}$$

and according to Eqs. (15) and (16)

$$\mathbf{a}_{\rm rad,tot}(s) = \frac{4\pi}{3c}\,\frac{\partial B(T)}{\partial T}\,\nabla T\,. \tag{23}$$

3. Radiative transfer equation for slowly moving 3D media

In order to obtain the transfer equation for a *moving* medium, we start from the static equation for a 3D medium (1) and apply to it the "simplified" Lorentz transformation

$$\lambda_0 \;=\; \lambda(1+\boldsymbol{\beta}\cdot\mathbf{n_0})\,, \tag{24}$$

$$\mathbf{n}_0 \;=\; \mathbf{n}\,, \tag{25}$$

which, in fact, is a Galilei transformation in combination with the linear Doppler formula. Quantities referring to the comoving frame are denoted by the subscript 0 here (in this section only).

By applying (24) and (25) we restrict ourselves to sufficiently small velocities $\boldsymbol{\beta} = \mathrm{v}/c$ so that we may assume for the Lorentz factor $1/\sqrt{1-\beta^2} = 1$, and furthermore neglect the aberration and advection, i.e. keep $\mathbf{n}$ unchanged. These assumptions are justified as long as the velocity change over the mean free path length $1/\chi_{\rm c}$ of the photons in the continuum (denoted by c) or, equivalently, over a distance corresponding to unit optical depth, $d\tau_c = \chi_{\rm c} ds \simeq 1$, in the continuum is sufficiently small, i.e. if

$$\frac{1}{\chi_{\rm c}}\left|\frac{d\beta}{ds}\right| = \left|\frac{d\beta}{d\tau_{\rm c}}\right| \ll 1\,. \tag{26}$$

In the diffusion limit of radiation fields this condition is usually fulfilled.

Proceeding with the derivation of the comoving-frame transfer equation, we now consider any vector $\mathbf{x}(s)$ to depend on the variables $\mathbf{x_0}$ *and* λ_0 rather than directly on the length variable s, i.e.

$$\mathbf{x}(s) = \mathbf{x}\Big(\mathbf{x_0}(s), \lambda_0(s)\Big)\,. \tag{27}$$

Then the nabla operator in Eq. (1) has to be replaced by

$$\nabla \quad\Rightarrow\quad \nabla + (\nabla\lambda_0)\;\frac{\partial}{\partial\lambda_0} \tag{28}$$

with

$$\nabla\lambda_0 = \lambda\,\nabla(\boldsymbol{\beta}\cdot\mathbf{n_0}) = \frac{\lambda_0}{1+\boldsymbol{\beta}\cdot\mathbf{n_0}}\,\nabla(\boldsymbol{\beta}\cdot\mathbf{n_0}) \tag{29}$$

according to Eq. (24). Introducing this expression into the transfer equation then yields

$$\begin{aligned}\mathbf{n}_0\cdot\nabla I + \frac{\lambda_0}{1+\boldsymbol{\beta}\cdot\mathbf{n}_0}\,\mathbf{n}_0\cdot\nabla(\boldsymbol{\beta}\cdot\mathbf{n}_0)\frac{\partial I}{\partial\lambda_0}&\\ = \mathbf{n}_0\cdot\nabla I + w_0\,\lambda_0\frac{\partial I}{\partial\lambda_0} &= -\chi(I-S)\end{aligned} \tag{30}$$

where

$$w_0 = \frac{1}{1+\boldsymbol{\beta}\cdot\mathbf{n}_0}\,\mathbf{n}_0\cdot\nabla(\boldsymbol{\beta}\cdot\mathbf{n}_0) \simeq \mathbf{n}_0\cdot\nabla(\boldsymbol{\beta}\cdot\mathbf{n}_0)\,. \tag{31}$$

Since in the following we are dealing exclusively with *comoving-frame* quantities, we for simplicity drop the subscript 0 in our notation so that our basic 3D radiative transfer equation now reads, in coordinate-free form,

$$\frac{dI}{d\mathbf{s}} + w\frac{\partial I}{\partial\xi} = -\chi(I-S) \tag{32}$$

with

$$w \simeq \mathbf{n}\cdot\nabla(\boldsymbol{\beta}\cdot\mathbf{n})\,. \tag{33}$$

Thus the motions of the medium enter the transfer equation only in the form of the "velocity gradient" w, and consequently all their effects upon the radiative quantities can be expressed in terms of w. We further emphasize that in a moving medium the *comoving* frame is the relevant "natural" description for the radiative transfer equation since in particular all thermodynamic quantities are defined in this frame.

Expressed in terms of w, the condition (26) for the diffusion limit now reads

$$|w| \ll \chi_{\rm c} \tag{34}$$

since $|w|$ is at most of the order of $d\beta/ds$.

Eq. (32) has the same *mathematical* structure as the *1D* comoving-frame transfer equation used by Baschek et al. (1997b) for plane-parallel and spherical media so that its analytical solution (Baschek et al. 1997a,b) can also be applied here. That equation was derived as limiting case for small β and for the directions $\mu = \pm 1$ from the full (spherically symmetric) relativistic transfer equation given by Mihalas & Weibel-Mihalas (1984). We note that the meaning of the coefficients in the two papers, however, is different: our χ in Eq. (32) reads $(\chi - 5w)$ in Baschek et al. (1997b), and their w is simply equal $d\beta/ds$.

In the diffusion limit, however, at which we aim in this Paper, we can neglect the term $5wI$ in the transfer equation since $\chi \geq \chi_{\rm c}$ and hence $|w| \ll \chi$ from the condition (34). (Incidentally, the "transport-type" term $5wI$, acting as an additional extinction, would not appear in the transfer equation if the relativistically invariant intensity $\lambda^5 I$ had been used instead of I, cf. also Wehrse & Baschek 1999.)

4. Solution in the limit of high optical depth

According to Baschek et al. (1997b) the solution of the transfer equation (Eq. 32) for *constant* w, a depth-*in*dependent extinction coefficient $\chi(\xi)$, and no incident radiation at the two boundaries $s = 0$ and $s = \tilde{s}$ of the layer considered, yields in the positive direction of s the intensity

$$\begin{aligned}\mathcal{I}^+(s,\xi;w) \;=\; &\int_0^s \exp\left(-\int_\ell^s \chi\left(\xi + w(\hat{\ell}-s)\right)d\hat{\ell}\right)\\ &\times\;\chi\left(\xi + w(\ell-s)\right)B\left(\ell,\xi + w(\ell-s)\right)d\ell\,,\end{aligned} \tag{35}$$

and – due to symmetry – in the opposite direction

$$\begin{aligned}\mathcal{I}^-(s,\xi;w) \;=\; &\int_s^{\tilde{s}} \exp\left(-\int_s^\ell \chi\left(\xi - w(\hat{\ell}-s)\right)d\hat{\ell}\right)\\ &\times\;\chi\left(\xi - w(\ell-s)\right)B\left(\ell,\xi - w(\ell-s)\right)d\ell\,.\end{aligned} \tag{36}$$

Here and in the following we introduce w as additional variable in the argument list of all quantities referring to *moving* media, while static quantities are denoted without w.

In order to apply the solution of Baschek et al. (1997b) to the diffusion limit we here need to demand only that χ – similarly as

β – does not change significantly over a mean free photon path in the continuum, i.e. we may abandon their strict assumption of χ being independent on s.

The solutions for $\mathcal{I}^+(s,\xi;w)$, $\mathcal{I}^-(s,\xi;w)$, and hence for $\mathcal{F}(s,\xi;w)$ can conveniently be written in terms of the *spectral thickness* (Baschek et al. 1997b)

$$\psi(\xi) = \int_{\xi_1}^{\xi} \chi(\zeta) d\zeta \tag{37}$$

with ξ_1 being an arbitrary logarithmic reference wavelength, a formalism which we will also use in this Paper for the derivation of radiative quantities in moving media.

In terms of ψ the flux for the linearized source function reads (cf. Sect. 2)

$$\begin{aligned}
&\mathcal{F}(s_0,\xi;w) = \\
&\Big(p(s_0,\xi) - q(s_0,\xi,\mathbf{n})\Big) \exp\left[\frac{-\psi(\xi)+\psi(\xi - w s_0)}{w}\right] \\
&-p(s_0,\xi)\exp\left[\frac{-\psi(\xi)+\psi(\xi - w(\tilde{s}-s_0))}{w}\right] \\
&+q(s_0,\xi,\mathbf{n})\int_{s_0}^{\tilde{s}} \exp\left[\frac{-\psi(\xi)+\psi(\xi + w(s_0-\ell))}{w}\right] d\ell \\
&+q(s_0,\xi,\mathbf{n})\int_{0}^{s_0} \exp\left[\frac{-\psi(\xi)+\psi(\xi - w(s_0-\ell))}{w}\right] d\ell \,.
\end{aligned} \tag{38}$$

For a fixed value of ξ we then obtain in the limit s_0, $(\tilde{s}-s_0) \to \infty$

$$\begin{aligned}
\mathcal{F}(s_0,\xi;w) &= 2q(s_0,\xi,\mathbf{n})\int_0^{\infty} \exp\left(-\frac{\psi(\xi)-\psi(\xi - w\ell)}{w}\right) d\ell \\
&= 2q(s_0,\xi,\mathbf{n})\int_0^{\infty} \exp\left(-\frac{1}{w}\int_{\xi - w\ell}^{\xi} \chi(\zeta) d\zeta\right) d\ell
\end{aligned} \tag{39}$$

(see also Fig. 2).

Eq. (39) is the *key equation* we derive in this Paper. It contains the effects of the differential motions within the medium and allows – as will be shown subsequently – to calculate the flux, the radiative acceleration, and a generalized expression for the Rosseland mean opacity.

From the definition (37) of the spectral thickness follows $\psi'(\xi) \equiv d\psi/d\xi = \chi(\xi) > 0$ since the extinction coefficient $\chi(\xi)$ is always positive. Hence $\psi(\xi)$ increases monotonically with ξ and $(\psi(\xi) - \psi(\xi - w\ell))/w > 0$ for either sign of w. If in addition $\psi(\xi)$ increases sufficiently fast with increasing ξ and hence with s_0, the integrals (39) exist. For this to occur, already a wavelength-independent continuum, resulting in a linear dependence of $\psi(\xi)$ on ξ, suffices. In this case there is a sufficiently large optical depth at all wavelengths – as is required by the diffusion approximation – and hence the validity of Eq. (39) is guaranteed. Furthermore we point out that in the limit of large optical depths the integrals over ℓ in (35) and (36) formally extend over an infinite interval; due to the effective cutoff properties of the exponential functions, however, they extend in practice – due to the assumption of a linearized depth-dependence of the source function – only over a distance of the order of $1/\chi_{\rm c}$.

At this place we convince ourselves that the argument of the exponential in Eq. (39) reduces to Eq. (18) in the limit $w \to 0$ since

$$\begin{aligned}
\lim_{w\to 0} \frac{\psi(\xi)-\psi(\xi - w\ell)}{w} &= \lim_{w\to 0} \frac{\psi(\xi)-\psi(\xi - w\ell)}{w\ell} \cdot \ell \\
&= \psi'(\xi)\cdot \ell = \chi(\xi)\,\ell \,.
\end{aligned} \tag{40}$$

5. Radiative quantities in the diffusion limit

We now turn to the description of the expressions for the flux and the radiative acceleration in a differentially moving medium with velocities β and gradients w.

The monochromatic flux in the diffusion limit for a linearized source function (Eq. 39) now reads

$$\begin{aligned}
&\mathcal{F}(s_0,\xi;w) \\
&\quad = 2g(s_0,\xi,\mathbf{n})\int_0^{\infty} \exp\left(-\frac{\psi(\xi)-\psi(\xi - ws)}{w}\right) ds \,.
\end{aligned} \tag{41}$$

In order to emphasize the effects of the motions we write the flux in the form

$$\begin{aligned}
\mathcal{F}(s_0,\xi;w) &= \frac{2g(s_0,\xi,\mathbf{n})}{\chi(\xi)} \cdot \Big[1+\theta(s_0,\xi;w)\Big] \\
&= \mathcal{F}(s_0,\xi)\cdot \Big[1+\theta(s_0,\xi;w)\Big] \,,
\end{aligned} \tag{42}$$

and hence the monochromatic acceleration as

$$\begin{aligned}
a_{\rm rad}(s_0,\xi;w) &= \frac{1}{c}\chi(\xi)\mathcal{F}(s_0,\xi;w) \\
&= a_{\rm rad}(s_0,\xi)\cdot \Big[1+\theta(s_0,\xi;w)\Big]
\end{aligned} \tag{43}$$

where $\mathcal{F}(s_0,\xi)$ and $a_{\rm rad}(s_0,\xi)$ are the static quantities (18) and (22), respectively. According to Eq. (41), the "w correction factor" is given by

$$\begin{aligned}
&1 + \theta(s_0,\xi;w) \\
&\quad = \chi(\xi)\int_0^{\infty} \exp\left(-\frac{\psi(\xi)-\psi(\xi - ws)}{w}\right) ds \,.
\end{aligned} \tag{44}$$

Analogously to the monochromatic expressions, we give the corresponding *wavelength-integrated* quantities in the form

$$\begin{aligned}
\mathcal{F}_{\rm tot}(s_0;w) &= \int_{-\infty}^{\infty} \mathcal{F}(s_0,\xi;w)\,{\rm e}^{\xi} d\xi \\
&= \mathcal{F}_{\rm tot}(s_0)\cdot \Big[1+\Theta(s_0;w)\Big] \,,
\end{aligned} \tag{45}$$

$$\begin{aligned}
a_{\rm rad,tot}(s_0;w) &= \frac{1}{c}\int_{-\infty}^{\infty} \chi(\xi)\mathcal{F}(s_0,\xi;w)\,{\rm e}^{\xi} d\xi \\
&= a_{\rm rad,tot}(s_0)\cdot \Big[1+\Xi(s_0;w)\Big] \,,
\end{aligned} \tag{46}$$

with $\mathcal{F}(s_0)_{\rm tot}$ and $a_{\rm rad,tot}(s_0)$ being the corresponding integrated static quantities. Then

$1 + \Theta(s_0;w)$

$$= \bar{\chi}_{\rm R}(s_0) \int_{-\infty}^{\infty} \frac{1}{\chi(\xi)} \Big[1 + \theta(s_0, \xi; w)\Big] G(s_0, \xi) d\xi$$
$$= \bar{\chi}_{\rm R}(s_0) \int_{-\infty}^{\infty} \int_0^{\infty} \exp\left(-\frac{\psi(\xi) - \psi(\xi - ws)}{w}\right) ds \times G(s_0, \xi) d\xi \,, \quad (47)$$

$$1 + \Xi(s_0; w) = \int_{-\infty}^{\infty} \Big[1 + \theta(s_0, \xi; w)\Big] G(s_0, \xi) d\xi$$
$$= \int_{-\infty}^{\infty} \int_0^{\infty} \exp\left(-\frac{\psi(\xi) - \psi(\xi - ws)}{w}\right) ds \times \chi(\xi) G(s_0, \xi) d\xi \quad (48)$$

with G as defined by Eq. (8). We point out that in our formalism – due to the additional factor $\chi(\xi)$ – the w-correction for the total radiative acceleration differs from that of the total flux although the corresponding monochromatic correction factors are identical. Note that – although G is independent of **n** – θ, Θ, and Ξ do depend on the direction via the **n**-dependence of w.

We may now introduce a *generalized* Rosseland opacity $\bar{\chi}_\beta$ for a differentially *moving medium* so that the total flux can be described analogously to the static case. We define $\bar{\chi}_\beta$ by the relation

$$\mathcal{F}_{\rm tot}(s_0; w) = \frac{2\, g(s_0, \mathbf{n})}{\bar{\chi}_\beta(s_0; w)} \,. \quad (49)$$

Note that $\bar{\chi}_\beta$ comprises, for $w = 0$, the conventional Rosseland mean $\bar{\chi}_\beta(s_0; 0) \equiv \bar{\chi}_{\rm R}(s_0)$. According to Eq. (45) the generalized Rosseland mean is then given by

$$\frac{1}{\bar{\chi}_\beta(s_0; w)} = \frac{1}{\bar{\chi}_{\rm R}(s_0)} \cdot \Big[1 + \Theta(s_0; w)\Big] \,. \quad (50)$$

When applying $\bar{\chi}_\beta$ one should keep in mind that it has been defined specifically for expressing total fluxes. However, it is *not* the appropriate generalization for the static mean opacities, which e.g. enter the local radiative energy balance, and are frequently also replaced by a Rosseland mean in the literature.

We note that our formulae which express the radiative quantities in terms of their static values can – in a straightforward manner – be applied only to *deterministic* extinction coefficients. The evaluation for stochastic line distributions is somewhat more involved since then the expectation values of all radiative quantities (including the static ones) are to be considered. This case will be discussed in a subsequent paper of this series.

6. Numerical results for a single Lorentzian line

In order to obtain some insight into the behavior of radiative fluxes in a medium of high optical depth according to the above equations we consider the simple case (which, however, already contains the essential features) of a continuum $\chi_{\rm c}$ that does not dependent on ξ and a *single* spectral line of Lorentzian shape at ξ_0 with damping constant γ (in the ξ-scale, cf. Wehrse et al. 1998), i.e.

$$\chi(\xi) = \chi_{\rm c} + A\, \frac{\gamma/(2\pi)}{(\xi - \xi_0)^2 + (\gamma/2)^2)} \,, \quad (51)$$

so that

$$\psi(\xi) = \chi_{\rm c}\, \xi + \frac{A}{\pi}\, \arctan\left(\frac{\xi - \xi_0}{\gamma/2}\right) + \text{const}\,. \quad (52)$$

In order check when the diffusion limit is reached we plot in Fig. 2 the value of the last integral in Eq. (38) as a function of s for $\chi_{\rm c} = 1$ and some values of the line strength A; the other integral behaves in the same way. It is seen that there is hardly any longer a variation for $s_0 > 1$, i.e. the diffusion limit is reached for $s_0 \simeq 1/\chi_{\rm c}$ at latest. By additional line absorption it may even be shifted to much smaller values.

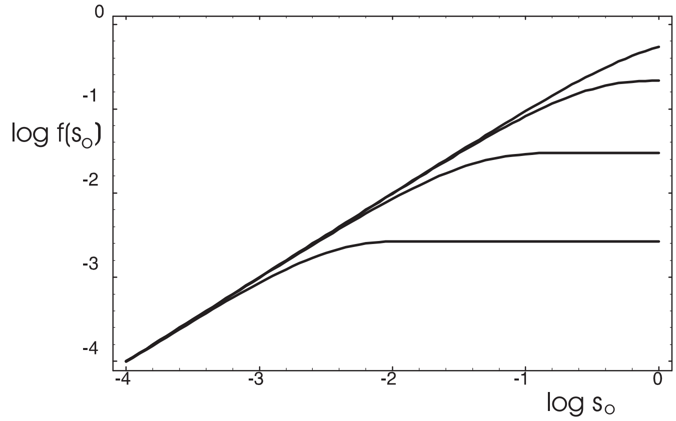

Fig. 2. The dependence of the last integral in Eq. (38) on s_0, calculated for a single Lorentzian lines on a continuum for various line strengths A, demonstrates the approach towards the diffusion limit; *upper curve:* very weak line, *lower curve:* strong line.

The dependence of the monochromatic flux $\mathcal{F}(s_0, \xi; w)$ in the line center on the strength of the continuum χ_c and of the line A (Fig. 3) for given constant w is basically the same as in the static case: an increase in the extinction leads to a decrease in the flux independent of the source of the extinction. This implies that lines have an influence on the flux only when they are are of sufficient strength. As is seen in Fig. 4, a reduction in the velocity gradient or in the damping width leads to a decrease in the monochromatic flux at the line center. However, the main variations occur only for small γ and w values since for large w the line is essentially smeared out and the information on the intrinsic γ is lost. This behavior holds only for the monochromatic flux at or close to the line center; in contrast, for the flux integrated over the line the situation is different since in moving configurations the influence of the line extends (cf. Fig. 5) much further in wavelength than in the static case. Fig. 5 also demonstrates that for a given distance $(\xi - \xi_0)$ from the line center the γ dependence of the flux may be quite complicated, since changes in the intrinsic line profile may or may not be compensated by Doppler shifts. Furthermore, it is seen that the line influence is – in accordance with Fig. 3 – under most conditions strongest at the line center.

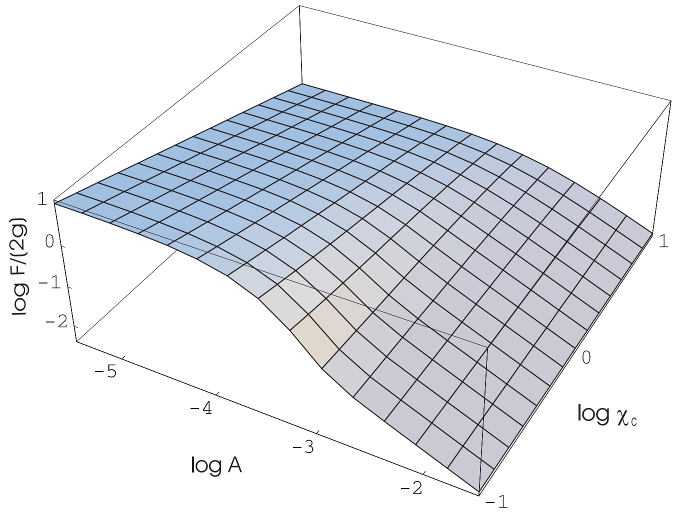

Fig. 3. Dependence of the monochromatic diffusion flux $\log\,\mathcal{F}(s_0,\xi;w)/(2g(s_0,\xi,\mathbf{n}))$ on A and $\chi_{\rm c}$ for a single Lorentzian line on a continuum for $\xi-\xi_0=0$, $w=10^{-4}$, and $\gamma=10^{-4}$. The behavior also reflects the effect of the motions on the wavelength-integrated flux and hence on the generalized Rosseland mean opacity.

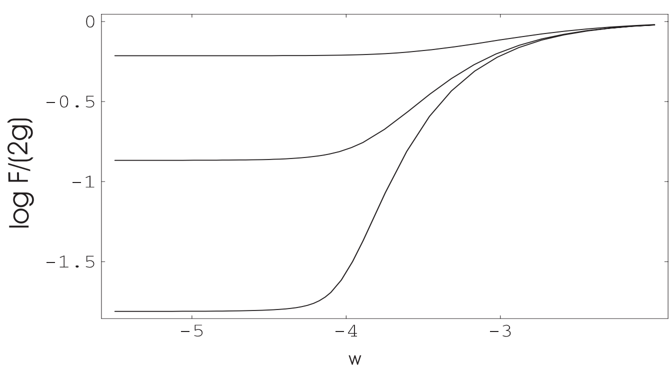

Fig. 4. $\log\,\mathcal{F}(s_0,\xi;w)/(2g(s_0,\xi,\mathbf{n}))$ as a function of w for $\gamma=10^{-3}$, 10^{-4}, and 10^{-5} (*top* to *bottom*) in the center ($\xi-\xi_0=0$) of a single Lorentzian line on a continuum with $\chi_{\rm c}=1$ and $A=0.001$ (cf. also Fig. 3).

7. Concluding remarks and outlook

Another equivalent way to interpret Eq. (41) is obtained by introducing the *mean* extinction coefficient at ξ over the *interval* Δ,

$$\bar{\chi}(\xi;\Delta)=\frac{1}{\Delta}\int\limits_{\xi-\Delta}^{\xi}\chi(\zeta)d\zeta\,, \tag{53}$$

instead of the spectral thickness. Then

$$\frac{\psi(\xi)-\psi(\xi-ws)}{w}=\frac{1}{ws}\int\limits_{\xi-ws}^{\xi}\chi(\zeta)d\zeta\cdot s=\bar{\chi}(\xi;ws)\cdot s \tag{54}$$

so that the radiative quantities such as the flux $\mathcal{F}$ or the radiative force for moving media can – in the deterministic case – completely be described *either* by the spectral thickness *or* by the set of the mean extinction coefficients $\bar{\chi}(\xi;ws)$ for the relevant ranges in ξ and ws.

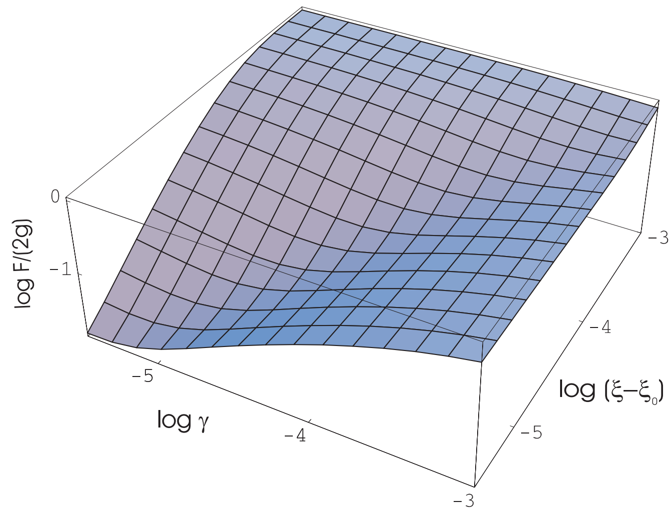

Fig. 5. $\log\,\mathcal{F}(s_0,\xi;w)/(2g(s_0,\xi,\mathbf{n}))$ as a function of $(\xi-\xi_0)$ and γ for $w=0.0001$, $\chi_{\rm c}=1$, and $A=0.001$ for a single Lorentzian line on a continuum (cf. also Fig. 3).

In order to elucidate the connection to the static diffusion, we consider the very special case that the mean extinction coefficient does *not* depend on the interval ws, i.e. $\bar{\chi}(\xi;ws)=\bar{\chi}(\xi)$. Then, using (54), the integration over depth in Eq. (41) can be performed, and with Eq. (49) leads to

$$\frac{1}{\bar{\chi}_\beta(s_0;w)}=\int_{-\infty}^{\infty}\frac{G(s_0,\xi)}{\bar{\chi}(\xi)}\,d\xi\,. \tag{55}$$

This result resembles the usual static Rosseland mean except that now for the moving medium the mean extinction coefficient $\bar{\chi}(\xi)$ over the interval ws replaces the "ordinary" monochromatic extinction coefficient $\chi(\xi)$.

In conclusion, we have derived this paper general expressions for the radiative flux and acceleration in arbitrarily shaped, optically very thick, and differentially moving media far from the surfaces. These expressions are basically rather simple but the integrals involved can be evaluated analytically only in very special cases. In addition, the dependencies on the input parameters are not immediately evident. We therefore have presented here only numerical results of the monochromatic flux for a single Lorentzian spectral line on a continuum which show that there is in fact a quite intricate interplay of the parameters. We have restricted the discussion of our examples on monochromatic quantities in order investigate the relative importance of the line core and the near and far wings.

In astronomical applications strictly monochromatic radiative quantities are only rarely of interest, they more or less serve as the basis for calculating the more important *wavelength-integrated* quantities.

In Paper II we show that in the limits of small and large velocity gradients much more insight can be gained. In particular, it is demonstrated that for $w=0$ Rosseland's result for the static case is regained. In addition, for isolated narrow

Lorentzian lines on a flat continuum the wavelength integrals can be obtained analytically. When there are many overlapping lines, convenient expressions can be obtained for large velocity gradients, whereas for small w the integration over ξ has to be carried out numerically. Since many wavelength points have to be considered and the integrand requires numerical differentiations of the extinction coefficient such calculations are quite demanding in programming and CPU time. A more satisfactory approach, however, is to describe the extinction coefficient by a Poisson point process (cf. Wehrse et al. 1998) since it allows to derive the expectation values for the flux and the acceleration in terms of the mean line separation and of the shapes and strengths of the lines as well as of the continuum. The formalism will be presented in a subsequent paper.

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