Astron. Astrophys. 360, 1086–1095 (2000)

ASTRONOMY
AND
ASTROPHYSICS

Profiles of the spectral lines formed in stochastic multicomponent atmosphere

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Received 23 November 1999 / Accepted 11 April 2000

Abstract. An analytical procedure is explored to construct the profiles of the spectral lines formed in a stochastic multicomponent atmosphere. A model one-dimensional problem is solved for the atmospheres in both local thermodynamic equilibrium (LTE) and non-LTE. The results previously obtained by the authors for the LTE atmosphere are generalized to include, in particular, correlations of the random variations in the physical properties of the structural elements. It is shown that the stochastic multicomponent atmosphere may produce a kind of double-peaked profile, which is usually associated with the multiple scattering effects.

The problem of the line formation in a stochastic non-LTE atmosphere is considered, assumpting the complete redistribution over frequencies with allowance for the absorption in continuum. For the line profile, a closed-form analytical expression is derived. The computed profiles are compared with those for an homogeneous deterministic atmosphere with preliminarily averaged physical properties. This allows us to conclude that the discrepancy between the line profiles relevant to these two cases may become significant, whereas the normalized profiles are fairly close to each other.

Key words: radiative transfer – Sun: atmosphere – Sun: prominences – stars: atmospheres

1. Introduction

In previous papers (Nikoghossian et al. 1997, 1998 - hereafter referred to as Paper I) we developed a theory to determine the statistical properties of the line radiation emerging from a stochastic multicomponent atmosphere. Two extreme situations were considered: a pure absorbing atmosphere, or atmosphere in LTE, and a pure scattering non-LTE atmosphere. In these investigations we were interested mainly in statistical properties of the observable radiation integrated over the whole frequency domain of the spectral line. The line profiles were calculated only for the LTE atmosphere. Allowing for the multiple scattering process is much more complicated because of coupling between various volumes of the medium and the frequency redistribution effects. Beginning with relatively simple models for the structural pattern of an atmosphere may, however, give a rough idea of the non-LTE effects upon the spectral line profile.

The main objective of this paper is to develop an analytical procedure for evaluating the profiles of the lines originated in a model stochastic multicomponent atmosphere as first proposed by Jefferies and Lindsey (1988). The solution of this problem given in Paper I for the LTE atmosphere revealed quantitative as well as qualitative discrepancies with respect to those formed in a homogeneous atmosphere with preliminarily averaged random characteristics. The question to be answered is what changes does multiple scattering introduce into this picture.

Our treatment is motivated by the study of the extreme ultraviolet (EUV) spectra of solar quiescent prominences, the composite fine structure of which is now well-established. Among other EUV lines, the hydrogen resonant line Ly-α is prominent because it gives information on the relatively cold regions of prominences. The vast observational material on this line leads us to seek an adequate interpretation of the observed intensities. The large opacity of the medium in Ly-α and smallness of the destruction probability indicate the importance of effects introduced by the radiation diffusion process. One of the reasons that makes the Ly-α formation problem even more complicated is the strongly inhomogeneous structure of atmosphere, the physical characteristics of which undergo random variations. This situation causes essential difficulties for theoretical treatment, which has not yet been handled separately.

As in Paper I, our approach to tackling the problem is based on the concept of photon escape probability (see, e.g., Sobolev 1963). This allows one to derive a closed-form analytical expression for the mean frequency profile of the spectral line for any value of the destruction coefficient ϵ. By mean profile we understand an average over a large number of realizations of the stochastic multicomponent atmosphere. The radiation diffusion in the medium is assumed to obey the complete frequency redistribution with allowance for the absorption in continuum.

The outline of the paper is as follows: We begin, in the opening Sect. 2, by formulating the problem and recalling in brevity the results obtained in Paper I for the LTE atmosphere, and needed for further discussion. Some generalizations, particularly the case of correlated variations of physical features

Send offprint requests to: Z. Mouradian of structural elements, are considered. Sect. 3 is devoted to the stochastic non-LTE atmosphere. A closed-form analytical expression for the observable emergent intensity is derived. The calculations are performed for the internal energy sources distributed over frequency as the line absorption profile. The results are discussed and summarized in Sect. 4.

2. Formulation of the problem. The LTE atmosphere

The model problem to be discussed below was first considered by Jefferies & Lindsey (1988). It possesses at least two advantages. Despite its relative simplicity, which makes it suitable for expository purposes, this model problem permits elucidation of the important specific features of the multicomponent stochastic problems. Secondly, beginning with this model case makes it is easier to find out the range of more general and realistic problems, for which the developed formalism remains in force. From now on we shall limit our attention to the one-dimensional problem, although the majority of the results may be extended to comprise 3D-problems as well.

Let us treat a composite atmosphere consisting of N structural elements each of which is characterized by the power of the radiative energy release B and optical thickness τ. The value of B is assumed to be constant within each individual component. We suppose that the quantities B and τ are random and correspondingly take one of the values B_i and τ_i $(i = 1, 2, ...n)$ with probability p_i, so that $\sum_{i=1}^{n} p_i = 1$. For simplicity's sake, we assume also that there is no radiation incident on the medium from outside, although it can be easily included if necessary.

Thus, suppose that we have the LTE atmosphere with the structure just described and are interested in the mean intensity $\langle I_N \rangle$ of emerging radiation. The assumption of LTE is crucial in that the requisite quantity may be found directly on the basis of solely physical considerations in this case. This simplification is due to the absence of scatterings, which leads in turn to the absence of the beams reflected from components. This means that the radiation intensity at any fixed point of the medium is a functional only of intensities at the points preceding the given one along the path of rays, so that the averaging procedure may be performed in parts to yield

$$\langle I_N \rangle = q\langle I_{N-1} \rangle + \langle I_1 \rangle, \tag{1}$$

where $q = \langle \exp(-\tau) \rangle$ is the mean opacity of atmosphere, and $\langle I_1 \rangle = \langle B[1 - \exp(-\tau)] \rangle$. The more general result reads

$$\langle I_N \rangle = q^k \langle I_{N-k} \rangle + \langle I_k \rangle, \tag{2}$$

and finally

$$\langle I_N \rangle = L_N \langle I_1 \rangle, \tag{3}$$

where $L_N = \left(1 - q^N\right) / (1 - q)$.

The more detailed and rigorous derivation of Eqs. (1)-(3) is given in Paper I[1]. Eqs. (1)-(3) were first obtained by Jefferies & Lindsey (1988) for the case $n = 2$. As noted in Paper I, the restriction imposed on n is not essential, and the results hold for arbitrary n. Moreover, it is easy to extend the above equations to cover more general situations, when B and τ are continuously distributed (rather than discrete) random quantities. Indeed, let $f(B, \tau)\, \mathrm{d}B\mathrm{d}\tau$ be the joint probability for finding the values of quantities B and τ within intervals B, $B + \mathrm{d}B$ and τ, $\tau + \mathrm{d}\tau$, respectively. The above reasoning concerning the LTE atmosphere then immediately allows us to write in place of Eq. (1)

$$\langle I_N \rangle = \int_0^\infty \mathrm{d}B \int_0^\infty f(B, \tau) \times \left[\langle I_{N-1} \rangle \exp(-\tau) + B(1 - \exp(-\tau))\right] \mathrm{d}\tau. \tag{4}$$

Now observing that

$$\langle I_1 \rangle = \int_0^\infty \mathrm{d}B \int_0^\infty f(B, \tau)\, B(1 - \exp(-\tau))\, \mathrm{d}\tau, \tag{5}$$

we are led to Eq. (1). The derivation of counterparts of Eqs. (2)-(3) is straightforward.

When the quantities B and τ are statistically independent, i.e., $f(B, \tau) = f_1(B)\, f_2(\tau)$, Eq. (4) is simplified to

$$\langle I_N \rangle = \langle B \rangle \left(1 - q^N\right), \tag{6}$$

In fact, Eq. (6) could be obtained directly by averaging the well-known expression for the intensity of observed radiation $I = B(1 - \exp(-N\tau))$. If the medium is illuminated from outside by radiation of intensity I_0 its contribution into observed intensity is determined by an additional item $I_0 q^N$ appearing in the right-hand side of Eqs. (1)-(4).

Up to here the random variations of physical properties of components were assumed to be non-correlated. It is easy to show that the consideration of correlated variations for the LTE atmosphere does not introduce additional difficulties of principle. For simplicity, we shall demonstrate it for the case when correlations are Markovian, so that the probability of finding the kth component at a certain state j (i.e., characterized by the pair of parameters B_j, τ_j) depends only on the state realized for the previous $(k-1)$th component along the path of the ray. It is well known (see Bharucha-Reid 1960) that such a random process is described by the probability array $p = (p_1, p_2, ...p_n)$, characterizing the probability distribution of the initial component, and by the Markovian transition matrix $T = (t_{ij})$, elements of which give the conditional probabilities of transition between the states i and j. We suppose that the process is stationary, i.e., the elements t_{ij} depend merely on the states i and j, rather than on the ordinal number of the structural element.

Consider a stochastic atmosphere consisting of N components, each of which may be found at n possible states. Again by performing the averaging process for LTE in parts, we shall employ Eq. (1) to derive the solution of the problem at hand. To this end, let us introduce the quantity $\langle I_N^{(k)} \rangle$, denoting the mean emergent intensity for configurations with the (Nth) element

[1] here we use the notation q instead of α adopted in Paper I, reserving the latter for the profile of the absorption coefficient introduced below.

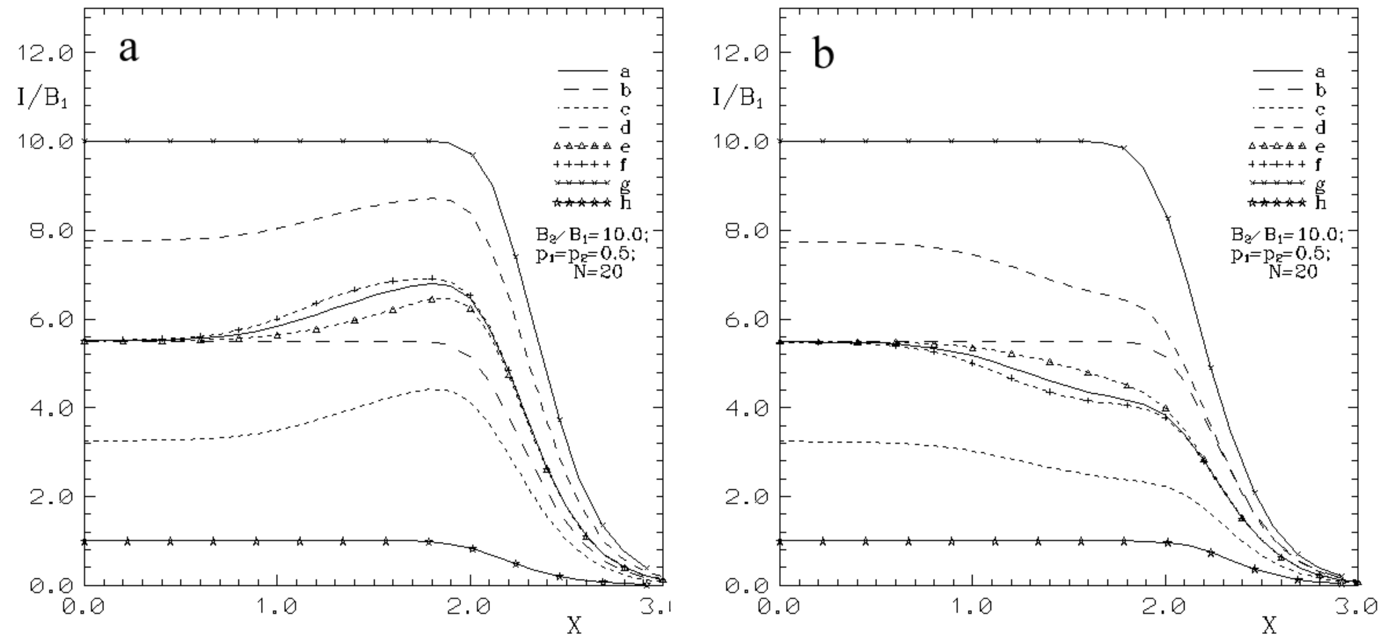

Fig. 1a and b. The profiles of the LTE lines for: **a** $\tau_1 = 5.0$, $\tau_2 = 10.0$, **b** $\tau_1 = 10.0$, $\tau_2 = 5.0$ and (a) stochastic atmosphere with pure random variations; (b) homogeneous atmosphere with averaged physical properties; (c)-(f) stochastic atmosphere with Markovian correlations between components defined by the transition matrices T_c,T_d,T_e,T_f; (g) homogeneous atmosphere with $B = 10$ and $\tau_0 = N\tau_2$; (h) homogeneous atmosphere with $B = 1$ and $\tau_0 = N\tau_1$.

being found at the state k. By using the Markovian property of the process the procedure similar to that when deriving Eq. (1) allows one to write

$$\langle I_N^{(k)} \rangle = \sum_{i=1}^{n} t_{ik} \left[\langle I_{N-1}^{(k)} \rangle \exp\left(-\tau_k\right) + p_i^{(N-1)} J_k \right], \qquad (7)$$

$$(N = 2, 3, ...)$$

where $J_k = B_k \left(1 - \exp\left(-\tau_k\right)\right)$, and $p_i^{(N)}$ is the probability of finding the Nth component at the state i. The probabilistic meaning of Eq. (7) is obvious: the averaging process comprises all configurations with the last element found at the kth state. The configurations differ one from the other by realization of the previous element. It is also evident that

$$\langle I_1^{(k)} \rangle = p_k J_k, \qquad p_k^{(1)} = p_k, \qquad (k = 1, 2, ...n) \,. \qquad (8)$$

Once determined, the quantities $\langle I_N^{(k)} \rangle$ enable one to find the requisite mean intensity $\langle I_N \rangle$ simply by summing up over all the states

$$\langle I_N \rangle = \sum_{i=1}^{n} \langle I_N^{(k)} \rangle. \qquad (9)$$

The final point to discuss in connection with the LTE atmosphere concerns the mean profiles of spectral lines. It is clear physically that the absence of scatterings rules out the possibility of redistribution of the line radiation over frequencies, so that the averaging over a set of different configurations may be carried out separately for each frequency. Thus, the generalization of Eqs. (1)-(9) to the frequency-dependent case is trivial, and needs only be mentioned in arguments.

Suppose, for instance, that the power of the energy sources for ith component of the LTE stochastic atmosphere has the form $B_i \alpha\left(x\right)$, where $\alpha\left(x\right)$ is the profile of the absorption coefficient, and x is the dimensionless frequency denoting the displacement from the centre of the line in the Doppler units. Now the quantities τ_i will be identified with the optical thickness of the i th structural element in the central frequency of the line. In accordance with all the above, in place of Eq. (3) we can write

$$\langle I_N\left(x\right) = L_N\left(x\right) \langle I_1\left(x\right) \rangle, \qquad (10)$$

where $\langle I_1\left(x\right) \rangle = \langle B\left[1 - \exp\left(-\alpha\left(x\right)\tau\right)\right] \rangle$; $L_N\left(x\right) = \left(1 - q^N\left(x\right)\right) / \left(1 - q\left(x\right)\right)$; and $q\left(x\right) = \langle \exp\left(-\alpha\left(x\right)\tau\right) \rangle$.

The remaining equations may be extended to the frequency-dependent case in an analogous manner.

Figs. 1a,b show the frequency profiles of radiation emerging from the LTE stochastic and multicomponent atmospheres. Calculations were performed for $n = 2$, though the results obviously hold for an arbitrary number of realizations with the same mean characteristics of a single structural component. For comparison, the intensities emerging from the proper homogeneous atmospheres are also given. We let $B_1 = 1$, i.e., all intensities are normalized to B_1. The opaque component in Fig. 1a is characterized by the larger value of B (which seems more likely for the solar quiescent prominences), as opposed to that in Fig. 1b. The curves (a) and (b) correspond to homogeneous atmospheres with $B_2 = \text{const}$, $\tau_0 = N\tau_2$ (τ_0 stands for the total optical thickness of an atmosphere), and $B_1 = \text{const}\ (= 1)$, $\tau_0 = N\tau_1$, respectively. The profile (g) refers to the stochastic atmosphere with pure random variations. The cases (c)-(f) ex-

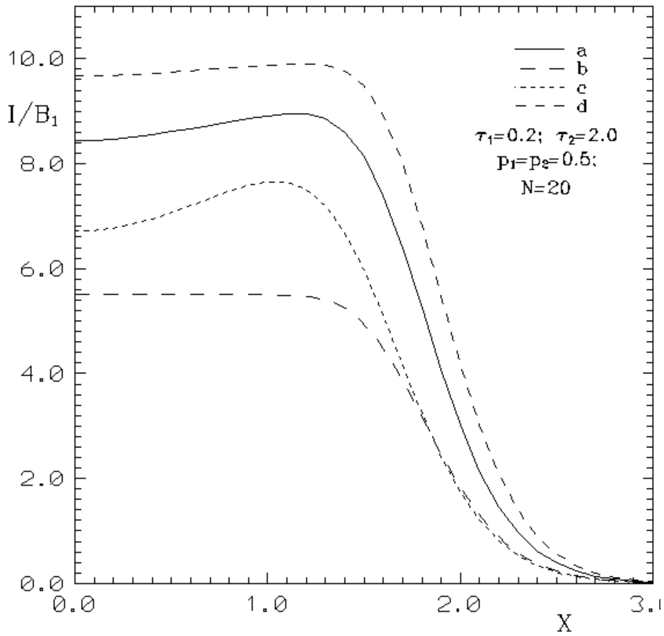

Fig. 2. The profiles of the LTE spectral lines for (a) stochastic atmosphere with pure random variations; (b) homogeneous atmosphere with preliminarily averaged physical properties; (c)-(f) stochastic atmosphere with Markovian correlations between components defined by the transition matrices T_c,T_d.

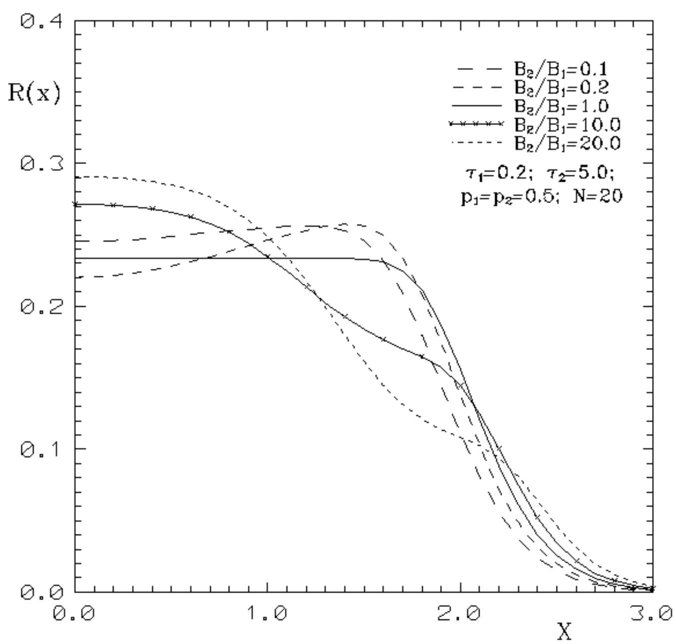

Fig. 3. The normalized profiles of the LTE spectral lines for various B_2/B_1 and values of other parameters as indicated.

hibit the effect of correlated variations defined by the Markovian transition matrices

$$T_c = \begin{pmatrix} 0.8 & 0.2 \\ 0.7 & 0.3 \end{pmatrix}, T_d = \begin{pmatrix} 0.2 & 0.8 \\ 0.3 & 0.7 \end{pmatrix}, \quad (11)$$

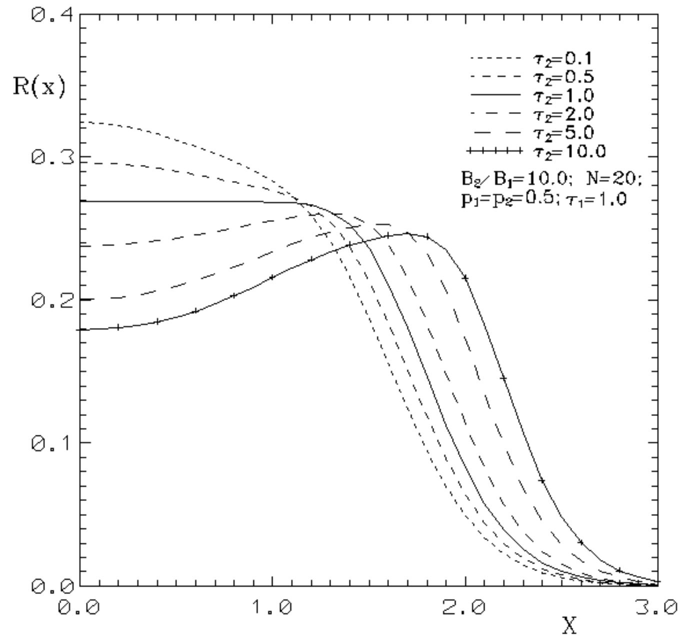

Fig. 4. The normalized profiles of the LTE spectral lines for various τ_2 and indicated values of other parameters.

$$T_e = \begin{pmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{pmatrix}, T_f = \begin{pmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \end{pmatrix},$$

respectively. The final two matrices are double stochastic (see, e.g. Bharucha-Reid 1960) and result in the profiles closest to those for the pure stochastic atmosphere.

Difficulties arising when solving the line formation problems for stochastic atmospheres often lead one to replace it by a certain deterministic problem for homogeneous atmosphere with averaged, in a certain sense, physical parameters. The profiles of the line formed in such an ‘averaged’ homogeneous atmosphere are referred to as case (b). Figs. 1a,b show that the resulting profiles differ from those for the pure stochastic atmosphere not only quantitatively but also qualitatively. The profiles for homogeneous atmospheres are characterized by a flat plateau in the core, the latter being wider as the total optical thickness of atmosphere is greater. In contrast, the profile relevant to the pure stochastic atmosphere exhibits more complex variation with frequency. The double-peaked shape of profiles for stochastic atmosphere shown in Fig. 1a deserves special attention. The importance of this result is that such profiles are usually associated with the multiple scattering process in a non-LTE homogeneous atmosphere (see also next section). Coincidence of the profiles (a) and (b) in a certain frequency domain of the line core is specific to the large optical thickness of components chosen in Figs. 1a and 1b. Further increase in τ_i extends this interval so that the discrepancy between two profiles becomes essential only for high values of x. On the other hand, by comparing the curves (a) and (b) in Fig. 2, we infer that, for smaller τ_i, the difference between these profiles may be significant throughout the line. Figs. 3,4 give an idea of variations of the normalized profiles, $R\left(x\right) = I\left(x\right) / \int I\left(x'\right) \mathrm{d}x'$, with optical thickness of structural elements and with the power of the internal energy sources.

3. The NLTE atmosphere

Once introduced into consideration, the multiple scattering process complicates the problem to a great extent. Now the intensity at any given point is a functional of the radiation field at any other point in the atmosphere. In addition, the frequency redistribution means the intensity of a given frequency depends on the radiation field at any part of the spectral line. The radiation diffusion within a medium leads to the appearance of reflected beams, therefore the mean statistical characteristics of a multicomponent atmosphere will be altered when adding a new component to it. Now the main idea underlying the derivations for the LTE atmosphere fails, since the averaging process can no longer be performed in parts.

The immediate objective of this section is to propose an algorithm for solving the model stochastic problem considered above and derive a closed-form analytical expression for the line profile. We shall show that the direct approach based on the combinatorial analysis and the classical theory of the radiative transfer allows us to obtain the exact solution of the problem for any distribution of the internal energy sources. Here we confine ourselves to considering the one-dimensional problem of the non-LTE line formation, although the approach may be easily extended to cover three-dimensional cases as well.

We begin by introducing the function $P(\tau, x', x, \tau_0)$ of the following probabilistic meaning. $P\mathrm{d}x$ is the probability that a photon of frequency x', *moving* in either direction at optical depth τ of an atmosphere of finite optical thickness τ_0, will escape it from the boundary $\tau = 0$ within frequency interval $(x, x + \mathrm{d}x)$. We shall distinguish it from the function $p(\tau, x, \tau_0)$ in use later, which also describes the photon exit from atmosphere, but only on condition that the photon was initially *absorbed* at optical depth τ. It is easily seen that if a certain layer of the medium found between optical depths τ' and τ'' contains isotropic energy sources of power $\varepsilon(\tau, x')$, the contribution of this layer into observed intensity will be

$$I_\varepsilon(0, x, \tau_0) = \int_{-\infty}^{\infty} \mathrm{d}x' \int_{\tau'}^{\tau''} \varepsilon(\tau, x') P(\tau, x', x, \tau_0)\, \mathrm{d}\tau = \int_{\tau'}^{\tau''} \widetilde{P}(\tau, x, \tau_0)\, \mathrm{d}\tau, \quad (12)$$

where

$$\widetilde{P}(\tau, x, \tau_0) = \int_{-\infty}^{\infty} \varepsilon(\tau, x') P(\tau, x', x, \tau_0)\, \mathrm{d}x'. \quad (13)$$

We defer the question of determining the function $\widetilde{P}$ until Sect. 3.2, and turn now to the combinatorial part of the problem.

3.1. The observed mean profile

Consider a stochastic scattering atmosphere consisting of N structural elements. We suppose that there exist two types of elements each realizing with probabilities p_1 and p_2. The elements are regarded as layers characterized by the power $2\varepsilon_q(\tau, x)$ and optical thickness $\tau_q\ (q = 1, 2)$ in the center of the line. As above, the contribution of each individual layer to the observed line profile is given by integrals of the type Eq. (12), with ε_q taken instead of ε (see Eq. (16) below). The limits of integrations depend on the place occupied by the layer within atmosphere (or more precisely, on the number of components of each type found between the boundary $\tau = 0$ and the layer under consideration), and the realized value of the total optical thickness. The last two points are pure combinatorial problems, and may be solved as follows.

For certainty, let the atmosphere be composed of $k \neq 0$ layers of the first kind and $N - k$ layers of the second kind. There exist $C_N^k = N!/k!\,(N-k)!$ configurations of such composition, each occurring with probability $p_1^k p_2^{N-k}$, and differing one from the other by the order of the layer arrangement. The optical thickness of such an atmosphere is $\tau_0 = k\tau_1 + (N-k)\,\tau_2$. Suppose that the mth layer (numbered from the boundary $\tau = 0$) is an element of the first kind. The probability of this event obviously is $k/N = C_{N-1}^{k-1}/C_N^k$. In addition, one can easily be convinced that the probability of finding exactly i components of the first kind, preceding the fixed mth layer, obeys the hypergeometric distribution (see Feller 1957), and is $C_{m-1}^i C_{N-m}^{k-i-1}/C_{N-1}^{k-1}$. The limits of variation of the index i are determined by the ordinal number m of a given layer and the total number k of components of the first kind. The upper limit, $\bar{i} = k - 1$, if $k \leq m$, and $\bar{i} = m - 1$, otherwise, which may be joined to write $\bar{i} = \min(k, m) - 1$. Similarly, we have $\underline{i} = m - 1 - \min(N-k, m-1)$ for the lower limit. Finally, we note that the fixed mth layer is located between optical depths τ_m and $\tau_m + \tau_1$, where $\tau_m = i\tau_1 + (m-1-i)\,\tau_2$.

Taking into account this reasoning, for the mean contribution $\langle I'_N(x)\rangle$ of components of the first kind into radiation of a N-component atmosphere, we find

$$\langle I'_N(x)\rangle = \sum_{k=1}^{N} p_1^k p_2^{N-k} \sum_{m=1}^{N} \sum_{i=\underline{i}}^{i=\bar{i}} C_{N-m}^{k-i-1} C_{m-1}^{i} \times \int_{\tau_m}^{\tau_m+\tau_1} \widetilde{P}_1(t, x, k\tau_1 + (N-k)\,\tau_2)\, \mathrm{d}t, \quad (14)$$

or

$$\langle I'_N(x)\rangle = \sum_{k=1}^{N} p_1^k p_2^{N-k} \sum_{m=1}^{N} \sum_{i=\underline{i}}^{i=\bar{i}} C_{N-m}^{k-i-1} C_{m-1}^{i} \times \int_{0}^{\tau_1} \widetilde{P}_1(\tau_m + t, x, k\tau_1 + (N-k)\,\tau_2)\, \mathrm{d}t, \quad (15)$$

where

$$\widetilde{P}_q(\tau, x, \tau_0) = \int_{-\infty}^{\infty} \varepsilon_q(\tau, x') P(\tau, x', x, \tau_0) \, \mathrm{d}x', \tag{16}$$

$$(q = 1, 2).$$

Similar derivation can be carried out for the components of the second kind to obtain

$$\langle I''_N(x) \rangle = \sum_{l=1}^{N} p_1^{N-l} p_2^l \sum_{m=1}^{N} \sum_{j=\underline{j}}^{j=\overline{j}} C_{N-m}^{l-j-1} C_{m-1}^{j} \times$$

$$\int_0^{\tau_2} \widetilde{P}_2(\tau_m + t, x, k\tau_1 + (N-k)\tau_2) \, \mathrm{d}t, \tag{17}$$

wherein $\overline{j} = \min(l, m) - 1$, $\underline{j} = m - 1 - \min(N - l, m - 1)$, and $\tau_m = (m - 1 - j)\tau_1 + \overline{j}\tau_2$. Note that the situations of the elements of one or another sort being absent are described by the last terms in the external summations in Eqs. (15) and (17).

Summing up Eqs. (15) and (16), we arrive at the requisite formula for the observed mean profile $\langle I_N(x) \rangle = \langle I'_N(x) \rangle + \langle I''_N(x) \rangle$, which, after some simple transformations, takes a more convenient form

$$\langle I_N(x) \rangle =$$

$$\sum_{s=0}^{N-1} p_1^s p_2^{N-s-1} \sum_{m=0}^{N-1} \sum_{i=\underline{i}}^{i=\overline{i}} C_{N-m-1}^{s-i} C_m^i Q_{Ns}^{im}(x), \tag{18}$$

where $\overline{i} = \min(s, m)$, $\underline{i} = m - \min(N - s, m)$, and

$$Q_{Ns}^{im}(x) = \langle \int_0^{\tau_q} \widetilde{P}_q(\overline{\tau}, x, \tau_{0q}) \, \mathrm{d}t \rangle, \tag{19}$$

where, for brevity, we have introduced notations: $\overline{\tau} = i\tau_1 + (m-i)\tau_2 + t; \tau_{0q} = \tau_q + s\tau_1 + (N - s - 1)\tau_2$. The symbol notation for the average over two sorts of components is understood in its conventional sense as

$$\langle f_q(\tau_q) \rangle = p_1 f_1(\tau_1) + p_2 f_2(\tau_2). \tag{20}$$

3.2. The photon exit probability

The final matter to be taken up to complete our derivation is the question of evaluating the function

$$\widetilde{P}(\tau, x, \tau_0) = \int_{-\infty}^{\infty} \varepsilon(\tau, x') P(\tau, x', x, \tau_0) \, \mathrm{d}x', \tag{21}$$

appearing in Eqs. (18)-(19). We begin by the function P, the probabilistic meaning of which was given at the outset of this section. The algorithm described below is similar to that adopted by Sobolev (1955) for function p.

The radiation diffusion we consider is determined by photon destruction coefficient ϵ and ratio β of the absorption coefficient in the continuum to that in the center of the line. Being applied to function P, the invariance principle (see Ambartsumian 1960 and Sobolev 1963) leads to the following equation

$$\frac{\partial P(\tau, x', x, \tau_0)}{\partial \tau} = -v(x) P(\tau, x', x, \tau_0) +$$

$$\int_{-\infty}^{\infty} \alpha(x'') p(0, x'', x, \tau_0) P(\tau, x', x'', \tau_0) \, \mathrm{d}x'' - \tag{22}$$

$$\int_{-\infty}^{\infty} \alpha(x'') p(\tau_0, x'', x, \tau_0) P(\tau_0 - \tau, x', x'', \tau_0) \, \mathrm{d}x'',$$

where $v(x) = \alpha(x) + \beta$.

Bearing in mind the probabilistic meaning of the function P, we find the boundary condition imposed at $\tau = 0$ to be

$$P(0, x', x, \tau_0) = \delta(x - x') + \rho(x', x, \tau_0), \tag{23}$$

where δ is the δ-function of Dirac and ρ the reflectance of the medium. The latter is introduced in such a way that $\rho(x', x, \tau_0) \, \mathrm{d}x$ gives the probability that a quantum of frequency x' incident on the medium will be reflected from it as a quantum in the frequency interval $(x, x + \mathrm{d}x)$.

From now on we confine ourselves to considering the complete frequency redistribution. Referring the reader to the above cited literature for details of the invariance technique, we note that in this case the quantities $p(0, x', x, \tau_0)$ and $p(\tau_0, x', x, \tau_0)$ are simply expressed in terms of Ambartsumian's functions φ and ψ[1]: $p(0, x', x, \tau_0) = (1 - \epsilon)\varphi(x, \tau_0)/2$, $p(\tau_0, x'', x, \tau_0) = (1 - \epsilon)\psi(x, \tau_0)/2$. The functions φ and ψ satisfy the following system of equations:

$$\phi(x, \tau_0) = A\alpha(x) + (1 - \epsilon)\frac{1}{2} \times$$

$$\int_{-\infty}^{\infty} \frac{\varphi(x, \tau_0)\varphi(x', \tau_0) - \psi(x, \tau_0)\psi(x', \tau_0)}{v(x) + v(x')} \alpha(x') \, \mathrm{d}x' \tag{24}$$

$$\psi(x, \tau_0) = A\alpha(x) \, \mathrm{e}^{-v(x)\tau_0} + (1 - \epsilon)\frac{1}{2} \times$$

$$\int_{-\infty}^{\infty} \frac{\varphi(x, \tau_0)\varphi(x', \tau_0) - \psi(x, \tau_0)\psi(x', \tau_0)}{v(x) - v(x')} \alpha(x') \, \mathrm{d}x',$$

where A is a normalizing factor equal to $\pi^{-1/2}$ for the Doppler broadening of line. This system of equations is a natural generalization of that presented by Sobolev (1955) for the case of

[1] we follow here Ambartsumian's original notations for these functions, although the Chandrasekhar notations X and Y (see Chandrasekhar 1960) are also widely used

$\beta \neq 0$. Knowing functions φ and ψ, we can determine the reflectance and transmittance of a finite atmosphere. In particular, for the reflectance ρ we have

$$\rho\left(x', x, \tau_0\right) = (1-\epsilon)\,\frac{1}{2A}\times \frac{\varphi\left(x, \tau_0\right)\varphi\left(x', \tau_0\right) - \psi\left(x, \tau_0\right)\psi\left(x', \tau_0\right)}{v\left(x\right) + v\left(x'\right)}. \tag{25}$$

Returning to Eq. (22), we see that for the complete frequency redistribution it acquires the form

$$\frac{\partial P\left(\tau, x', x, \tau_0\right)}{\partial \tau} = -v\left(x\right) P\left(\tau, x', x, \tau_0\right) + (1-\epsilon)\,\frac{1}{2}\varphi\left(x, \tau_0\right)\int\limits_{-\infty}^{\infty}\alpha\left(x''\right) P\left(\tau, x', x'', \tau_0\right) \mathrm{d}x'' - (1-\epsilon)\,\frac{1}{2}\psi\left(x, \tau_0\right)\int\limits_{-\infty}^{\infty}\alpha\left(x''\right) P\left(\tau_0-\tau, x', x'', \tau_0\right) \mathrm{d}x''. \tag{26}$$

Multiplying Eq. (26) by $\varepsilon_q\left(\tau, x'\right)$ and integrating over all frequencies, by virtue of Eq. (16), we arrive at the following equation for the function $\widetilde{P}$:

$$\frac{\partial \widetilde{P}\left(\tau, x, \tau_0\right)}{\partial \tau} = -v\left(x\right)\widetilde{P}\left(\tau, x, \tau_0\right) + (1-\epsilon)\,\frac{1}{2}\varphi\left(x, \tau_0\right) R\left(\tau, \tau_0\right) - (1-\epsilon)\,\frac{1}{2}\psi\left(x, \tau_0\right) R\left(\tau_0-\tau, \tau_0\right), \tag{27}$$

where

$$R\left(\tau, \tau_0\right) = \int\limits_{-\infty}^{\infty}\alpha\left(x'\right)\widetilde{P}\left(\tau, x', \tau_0\right) \mathrm{d}x'. \tag{28}$$

Using Eq. (23) the boundary condition reads

$$\widetilde{P}\left(0, x, \tau_0\right) = \varepsilon\left(0, x\right) + \int\limits_{-\infty}^{\infty}\varepsilon\left(0, x'\right)\rho\left(x', x, \tau_0\right) \mathrm{d}x'. \tag{29}$$

The formal solution of Eq. (26) is

$$\widetilde{P}\left(\tau, x, \tau_0\right) = \widetilde{P}\left(0, x, \tau_0\right) \mathrm{e}^{-v(x)\tau} + (1-\epsilon)\,\frac{1}{2}\int\limits_{0}^{\tau} R\left(t, \tau_0\right) \mathrm{e}^{-v(x)(\tau-t)} \mathrm{d}t - -(1-\epsilon)\,\frac{1}{2}\int\limits_{0}^{\tau} R\left(\tau_0-t, \tau_0\right) \mathrm{e}^{-v(x)(\tau-t)} \mathrm{d}t. \tag{30}$$

All that remains to determine the required function $\widetilde{P}$ is to find $R\left(\tau, \tau_0\right)$. The latter satisfies an integral equation, which is obtained from Eq. (30) by multiplying it by $\alpha\left(x'\right)$, and integrating the result over all frequencies

$$R\left(\tau, \tau_0\right) = F\left(\tau, \tau_0\right) + (1-\epsilon)\,\frac{1}{2}\int\limits_{0}^{\tau}\Phi\left(\tau-t, \tau_0\right) R\left(t, \tau_0\right) \mathrm{d}t - (1-\epsilon)\,\frac{1}{2}\int\limits_{0}^{\tau}\Psi\left(\tau-t\right) R\left(\tau_0-t, \tau_0\right) \mathrm{d}t, \tag{31}$$

where the following notations are used:

$$\Phi\left(\tau, \tau_0\right) = \int\limits_{-\infty}^{\infty}\alpha\left(x\right)\varphi\left(x, \tau_0\right) \mathrm{e}^{-v(x)\tau} \mathrm{d}x, \quad \Psi\left(\tau, \tau_0\right) = \int\limits_{-\infty}^{\infty}\alpha\left(x\right)\psi\left(x, \tau_0\right) \mathrm{e}^{-v(x)\tau} \mathrm{d}x, \quad F\left(\tau, \tau_0\right) = \int\limits_{-\infty}^{\infty}\alpha\left(x\right) P\left(0, x, \tau_0\right) \mathrm{e}^{-v(x)\tau} \mathrm{d}x. \tag{32}$$

We now have covered all the prerequisites for describing the algorithm for solving the transfer problem at hand, namely for computation of integrals given by Eq. (19). We start by calculating the functions φ and ψ from Eqs. (24), constructing the functions Φ, Ψ and F in parallel. This is the only point that needs comments because of the accuracy required. Several methods have been proposed to this end (see, Sobolev 1957); however, the iterative scheme applied to Eqs. (19) appears to be more accurate and not time-consuming. The integrals in these equations are replaced with the Hermite-Gaussian quadrature sums, the order of which should be chosen as high as possible to provide the accuracy needed for the large values of optical thickness. The apparent difficulty connected with singularity involved in the second of Eqs. (19) may be overcome by taking the zeros of Hermite polynomials of the *different* order as the frequency grid of the external and internal variables. The further steps in the solution of Eq. (31) and subsequent calculations using Eq. (30) do not encounter any difficulty of principle.

3.3. Special cases

Up to now we have not specified the analytical form for the power of the internal energy sources. In this paper we are interested in the sources of the form $\varepsilon_q\left(\tau, x\right) = B_q\alpha\left(x\right)$. In view of Eq. (21), Eq. (19) reads

$$Q_{Ns}^{im}\left(x\right) = \langle B_q\int\limits_{0}^{\tau_q}\widetilde{P}\left(\overline{\tau}, x, \tau_{0q}\right) \mathrm{d}t\rangle. \tag{33}$$

where the function

$$\widetilde{P}\left(\tau, x, \tau_0\right) = \int\limits_{-\infty}^{\infty}\alpha\left(x'\right) P\left(\tau, x', x, \tau_0\right) \mathrm{d}x' \tag{34}$$

is determined from Eq. (30). The boundary value of this function appearing in Eq. (30) is simplified by virtue of Eqs. (29), (24) and (25) to $\widetilde{P}\left(0, x, \tau_0\right) = \varphi\left(x, \tau_0\right)$. Finally, for the function F in Eq. (31), we obtain $F\left(\tau, \tau_0\right) = \Phi\left(\tau, \tau_0\right)$.

Several special assumptions concerning the physical properties of atmosphere lead to further simplifications of the results as demonstrated below in a. through d.

a. Let $B_1 \neq B_2$, and τ_q be τ common for all components. Then we find from Eq. (33)

$$Q_{Ns}^{im}(x) = \langle B_q \rangle \int_0^{\tau} \widetilde{P}(m\tau + t, x, N\tau)\,\mathrm{d}t, \tag{35}$$

so that the quantities $Q_{Ns}^{im}(x)$ no longer depend on i and s. Bearing in mind the probabilistic nature of the hypergeometric distribution, the summation over i in Eq. (18) is performed immediately to yield

$$\sum_{i=\underline{i}}^{\bar{i}} C_{N-m-1}^{s-i} C_m^i = C_{N-1}^s. \tag{36}$$

Thus, Eq. (18) takes the form

$$\langle I_N(x) \rangle = \langle B_q \rangle \times$$
$$\sum_{s=0}^{N-1} C_{N-1}^s p_1^s p_2^{N-s-1} \sum_{m=0}^{N-1} \int_0^{\tau} \widetilde{P}(m\tau + t, x, N\tau)\,\mathrm{d}t, \tag{37}$$

or finally

$$\langle I_N(x) \rangle = \langle B_q \rangle \int_0^{N\tau} \widetilde{P}(t, x, N\tau)\,\mathrm{d}t. \tag{38}$$

If we suppose in addition that $B_1 = B_2 = B$, Eq. (38) converts, as could be expected, into the solution I^* of the deterministic problem for homogeneous atmosphere of optical thickness $\tau_0 = N\tau$

$$I^*(x, \tau_0, B) = B \int_0^{\tau_0} \widetilde{P}(t, x, \tau_0)\,\mathrm{d}t. \tag{39}$$

Now we observe that Eq. (38) may be rewritten as follows:

$$\langle I_N(x) \rangle = I^*(x, N\tau, \langle B_q \rangle), \tag{40}$$

i.e., the problem in this special case is reduced to the one for a homogeneous atmosphere. This facilitates calculations to a great extent, particularly considering the fact that the quantity I^* is expressed directly in terms of the functions φ and ψ (see Sobolev 1955)

$$I^*(x, \tau_0, B) = \frac{B}{Av(x)} \times \frac{\varphi(x, \tau_0) - \psi(x, \tau_0)}{1 - (1-\epsilon)\frac{1}{2}\left[\varphi_0(\tau_0) - \psi_0(\tau_0)\right]}, \tag{41}$$

where

$$\phi_0(\tau_0) = \int_{-\infty}^{\infty} \frac{\alpha(x)\,\varphi(x, \tau_0)}{v(x)}\mathrm{d}x,$$

$$\psi_0(\tau_0) = \int_{-\infty}^{\infty} \frac{\alpha(x)\,\psi(x, \tau_0)}{v(x)}\mathrm{d}x.$$

b. Let us now suppose that $B_1 = B_2 = B$, and $\tau_1 = \tau_2$; i.e., the energy sources are uniformly distributed in the atmosphere so that the configurations may differ one from the other merely by the realized value of the optical thickness. For fixed N and k, there obviously exist $N+1$ of those configurations, while others differ only by arrangement of the components. Consequently we can write

$$\langle I_N(x) \rangle = B \sum_{s=0}^{N} C_N^s p_1^s p_2^{N-s} \times$$
$$\int_0^{s\tau_1 + (N-s)\tau_2} \widetilde{P}(t, x, s\tau_1 + (N-s)\tau_2)\,\mathrm{d}t = \tag{42}$$
$$\sum_{s=0}^{N} C_N^s p_1^s p_2^{N-s} I^*(x, s\tau_1 + (N-s)\tau_2, B).$$

As a result, in this special case the mean observed intensity is also expressed in terms of intensity I^* for homogeneous atmosphere, and the calculations are limited by solving the system of equations (24) numerically.

c. In the LTE limit when $\epsilon = 1$, Eq. (30) yields $\widetilde{P}(\tau, x, \tau_0) = \widetilde{P}(0, x, \tau_0)\exp[-v(x)\tau]$, or with use of Eqs. (29), (24), (25):

$$\widetilde{P}(\tau, x, \tau_0) = \alpha(x)\,\mathrm{e}^{-v(x)\tau}. \tag{43}$$

Substituting Eq. (43) into Eq. (33), and availing ourselves of the notations adopted in Eq. (10), we can rewrite Eq. (18) as

$$\langle I_N(x) \rangle = \langle I_1(x) \rangle \frac{\alpha(x)}{v(x)} \times$$
$$\sum_{s=0}^{N-1} p_1^s p_2^{N-s} \sum_{m=0}^{N-1} \sum_{i=\underline{i}}^{\bar{i}} C_{N-m-1}^{s-i} C_m^i q_1^i(x)\, q_2^{m-i}(x), \tag{44}$$

where $q_j(x) = \exp[-v(x)\tau_j]$, $(j = 1, 2)$. As calculations for $\beta = 0$ show, Eq. (44) is equivalent to Eq. (10) obtained by other means. This implies the useful identity

$$\sum_{s=0}^{N-1} p_1^s p_2^{N-s} \sum_{m=0}^{N-1} \sum_{i=\underline{i}}^{\bar{i}} C_{N-m-1}^{s-i} C_m^i q_1^i(x)\, q_2^{m-i}(x) =$$
$$\frac{1 - q^N(x)}{1 - q(x)}, \tag{45}$$

where $(q(x) = \langle q_j(x) \rangle)$, the rigorous mathematical proof of which is far from trivial and may be of interest from the point of view of combinatorial analysis.

3.4. Numerical results

Figs. 5a,b display the profiles of the non-LTE lines formed in a stochastic multicomponent atmosphere. For comparison, the profiles for the homogeneous atmosphere with preliminarily averaged physical parameters are given. As in Sect. 2, two mutually opposite situations are considered depending on whether

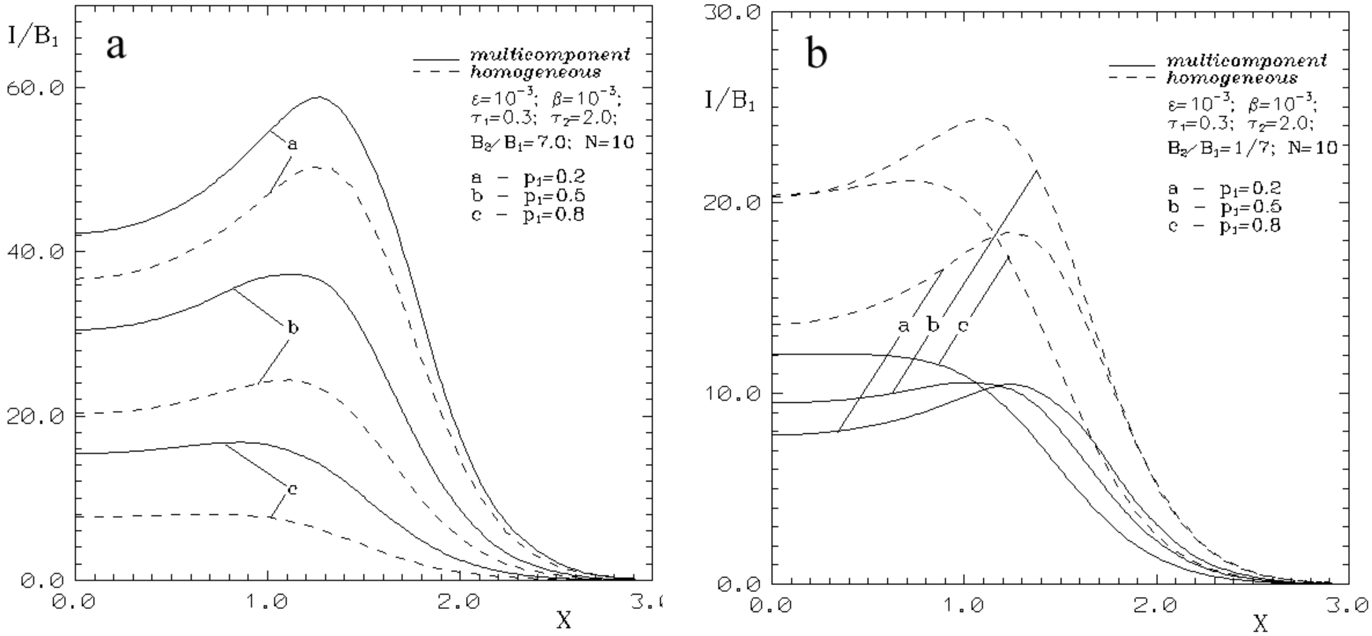

Fig. 5a and b. The profiles of the NLTE spectral lines for: **a** $B_2/B_1 = 7.0$, **b** $B_2/B_1 = 1/7$ and indicated values of other parameters.

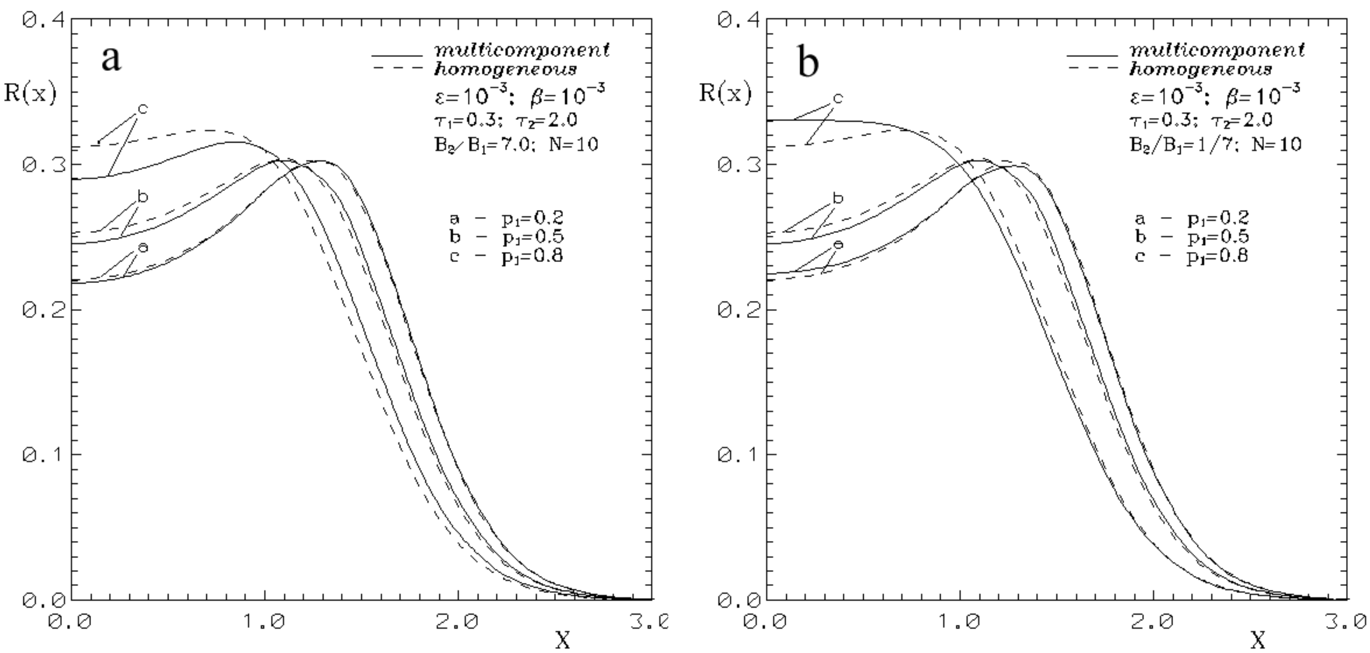

Fig. 6a and b. The normalized profiles of the NLTE spectral lines for: **a** $B_2/B_1 = 7.0$, **b** $B_2/B_1 = 1/7$ and indicated values of other parameters.

more (Fig. 5a) or less (Fig. 5b) powerful energy sources are contained in the relatively opaque components. For the mean values of optical thickness $\langle\tau_0\rangle = N\langle\tau\rangle \geq 6 \div 8$ (depending on ϵ and β, typical for resonant lines), the line profiles possess, as a rule, a double-peaked shape, as is the case for homogeneous atmospheres. We see that the profiles for the averaged homogeneous atmosphere reproduce the actual profiles fairly well, while the differences between them may be regarded as essential in the integral sense. The great difference in behavior of these two types of profiles relative to each other in Fig. 5a, as compared to that in Fig. 5b, deserves attention. In Fig. 5a the total energy radiated by the stochastic atmosphere is greater than that radiated by the averaged homogeneous atmosphere, while the situation presented in Fig. 5b is diametrically opposed. Nevertheless, as can be inferred from Figs. 6a,b, in both cases the normalized profiles differ only slightly. In conclusion all the described effects are the more pronounced the smaller the role played by thermalization of quanta in the medium.

4. Concluding remarks

An exact analytical solution has been found for a model problem of the spectral line formation in both the LTE and non-LTE stochastic multicomponent atmospheres. The most important results obtained may be summarized as follows.

i. The line profiles formed in LTE stochastic atmosphere may have rather complicated shapes and, in general, differ significantly from those specific to the averaged homogeneous atmosphere. Particularly, it can produce double-peaked profiles which are usually associated with multiple scattering effects.
ii. The influence of the Markovian correlations between structural elements on the line profile is revealed.
iii. The line formation problem for non-LTE stochastic atmospheres considered in the paper is the only case known to the authors of an explicit solution of this kind of radiative transfer problem. It has been shown that the normalized profiles of lines formed in the averaged homogeneous atmosphere fit the actual profiles fairly well while, in energetic sense, they may differ considerably. This is important in connection with astrophysical problems that use the observed spectra for diagnostics of the emitting atmosphere. Remaining symmetrical, the profiles of the line-intensities outgoing from a stochastic multicomponent atmosphere may, however, differ significantly from those for deterministic homogeneous atmospheres. This effect needs to be taken into account along with the effects due to the mutual motions of components (Suemoto 1977) and due to the velocity field within each of them (Shine 1975).

The evident advantage of the model problem discussed here is that it provides an important insight into specificity of the line radiation in an inhomogeneous atmosphere with randomly varied properties and it allows for various generalizations. One obvious and immediate generalization is the treatment of three-dimensional versions of the problem with allowance for partial redistribution over frequencies which can be attained without any additional complication of principle. Another possible generalization for non-LTE atmospheres is the consideration of the larger number n of possible realizations for the random parameters. The disadvantage of the solution obtained for this case is the drastic increase in the number of operations with increasing N and n, which makes the computational process time-consuming. This leads to pressure for further development of the theory to investigate the asymptotic properties of the solution for the large values of these parameters.

Acknowledgements. The authors are indebted to Dr. H.A. Haruthyunian for placing at our disposal the highly accurate values of the frequency grids and corresponding weights for the high-order Hermite-Gaussian quadrature sums, as well as to the referee (Dr. V. Grinin) for helpful comments. The research was supported by NATO under grant CRG 960346 (A.G.N.).

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