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Stellar evolution with rotation

VI. The Eddington and Ω–limits, the rotational mass loss for OB and LBV stars

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Abstract. Several properties of massive stars with large effects of rotation and radiation are studied. For stars with shellular rotation, i.e. stars with a constant angular velocity Ω on horizontal surfaces (cf. Zahn 1992), we show that the equation of stellar surface has no significant departures with respect to the Roche model; high radiation pressure does not modify this property. Also, we note that contrarily to some current expressions, the correct Eddington factors Γ in a rotating star explicitely depend on rotation. As a consequence, the maximum possible stellar luminosity is reduced by rotation.

We show that there are 2 roots for the equation giving the rotational velocities at break–up: 1) The usual solution, which is shown to apply when the Eddington ratio Γ of the star is smaller than formally 0.639. 2) Above this value of Γ, there is a second root, inferior to the first one, for the break–up velocity. This second solution tends to zero, when Γ tends towards 1. This second root results from the interplay of radiation and rotation, and in particular from the reduction by rotation of the effective mass in the local Eddington factor. The analysis made here should hopefully clarify a recent debate between Langer (1997, 1998) and Glatzel (1998).

The expression for the global mass loss–rates is a function of both Ω and Γ, and this may give raise to extreme mass loss–rates ($\Omega\Gamma$–limit). In particular, for O–type stars, LBV stars, supergiants and Wolf–Rayet stars, even slow rotation may dramatically enhance the mass loss rates. Numerical examples in the range of 9 to 120 $M_\odot$ at various $T_{\rm eff}$ are given.

Mass loss from rotating stars is anisotropic. Polar ejection is favoured by the higher $T_{\rm eff}$ at the polar caps ($g_{\rm eff}$–effect), while the ejection of an equatorial ring is favoured by the opacity effect (κ–effect), if the opacity grows fastly for decreasing $T_{\rm eff}$.

Key words: stars: rotation – stars: evolution – stars: mass-loss

1. Introduction

Recent models of stellar evolution with rotation (Meynet & Maeder 2000) have shown that rotation heavily modifies all the model outputs for massive stars. In the course of the above mentioned work, it was realized that several basic points in the stellar physics need to be further clarified, since they may have some important consequences on the evolution. These points concern in particular the correct expression of the break–up velocities and the dependence of the mass loss rates $\dot{M}$ on the observed rotation velocities v. These problems are of great concern for the most luminous stars close to the Eddington limit, like OB stars, supergiants, LBV and WR stars.

There is an interesting debate in recent litterature about what is the correct expression for the critical velocity and what is the dependence of the $\dot{M}$–rates on the rotation velocities v. The critical rotation velocity of a star is often written as $v^2_{\rm crit} = \frac{GM}{R}(1-\Gamma)$, where $\Gamma = L/L_{\rm Edd}$ is the ratio of the stellar luminosity to the Eddington luminosity. With this expression, Langer (1997, 1998) suggests that “no matter the rotation rate may be, it (the star) will arrive at critical rotation well before Γ = 1 is actually reached”. Consequently, Langer introduces the concept that the stars generally reach the break–up limit, i.e. the Ω–limit, earlier in evolution than the Γ–limit.

This point of view was disputed by Glatzel (1998), who stressed that the Ω–limit is an artefact based on the disregard of gravity darkening and on the assumption of a uniform brightness over the surface of rotating stars. Glatzel concludes that the Eddington factor has no effect on the critical rotation. This problem needs to be further examined and this is what is done here.

Another important issue is the dependence of the mass loss rates $\dot{M}$ on rotation velocities v (cf. Maeder & Meynet 2000). On one side, Langer (1997, 1998), Heger et al. (2000), Meynet & Maeder (2000) are using a relation $\dot{M}$ vs. v from Friend & Abbott (1986) which formally leads to infinite mass loss rates at break–up velocities. On the other side, Owocki et al. (1996, 1998), Owocki & Gayley (1997) and Glatzel (1998) show that even at extreme rotation the mass loss rates remain finite and do not differ too much from the case of zero–rotation. There also, some further analysis is needed.

In a rotating star, the total gravity is the sum of the gravitational, centrifugal and radiative accelerations:

$$\boldsymbol{g}_{\rm tot} = \boldsymbol{g}_{\rm eff} + \boldsymbol{g}_{\rm rad} = \boldsymbol{g}_{\rm grav} + \boldsymbol{g}_{\rm rot} + \boldsymbol{g}_{\rm rad}. \quad (1.1)$$

For purpose of clarity, we adopt the following definitions:

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– We speak of the Eddington or Γ–limit, when rotation effects can be neglected and $\boldsymbol{g}_{\mathbf{rad}} + \boldsymbol{g}_{\mathbf{grav}} = \mathbf{0}$, which implies that

$$\Gamma = \frac{\kappa L}{4\pi c G M} \ \rightarrow \ 1. \tag{1.2}$$

In that case $L = L_{\rm EDD} = 4\pi c G M/\kappa$. The opacity κ considered here is the total opacity, unless we specify it differently (cf. Sect. 4.2).

– The break–up or Ω–limit is reached, for a star with an angular velocity Ω at the surface, when the effective gravity $\boldsymbol{g}_{\mathbf{eff}} = \boldsymbol{g}_{\mathbf{grav}} + \boldsymbol{g}_{\mathbf{rot}} = \mathbf{0}$ and in addition when radiation pressure effects can be neglected.

– *The $\Omega\Gamma$–limit is reached when the total gravity $\boldsymbol{g}_{\mathbf{tot}} = \mathbf{0}$, with significant effects of both rotation and radiation.* This is the general case, that we study here. It should lead to the two above cases in their respective limits.

In Sect. 2, we examine the surface gravity, the Eddington factors and the limiting luminosity. In Sect. 3, the expression of the break–up velocities are considered, while in Sect. 4 we examine the mass loss rates. The equation of the surface with account of rotation and radiative acceleration is discussed in an Appendix.

2. Surface gravity, Eddington factors and limiting luminosity

2.1. The von Zeipel theorem close to the $\Omega\Gamma$–limit

The von Zeipel theorem (1924) expresses that the radiative flux $\boldsymbol{F}$ at some colatitude ϑ in a rotating star is proportional to the local effective gravity $\boldsymbol{g}_{\mathbf{eff}}$. In a previous work (Maeder 1999), we have generalized this theorem to the case of shellular rotation proposed by Zahn (1992). Shellular rotation results from strong horizontal turbulence which reduces the latitudinal dependence of rotation and makes the angular velocity Ω constant on an isobar. Here, we shall consider the case of stars with shellular rotation, where the $\Omega\Gamma$–limit may play a role.

As shown by Langer (1997), stars close to the Eddington limit tend to develop convection in the outer layers (cf. also Maeder 1980). However, in the outer layers the convective flux is generally negligible and the main transport mechanism is radiative transfer, a point also emphasized by Glatzel (1998). As a numerical example, in a 60 $M_\odot$ model (Meynet & Maeder 2000) at the end of the MS phase with log $L/L_\odot$= 5.89 and log $T_{\rm eff}$ =4.34, the convective flux is negligible down to a fractional radius r/R of 0.85. In a 120 $M_\odot$ with log $L/L_\odot$ = 6.32 and log $T_{\rm eff}$ =4.35 at the end of the MS, the convective flux is negligible down to r/R = 0.63. Thus, the basic condition to apply the von Zeipel theorem to stars close to the $\Omega\Gamma$–limit is fulfilled, since the flux is essentially radiative.

The expression of the flux $\boldsymbol{F}$ (Maeder 1999) for a star with angular velocity Ω on the isobaric stellar surface (cf. Appendix) is

$$\boldsymbol{F} = -\frac{L(P)}{4\pi G M_\star} \boldsymbol{g}_{\mathbf{eff}} [1+\zeta(\vartheta)] \quad \text{with} \tag{2.3}$$

$$M_\star = M\left(1 - \frac{\Omega^2}{2\pi G \rho_{\rm m}}\right) \quad \text{and} \tag{2.4}$$

$$\zeta(\vartheta) = \left[\left(1 - \frac{\chi_T}{\delta}\right)\Theta + \frac{H_T}{\delta}\frac{d\Theta}{dr}\right] P_2(\cos\vartheta). \tag{2.5}$$

There, $\rho_{\rm m}$ is the internal average density, $\chi = 4acT^3/(3\kappa\rho)$ and χ_T is the partial derivative with respect to T. The quantity Θ is defined by $\Theta = \frac{\tilde{\rho}}{\bar{\rho}}$, i.e. the ratio of the horizontal density fluctuation to the average density on the isobar, which is given by $\frac{\tilde{\rho}}{\bar{\rho}} = \frac{1}{3}\frac{r^2}{\bar{g}}\frac{d\Omega^2}{dr}$ where $\bar{g}$ is the average gravity on an isobar (cf. Zahn 1992). One has the thermodynamic coefficients $\delta = -(\partial \ln\rho/\partial\ln T)_{P,\mu}$, H_T is the temperature scale height. The term $\zeta(\vartheta)$, which expresses the deviations of the von Zeipel theorem due to the baroclinicity of the star, is generally very small (cf. Maeder 1999). Let us emphasize that the flux is proportional to $\boldsymbol{g}_{\mathbf{eff}}$ and not to $\boldsymbol{g}_{\mathbf{tot}}$. This results from the fact that the equation of hydrostatic equilibrium is $\frac{\boldsymbol{\nabla} P}{\rho} = -\boldsymbol{g}_{\mathbf{eff}}$. The effect of radiation pressure is already counted in the expression of P, which is the total pressure. We may call $M_\star$ the effective mass, i.e. the mass reduced by the centrifugal force. This is the complete form of the von Zeipel theorem in a differentially rotating star with shellular rotation, whether or not one is close to the Eddington limit.

2.2. Expressions of the gravity and of the local Eddington factor

Let us express the total gravity at some colatitude ϑ, taking into account the radiative acceleration

$$\boldsymbol{g}_{\mathbf{rad}} = \frac{1}{\rho}\boldsymbol{\nabla} P_{\rm rad} = \frac{\kappa(\vartheta)\boldsymbol{F}}{c} \ , \tag{2.6}$$

thus one has with Eqs. (1.1), (2.3) and (2.4)

$$\boldsymbol{g}_{\mathbf{tot}} = \boldsymbol{g}_{\mathbf{eff}} \left[1 - \frac{\kappa(\vartheta) L(P)[1+\zeta(\vartheta)]}{4\pi c G M (1 - \frac{\Omega^2}{2\pi G\rho_{\rm m}})}\right]. \tag{2.7}$$

The rotation effects appear both in $\boldsymbol{g}_{\mathbf{eff}}$ and in the term in brackets. When we write $\kappa(\vartheta)$, we mean that in a rotating star, the local $T_{\rm eff}$ and gravity vary with latitude and so does the opacity.We may also consider the local limiting flux. The condition $\boldsymbol{g}_{\mathbf{tot}} = \mathbf{0}$ in Eq. (1.1) with Eq. (2.6) for $\boldsymbol{g}_{\mathbf{rad}}$ allows us to define a limiting flux,

$$\boldsymbol{F}_{\mathbf{lim}}(\vartheta) = -\frac{c}{\kappa(\vartheta)} \boldsymbol{g}_{\mathbf{eff}}(\vartheta) \ . \tag{2.8}$$

From that we may define the ratio $\Gamma_\Omega(\vartheta)$ of the actual flux $F(\vartheta)$ to the limiting local flux in a rotating star,

$$\Gamma_\Omega(\vartheta) = \frac{F(\vartheta)}{F_{\rm lim}(\vartheta)} = \frac{\kappa(\vartheta)\ L(P)[1+\zeta(\vartheta)]}{4\pi c G M\left(1 - \frac{\Omega^2}{2\pi G \rho_{\rm m}}\right)} \ . \tag{2.9}$$

As a matter of fact, $\Gamma_\Omega(\vartheta)$ is the local Eddington ratio. For zero rotation $\Gamma_\Omega(\vartheta) = \Gamma$ as given by Eq. (1.2). Using relation (2.9), we may write the Eq. (2.7) for the total gravity as

$$\boldsymbol{g}_{\mathbf{tot}} = \boldsymbol{g}_{\mathbf{eff}} \left[1 - \Gamma_\Omega(\vartheta)\right] \ . \tag{2.10}$$

This shows that the expression for the total acceleration in a rotating star is similar to the usual one, except that Γ is replaced by the local value $\Gamma_\Omega(\vartheta)$. Indeed, contrarily to expressions such as $\boldsymbol{g_{\rm tot}} = \boldsymbol{g_{\rm eff}}\,(1-\Gamma)$ often found in literature, we see that the appropriate Eddington factor (2.9) also depends on the angular velocity Ω on the isobaric surface.

From (2.9), we note that over the surface of a rotating star, which has a varying gravity and $T_{\rm eff}$, $\Gamma_\Omega(\vartheta)$ is the highest at the latitude where $\kappa(\vartheta)$ is the largest, (if we neglect the effects of $\zeta(\vartheta)$, which is justified in general). If the opacity increases with decreasing T as in hot stars, the opacity is the highest at the equator and there the limit $\Gamma_\Omega(\vartheta) = 1$ may be reached first. Thus, it is to be stressed that if the limit $\Gamma_\Omega(\vartheta) = 1$ happens to be met at the equator, it is not because $\boldsymbol{g_{\rm eff}}$ is the lowest there, but because the opacity is the highest! Indeed, both dependences in $\boldsymbol{g_{\rm eff}}$ have cancelled each other in the ratio given by Eq. (2.9).

2.3. The luminosity at the $\Omega\Gamma$–limit

The $\Omega\Gamma$–limit is reached, when the local Eddington ratio $\Gamma_\Omega(\vartheta) = 1$ at some colatitude ϑ. The condition $\Gamma_\Omega(\vartheta) = 1$ allows us to define a limiting luminosity $L_{\Omega\Gamma}$ at the $\Omega\Gamma$–limit, i.e. when both the effects of radiative acceleration and rotation are important. From (2.9) we have

$$L_{\Omega\Gamma} = \frac{4\pi cGM}{\kappa(\vartheta)\,[1+\zeta(\vartheta)]}\left(1-\frac{\Omega^2}{2\pi G\rho_{\rm m}}\right)\,. \tag{2.11}$$

It means that for a certain angular velocity Ω on the isobaric surface, the maximum permitted luminosity of a star is reduced by rotation, with respect to the usual Eddington limit (cf. Sect. 1). This conclusion was also reached by Glatzel (1998). In the above relation, $\kappa(\vartheta)$ is the largest value of the opacity on the surface of the rotating star. For O–type stars with photospheric opacities dominated by electron scattering, the opacity κ is the same everywhere on the star. For the equation of the surface discussed in the Appendix, the maximum value of $\frac{\Omega^2}{2\pi G\rho_{\rm m}} = 0.361$, (with more digits it is 0.360747).

3. The break–up velocities

We have seen above that rotation may be considered as reducing the maximum possible luminosity for a star. An alternative way to consider the problem is to ask the question: what happens to the break–up velocity for a star close to the Eddington limit? Most authors (Langer 1997, 1998, 1999; Lamers et al. 1999; Heger et al. 2000) write this critical velocity $v_{\rm crit}$ like

$$v^2_{\rm crit} = \frac{GM}{R}(1-\Gamma)\,. \tag{3.12}$$

This relation is true if we assume that the brightness of the rotating star is uniform over its surface, which is in contradiction with von Zeipel's theorem. Surprisingly, some authors use this relation simultaneously with the von Zeipel theorem. Eq. (3.12), which we do not support, in agreement with Glatzel (1998), implies that the break–up velocity is reduced by the proximity to the Eddington limit. The problem needs to be further studied carefully.

The critical velocity is reached when somewhere on the star one has $\boldsymbol{g_{\rm tot}} = \boldsymbol{0}$, i.e. according to (2.10)

$$\boldsymbol{g_{\rm eff}}\;[1-\Gamma_\Omega(\vartheta)] = \boldsymbol{0}. \tag{3.13}$$

This equation has two roots. The first one $v_{\rm crit,1}$ is given by the usual condition $\boldsymbol{g_{\rm eff}} = \boldsymbol{0}$, which implies the equality $\Omega^2 R^3_{\rm eb}/(GM) = 1$ at the equator (cf. Eq. A2 in Appendix). This corresponds to an equatorial critical velocity

$$v_{\rm crit,1} = \Omega\,R_{\rm eb} = \left(\frac{2}{3}\frac{GM}{R_{\rm pb}}\right)^{\frac{1}{2}}\,. \tag{3.14}$$

$R_{\rm eb}$ and $R_{\rm pb}$ are respectively the equatorial and polar radius at the break–up velocity and they obey to the surface equation. We notice that the critical velocity $v_{\rm crit,1}$ is independent on the Eddington factor. To this extent, this is in agreement with Glatzel (1998). The basic physical reason for this independence is quite clear: the radiative flux decreases at the equator, when the effective gravity decreases.

Eq. (3.13) has a second root, which is given by the condition $\Gamma_\Omega(\vartheta)$ = 1. As seen above, this condition will in general be met at the equator first. We thus have to search for the corresponding critical velocity $v_{\rm crit,2}$ for a given value of the stellar luminosity. This second root has to be compared to the first one. For given values of M and L, the lowest of the two roots $v_{\rm crit,1}$ and $v_{\rm crit,2}$ is the significant one, since as soon as it will be reached the matter at the surface of the star is no longer bound. The condition $\Gamma_\Omega(\vartheta)$ = 1 gives, if we neglect $\zeta(\vartheta)$,

$$\frac{\kappa(\vartheta)L(P)}{4\pi cGM} = 1-\frac{\Omega^2}{2\pi G\rho_{\rm m}}. \tag{3.15}$$

Let us write

$$\frac{\Omega^2}{2\pi G\rho_{\rm m}} \;=\; \frac{16}{81}\omega^2 V'(\omega)\quad\text{with} \tag{3.16}$$

$$V'(\omega) = \frac{V(\omega)}{\frac{4}{3}\pi R^3_{\rm pb}}\quad\text{and}\quad \omega^2 = \frac{\Omega^2 R^3_{\rm eb}}{GM}. \tag{3.17}$$

The quantity ω is the fraction of the angular velocity at the classical break–up given by Eq. (3.14). The density $\rho_{\rm m}(\omega) = M/V(\omega)$, where the stellar volume $V(\omega)$ depends on rotation. The quantity $V'(\omega)$ is the ratio of the actual volume of a star with rotation ω to the volume of a sphere of radius $R_{\rm pb}$. $V'(\omega)$ is obtained by the integration of the solutions of the surface equation (A6) for a given value of the parameter ω. At break–up velocity $v_{\rm crit,1}$, the value of $V'(\omega)$ = 1.829, which gives the maximum value of $\frac{\Omega^2}{2\pi G\rho_{\rm m}}$ = 0.361. If we call $\Gamma_{\rm max}$ the maximum Eddington ratio $\kappa(\vartheta)L(P)/(4\pi cGM)$ over the surface (in general at the equator), Eq. (3.15) can thus be written

$$\frac{16}{81}\;\omega^2\;V'(\omega)\;=\;1-\Gamma_{\rm max}\,. \tag{3.18}$$

For a given value of $\Gamma_{\rm max}$, one must search the value of ω which satisfies this equation. This is easily obtained by solving numerically the surface equation. For a given large enough $\Gamma_{\rm max}$ (i.e. larger than 0.639), the obtained ω–value is lower than 1,

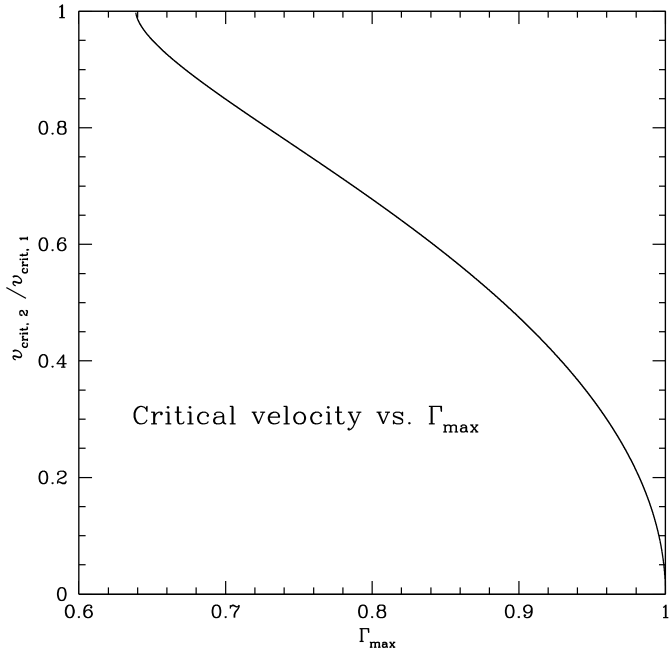

Fig. 1. The critical velocity $v_{\rm crit,2}$ expressed as a fraction of $v_{\rm crit,1}$ plotted as a function of the Eddington factor $\Gamma_{\rm max}$, which is the largest value of the Eddington factor reached where the opacity is the largest over the stellar surface. We notice that when the Eddington factor tends towards unity, the critical velocity goes down to zero.

and this implies a corresponding critical velocity $v_{\rm crit,2}$ given by

$$v^2_{\rm crit,2} = \Omega^2 R^2_{\rm e}(\omega) = \frac{81}{16}\,\frac{1-\Gamma_{\rm max}}{V'(\omega)}\frac{GM}{R^3_{\rm eb}}R^2_{\rm e}(\omega) = \frac{9}{4}\,v^2_{\rm crit,1}\,\frac{1-\Gamma_{\rm max}}{V'(\omega)}\,\frac{R^2_{\rm e}(\omega)}{R^2_{\rm pb}}\,, \qquad (3.19)$$

where $\mathrm{R_e}(\omega)$ is the equatorial radius for a given value of the rotation parameter ω. Fig. 1 illustrates the results. We notice that the second root $v_{\rm crit,2}$, expressed as a fraction of the first root $v_{\rm crit,1}$ given by (3.14), tends towards zero when the Eddington factor $\Gamma_{\rm max}$ tends towards 1. The first root defined by (3.14) can still be written, but the second root defined by the condition $\Gamma_\Omega(\vartheta) = 1$ is met first. This reduction of the critical velocity with respect to the classical expression only occurs for Eddington factors larger than 0.639, since the maximum value of $\frac{\Omega^2}{2\pi G\rho_{\rm m}}$ is 0.361 (cf. Eq. 3.15).

This second root results physically from both effects of rotation and radiation: for $\Gamma_{\rm max} > 0.639$, a zero value of $\boldsymbol{g_{\rm tot}}$ can be achieved for non–extreme rotations. This enters through the reduction due to rotation of the effective mass $\mathrm{M}_\star$, which is the significant mass in the local Eddington factor $\Gamma_\Omega(\vartheta)$.

For $\Gamma_{\rm max} < 0.639$, Eq. (3.18) has no solution. Physically this means that when the star is sufficiently far from the Eddington limit, the reduction of the effective mass $\mathrm{M}_\star$ by rotation is not sufficient to bring $\Gamma_\Omega(\vartheta)$ to 1. In that case, Eq. (3.13) has only one root given by the classical Eq. (3.14).

We hope that these results clarify the debate between Glatzel (1998) and Langer (1997). On one hand, we see that the claim by Langer that stars close to the Eddington limit have a lower rotation limit is correct, even if the Eq. (3.12) by Langer is not the right one. On the other side, Glatzel has claimed that the Eddington factor does not affect the break–up velocity, we see that this is true in general for most stars for the reasons given above, however for $\Gamma_{\rm max} \geq 0.639$ this statement does not apply.

The moment when stars reach their critical velocities is far from being an academic one, since when this occurs large mass loss enhancements may result, a point which is examined below.

4. The mass loss rates as a function of Ω and Γ

4.1. Present context

The effects of rotation on the mass loss rates have been studied both observationally and theoretically. Observationally, very large changes of the $\dot{M}$–rates, i.e. up to 2–3 orders of a magnitude, were suggested by Vardya (1985). However, Nieuwenhuijzen & de Jager (1988) claimed with reason that the correlation found by Vardya was largely the reflect of the distribution of the mass loss rates and rotational velocities over the HR diagram. When disentangling the various effects, Nieuwenhuijzen and de Jager found much smaller effects of rotation. However, they noticed that the $\dot{M}$–rates of the Be–stars are larger by about a factor 10^2. Since Be–stars are fast rotating stars, we may wonder whether the effects of rotation on the mass loss rates are really so negligible, as these last authors considered them. Certainly further observational studies are also required.

On the theoretical side, Pauldrach et al. (1986), Friend & Abbott (1986) find only a moderate increase of the $\dot{M}$–rates, of about 30% for $v = 350$ km/s. Friend and Abbott find an increase of the $\dot{M}$–rates which can be fitted by the relation (Langer 1998; Heger & Langer 1998)

$$\dot{M}(v) \;=\; \dot{M}(v=0)\left(\frac{1}{1-\frac{v}{v_{\rm crit}}}\right)^{\xi} \qquad (4.20)$$

with ξ = 0.43. This expression, often used in evolutionary models, is based on wind models which do not account for the von Zeipel theorem. We notice that Eq. (4.20) diverges at break–up, while as shown by Glatzel (1998) and Owocki et al. (1996, 1998), the stellar mass loss rates should not diverge at the Ω–limit, (however see Sect. 4.3).

When a proper account of the gravity darkening is made, there are two main terms contributing to the anisotropic mass loss rates from a rotating star (cf. Maeder 1999). 1) The "$\mathrm{g_{eff}}$–effect" which favours polar ejection, since the polar caps of a rotating star are hotter. 2) The "opacity or κ–effect", which may favour an equatorial ejection, when the opacity is large enough at the equator due to the lower $T_{\rm eff}$.

In O–type stars, since opacity is due mainly to the T–independent electron scattering, the $\mathrm{g_{eff}}$–effect is likely to dominate, raising a fast highly ionized polar wind. In B– and later type stars, the opacity effect may favour a dense equatorial wind and ring formation, with low terminal velocities and low ionization. Recently Petrenz & Puls (2000) have constructed 2–D models of line driven winds for rotating O–type stars. They found that the mass loss from hot star is essentially polar due to the $g_{\rm eff}$–effect. Quantitatively their results differ very little from previous works by Pauldrach et al. (1986).

4.2. Expression of the $\dot{M}$–rates as a function of Ω and Γ

It is worth to further examine the consequences of the above results on the dependence of the mass loss rates on rotation. According to the radiative wind theory (cf. Castor et al. 1975; Pauldrach et al. 1986; Kudritzki et al. 1989; Puls et al. 1996), we may write the mass loss fluxes $\Delta \dot{M}/\Delta\sigma$ by surface elements $\Delta\sigma$

$$\frac{\Delta \dot{M}(\vartheta)}{\Delta\sigma} \simeq (k\alpha)^{1/\alpha} \left(\frac{1-\alpha}{\alpha}\right)^{\frac{1-\alpha}{\alpha}} F(\vartheta)^{1/\alpha} g_{tot}^{1-\frac{1}{\alpha}}(\vartheta) \quad (4.21)$$

where k and α are the force multiplier parameters. At some temperatures, the ionisation equilibrium of the stellar wind is changing abruptly and so does the opacity of the plasma. Consequently, the values of the force multipliers undergo rapid transitions for certain values of $T_{\rm eff}$, particularly at 21 000 K and maybe also at 10 000 K (cf. Lamers et al. 1995; Lamers 1997). Such fast transitions of the wind properties are called by Lamers a bi–stability of the stellar winds, since near the transition limit the wind can exist in two states. There are both empirical and theoretical determinations of α, however they lead to rather different values (cf. Lamers et al. 1995). The empirical ones, based on the values of the observed terminal velocities, are in general smaller than the theoretical estimates. As empirical values, Lamers et al. (1995) obtain, for example, α = 0.52 for $4.70 \geq \log T_{\rm eff} \geq 4.35$, (type B1.5 or earlier); α = 0.24, 0.21, 0.17, 0.15 for $\log T_{\rm eff}$ = 4.30 (type B2.5), 4.20 (B5), 4.00 (B9.5), 3.90 (A7) respectively. These transitions may produce jumps in the mass loss rates, with the high rates on the low side of the transition. As $T_{\rm eff}$ is decreasing from the pole to the equator, one may thus expect, on the surface of a fast rotating star of type B or later, the occurence of some bi–stability limits and the corresponding variations of the force multipliers and of the mass loss rates. In a star, where a bi–stability limit is crossed at some latitude, a steep increase of the mass flux will happen between this latitude and the equator, possibly leading to a huge equatorial ring.

With the expressions of the flux (2.3) and of $\boldsymbol{g_{\rm tot}}$ (2.10), we get for the mass flux

$$\frac{\Delta \dot{M}(\vartheta)}{\Delta\sigma} \simeq A \left[\frac{L(P)}{4\pi G M_\star(P)}\right]^{\frac{1}{\alpha}} \frac{g_{\rm eff}[1+\zeta(\vartheta]^{\frac{1}{\alpha}}}{(1-\Gamma_\Omega(\vartheta))^{\frac{1}{\alpha}-1}}$$

$$\text{with} \quad A = (k\alpha)^{\frac{1}{\alpha}} \left(\frac{1-\alpha}{\alpha}\right)^{\frac{1-\alpha}{\alpha}} , \quad (4.22)$$

with $M_\star$ given by Eq. (2.4). We notice the gravity–effect, which favours mass loss at the pole, where the total gravity is higher, and the κ–effect which favours high mass loss where α is small. The proximity to the Eddington limit will enhance the mass flux due to the term $\Gamma_\Omega(\vartheta)$, while rotation enhances the mass flux through both the terms $M_\star$ and $\Gamma_\Omega(\vartheta)$. In the theory of radiatively driven winds, the total opacity at a given optical depth is expressed with the force multipliers in terms of the electron scattering opacity $\kappa_{\rm es}$. This means that in Eq. (4.22), $\Gamma_\Omega(\vartheta)$ is just

$$\Gamma_\Omega(\vartheta) = \frac{\kappa_{\rm es}\; L(P)[1+\zeta(\vartheta)]}{4\pi c G M \left(1-\frac{\Omega^2}{2\pi G \rho_{\rm m}}\right)} . \quad (4.23)$$

The dependence on latitude of $\Gamma_\Omega(\vartheta)$ would only come through the term $\zeta(\vartheta)$.

4.3. Dependence of the global mass loss rates on rotation

Let us estimate how the global mass loss rates depend on the rotation velocities and on the proximity of the $\Omega\Gamma$ limit. For that we henceforth neglect the small corrective term $\zeta(\vartheta)$ in the expression of the flux. We have, if $\Sigma(\omega)$ is the total surface

$$\frac{\dot{M}}{\Sigma(\omega)} \simeq A \left[\frac{L(P)}{4\pi G M_\star}\right]^{\frac{1}{\alpha}} \overline{\left(\frac{g_{\rm eff}}{(1-\Gamma_\Omega)^{\frac{1}{\alpha}-1}}\right)} \simeq$$

$$A \left[\frac{L(P)}{4\pi G M_\star}\right]^{\frac{1}{\alpha}} \frac{\overline{g_{\rm eff}}}{(1-\Gamma_\Omega)^{\frac{1}{\alpha}-1}} , \quad (4.24)$$

since Γ_Ω is independent on ϑ, and with appropriate α– and A–values. For the average effective gravity, we have

$$\overline{g_{\rm eff}} = \frac{\int\int \boldsymbol{g_{\rm eff}} \cdot \boldsymbol{d\sigma}}{\Sigma(\omega)} = \frac{4\pi G M_\star}{\Sigma(\omega)} , \quad (4.25)$$

after integration over the stellar surface which is an isobar (cf. Appendix). This leads to the following expression for the total mass loss rate from the star

$$\dot{M} \simeq \frac{A\; L(P)^{\frac{1}{\alpha}}}{(4\pi G M)^{\frac{1}{\alpha}-1} \left[1-\frac{\Omega^2}{2\pi G \rho_{\rm m}}\right]^{\frac{1}{\alpha}-1} (1-\Gamma_\Omega)^{\frac{1}{\alpha}-1}} . \quad (4.26)$$

This relation expresses how the total mass loss rate from a star depends on mass, luminosity, Eddington factor and rotation, (see Fig. A1 for a simple expression of the rotation parameter). If we omit rotation, Eq. (4.26) is identical to the typical relations used in literature (cf. Pauldrach et al. 1986; Lamers 1997). The amplitude of the effects very much depends on the opacity and in particular on the value of the force multiplier α.

Let us consider a rotating star with angular velocity Ω and a non–rotating star of the same mass M at about the same location in the HR diagram. The ratio of their mass loss rates can be written,

$$\frac{\dot{M}(\Omega)}{\dot{M}(0)} = \frac{(1-\Gamma)^{\frac{1}{\alpha}-1}}{\left[1-\frac{\Omega^2}{2\pi G \rho_{\rm m}}\right]^{\frac{1}{\alpha}-1} (1-\Gamma_\Omega)^{\frac{1}{\alpha}-1}} , \quad (4.27)$$

where Γ is the Eddington ratio corresponding to electron scattering opacity for the non–rotating star. From Eq. (4.23), we have the relation

$$\Gamma_\Omega = \frac{\Gamma}{1-\frac{\Omega^2}{2\pi G \rho_{\rm m}}} . \quad (4.28)$$

Table 1. Values of Γ at the end of the MS for various initial stellar masses, and of the ratios $\dot{M}(\Omega)/\dot{M}(0)$ of the mass loss rates for a star at break–up rotation to that of a non–rotating star of the same mass and luminosity at $\log T_{\rm eff} \geq 4.35$, at $\log T_{\rm eff}$ = 4.30, 4.00 and 3.90. The empirical force multipliers α by Lamers et al. (1995) are used.

$M_{\rm ini}$	Γ	$\frac{\dot{M}(\Omega)}{\dot{M}(0)}$ α = 0.52	$\frac{\dot{M}(\Omega)}{\dot{M}(0)}$ α = 0.24	$\frac{\dot{M}(\Omega)}{\dot{M}(0)}$ α = 0.17	$\frac{\dot{M}(\Omega)}{\dot{M}(0)}$ α = 0.15
120	0.903	∞	∞	∞	∞
85	0.691	∞	∞	∞	∞
60	0.527	3.78	96.2	1130	3526
40	0.356	2.14	13.6	55.3	106.0
25	0.214	1.76	7.02	20.1	32.6
20	0.156	1.67	5.87	15.2	23.6
15	0.097	1.60	5.04	12.1	18.1
12	0.063	1.57	4.68	10.8	15.8
9	0.034	1.54	4.41	9.8	14.2

We get finally

$$\frac{\dot{M}(\Omega)}{\dot{M}(0)} = \frac{(1-\Gamma)^{\frac{1}{\alpha}-1}}{\left[1-\frac{\Omega^2}{2\pi G\rho_{\rm m}}-\Gamma\right]^{\frac{1}{\alpha}-1}} \; . \tag{4.29}$$

If $\Omega = 0$, this ratio is of course equal to 1. This ratio, which is the main result of this work, can also be expressed with the ratio $v/v_{\rm crit,1}$ of the rotational velocity v to the critical velocity given by the usual Eq. (3.14). From Eq. (A7) in the Appendix, we have $\frac{\Omega^2}{2\pi G\rho_{\rm m}} \simeq \frac{4}{9}\frac{v^2}{v^2_{\rm crit,1}}$ over a large range of values (cf. Fig. A1), thus one can write

$$\frac{\dot{M}(\Omega)}{\dot{M}(0)} \simeq \frac{(1-\Gamma)^{\frac{1}{\alpha}-1}}{\left[1-\frac{4}{9}(\frac{v}{v_{\rm crit,1}})^2-\Gamma\right]^{\frac{1}{\alpha}-1}} \; . \tag{4.30}$$

For a star with a small Eddington factor, it simplifies to

$$\frac{\dot{M}(\Omega)}{\dot{M}(0)} \simeq \frac{1}{\left[1-\frac{4}{9}(\frac{v}{v_{\rm crit,1}})^2\right]^{\frac{1}{\alpha}-1}} \; . \tag{4.31}$$

Eq. (4.31) shows that the effects of rotation on the $\dot{M}$–rates remain moderate in general. This is in agreement with the results by Owocki et al. (1996), by Owocki & Gayley (1997) and by Glatzel (1998), and also with more elaborate non–LTE 1–D and 2–D models by Pauldrach et al. (1986), Petrenz & Puls (2000). However, this is only true for stars far enough from the Eddington limit. *When Γ is significant, rotation may drastically increase the mass loss rates as shown by (4.29) or (4.30).* This is particularly the case for low values of α, i.e. for stars with log $T_{\rm eff} \leq 4.30$. In the extreme cases where $\Gamma > 0.639$, a moderate rotation may even make the denominator of (4.29) or (4.30) to vanish, thus leading to extreme mass loss.

Table 1 shows some numerical results based on Eq. (4.29). For different initial stellar masses in the Geneva models at Z =0.02 (Schaller et al. 1992), the Γ factors at the end of the Main Sequence (MS) phase are given, as well as the predicted ratios $\dot{M}(\Omega)/\dot{M}(0)$ of the mass loss rates for a star at break–up rotation and for a non–rotating star of the same mass and luminosity. These ratios are given at $\log T_{\rm eff} \geq 4.35$ ($\alpha = 0.52$), at $\log T_{\rm eff}$ = 4.30 ($\alpha = 0.24$), at $\log T_{\rm eff}$ = 4.00 ($\alpha = 0.17$) and at $\log T_{\rm eff}$ = 3.90 ($\alpha = 0.15$) for the same value of Γ. This covers the range of the typical $T_{\rm eff}$ of OB and LBV stars, the differences with $T_{\rm eff}$ result from the differences in the α–parameter. The indication ∞ in Table 1 means that the bracket term in (4.29) or (4.30) may vanish at maximum rotation, which leads to extreme mass outflows.

The ratios $\dot{M}(\Omega)/\dot{M}(0)$ keep quite moderate even at extreme rotation for MS stars up to 40 $M_\odot$, while for MS stars above 60 $M_\odot$ they can become very large. These ratios may also be very large for B–type supergiants and LBV stars. In particular, we notice that for stars close to the Humphreys–Davidson limit the ratios $\dot{M}(\Omega)/\dot{M}(0)$ may diverge. Such stars are typically at the $\Omega\Gamma$–limit. For log $T_{\rm eff} \leq 4.30$, the force multiplier α is also very small, which favours extreme mass loss. On the whole, it is striking that the domain where $\dot{M}(\Omega)/\dot{M}(0)$ has the possibility to diverge so closely corresponds to the observed domain of LBV stars. The present results will enable us to better specify the changes of the $\dot{M}$ rates in massive star models.

5. Conclusion

We conclude that the concept of an $\Omega\Gamma$–limit reached during the evolution of the most massive stars is not an artefact, but the existence of this limit is confirmed by consistent developments based on the von Zeipel theorem. However, we emphasize that the expression currently used for the critical velocity is not correct. We have also clarified the dependence of the mass loss rates on the rotation velocities in the general case.

We can make the following remarks on the various limits:

1. *The Γ–limit:* The mass loss rates grow steeply as the Eddington limit is approached, even in absence of rotation. This is a well known result of the classical wind theory.
2. *The Ω–limit:* We see that the case of only rotational effects does not apply for O–type stars and even for the early B–type stars, since they always have a significant Γ–value. Only for spectral types later than B3 on the MS, the Γ term can be ignored. In the framework of the radiative wind theory, the growth of the mass loss rates remains limited.
3. *The $\Omega\Gamma$–limit:* This general case is met for rotating OB stars, LBV stars, supergiants and Wolf–Rayet stars, because both Γ and rotation are important. The bracket in (4.29) is reduced by rotation and by the proximity to the Eddington limit. As shown by Table 1, both effects produce steep enhancements of the mass loss rates, especially for lower $T_{\rm eff}$ since α is lower. This may explain the very large mass loss rates for LBV stars, blue and yellow supergiants (cf. de Jager et al. 1988). If the ratio $\Gamma = \frac{\kappa_{\rm es} L}{4\pi c G M}$ is bigger than 0.639, the break–up limit is reached for reduced rotation velocities, as illustrated by Fig. 1. Then, extremely high mass loss rates may occur, a situation likely corresponding to the case of the LBV stars and maybe also to some WR stars.

Some words of caution are necessary. The $\dot{M}$–rates given here are the values predicted in the framework of the radiative wind theory. It is probable that close to break–up several other effects not included here may intervene, such as important horizontal fluxes, formally vanishing T– and P– gradients, instabilities, etc... Also, we may point out that if the flux vanishes, the radiative wind theory should not apply. Thus, for the detailed physics of the break–up, more complex analyses are certainly needed.

Finally, we note that it was generally believed that in addition to L and $T_{\rm eff}$, the mass loss rates only depend on metallicity Z. We see here another dependence which is quite significant and may introduce some scatter in the values of the $\dot{M}$–rates. Thus, we may expect that for a given initial mass the evolution is very different according to rotation, due to both rotational mixing, meridional circulation (Maeder & Zahn 1998) and to the induced differences in the mass loss rates.

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Appendix A: the equation of the stellar surface in a rotating star with high radiation pressure

Shellular rotation, with an angular velocity Ω constant on horizontal surfaces, was proposed by Zahn (1992). This rotation law results from strong horizontal geostrophic–like turbulence which homogeneizes rotation on the horizontal surfaces. As noted by Meynet & Maeder (1997), the isobars for shellular rotation are identical to the equipotentials of the conservative case, which are

$$\Psi = \frac{GM}{r(\vartheta)} + \frac{1}{2}\Omega^2 r^2(\vartheta)\sin^2\vartheta = \text{const.} \tag{A.1}$$

The components of the effective gravity are

$$\begin{aligned} g_{\text{eff},r} &= \frac{\partial \Phi}{\partial r} + \Omega^2 r \sin\vartheta \\ g_{\text{eff},\vartheta} &= \frac{1}{r}\frac{\partial \Phi}{\partial \vartheta} + \Omega^2 r \sin\vartheta\cos\vartheta \end{aligned} \tag{A.2}$$

where $\Phi = GM/r$. In vectorial form, one can write

$$\boldsymbol{\nabla} P = -\rho \boldsymbol{g_{\rm eff}} = -\rho(\boldsymbol{\nabla}\Psi - r^2\sin^2\vartheta\ \Omega\boldsymbol{\nabla}\Omega)\ . \tag{A.3}$$

This equation is interesting: it shows that if Ω is constant on isobars, Ψ is also constant on isobars. Moreover, for a motion $d\boldsymbol{s}$, the equation of the surface must satisfy

$$\boldsymbol{g_{\rm tot}} \cdot d\boldsymbol{s} = 0\ . \tag{A.4}$$

Since we have

$$\boldsymbol{g_{\rm tot}} = \boldsymbol{g_{\rm eff}}\left[1 - \Gamma_\Omega\right]\ , \tag{A.5}$$

this implies that the surface is perpendicular to $\boldsymbol{g_{\rm tot}}$, thus to $\boldsymbol{g_{\rm eff}}$ and to the P–gradient. The surface is an isobar, also if the radiation pressure is important. The equation of the surface can therefore be represented by Eq. (A1) and the procedure to calculate is quite conventional, i.e.

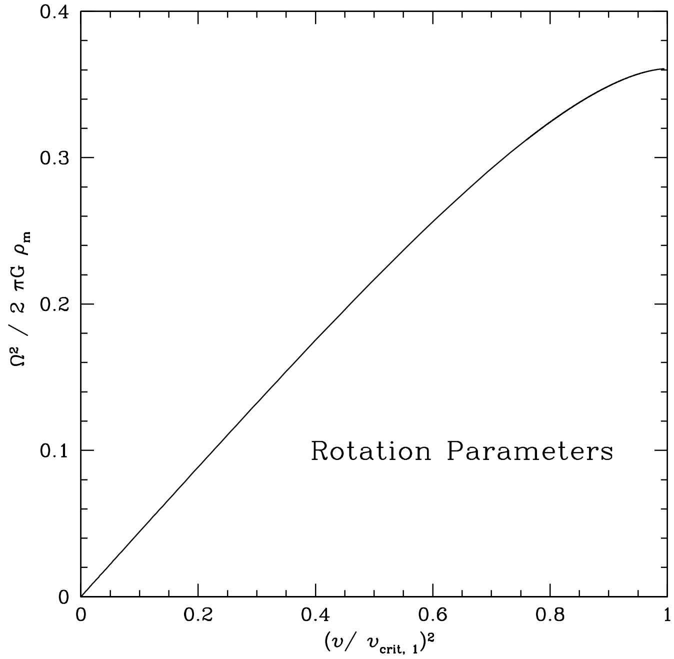

Fig. A.1. The rotation parameter $\frac{\Omega^2}{2\pi G\rho_{\rm m}}$ as a function of the square of the ratio of the rotational velocity to the critical velocity (3.14). The maximum value of $\frac{\Omega^2}{2\pi G\rho_{\rm m}}$ = 0.361, as seen above.

$$\frac{1}{x} + \frac{4}{27}\omega^2 x^2 \sin^2\vartheta = 1 \tag{A.6}$$

with $x = \frac{R}{R_{\rm p}}$. At break–up, the equatorial radius $R_{\rm eb}$ equals 1.5 times the polar radius $R_{\rm pb}$.

In the above demonstration, there is no need of the assumption $\Omega = \Omega(r)$, we just need the assumption that Ω is constant on horizontal surface, which is less restrictive. This is true, whether the radiative acceleration is important or not.

The critical rotation parameter naturally appearing in this work was the ratio $\frac{\Omega^2}{2\pi G\rho_{\rm m}}$. Account must be given to the change of the average density of the star with rotation. We can express this ratio by (3.16) or in term of the actual rotational velocity v and of the critical velocity $v_{\rm crit,1}$ (3.14),

$$\frac{\Omega^2}{2\pi G\rho_{\rm m}} = \frac{4}{9}\frac{v^2}{v^2_{\rm crit,1}} V'(\omega)\frac{R^2_{\rm pb}}{R^2_{\rm e}(\omega)}\ . \tag{A.7}$$

The relation between $\frac{\Omega^2}{2\pi G\rho_{\rm m}}$ and the ratio $\frac{v^2}{v^2_{\rm crit,1}}$ is illustrated in Fig. A1. The product $V'(\omega)\frac{R^2_{\rm pb}}{R^2_{\rm e}(\omega)}$ has a limited range of variation, being equal to 1 for zero rotation and to 0.813 at break–up velocity. This means that for a crude estimate at low or moderate velocities, one may just ignore this product in Eq. (A7).

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