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The dynamo effect of magnetic flux tubes

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Abstract. It is shown that toroidal magnetic flux tubes in a rotating star do not provide a *net* dynamo effect, even if they are subject to random forcing, unless the tubes are unstable to small displacements. The dynamo effect is produced by transverse helical waves that propagate along the flux tube, resulting in an electric current (anti-) parallel to the unperturbed magnetic field, equivalent to an *alpha effect*. For unstable flux tubes, the alpha effect is enabled by a systematic phase difference between the velocity and magnetic field perturbations. For stable flux tubes subject to random forcing, phase differences do occur, but they vanish in the mean. The requirement for instability is met if the magnetic field strength exceeds a threshold value. Therefore a stellar dynamo based on flux tubes is not self-excited, but needs triggering until the magnetic threshold is surpassed.

Key words: Magnetohydrodynamics (MHD) – stars: magnetic fields – stars: late-type – Sun: magnetic fields

1. Introduction

A key issue of stellar dynamo theory concerns the maintenance of the poloidal magnetic field. While a toroidal magnetic field can be generated from poloidal components by differential rotation, the reverse process requires an *alpha effect* of some kind. Schematically, the alpha effect can be thought of as a process by which poloidal loops are formed around an initially toroidal magnetic field. Through the Coriolis force, rotation can provide such loops with a systematic orientation, and this can result in the generation of a net large-scale poloidal magnetic field.

The classical kinematic alpha effect is based on passive advection of the magnetic field by ambient convection, and the magnetic field strengths that it can produce are on the order of the equipartition field strength with respect to ambient convection. If the magnetic field strength is larger, advection is inhibited by Lorentz forces. For this reason, the kinematic alpha effect does not function in the overshoot region of the Sun and late-type stars, where a strong toroidal field ($B \approx 10^5$ G) can be stored (Caligari et al. 1995), but it can operate in sections of the convection zone above it, where the magnetic field is weaker. One possible scenario for dynamo action in late-type stars is therefore based on a spatial separation between the differential rotation, which occurs near the overshoot region, and the alpha effect in the convection zone (Parker 1993, Tobias 1996, Charbonneau & MacGregor 1997, Ossendrijver & Hoyng 1997). The two regions communicate through flux transport by downdrafts and/or overshooting convection.

Another scenario is based on a buoyancy instability of magnetic flux tubes. Magnetic buoyancy can result in a *dynamic* alpha effect which is not counteracted by strong fields but actually requires them. In the context of the thin flux tube approximation, the alpha effect due to a single unstable tube mode was calculated by Ferriz-Mas et al. (1994). MHD simulations have confirmed that in a rotating medium the magnetic buoyancy instability gives rise to an alpha effect (Brandenburg & Schmitt 1998). In a different guise, the idea of a buoyancy-driven solar dynamo was proposed already by Babcock (1961) and Leighton (1969). With a magnetic alpha effect, the question arises what happens when the magnetic field strength is below the instability threshold. Is there a residual alpha effect due to the random motions that perturb the stable flux tubes, or is there no alpha effect? In the present paper, this question is answered on the basis of a thin flux tube calculation, and the result is negative. Finally, some consequences of this result are briefly discussed.

2. The alpha effect of thin toroidal magnetic flux tubes

In the magnetic induction equation,

$$\left\{\frac{\partial}{\partial t} - \eta\nabla^2\right\}\boldsymbol{B} = \nabla \times (\boldsymbol{U} \times \boldsymbol{B}), \tag{1}$$

the alpha effect is represented by the term $\nabla \times (\boldsymbol{u} \times \boldsymbol{b})$, where $\boldsymbol{u} \times \boldsymbol{b}$ is the electromotive force (EMF) resulting from perturbations in the velocity, $\boldsymbol{u} = \boldsymbol{U} - \boldsymbol{U}_0$, and the magnetic field, $\boldsymbol{b} = \boldsymbol{B} - \boldsymbol{B}_0$. The goal is to calculate the mean EMF resulting from small perturbations of thin flux tubes around an initial equilibrium state defined by $\boldsymbol{r}(\boldsymbol{a}, 0) = \boldsymbol{a} = R_0\, \boldsymbol{e}_R + z_0\, \boldsymbol{e}_z$ and $\boldsymbol{B}(\boldsymbol{a}, 0) = B_0\, \boldsymbol{e}_\phi$. Here R, ϕ and z denote cylindrical coordinates; see Fig. (1) for the geometry. Since B_0 is a constant, the mean EMF has no contribution arising from gradients of the mean magnetic field, so that the alpha tensor can be defined through

$$\langle \boldsymbol{u} \times \boldsymbol{b} \rangle = \alpha \boldsymbol{B}_0. \tag{2}$$

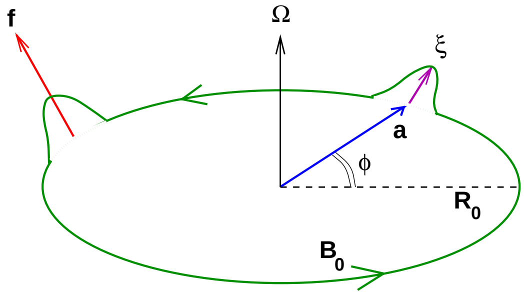

Fig. 1. Geometry of thin toroidal flux tubes. The equilibrium state is a circular tube with radius R_0 and field strength B_0, that lies in a plane parallel to the equatorial plane; $\boldsymbol{\xi}$ denotes the Lagrangian displacement, and $\boldsymbol{f}$ is an external force.

For a purely toroidal axisymmetric initial field, three components of α, namely $\alpha_{i\phi} = \langle \boldsymbol{u} \times \boldsymbol{b} \rangle_i / B_0$ ($i = R, \phi, z$), can be determined. Of these, $\alpha_{\phi\phi}$ is most important for the generation of the poloidal magnetic field. The Lagrangian displacement from the equilibrium configuration is defined as

$$\boldsymbol{r}(\phi, t) = \boldsymbol{a}(\phi) + \boldsymbol{\xi}(\phi, t); \qquad \boldsymbol{B}(\boldsymbol{r}, t) = \boldsymbol{B}_0 + \boldsymbol{b}(\boldsymbol{\xi}, t), \tag{3}$$

where ϕ stands for ϕ_0, i.e. longitude along the unperturbed tube. To good approximation, the flux tube's magnetic field is given by the Cauchy solution of the induction equation of ideal MHD (Eq. 1 with $\eta = 0$),

$$\frac{B_i(\boldsymbol{r}, t)}{\rho(\boldsymbol{r}, t)} = \sum_j \frac{\partial r_i(\boldsymbol{a}, t)}{\partial a_j} \frac{B_j(\boldsymbol{a}, 0)}{\rho(\boldsymbol{a}, 0)}. \tag{4}$$

The perturbations are assumed to be small, and therefore only terms of $\mathcal{O}(|\boldsymbol{\xi}|) \ll |\boldsymbol{a}|$ are retained, yielding

$$\boldsymbol{b}(\boldsymbol{r}, t) = (\boldsymbol{B}_0 \cdot \nabla_{\boldsymbol{a}})\, \boldsymbol{\xi}(\boldsymbol{a}, t) = \frac{B_0}{R_0} \frac{\partial \boldsymbol{\xi}}{\partial \phi}. \tag{5}$$

Since $\boldsymbol{u} = \partial \boldsymbol{\xi} / \partial t$, the EMF can now be expressed solely in terms of $\boldsymbol{\xi}$:

$$\boldsymbol{u} \times \boldsymbol{b} = \frac{B_0}{R_0} \frac{\partial \boldsymbol{\xi}}{\partial t} \times \frac{\partial \boldsymbol{\xi}}{\partial \phi}. \tag{6}$$

What remains to be done is to solve an equation of motion for $\boldsymbol{\xi}$, insert the result into (6), and, in the case of randomly forced flux tubes, perform an average.

3. Single eigenmodes

The linearised equation of motion for a toroidal thin magnetic flux tube in a rotating, stratified stellar convection zone was derived e.g. by Ferriz-Mas & Schüssler (1995), and can be written as

$$\mathsf{R}\, \boldsymbol{\xi} = 0, \tag{7}$$

where R is a differential operator containing terms up to second order in $\partial/\partial t$ and $\partial/\partial \phi$. Ferriz-Mas et al. (1994) calculated the alpha effect of a single eigenmode. Their result can be summarized as follows. The eigenmodes of R are found by inserting the Fourier ansatz

$$\boldsymbol{\xi} = \mathrm{Re}\, \tilde{\boldsymbol{\xi}}\, \mathrm{e}^{\mathrm{i}(m\phi + \omega_{km} t)} \tag{8}$$

and solving the resulting sixth-order dispersion relation (Appendix A). The eigenfrequencies, ω_{km} ($k = 1, 2, \cdots, 6$), are either real in the case of stable modes, or occur in a complex conjugate pair, which signifies instability. Allowing for a complex frequency, $\omega_{km} = \mathrm{Re}\, \omega_{km} - \mathrm{i}\gamma_{km}$, the EMF (6) can be written

$$\boldsymbol{u} \times \boldsymbol{b} = \frac{\mathrm{i} m \gamma_{km}}{2} \frac{B_0}{R_0}\, \tilde{\boldsymbol{\xi}}_{km} \times \tilde{\boldsymbol{\xi}}^*_{km}\, \mathrm{e}^{2\gamma_{km} t}, \tag{9}$$

where $\tilde{\boldsymbol{\xi}}_{km}$ is an eigenvector of the matrix $\hat{\mathsf{R}}$ (Appendix A). It follows that an alpha effect is possible for a single eigenmode under two conditions. First, the mode must be non-axisymmetric ($m \neq 0$), so that the Coriolis force can twist the tube. Second, the mode must be unstable ($\gamma_{km} \neq 0$). Only then can velocity and magnetic field perturbations have a phase difference, so that they are not parallel.

4. Stable flux tubes with random forcing

Stable flux tubes can also yield a nonvanishing EMF, because phase differences between the velocity and magnetic field perturbations can arise if different eigenmodes are superposed. Consider therefore a stable flux tube under the influence of an external force, which can be thought of as convective overshooting. The linearised equation of motion of a forced flux tube is[1]

$$\mathsf{R}\, \boldsymbol{\xi} = \boldsymbol{f}, \tag{10}$$

where $\boldsymbol{f}(\phi, t)$ is a forcing function to be specified later. A convenient way of solving $\boldsymbol{\xi}$ is to apply a Fourier transform, which is allowed since only stable flux tubes are considered:

$$\begin{aligned} \hat{\boldsymbol{\xi}} &= \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dt\, \boldsymbol{\xi}\, \mathrm{e}^{-\mathrm{i}(m\phi + \omega t)} \qquad \Leftrightarrow \\ \boldsymbol{\xi} &= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega\, \hat{\boldsymbol{\xi}}\, \mathrm{e}^{\mathrm{i}(m\phi + \omega t)}, \end{aligned} \tag{11}$$

and similarly for $\boldsymbol{f}$. The Fourier transform of (10) provides

$$\hat{\mathsf{R}}\, \hat{\boldsymbol{\xi}} = \hat{\boldsymbol{f}}, \qquad \Leftrightarrow \qquad \hat{\boldsymbol{\xi}} = \hat{\mathsf{I}}\, \hat{\boldsymbol{f}}, \tag{12}$$

where the matrix $\hat{\mathsf{I}}$ is the inverse of $\hat{\mathsf{R}}$ (Appendix A). Hence the flux tube displacement is

$$\boldsymbol{\xi} = \frac{1}{2\pi} \sum_m \int d\omega\, \hat{\mathsf{I}}\, \hat{\boldsymbol{f}}\, \mathrm{e}^{\mathrm{i}(m\phi + \omega t)}. \tag{13}$$

On symmetry grounds, the random forcing term should be *homogeneous*, i.e. its statistical properties should be independent of ϕ and t. Consider therefore a force with mean value $\langle \boldsymbol{f} \rangle = 0$ and a correlation function that satisfies

$$\langle f_i(\phi, t)\, f_j(\phi', t') \rangle = F_{ij}\, \psi(\phi - \phi')\, \chi(t - t'), \tag{14}$$

where $\langle \cdots \rangle$ denotes an ensemble average over the realisations of $\boldsymbol{f}$, and F is a constant correlation matrix. The actual shape of

[1] The present analysis covers only the general case of *additive* forcing in the equation of motion, thus excluding more exotic types of forcing, e.g. through variation of parameters.

ψ and χ is irrelevant; if $\boldsymbol{f}$ has a finite correlation length $\phi_{\rm c}$ and correlation time $t_{\rm c}$, then ψ and χ are peaked functions centered at the origin with a width of order $\phi_{\rm c}$ and $t_{\rm c}$, respectively. Inserting (13) into (6), the ϕ-component of the mean EMF can be written

$$\langle \boldsymbol{u} \times \boldsymbol{b} \rangle_\phi = \frac{1}{(2\pi)^2} \frac{B_0}{R_0} \sum_{ij} \sum_{mn} \int d\omega_1 d\omega_2 \, (m\omega_2 - n\omega_1) \times \hat{I}_{zi}(m,\omega_1)\, \hat{I}^*_{Rj}(n,\omega_2)\, \langle \hat{f}_i(m,\omega_1) \hat{f}^*_j(n,\omega_2) \rangle \times {\rm e}^{{\rm i}(m-n)\phi + {\rm i}(\omega_1-\omega_2)t}. \tag{15}$$

Similar expressions hold for the other components of the EMF. As is shown in Appendix B, the Fourier transform of the correlation function (14) is

$$\langle \hat{f}_i(m,\omega_1) \hat{f}^*_j(n,\omega_2) \rangle = 2\pi F_{ij}\, \hat{\psi}(m)\, \hat{\chi}(\omega_1) \times \delta(\omega_1 - \omega_2)\, \delta_{mn}. \tag{16}$$

If this expression is inserted into (15), it immediately follows that the mean EMF vanishes for any homogeneous random forcing term. In other words, stable flux tubes subject to random forcing do not provide an alpha effect.

This result is general in that it applies for *arbitrary* additive random forcing, provided that the force is statistically homogeneous (14), and that the resulting perturbations are small so that the linearised equation of motion (10) is valid. Inspection of Eq. (15) suggests that the alpha effect vanishes because all of its Fourier components are subject to destructive phase mixing. In order to clarify the situation, forcing by randomly distributed delta pulses is considered in more detail. Random delta pulses are a particularly suitable example of homogeneous forcing, since the resulting flux tube displacements can be calculated analytically.

4.1. Randomly distributed delta pulses

Consider a series of randomly distributed delta pulses, i.e.

$$\boldsymbol{f} = \sum_p \boldsymbol{g}_p\, \delta(\phi - \phi_p)\, \delta(t - t_p), \tag{17}$$

where $\boldsymbol{g}_p$ is a random vector, which can be assumed real without loss of generality since a possible imaginary part of $\boldsymbol{f}$ has no effect on the real part of $\boldsymbol{\xi}$.

Before calculating the mean EMF, it is shown that random delta pulses are a form of homogeneous forcing. The ensemble average of the correlation function, which involves the random parameters $\{\boldsymbol{g}_r, \phi_r, t_r\}$, factorizes as follows:

$$\langle f_i(\phi,t)\, f_j(\phi',t') \rangle = \sum_{pq} \langle \delta(\phi-\phi_p)\, \delta(\phi'-\phi_q) \rangle_{\{\phi_r\}} \times \langle g_{ip}\, g_{jq} \rangle_{\{\boldsymbol{g}_r\}}\, \langle \delta(t-t_p)\, \delta(t'-t_q) \rangle_{\{t_r\}}. \tag{18}$$

Assuming consecutive pulses to be independent, one has

$$\langle g_{ip}\, g_{jq} \rangle_{\{\boldsymbol{g}_r\}} = G_{ij}\, \delta_{pq}, \tag{19}$$

where G describes correlations between the components of $\boldsymbol{g}$. The other averages can now be performed using $p = q$. After rearranging products of delta functions that have a common variable in the argument according to $\delta(\phi-\phi_p)\,\delta(\phi'-\phi_p) = \delta(\phi-\phi')\,\delta(\phi-\phi_p)$ and $\delta(t-t_p)\,\delta(t'-t_p) = \delta(t-t')\,\delta(t-t_p)$, the remaining random terms are averaged, assuming a pulse distribution that is homogeneous in ϕ and t (*shot noise*). The spatial average yields

$$\langle \delta(\phi-\phi_p) \rangle_{\{\phi_r\}} = \frac{1}{2\pi} \int_0^{2\pi} d\phi_p\, \delta(\phi-\phi_p) = \frac{1}{2\pi}. \tag{20}$$

The time average is performed by envoking Campbell's theorem, which concerns the mean effect of an infinite number of independent signals, occurring at random unordered times $t_1, t_2, \cdots$. If the signals are described by narrow peaks $\zeta(t-t_p)$ centered at $t = t_p$, then the theorem states that

$$\sum_p \langle \zeta(t-t_p) \rangle_{\{t_r\}} = \nu \int dt\, \zeta(t), \tag{21}$$

where $1/\nu$ is the average time between two pulses. The proof is easy and can be found in Rice (1954). Applying Campbell's theorem, one finds

$$\sum_p \langle \delta(t-t_p) \rangle_{\{t_r\}} = \nu \int dt\, \delta(t) = \nu. \tag{22}$$

Altogether, the correlation function for randomly distributed delta pulses becomes

$$\langle f_i(\phi,t)\, f_j(\phi',t') \rangle = \frac{\nu G_{ij}}{2\pi}\, \delta(\phi-\phi')\, \delta(t-t'). \tag{23}$$

This is a special case of (14), which proves that random delta pulses are a form of homogeneous forcing.

4.2. A single delta pulse

Consider a forcing term consisting of a single delta pulse,

$$\boldsymbol{f} = \boldsymbol{g}\, \delta(\phi-\phi_1)\, \delta(t-t_1) \Leftrightarrow \hat{\boldsymbol{f}} = \frac{1}{2\pi} \boldsymbol{g}\, {\rm e}^{-{\rm i}(m\phi_1 + \omega t_1)}. \tag{24}$$

The flux tube displacement (13) is obtained through complex integration in the ω-plane along a contour C consisting of a line segment C_1 parallel to the real axis and a large semicircle C_2. The integrand has poles on the real axis located at $\omega_{1m}, \omega_{2m}, \cdots, \omega_{6m}$. If $t > t_1$, C_2 has to be placed in the upper halfplane ($\mathrm{Im}\,\omega > 0$) for convergence; if $t < t_1$, one must put C_2 in the lower halfplane ($\mathrm{Im}\,\omega < 0$). In both cases C_1 must pass below the poles in order to obtain the retarded solution and not the advanced. Cauchy's theorem then yields

$$\xi_i = \frac{{\rm i}H(t-t_1)}{2\pi} \sum_j \sum_m \sum_k {\rm Res}_k \{ \hat{I}_{ij}(m,\omega) \}\, g_j \times {\rm e}^{{\rm i}m(\phi-\phi_1) + {\rm i}\omega_{km}(t-t_1)}, \tag{25}$$

where k labels the first-order poles (degenerate modes, if present, do not contribute to $\boldsymbol{\xi}$). In order to simplify the result, the following notation is introduced:

$$\Delta t = t - t_1, \quad \Delta\phi = \phi - \phi_1, \tag{26}$$

and

$$\tilde{\xi}_{ikm} = \frac{\mathrm{i}}{2\pi} \sum_j \mathrm{Res}_k\{\hat{I}_{ij}(m,\omega)\}\, g_j, \tag{27}$$

where i labels the components of the complex vector $\tilde{\boldsymbol{\xi}}_{km}$, which describes the response of mode (k,m) to the force $\boldsymbol{g}$. Reality of $\boldsymbol{\xi}$ is ensured because $\tilde{\boldsymbol{\xi}}^*_{km} = \tilde{\boldsymbol{\xi}}_{k,-m}$ (Appendix A). The displacement vector becomes

$$\boldsymbol{\xi} = H(\Delta t) \sum_m \sum_k \tilde{\boldsymbol{\xi}}_{km}\, \mathrm{e}^{\mathrm{i}(m\Delta\phi + \omega_{km}\Delta t)}, \tag{28}$$

and the velocity and magnetic field perturbations are

$$\boldsymbol{u} \;=\; \mathrm{i}H(\Delta t) \sum_m \sum_k \omega_{km}\, \tilde{\boldsymbol{\xi}}_{km}\, \mathrm{e}^{\mathrm{i}(m\Delta\phi + \omega_{km}\Delta t)}, \tag{29}$$

$$\boldsymbol{b} \;=\; \mathrm{i}H(\Delta t)\, \frac{B_0}{R_0} \sum_m \sum_k m\, \tilde{\boldsymbol{\xi}}_{km}\, \mathrm{e}^{\mathrm{i}(m\Delta\phi + \omega_{km}\Delta t)}. \tag{30}$$

Hence the EMF is given by

$$\begin{aligned} \boldsymbol{u} \times \boldsymbol{b} = H(\Delta t) \frac{B_0}{R_0} \sum_{mn} \sum_{kl} n\omega_{km}\, \tilde{\boldsymbol{\xi}}_{km} \times \tilde{\boldsymbol{\xi}}^*_{ln} \\ \times\, \mathrm{e}^{\mathrm{i}(m-n)\Delta\phi + \mathrm{i}(\omega_{km} - \omega_{ln})\Delta t}, \end{aligned} \tag{31}$$

which in general does not vanish identically, since $\boldsymbol{u}$ and $\boldsymbol{b}$ are not parallel. Next the ensemble average of the EMF over the realisations of $\boldsymbol{g}$, ϕ_1, and t_1 is performed, but in order to keep the expressions transparent, $\langle \tilde{\boldsymbol{\xi}}_{km} \times \tilde{\boldsymbol{\xi}}_{ln} \rangle_{\boldsymbol{g}}$ is not elaborated. The remaining random factors yield

$$\langle \mathrm{e}^{\mathrm{i}(m-n)\Delta\phi} \rangle_{\phi_1} = \frac{1}{2\pi} \int_0^{2\pi} d\phi_1\, \mathrm{e}^{\mathrm{i}(m-n)(\phi-\phi_1)} = \delta_{mn}, \tag{32}$$

and

$$\begin{aligned} \langle H(\Delta t)\, \mathrm{e}^{\mathrm{i}(\omega_{km} - \omega_{ln})\Delta t} \rangle_{t_1} &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt_1\, H(t - t_1) \\ &\quad \times \mathrm{e}^{\mathrm{i}(\omega_{km} - \omega_{ln})(t - t_1)} \\ &= \begin{cases} \frac{1}{2} & (\omega_{km} = \omega_{ln}) \\ 0 & (\omega_{km} \neq \omega_{ln}). \end{cases} \end{aligned} \tag{33}$$

This implies that all spatially and temporally periodic terms vanish, leaving only the constant contributions. Since all eigenmodes appearing in (31) are non-degenerate for a given m, (32) and (33) imply $k = l$, so that

$$\begin{aligned} \langle \boldsymbol{u} \times \boldsymbol{b} \rangle &= \frac{1}{2} \frac{B_0}{R_0} \sum_m \sum_k m\omega_{km}\, \langle \tilde{\boldsymbol{\xi}}_{km} \times \tilde{\boldsymbol{\xi}}^*_{km} \rangle_{\boldsymbol{g}} \\ &= \frac{1}{2} \frac{B_0}{R_0} \sum_{m>0} \sum_k m\omega_{km}\, \langle \tilde{\boldsymbol{\xi}}_{km} \times \tilde{\boldsymbol{\xi}}^*_{km} + \tilde{\boldsymbol{\xi}}^*_{km} \times \tilde{\boldsymbol{\xi}}_{km} \rangle_{\boldsymbol{g}} \\ &= 0. \end{aligned} \tag{34}$$

The explanation why forced stable flux tubes do not provide an alpha effect can now be summarized as follows. Terms that consist of *mixed* modes, i.e. $k \neq l$ or $m \neq n$, are spatially or temporally periodic and have random phases, so that their mean effect vanishes due to destructive phase mixing. Hence the mean EMF reduces to a sum over contributions of *single* modes, say, $\langle \boldsymbol{u} \times \boldsymbol{b} \rangle = \sum_{m>0} \sum_k \langle \boldsymbol{u}_{km} \times \boldsymbol{b}_{km} \rangle$. These contributions also vanish, because there is no phase difference between velocity and magnetic field perturbations for stable modes.

5. Conclusion

For magnetic field strengths below the threshold for the buoyancy instability, flux tubes in a stellar overshoot layer are stable, and, if the field strength is somewhat above the equipartition value, they can be perturbed by ambient fluid motions without being shredded. The calculations presented here show that such random forcing of stable flux tubes does not result in a net alpha effect. The reason for the vanishing of alpha is the absence of systematic phase differences between velocity and magnetic field perturbations, which leads to destructive phasemixing. Only for unstable flux tubes, systematic phase differences do exist, and a net alpha effect is possible.

The result holds for arbitrary, but sufficiently weak, statistically homogeneous random forcing. In particular, forces that are strong enough to cause a breakdown of the flux tube's stable equilibrium are excluded. Fluxtubes in solar-type stars can be stably stored only within certain latitude belts in the subadiabiatic overshoot layer underneath the convection zone, whose boundaries depend on the magnetic field strength and other parameters. At other latitudes and throughout the outer convection zone the flux tubes are unstable to small displacements. One could thus imagine an upward or meridional force that pushes the tubes so strongly that some loops reach an unstable locus. Subsequently these loops could develop systematic phase differences between velocity and magnetic field disturbances, resulting in a net alpha effect. But clearly the linearisation has become invalid here (and possibly much earlier already). Therefore such a situation is not covered by the present analysis, which assumes that the flux tube's equilibrium position remains stable in spite of the forcing. The observed virtual absence of sunspots during the Maunder minimum suggests that this assumption is indeed valid during grand minima of stellar activity. Perhaps a decrease in the magnetic field strength may explain this enhanced stability.

It may be surprising at first sight that the result holds for *any* stable toroidal flux tube, seemingly irrespective of the precise nature of the forces that act on the tube (gravity, Coriolis force, magnetic buoyancy), and independent of the strength of the random forcing compared to the other forces, as long as it is weak. This can be attributed to the symmetry of the linearised equation of motion (the matrix $\hat{\mathsf{R}}$), which is such that a flux tube is stable if and only if all eigenmodes are purely oscillatory. As long as the randomly forced tube is stable, its electromotive force is subject to destructive phasemixing, irrespective of the strength of the random forcing.

A consequence of the vanishing of the alpha effect for randomly perturbed stable flux tubes is that a stellar dynamo based on the buoyancy-driven alpha effect of magnetic flux tubes can-not be self-excited. The dynamo switches off if a significant number of flux tubes fail the instability criterion. Switching it on again is possible if a source of magnetic flux can be identified that operates during off phases. The obvious candidate is the convection zone, where turbulent dynamo action generates a small-scale, fluctuating magnetic field, which has a strength of at most the equipartition value. Overshooting convection or downdrafts transport some of the magnetic flux downward into the overshoot layer, where it can be stably stored. Depending on the sign of the field, this leads to accumulation or annihilation of flux, and may trigger occasional transitions from the on-state to the off-state or vice versa. The analysis therefore confirms that the buoyancy-driven alpha effect may explain the occurrence of grand minima of stellar activity, as was suggested by Ferriz-Mas et al. (1994).

Hence, ironically, the inability of stably stored magnetic flux tubes to produce an alpha effect when they are perturbed by a random external force fits well in a dynamo scenario based on the buoyancy-driven alpha effect of flux tubes (Schüssler 1993). The main advantage of such a scenario is that the strong super-equipartition magnetic fields that are thought to exist near the base of the convection zone of late-type stars do not counteract dynamo action, as would be the case with a kinematic alpha effect, but in fact produce it. The flux tube scenario is reminiscent of the models of Babcock (1961), Leighton (1969), Durney (1997), and Dikpati & Charbonneau (1999), the main difference being that the generation of the poloidal magnetic field does not take place at the solar surface, but in the magnetic storage layer below the convective envelope.

Numerical simulations of a one-dimensional mean-field model by Schmitt et al. (1996) have illustated that a buoyancy-driven alpha effect can explain the occurrence of grand minima in solar-type stars. A more elaborate two-dimensional model of the solar dynamo that incorporates downdrafts and small-scale dynamo action in the convection zone is presented elsewhere (Ossendrijver 2000).

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Appendix A: properties of stable eigenmodes

In the calculation of alpha, several symmetry properties of the eigenmodes were used. These properties can be easily verified by considering the matrix $\hat{\mathsf{R}}$, for which Ferriz-Mas & Schüssler (1995) have presented the following expression with respect to the basis $\{\boldsymbol{e}_R, \boldsymbol{e}_\phi, \boldsymbol{e}_z\}$:

$$\hat{\mathsf{R}} = \begin{pmatrix} -\omega^2 + 2m^2 f^2 - C_R & -2\mathrm{i}\omega\Omega - \mathrm{i}mC_\phi & -C_z \\ 2\mathrm{i}\omega\Omega - \mathrm{i}mA_R & -\omega^2 + 2m^2 f^2 c_T^2/v_A^2 & -\mathrm{i}mA_z \\ -D_R & -\mathrm{i}mD_\phi & -\omega^2 + 2m^2 f^2 - D_z \end{pmatrix} \quad \text{(A.1)}$$

For the meaning of the various constants, the reader is referred to Ferriz-Mas & Schüssler (1995). At present it suffices that they are all real, so that the determinant can be written

$$\begin{aligned} \Delta &= -(\omega^6 + c_4\omega^4 + c_3\omega^3 + c_2\omega^2 + c_1\omega + c_0) \\ &= -(\omega - \omega_{1m})(\omega - \omega_{2m}) \cdots (\omega - \omega_{6m}), \end{aligned} \quad \text{(A.2)}$$

with real coefficients c_k. Hence the eigenfrequencies ω_{km} are either real or occur in a complex conjugate pair, and, if ω is real, then Δ is also real. Since the presence of a complex eigenfrequency always means instability, while only stable flux tubes are considered, all eigenfrequencies are assumed to be real. This is true e.g. for a sufficiently weak magnetic field. Furthermore, the following properties hold for real ω:

$$\hat{\mathsf{R}}^*(m,\omega) = \hat{\mathsf{R}}(-m,-\omega), \quad \text{(A.3)}$$

$$\Delta(-m,-\omega) = \Delta(m,\omega), \quad \text{(A.4)}$$

$$\omega_{k,-m} = -\omega_{km}. \quad \text{(A.5)}$$

Here (A.3) is obvious from (A.1), (A.4) follows from (A.3), and (A.5) follows from (A.4). The inverse of $\hat{\mathsf{R}}$ is given by $\hat{I}_{ij} = (-1)^{i+j} \det(\hat{\mathsf{r}}_{ji})/\Delta$, where $\hat{\mathsf{r}}_{ij}$ is a 2×2 matrix obtained from $\hat{\mathsf{R}}$ by omitting row i and column j. It follows immediately from (A.3) that $\hat{I}^*_{ij}(m,\omega) = \hat{I}_{ij}(-m,-\omega)$. Using these results, it can finally be shown that

$$\mathrm{Res}_k\{\hat{I}^*_{ij}(m,\omega)\} = -\mathrm{Res}_k\{\hat{I}_{ij}(m,\omega)\}, \quad \text{(A.6)}$$

which ensures the reality of $\boldsymbol{\xi}$ (28).

Appendix B: Fourier transform of the correlation function for homogeneous forcing

The Fourier transform of the correlation function of the force $\boldsymbol{f}$ is defined as

$$\langle \hat{f}_i(m,\omega_1)\hat{f}^*_j(n,\omega_2)\rangle = \frac{1}{(2\pi)^2}\int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi_2 \int dt_1 \int dt_2 \times \langle f_i(\phi_1,t_1) f_j(\phi_2,t_2)\rangle\, \mathrm{e}^{-\mathrm{i}(m\phi_1 - n\phi_2) - \mathrm{i}(\omega_1 t_1 - \omega_2 t_2)}. \quad \text{(B.1)}$$

Statistical homogeneity is assumed (14), and a change of variables is made for t_1 and t_2 according to

$$\begin{cases} T_1 = t_1 + t_2 \\ T_2 = t_1 - t_2 \end{cases} \Leftrightarrow \begin{cases} t_1 = (T_1 + T_2)/2 \\ t_2 = (T_1 - T_2)/2, \end{cases} \quad \text{(B.2)}$$

with similar transformations applying for $(\phi_1,\phi_2) \to (\Phi_1,\Phi_2)$, $(\omega_1,\omega_2) \to (\Omega_1,\Omega_2)$ and $(m,n) \to (M,N)$. The integration range of Φ_2 and Φ_1 is the diamond A enclosed by straight lines $\Phi_1 + \Phi_2 = 0, 4\pi$ and $\Phi_1 - \Phi_2 = 0, 4\pi$. Using the identities $m\phi_1 - n\phi_2 = (M\Phi_2 + N\Phi_1)/2$ and $\omega_1 t_1 - \omega_2 t_2 = (\Omega_1 T_2 + \Omega_2 T_1)/2$, one finds

$$\begin{aligned} \langle \hat{f}_i(m,\omega_1)\hat{f}^*_j(n,\omega_2)\rangle &= \frac{F_{ij}}{4(2\pi)^2}\int_A d\Phi_1 d\Phi_2\, \psi(\Phi_2) \\ &\times \mathrm{e}^{-\mathrm{i}(M\Phi_2 + N\Phi_1)/2}\int dT_1\, \mathrm{e}^{-\mathrm{i}\Omega_2 T_1/2} \\ &\times \int dT_2\, \chi(T_2)\, \mathrm{e}^{-\mathrm{i}\Omega_1 T_2/2}. \end{aligned} \quad \text{(B.3)}$$

The integrations over T_1 and T_2 are independent and yield

$$\int dT_1\,\mathrm{e}^{-\mathrm{i}\Omega_2 T_1/2} \;=\; 2\pi\,\delta(\Omega_2/2) = 4\pi\,\delta(\Omega_2), \qquad \text{(B.4)}$$

$$\int dT_2\,\chi(T_2)\,\mathrm{e}^{-\mathrm{i}\Omega_1 T_2/2} \;=\; \sqrt{2\pi}\,\hat{\chi}(\Omega_1/2), \qquad \text{(B.5)}$$

where $\hat{\chi}$ denotes the Fourier transform of χ. Next comes $I = \int_A d\Phi_1 d\Phi_2\,\psi(\Phi_2)\exp\{-\mathrm{i}(M\Phi_2+N\Phi_1)/2\} = I_1+I_2$, where the respective integration domains of I_1 and I_2 are defined by

$$\int_A d\Phi_1 d\Phi_2 = \int_{-2\pi}^{0} d\Phi_2 \int_{-\Phi_2}^{4\pi+\Phi_2} d\Phi_1 + \int_0^{2\pi} d\Phi_2 \int_{\Phi_2}^{4\pi-\Phi_2} d\Phi_1. \qquad \text{(B.6)}$$

The integrations over Φ_1 yield

$$\int_{-\Phi_2}^{4\pi+\Phi_2} d\Phi_1\,\mathrm{e}^{-\mathrm{i}N\Phi_1/2} \;=\; \begin{cases} \dfrac{\mathrm{e}^{-\mathrm{i}N\Phi_2/2}-\mathrm{e}^{\mathrm{i}N\Phi_2/2}}{-\mathrm{i}N/2} & \text{(X)},\\ 4\pi+2\Phi_2 & \text{(Y)}, \end{cases} \qquad \text{(B.7)}$$

$$\int_{\Phi_2}^{4\pi-\Phi_2} d\Phi_1\,\mathrm{e}^{-\mathrm{i}N\Phi_1/2} \;=\; \begin{cases} -\dfrac{\mathrm{e}^{-\mathrm{i}N\Phi_2/2}-\mathrm{e}^{\mathrm{i}N\Phi_2/2}}{-\mathrm{i}N/2} & \text{(X)},\\ 4\pi-2\Phi_2 & \text{(Y)}. \end{cases} \qquad \text{(B.8)}$$

Here X is short for $N \neq 0$, and Y is short for $N = 0$. First consider the case $N \neq 0$:

$$\begin{aligned} I_1 \;&=\; -\frac{2}{\mathrm{i}N}\int_{-2\pi}^{0} d\Phi_2\,\psi(\Phi_2)\,\mathrm{e}^{-\mathrm{i}M\Phi_2/2}\Big\{\mathrm{e}^{-\mathrm{i}N\Phi_2/2}-\mathrm{e}^{\mathrm{i}N\Phi_2/2}\Big\}\\ &=\; -\frac{2}{\mathrm{i}N}\int_0^{2\pi} d\phi\,\psi(\phi)\,\mathrm{e}^{-\mathrm{i}M\phi/2}\Big\{\mathrm{e}^{-\mathrm{i}N\phi/2}-\mathrm{e}^{\mathrm{i}N\phi/2}\Big\}\\ &=\; -I_2 \quad (N\neq 0), \end{aligned} \qquad \text{(B.9)}$$

where the integration variable was changed to $\phi = \Phi_2 + 2\pi$, and the identities $\psi(\Phi_2) = \psi(\phi)$ and $\exp\{-\mathrm{i}\pi(M+N)\} = \exp\{-\mathrm{i}\pi(M-N)\} = 1$ were used. Hence I vanishes if $N \neq 0$. If $N = 0$, then

$$\begin{aligned} I_1 \;&=\; \int_{-2\pi}^{0} d\Phi_2\,\psi(\Phi_2)\,(4\pi+2\Phi_2)\,\mathrm{e}^{-\mathrm{i}m\Phi_2}\\ &=\; 2\int_0^{2\pi} d\phi\,\psi(\phi)\,\phi\,\mathrm{e}^{-\mathrm{i}m\phi} \quad (N=0), \end{aligned} \qquad \text{(B.10)}$$

where a change of variables $\phi = \Phi_2 + 2\pi$ was made. Similarly,

$$I_2 = \int_0^{2\pi} d\Phi_2\,\psi(\Phi_2)\,(4\pi-2\Phi_2)\,\mathrm{e}^{-\mathrm{i}m\Phi_2} \quad (N=0), \qquad \text{(B.11)}$$

so that

$$I = 4\pi\int_0^{2\pi} d\phi\,\psi(\phi)\,\mathrm{e}^{-\mathrm{i}m\phi} = 2\,(2\pi)^{3/2}\hat{\psi}(m) \quad (N=0), \qquad \text{(B.12)}$$

where $\hat{\psi}$ is the Fourier transform of ψ. Inserting (B.4), (B.5) and (B.12) into (B.3), one obtains

$$\begin{aligned} \langle \hat{f}_i(m,\omega_1)\hat{f}_j^*(n,\omega_2)\rangle &= 2\pi\,F_{ij}\,\hat{\psi}(m)\,\hat{\chi}(\omega_1)\\ &\quad\times\delta(\omega_1-\omega_2)\,\delta_{mn}. \end{aligned} \qquad \text{(B.13)}$$

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