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ASTRONOMY AND ASTROPHYSICS

Radial stellar oscillations under the influence of the dynamics of the atmosphere — a one-dimensional approach

I. Linear adiabatic oscillations of a special model

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Abstract. The dynamics of the quiet solar atmosphere are highly nonlinear. Both the standing waves of solar oscillations and acoustic waves generated in the upper convection zone become nonlinear in the atmosphere and transform into shock waves. Interactions of shock waves, the formation of contact discontinuities, and interactions of shocks with these discontinuities will occur. The strong nonlinear dynamics of the atmosphere should influence high order p-modes of the Sun. In this series of papers we shall deal with fundamental properties of the interaction of the interior of a star with its atmosphere. According to the state of numerical techniques, we must restrict ourselves to radial oscillations or to the vertical dynamics of the atmosphere, respectively. As the nonlinear dynamics of the atmosphere governs the problem, we use a simple equilibrium model of the Sun or a star. For simplicity, we do not take a radial model but a plane layer model. Our particular “standard model” is a layer with nearly constant density in the interior and a smoothly matched isothermal atmosphere. The structure of this configuration is fitted to the structure of the Sun. In the present paper we present the equilibrium model and solutions of its linear adiabatic wave equation. The equilibrium configuration has been selected so, that the wave equation can be transformed to the equation of the associated Legendre functions. We determine the discrete eigenfrequencies, the modes, and the eigenfunctions of the continuous frequency spectrum. Resonances of the continuum are discussed. Also a set of discrete complex frequencies exists. The corresponding waves are not damped modes but limiting cases of instationary waves. The influence of an isothermal corona with a discontinuous transition layer on the frequency spectrum is investigated. We find strong resonances at frequencies between the discrete frequencies of the corona-free model.

Key words: hydrodynamics – waves – Sun: atmosphere – Sun: oscillations – stars: oscillations

1. Introduction

Observations of velocity fields and brightness fluctuations of the quiet solar atmosphere yield results which give evidence of the existence of standing waves in the upper atmosphere of the Sun. Travelling waves are observed only in the photosphere and lower chromosphere. In the upper chromosphere, however, the observational findings indicate non-propagating waves. Reviews given by Deubner (1998) and Deubner & Steffens (2000) deal with this problem. However, results of the linear theory have been used to interprete the observational data.

The theory of linear radial and nonradial adiabatic pulsations of spherically symmetric stars is well-founded (cf., e.g. Ledoux & Walraven 1958, Unno et al. 1989). The mathematical theory of radial pulsations is elegant provided a zero-pressure boundary condition is placed at the surface of the star. In this case, the pulsation equation forms a Sturm-Liouville type eigenvalue problem. Also the mathematical theory of nonradial oscillations with zero-pressure boundary conditions is attractive. Here, for the study of higher order modes, Cowling’s approximation may be used. Further, for high order p-modes, it is sufficient to study only the outer parts of a star or the Sun by a plane layer approximation with constant gravity. If, however, an atmosphere is matched to the convection zone or if the zero pressure boundary condition is replaced by a radiation condition, so that waves can propagate outwards, the spectrum becomes continuous above a certain frequency like the cut-off frequeny of the isothermal atmosphere. Further, the atmosphere or the boundary conditions influence the frequencies and the form of the remaining discrete modes. References to papers dealing with p-modes of a plane layer with constant gravity (the outer convection zone with an overlying atmosphere) are given by Schmitz & Steffens (1999).

The linear dynamics of an atmosphere are also determined by resonance oscillations excited by pulses or waves. In the case of an isothermal atmosphere, the frequency of the resonance oscillation is the acoustic cut-off frequency. The presence of gravity alone does not cause a resonance oscillation. Schmitz & Fleck (1995) have shown that the occurence of this oscillation depends also on the form of the temperature stratification.

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The main problem of the linear theory is the fact that linear waves in an atmosphere become nonlinear and form shock-waves. This behaviour concerns running waves in the acoustic domain of the diagnostic diagram as well as standing waves with frequencies below the acoustic cut-off frequency. Also the waves of the tail of a propagating strong pulse can transform into shock-waves (Holweg 1982). Calculations of two-dimensional finite-amplitude waves performed with an appropriate wave code (Schmitz 1986) always result in the formation of shocks. The linear theory works only in the limit of really small wave amplitudes. Also finite-amplitude standing waves in gases without an external gravity quickly form shock-waves (Bechert 1940).

In theory, in an atmosphere, both long period evanescent waves and short period acoustic waves generated by turbulent convection transform into shock waves. Then, contact discontinuities are formed by shock-overtaking, and shocks interact with these discontinuities. Because of the complexity of these dynamical processes, numerical calculations of the nonlinear dynamics of a non-magnetic atmosphere have been restricted to purely vertical motions (e.g. Schmitz et al. 1985, Schmitz & Fleck 1993, Fleck & Schmitz 1993, Carlson & Stein 1998). At present, it is not possible to tackle the problem of a nonlinear three-dimensional atmospheric wave field with its shock-fronts and the interactions numerically.

The one-dimensional simulations use isolated atmospheres with a given boundary condition (moving piston) at the bottom. This boundary condition determines the dynamics of the atmosphere without any reaction to the motion of the atmosphere. In practice, downwards propagating waves often conflict with the fixed motion of the piston.

This series of papers shall deal with mutual interactions between the linear dynamics of the interior of a star and the (nonlinear) dynamics of its atmosphere. Given the problems just mentioned, we shall study only radial motions or radial pulsations.

From a hydrodynamical point of view, the atmosphere is the most complicated part of a star. For this reason, we retain the plane atmosphere. As there are numerical codes for the calculation of vertically propagating shock waves and profound knowledge of the linear and nonlinear dynamics of a plane atmosphere, it is obvious to use this atmospheric model. As, in comparision with the atmosphere, the interior of the star is a linear hydrodynamical system, we shall describe also the interior by a plane approximation. Instead of the radial dynamics of a sphere, we study the vertical dynamics of a plane self-gravitating layer. This approximation enables a compact and numerically optimal formulation of the (nonlinear) hydrodynamical equations and the gravitation by use of the column mass as an independent variable (Schmitz & Wolf 1986). Another point of view supporting the plane layer approximation is the existence of the analytic model presented in this paper, which has no corresponding counterpart in the case of spherical symmetry.

In the one-dimensional case, the vertical dynamics of a plane layer and the radial dynamics of a sphere should not differ widely. For linear oscillations, the vertical dynamics of the layer and the radial dynamics of the sphere are closely related. This property is due to the Sturm-Liouville eigenvalue problem. The most significant properties of the solutions are: the behaviour of the zeros of the eigenfunctions; the orthogonality of the eigenfunctions; and that the order of the eigenvalues does not depend upon the detailed internal structure and spatial symmetry. However, for a nearly unstable star with $\gamma \approx 4/3$ a plane approximation would fail as the plane layer is stable for all values of γ.

In the present paper, we study the linear dynamics of an analytical model. This model is homogeneous in the interior, and is smoothly matched to an isothermal atmosphere. With a zero-pressure boundary condition instead of the atmosphere, the homogeneous layer corresponds to the homogeneous compressible model of Pekeris (1938). The adiabatic wave equation of the plane configuration can be reduced to the equation of the associated Legendre functions. Thereby, all kinds of waves and oscillations can be described in closed form. The results of the linear theory are the basis for investigations of the instationary and nonlinear behaviour of the model.

The paper is organized as follows: In Sect. 2 we present and discuss the equilibrium configuration and the basic parameters. Sect. 3 deals with the adiabatic wave equation and its reduction to the equation of the associated Legendre functions. The general solution of the wave equation and some basic properties of Legendre functions are treated in Sect. 4. Sect. 5 deals with the discrete modes of the configuration. The continuous spectrum is considered in Sect. 6. There, we also study the occurence of resonances. In Sect. 7 we present solutions with complex frequencies and comment upon their meaning. The influence of the hot isothermal corona on the spectrum is analyzed in Sect. 8. Some relations for Legendre functions are given in an appendix.

2. The equilibrium model

Let z be the outwards directed geometrical coordinate, p the pressure, ρ the density, T the temperature, and a the isothermal sound speed. Let m be half the column mass of the configuration, defined by $m = \int_0^z \rho(z)\, dz$. The equilibrium equation reads

$$\frac{1}{\rho}\frac{d}{dz}\left(\frac{1}{\rho}\frac{dp}{dz}\right) = \frac{d^2 p}{dm^2} = -4\pi G\ . \tag{1}$$

From this equation we obtain

$$p(m) = p_0\,(1 - \frac{m^2}{M^2})\,, \tag{2}$$

where $M = \int_0^\infty \rho(z)\, dz$ is half the total column mass and

$$p_0 = 2\pi G M^2\ . \tag{3}$$

We now introduce the relative mass $x = m/M$. We have

$$p(x) = p_0\,(1 - x^2)\ . \tag{4}$$

These relations are familiar as plane self-gravitating layers, and play a role for the physics of the interstellar gas and for the

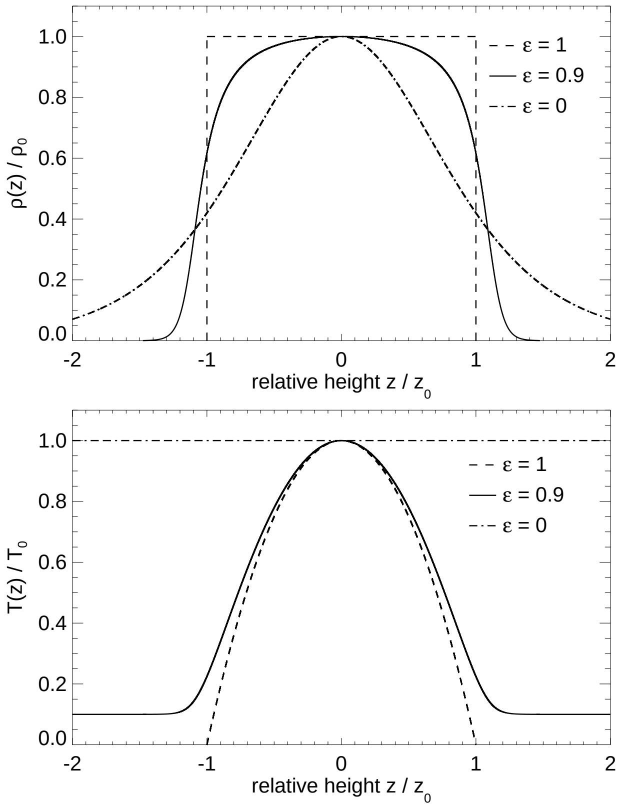

Fig. 1. Equilibrium densities and temperatures for $\epsilon = 0.9$ and the limiting cases $\epsilon = 0$ and $\epsilon = 1$

vertical structure of rotating gaseous disks or disk-galaxies. Now we choose the following sound speed stratification:

$$a^2(x) = a_0^2(1 - \epsilon\, x^2)\,, \qquad \text{with} \qquad 0 \le \epsilon \le 1 \tag{5}$$

where a_0 is the sound speed at the center $z = 0$. With the equation of state of the classical ideal gas, $p = a^2\,\rho$, where $a^2 = RT/\mu$, the density reads

$$\rho(x) = \frac{p_0}{a_0^2}\frac{(1-x^2)}{(1-\epsilon x^2)}\,. \tag{6}$$

For the geometrical coordinate $z = \int_0^m \frac{1}{\rho}\, dm$ we obtain

$$z = z_0\,[\,(1-\epsilon)\,\mathrm{Artanh}\, x + \epsilon\, x\,]\,, \tag{7}$$

where the effective thickness $2\,z_0$ is given by

$$z_0 = \frac{a_0^2}{2\pi G M}\,. \tag{8}$$

The gravity stratification is

$$g(m) = 4\pi\, G\, m\,, \tag{9}$$

so that the surface gravity is $g_\infty = 4\pi G M$. From the constant gravity g_∞ and the isothermal sound speed a_∞ of the atmosphere we obtain

$$a_0^2 = a_\infty^2\,/\,(1-\epsilon)\,, \quad M = \frac{g_\infty}{4\pi\, G}\,, \quad z_0 = \frac{2a_0^2}{g_\infty}\,. \tag{10}$$

The pressure scale height of the isothermal atmosphere is

$$H = H_a = a_\infty^2/g_\infty\,. \tag{11}$$

For $\epsilon = 1$, the density becomes constant. Then, the thickness of the configuration $2\,z_0$ is finite. In this case, we have $z = z_0\, x$ so that $z = z_0$ at $m = M$. For $\epsilon = 0$ the configuration is isothermal. Here, we obtain

$$\rho = \rho_0(1-x^2) \quad \text{with} \quad x = \tanh(\frac{z}{z_0}) \tag{12}$$

which is Spitzer's (1942) solution for the isothermal layer.

To approach the structure of a star with an isothermal atmosphere we have to take $1-\epsilon \ll 1$. Fig. 1 displays the temperature and the density as functions of the geometrical coordinate z for $\epsilon = 0.0,\ 0.9$, and 1.0.

In the following, we use data of the Sun: The central temperature is $T_0 = 1.5\,10^7$ K, the surface gravity is $g_0 = 2.74\,10^4$ cm/s^2. For the temperature of the atmosphere, we take $T_\infty = 4500$ K. Then, from the ratio of T_0 and T_∞, we obtain:

$$\epsilon = 0.9997\,.$$

From this value, we get $z_0 = 7.6\,10^{10}$ cm, a result in good agreement with the radius $R = 7\,10^{10}$ cm of the Sun.

3. The adiabatic wave equation

Let u be the velocity, c the adiabatic sound speed, ρ the density, p the pressure, γ the constant adiabatic exponent, and Δp the Lagrangian pressure perturbation. In Lagrangian representation, the hydrodynamic equations of vertical adiabatic motions read

$$\frac{\partial u}{\partial t} = -\,\frac{\partial\, p}{\partial m} - 4\pi\, G\, m\,, \tag{13}$$

$$\frac{\partial p}{\partial t} = -\,c^2\,\rho^2\,\frac{\partial\, u}{\partial m}\,. \tag{14}$$

By linearization of these equations we obtain

$$\frac{\partial u}{\partial t} = -\,\frac{\partial \Delta p}{\partial m}\,, \tag{15}$$

$$\frac{\partial \Delta p}{\partial t} = -\,c^2\,\rho^2\,\frac{\partial\, u}{\partial m}\,, \tag{16}$$

where now $c(m)$ and $\rho(m)$ are undisturbed quantities. We have $c^2 = \gamma\, a^2$. The resulting wave equation of the Lagrangian pressure perturbation is

$$\frac{\partial^2 \Delta p}{\partial\, t^2} = c^2\,\rho^2\,\frac{\partial^2\,\Delta p}{\partial m^2}\,. \tag{17}$$

Using the equilibrium configuration of Sect. 2, we get

$$(1-\epsilon\, x^2)\,\frac{\partial^2 \Delta p}{\partial\, t^2} = \gamma\,\frac{p_0^2}{a_0^2}\,(1-x^2)^2\,\frac{\partial^2\,\Delta p}{\partial m^2}\,. \tag{18}$$

Now, instead of the mass m we use the relative mass x, and separate the time dependence by $\exp[\,i\,\omega\, t\,]$ to obtain:

$$-\,\sigma^2\,(1-\epsilon\, x^2)\,\Delta p = (1-x^2)^2\,\frac{d^2\,\Delta p}{d\, x^2}\,, \tag{19}$$

where σ is a dimensionless frequency defined by

$$\omega^2 = \sigma^2\,\Omega^2 \quad \text{with} \quad \Omega^2 = \frac{\gamma\, g_\infty^2}{4\, a_0^2}\,. \tag{20}$$

We now denote the time-independent amplitude by Δp, and put

$$\Delta p(x) = p_*\,\sqrt{1-x^2}\;y(x)\,. \tag{21}$$

Here and in the following, the quantity p_* is an arbitrary pressure. The wave equation can be transformed to the differential equation of the associated Legendre functions

$$(1-x^2)^2\,\frac{d^2y}{dx^2} - 2\,x(1-x^2)\,\frac{dy}{dx} + \big(\,\nu(\nu+1)\,(1-x^2) - \mu^2\,\big) = 0\,, \tag{22}$$

where the degree ν and the order μ are given by

$$\nu\,(\nu+1) = \epsilon\,\sigma^2 \quad \text{and} \quad \mu^2 = 1-(1-\epsilon)\,\sigma^2\,. \tag{23}$$

We obtain

$$\nu = -\,\frac{1}{2} + \frac{1}{2}\,\sqrt{1+4\,\epsilon\,\sigma^2} \tag{24}$$

and

$$\mu = \sqrt{1-(1-\epsilon)\,\sigma^2}\,. \tag{25}$$

As $P^{\pm\mu}_{-\nu-1}(x) = P^{\pm\mu}_{\nu}(x)$, we have dropped the minus sign of the root in the expression of ν. The dimensionless cut-off frequency σ_∞ of the atmosphere is

$$\sigma_\infty = \frac{1}{\sqrt{1-\epsilon}}\,. \tag{26}$$

4. The general solution of the wave equation

In the following, ν and μ are real or complex numbers. Solutions of the Legendre differential equation are the associated Legendre functions of the first kind, $P^{+\mu}_\nu(x)$ and $P^{-\mu}_\nu(x)$, and the second kind, $Q^\mu_\nu(x)$. We use the following representation of the general solution:

$$y(x) = C_1\,\left[\,P^{+\mu}_\nu(x) + C_2\,P^{-\mu}_\nu(x)\,\right]\,. \tag{27}$$

From the condition of symmetry $y(-x) = y(+x)$, we obtain

$$y(x) = A\,\left[\,f(\nu,+\mu)\,P^{+\mu}_\nu(x) - f(\nu,-\mu)\,P^{-\mu}_\nu(x)\,\right] \tag{28}$$

where A is a complex constant and

$$f(\nu,\mu) = 2^{-\mu}\Gamma(\frac{1+\nu-\mu}{2})\Gamma(\frac{2+\nu-\mu}{2})\sin(\frac{\pi[\nu-\mu]}{2})\,. \tag{29}$$

Details are given in the appendix. Further, we have (Gradshteyn & Ryzhik 1980):

$$P^{\pm\mu}_\nu(-x) = P^{\pm\mu}_\nu(x) \quad \text{for} \quad \nu\pm\mu = 0,2,4,... \tag{30}$$

for real or complex ν and μ. In the following, we need the representation (Abramowitz & Stegun 1965)

$$P^\mu_\nu(x) = \frac{1}{\Gamma(1-\mu)}\left[\frac{1+x}{1-x}\right]^{\mu/2}{}_2F_1(-\nu,\nu+1;1-\mu;\frac{1-x}{2})\,. \tag{31}$$

At $x=0$ we have (Abramowitz & Stegun1965)

$$P^\mu_\nu(0) = 2^{+\mu}\pi^{-1/2}\,\cos[\frac{\pi(\nu+\mu)}{2}]\,\frac{\Gamma(\frac{1+\nu-\mu}{2})}{\Gamma(\frac{2+\nu+\mu}{2})}\,. \tag{32}$$

From Eq. (31) we obtain

$$P^\mu_\nu(x) \propto\,(1-x)^{-\,\mu/2} \quad \text{for} \quad x\to+1\,. \tag{33}$$

From Eq. (7) we get $z = z_0\,(1-\epsilon)\,\mathrm{Artanh}\,x$ for $z\to\infty$ or $x\to 1$. Therefore, $1-x = 2\,\exp(-z/H)$ for $z\to\infty$ or $x\to 1$ where $H = z_0\,(1-\epsilon)$ is the pressure scale height of the isothermal atmosphere. Finally, we have

$$P^\mu_\nu(x) \propto \exp(z\,\mu/2\,H) \quad \text{for} \quad z\to\infty\,. \tag{34}$$

5. The discrete spectrum

For $\sigma < \sigma_\infty$, the parameters ν and μ are real. Modes are selected by the boundary condition

$$y \propto \frac{\Delta p}{\sqrt{p}} \to 0 \quad \text{as} \quad z\to\pm\infty \quad \text{or} \quad x\to\pm 1\,, \tag{35}$$

which represents the behaviour of evanescent waves in an isothermal atmosphere. Now let us assume that $\mu > 0$. We have (Erdélyi 1953):

$$P^{+\mu}_\nu(x) \propto (1\mp x)^{-\,\mu/2} \quad \text{for} \quad x\to\pm 1 \tag{36}$$

and

$$P^{-\mu}_\nu(x) \propto (1\mp x)^{+\,\mu/2} \quad \text{for} \quad x\to\pm 1\,. \tag{37}$$

The solution fulfilling the condition $y\to 0$ for $x\to\pm 1$ is $P^{-\mu}_\nu(x)$. Thus, in Eq. (28) we have to put $f(\nu,\mu) = 0$. This condition is fulfilled by

$$\nu - \mu = j = 2\,n \quad \text{with} \quad n = 0,1,2,...\,. \tag{38}$$

We finally obtain:

$$y(x) = P^{-\mu}_\nu(x) \quad \text{with} \quad \nu-\mu = 2\,n\,. \tag{39}$$

Eq. (30) shows this solution is symmetric with respect to x.

Squaring the condition $-\mu = j-\nu$, i.e.

$$-\,\sqrt{1-(1-\epsilon)\,\sigma^2} = j - \frac{1}{2}\,\sqrt{1+4\,\epsilon\,\sigma^2} + \frac{1}{2} \tag{40}$$

we finally obtain the relation

$$\sigma^4 + \sigma^2\,[(2j+1)^2(1-\epsilon) - 2(1+j+j^2)] + (j^2+j+1)^2 - (2j+1)^2 = 0\,. \tag{41}$$

The root σ^2 fulfilling the condition $\nu-\mu = j = 2\,n$ is

$$\sigma^2 = \frac{1}{2}\,[2(j^2+j+1) - (1-\epsilon)(2j+1)^2 + (2j+1)\,\sqrt{1+3\epsilon+\epsilon(\epsilon-1)(2j+1)^2}\,]\,. \tag{42}$$

The second root σ^2 fulfils the condition $\nu+\mu = j$.

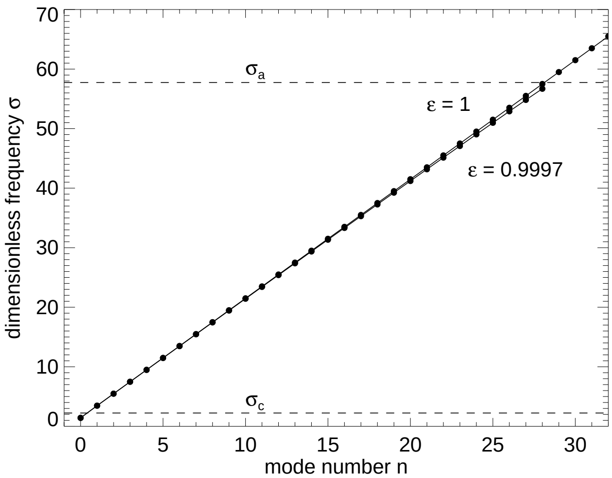

Fig. 2. Discrete dimensionless frequencies of a Sun-like configuration. The dashed lines indicate the cut-off frequencies σ_a of the atmosphere and σ_c of the isothermal corona considered in Sect. 8.

5.1. *The general case* $0 < \epsilon < 1$

Fig. 2 shows the dimensionless discrete frequencies for $\epsilon = 1$ and $\epsilon = 0.9997$. We have $\sigma^2 = 2$ for $n = 0$ and $\epsilon = 1.0$, and $\sigma^2 = 1.9997$ for $n = 0$ and $\epsilon = 0.9997$. Only higher harmonics are slightly affected by the isothermal atmosphere. For $\epsilon = 0.9997\,(0.9998)$ there are $29\,(36)$ modes beneath the acoustic cut-off frequency. Reports of precise observations of the radial solar eigenmodes, and accurate measurements of their frequencies, are abundant in the literature (Lazrek et al. 1997, Toutain et al. 1998, Rabello-Soares & Appourchaux 1999, Chaplin et al. 1999, Thiery et al. 2000). Modern helioseismic data indicate that ~ 39 radial resonances are observed up to the solar acoustic cut-off frequency at ≈ 5.5 mHz. Knölker (1983) who calculated radial pulsation frequencies of a solar model numerically, found 34 frequencies.

The pressure perturbations of the modes are given by

$$\Delta p(x) = p_* \sqrt{1-x^2}\, P_\nu^{-\mu}(x)\ . \tag{43}$$

Fig. 3 shows the pressure perturbations of the first three modes. The frequency of the fundamental mode of our simple model is $\omega = 7.8\,10^{-4}\ \mathrm{s}^{-1}$. The corresponding period $P = 2.2$ h is twice the period of the Sun.

5.2. *The special cases* $\epsilon = 1$ *and* $\epsilon = 0$

The case $\epsilon = 1$ describes a layer with constant density. For $\epsilon = 1$, where $\mu = 1$, we obtain

$$\sigma^2 = 2\,(n+1)\,(2n+1) \text{ with } n = 0, 1, 2, \ldots \tag{44}$$

or

$$\omega^2 = 4\pi\, G\, \rho_0\, \gamma\, (n+1)\,(2n+1) \text{ with } n = 0, 1, 2, \ldots\ . \tag{45}$$

This expression corresponds to

$$\omega^2 = 4\pi G \rho_0\, \frac{1}{3}\, [\gamma[n(2n+5)+3]-4\,] \text{ with } n = 0, 1, 2, \ldots \tag{46}$$

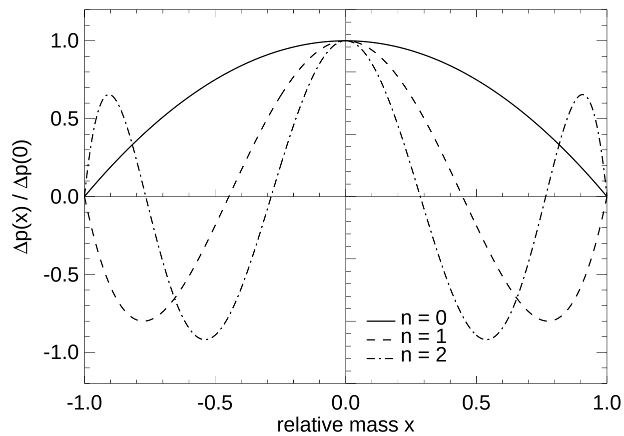

Fig. 3. The pressure perturbations of the three lowest normal modes as functions of the relative mass x

which gives the frequencies of the spherically symmetric, homogenous model of Pekeris (1938). The pressure perturbations of the modes of the homogeneous layer are given by

$$\Delta p(x) = p_* \sqrt{1-x^2}\, P_{2n+1}^1(x) \tag{47}$$

or

$$\Delta p(x) = p_*\, (1-x^2)\, \frac{d}{dx}\, P_{2n+1}(x)\ . \tag{48}$$

The case $\epsilon = 0$ describes Spitzer's (1942) isothermal layer. For $\epsilon = 0$, where $\nu = 0$, we obtain only one real value $\sigma^2 = 1$. With $\nu = 0$ and $\mu = 0$, i. e. $P_0^0(x) = 1$, we have:

$$\Delta p(x) = p_* \sqrt{1-x^2}\ . \tag{49}$$

The frequency of this singular mode is the cut-off frequency of the isothermal layer. Simon (1965) has studied the linear adiabatic dynamics of this configuration.

6. The continuous spectrum

For real frequencies above the acoustic cut-off frequency of the isothermal atmosphere, the spectrum is continuous. The degree ν is real, the order μ is imaginary. Let

$$\mu = i\,\alpha \quad \text{with} \quad \alpha = \sqrt{(1-\epsilon)\,\sigma^2 - 1}\ . \tag{50}$$

We evaluate $y(x)$ as given by Eq. (28). From Eq. (31) we obtain:

$$P_\nu^{i\alpha}(x) = \frac{1}{\Gamma(1-i\alpha)} \left[\frac{1+x}{1-x}\right]^{i\alpha/2} {}_2F_1(-\nu, \nu{+}1; 1{-}i\alpha; \frac{1-x}{2}). \tag{51}$$

As ${}_2F_1(.,.;1{-}i\alpha;x) = {}_2F_1^*(.,.;1{+}i\alpha;x)$ and $\Gamma(\zeta^*) = \Gamma^*(\zeta)$, the functions $P_{+\nu}^{i\alpha}(x)$ and $P_\nu^{-i\alpha}(x)$ are complex conjugate. Then, also the functions $f(\nu, +i\alpha)$ and $f(\nu, -i\alpha)$ are complex conjugate. Therefore, if $A = i\,B$ with real B, the general solution y(x) is real. The result is a standing wave with the pressure perturbation

$$\Delta p(x,t) = p_* \sqrt{1-x^2}\, y(x)\, \sin(\omega\, t)\ . \tag{52}$$

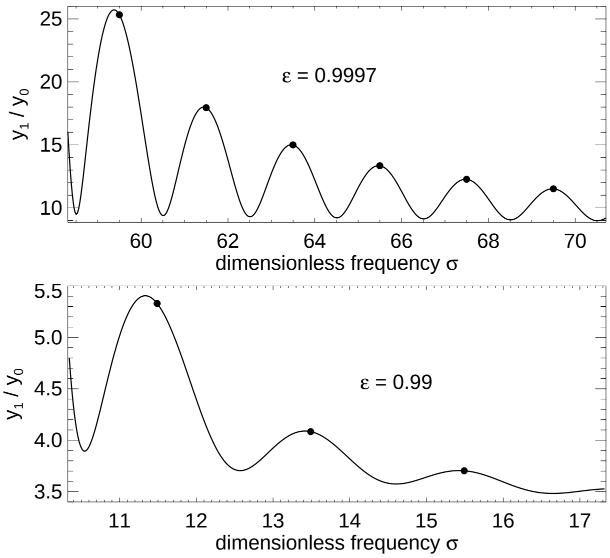

Fig. 4. Resonances above the cut-off frequency for a Sun-like configuration $T_0/T_\infty = 3333$ ($\epsilon = 0.9997$) and a configuration with $T_0/T_\infty = 100$. The solid dots indicate the discrete frequencies of the atmosphere-free layer.

Now we calculate the ratio of the amplitude in the atmosphere and the amplitude at the center of the configuration. Inserting Eq. (32) into the general solution (28), putting $B = 1$, and using addition formulas of the circular functions, we get

$$y(0) = -\,\pi^{-1/2}\,\sin(\pi\mu)\,\Gamma(\frac{1+\nu+\mu}{2})\Gamma(\frac{1+\nu-\mu}{2})\,. \quad (53)$$

For $\mu = i\,\alpha$ we obtain

$$y(0) = -\,i\,\pi^{-1/2}\,\sinh(\pi\alpha)\,\left|\Gamma(\frac{1+\nu+i\alpha}{2})\right|^2\,. \quad (54)$$

Let us now discuss the behaviour of the solution for $z \to \infty$ or $x \to 1$. From Eq. (51) we obtain

$$P_\nu^{i\alpha}(x) \to \frac{1}{\Gamma(1-i\alpha)}\left[\frac{1+x}{1-x}\right]^{i\alpha/2} \quad \text{for} \quad x \to 1\,. \quad (55)$$

Thus, in the limit $x \to 1$ we get

$$y = f(\nu, i\alpha)\,\left[2\,(1-x)\right]^{i\alpha} - f(\nu, -i\alpha)\,\left[2\,(1-x)\right]^{-i\alpha}\,. \quad (56)$$

Let y_1 be the amplitude of this oscillating function. We obtain: $y_1 = 2\,|f(\nu, i\,\alpha)|$, and finally $y_1 =$

$$\sqrt{\frac{2\sinh(\pi\alpha)}{\alpha}}\sqrt{\cosh(\pi\alpha)-\cos(\pi\nu)}\sqrt{\nu^2+\alpha^2}|\Gamma(\nu+i\alpha)|.\quad (57)$$

Let $y_0 = |y(0)|$. The ratio of the amplitude in the atmosphere and the amplitude at the center is

$$\frac{y_1}{y_0} = \frac{\sqrt{2\pi}\sqrt{\nu^2+\alpha^2}|\Gamma(\nu+i\alpha)|}{2^\nu\sqrt{\alpha}\left|\Gamma(\frac{1+\nu+i\alpha}{2})\right|^2}\sqrt{\frac{\cosh(\pi\alpha)-\cos(\pi\nu)}{\sinh(\pi\alpha)}}. \quad (58)$$

Fig. 4 shows this ratio for $\epsilon = 0.9997$ and $\epsilon = 0.99$ as a function of the dimensionless frequency σ. The dots indicate the positions of the discrete frequencies of the case $\epsilon = 1$. At the discrete frequencies of the configuration with constant density, resonances occur. Such resonances are familiar from quantum mechanical systems.

7. Quasi-stationary waves

Now we present solutions with complex frequencies. We consider only the range $z > 0$. The pressure perturbation of a outgoing progressive wave in an isothermal atmosphere is given by

$$\Delta p = p_*\,\exp(-\,z/2\,H + i\,\omega\,t - i\,k\,z)\,. \quad (59)$$

In the case of a complex frequency ω, also the wave number k is complex. We put $\omega = (i\,\beta \pm \alpha)\,\Omega$ and $k = (i\,q \pm r)\,/2H$. An outwards travelling, time damped wave is represented by

$$\Delta p = p_*\exp(-z/2H - \beta\Omega t + qz/2H \pm i\alpha\Omega t \mp irz/2H) \quad (60)$$

with $\alpha, \beta, q, r > 0$. Written in terms of the relative mass x, this expression reads

$$\Delta p = p_*\exp(-z/2H -\ \beta\,\Omega\,t \pm i\,\alpha\,\Omega\ t)\,(1-x)^{-\mu/2} \quad (61)$$

with $\mu = q \mp i\,r$. Because of Eq. (33) we may write

$$\Delta p = p_*\exp(-z/2H - \beta\Omega t \pm i\alpha\Omega t)\,P_\nu^\mu(x) \text{ for } x \to 1\,. \quad (62)$$

Comparison with the general solution (28) shows that $f(\nu, -\mu) = 0$, as opposed to the case of real discrete modes where $f(\nu, +\mu) = 0$. Then, from Eq. (29) we obtain the condition

$$\nu + \mu = j = 2\,n \quad \text{with} \quad n = 0, 1, 2, ... \quad (63)$$

Squaring the equation $\mu = j - \nu = 2\,n - \nu$ we finally obtain the dispersion relation (41), from which now

$$\omega^2 = \alpha_0 \pm\ i\,\beta_0 \quad \text{with} \quad \alpha_0 > 0,\ \beta_0 > 0 \quad (64)$$

where

$$\alpha_0 = \frac{\Omega^2}{2}\,[2(j^2+j+1) - (1-\epsilon)(2j+1)^2] \quad (65)$$

and

$$\beta_0 = \frac{\Omega^2}{2}\,[\,(2j+1)\,\sqrt{\epsilon(1-\epsilon)(2j+1)^2 - 1 - 3\epsilon}\,] \quad (66)$$

for $\epsilon(1-\epsilon)(2j+1)^2 - 1 - 3\epsilon > 0$. From these two roots ω^2 we obtain two roots ω describing time damped waves:

$$\omega = (\,\pm\,\alpha + i\,\beta\,)\,\Omega \quad \text{with} \quad \alpha > 0,\ \beta > 0\,. \quad (67)$$

The corresponding values of μ given by Eq. (23) are

$$\mu = q \mp\ i\,r \quad \text{with} \quad q > 0,\ r > 0\,. \quad (68)$$

Therefore, solutions are

$$\Delta p = p_*(1-x^2)^{1/2}\exp(\,[-\beta + i\,\alpha]\,\Omega\,t\,)\,P_{2\,n-q+i\,r}^{q-i\,r}(x) \quad (69)$$

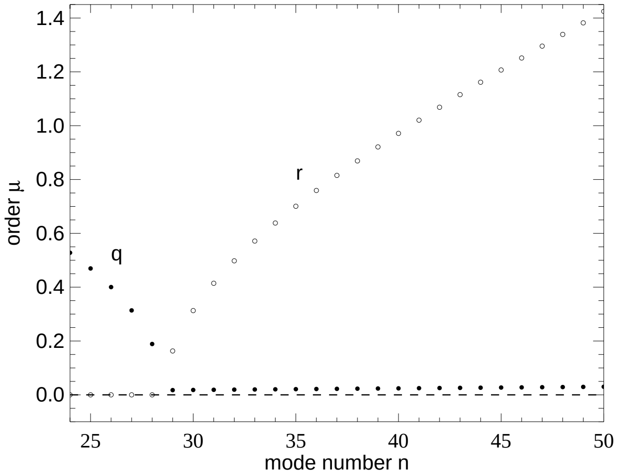

Fig. 5. Steepening parameter q (solid dots) and wavelength parameter r (open dots) of quasi-stationary waves as functions of the order n. We have $\mu = q \pm i\,r$.

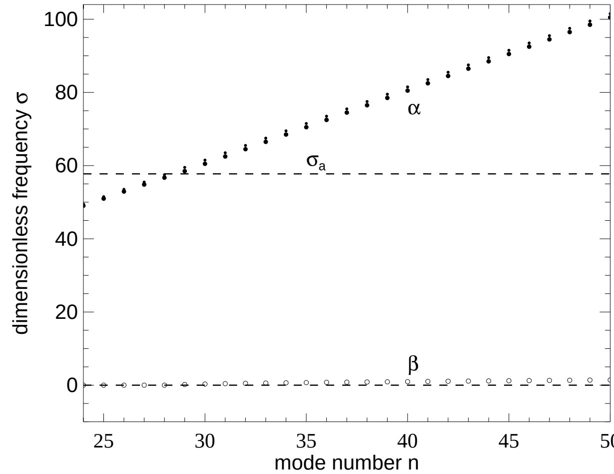

Fig. 6. Time damping coefficient β (open dots) and frequency α (heavy solid dots) of quasi-stationary waves as functions of the order n. We have $\sigma = \alpha + i\,\beta$. The small dots indicate the real discrete frequencies of the atmosphere-free layer.

and the complex conjugate

$$\Delta p = p_*(1-x^2)^{1/2} \exp(\,[-\beta - i\,\alpha]\,\Omega\,t\,)\; P^{q+i\,r}_{2\,n-q-i\,r}(x)\,. \quad (70)$$

From these complex solutions, real solutions can be constructed. The asymptotic form of these real solutions is

$$\Delta p = p_* \exp[-\frac{z}{2H}\,]\;\exp[\frac{q\,z}{2H} - \beta\,\Omega\,t]\;\sin(\alpha\,\Omega\,t - \frac{r\,z}{2H}) \quad (71)$$

The asymptotic form of the complex displacement $\xi(z,t)$ is

$$\xi \sim \exp[+\frac{z}{2H}]\;\exp[\frac{q\,z}{2H} - \beta\Omega\,t\,]\;\exp(\pm i\,[\alpha\Omega\,t - \frac{rz}{2H}])\,. \quad (72)$$

Fig. 5 shows the parameters q and r, Fig. 6 the coefficients β and α.

For plane convection zones with atmospheres, solutions of the wave equation of three-dimensional waves with complex frequencies have been calculated by Hindman & Zweibel (1994) and Schmitz & Steffens (2000).

Solutions of (general) wave equations with complex discrete frequencies due to suitable boundary conditions are often called quasi-normal modes. They play a role in wave equations of open or unbounded systems. Usually, these solutions do not form a complete set of normal modes.

In our case, adiabatic oscillations with complex frequencies must be interpreted as follows: For $z \to \infty$, the solutions behave as

$$\xi(z,t) \to \xi_0\; e^{i\,\omega t}\, e^{z/(2H)}\, e^{i\,k\,z} \quad \text{with} \quad \omega = \omega(k)\,. \quad (73)$$

Only for real k, the eigenfunctions of the continuous spectrum are integrable in a generalized sense. As $\rho(z) \sim \rho_0 \exp(-\,z/H)$ for $z \to \infty$, we have

$$\int_0^\infty \rho\;\xi_k\,\xi^*_{k'}\,dz \propto \delta(k-k')\,. \quad (74)$$

In the distribution sense, the eigenfunctions are orthogonal. This does not hold for complex wave numbers k with $Re(k) > 0$, where the above integral diverges. Such solutions cannot be normalized. Even if we would restrict ourselves to the interior of the layer, two conditions for a complete set of proper modes are not fulfilled: The discontinuity condition, and the no-tail condition. Problems of proper modes of open systems are dealt with by Ching et al. (1998).

Adiabatic waves with complex frequencies must be interpreted like the corresponding waves in quantum mechanics. There, complex values of the energy are used to describe non-stationary states of a system. In the exterior region of a leaky, one-dimensional potential, the time-damped wave function increases exponentially with respect to the spatial coordinate. Such a state is called a quasi-stationary state, and it is pointed out that this non-integrable state approximates an instationary state which is integrable. The problem is dealt with by Blochinzew (1957), Macke (1959) and Landau & Lifshitz (1959). Also in our case, quasi-stationary waves should be considered as approximations to instationary waves.

8. The influence of a hot, static corona

We study the effect of an isothermal corona on the frequency spectrum of the configuration. The corona is matched to the atmosphere by a temperature jump. The temperature of the corona is $T_c = 1.5\,10^6$ K, its mean molecular weight is $\mu_c = 0.6$. The position of the transition layer is $z_c = 1.00390\,z_0 = 7.61576\,10^8$ m, lying 2000 km above the position $z = 1.00127\,z_0 = 7.59576\,10^8$ m where $T = 5800$ K. Fig. 7 shows this configuration. We denote the pressure scale heights of the corona and of the atmosphere by H_c and H_a, the corresponding cut-off frequencies by ω_c and ω_a or σ_c and σ_a. Let

$$\kappa_a = \frac{1}{2H_a}\,\sqrt{1-\sigma^2/\sigma_a^2} \text{ and } k_a = \frac{1}{2H_a}\,\sqrt{\sigma^2/\sigma_a^2 - 1}\,, \quad (75)$$

$$\kappa_c = \frac{1}{2H_c}\,\sqrt{1-\sigma^2/\sigma_c^2} \text{ and } k_c = \frac{1}{2H_c}\,\sqrt{\sigma^2/\sigma_c^2 - 1}\,. \quad (76)$$

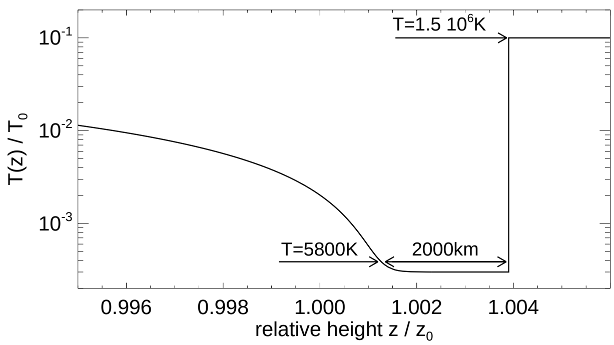

Fig. 7. Addition of an isothermal corona to a Sun-like configuration. The relative temperature as a function of the relative height.

Let Δp_a be the Lagrangian pressure perturbation of the atmosphere, Δp_c that of the corona. Because of the symmetry, it is sufficient to consider only the case $z > 0$. We have

$$\Delta p_a = e^{-z/2H_a}[A_a \sin(k_a z) + B_a \cos(k_a z)] \text{ for } \omega > \omega_a, \quad (77)$$

$$\Delta p_c = e^{-z/2H_c}[A_c \sin(k_c z) + B_c \cos(k_c z)] \text{ for } \omega > \omega_c, \quad (78)$$

$$\Delta p_a = e^{-z/2H_a}[C_a\, e^{-\kappa_a z} + D_a\, e^{+\kappa_a z})] \text{ for } \omega < \omega_a, \quad (79)$$

$$\Delta p_c = e^{-z/2H_c}\; C_c\, e^{-\kappa_c z} \text{ for } \omega < \omega_c\,. \quad (80)$$

The amplitudes A_a, B_a, C_a, D_a are related to the parameters of the solutions (43) and (52). The dimensionless cut-off frequencies of the corona and the atmosphere are $\sigma_c = 2.236$ and $\sigma_a = 57.735$.

Three cases have to be considered:

a) $\omega < \omega_c$. In this case, the pressure perturbation of the corona is given by Eq. (80). As can be seen from Fig. 2, only the fundamental mode is left. By the existence of the corona the frequency and the form of this mode are slightly changed.

b) $\omega_a < \omega$. In this case, the waves are acoustic in the atmosphere and in the corona. The pressure perturbation of the atmosphere is given by Eq. (77), that of the corona by Eq. (78). The amplitudes A_a and B_a of the atmospheric parts are fixed, the amplitudes A_c and B_c have to be determined by the conditions for continuity. Therefore, the solution in the atmosphere is not affected by the corona. The resonances shown in Fig. 4 are unchanged.

c) $\omega_c < \omega < \omega_a$. In this case, waves are evanescent in the atmosphere and oscillatory in the corona. The frequency spectrum is continuous. The pressure perturbation of the atmosphere is given by Eq. (79), that of the corona by Eq. (78). Now the solution in the atmosphere is affected by the existence of the corona. We have determined the quantities A_c, B_c, C_a, and D_a by the conditions for continuity. Finally we have studied the ratio of the amplitudes in the atmosphere and at the center. Fig. 8 shows this ratio at the transition layer and at $T = 5800$ K.

The dots at the abscissa mark the discrete frequencies of the corona-free model. The resonances are maximum at the intermediate positions. The explanation is simple:

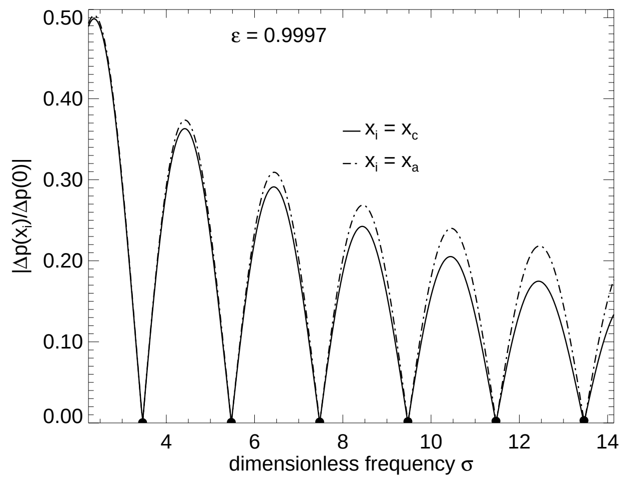

Fig. 8. Resonances above the cut-off frequency of the corona. Relative pressure perturbations as functions of the dimensionless frequency σ at the position x_a where $T = 5800$ K (solid line) and at the position x_c of the transition layer (dashed-dotted line). Dots at the abscissa mark the discrete frequencies of the corona-free model.

Without the corona, the behaviour of the eigenfunctions in the atmosphere is given by

$$\Delta p_a = C_a \exp(-z/2H_a - \kappa_a z) \quad \text{for} \quad \omega = \omega_{2n}\,. \quad (81)$$

The divergent solutions behave as

$$\Delta p_a = D_a \exp(-z/2H_a + \kappa_a z) \quad \text{for} \quad \omega \neq \omega_{2n}\,. \quad (82)$$

When the atmosphere is matched by a corona, these solutions appear in the atmosphere and dominate the total solution.

9. Conclusions

We have presented a simple, one-dimensional stellar model, and have solved its linear adiabatic wave equation. The equilibrium configuration consists of an essentially homogeneous layer with a smoothly matched isothermal atmosphere. The plane approximation enables application of numerical codes written for the calculation of the dynamics of plane atmospheres. Its also leads to an adiabatic wave equation the solutions of which can be given in closed form.

The plane configuration was fitted to the structure of the Sun. In this case, the number of discrete pulsation frequencies of the layer roughly equals the number of the frequencies of radial pulsations of the Sun. We find that the frequencies of the atmosphere-free homogeneous layer are changed only marginally by the addition of the atmosphere. Practically, only a few frequencies immediately below the cut-off frequency are shifted.

The continuous spectrum above the acoustic cut-off frequency shows resonances. The frequencies of the resonances nearly coincide with the corresponding frequencies of the atmosphere-free configuration. However, the resonances are not strong so that they problably do not play a significant role.

There is an infinite number of discrete complex frequencies with real parts above the acoustic cut-off frequency. The time-dependent solutions belonging to these frequencies represent damped oscillations with outwards travelling atmospheric waves. Such solutions are common in the theory of open or unbounded systems. They are not proper modes as they are not normalizable and do not form a complete set of basis functions. As in the case of quantum-mechanical systems these solutions are interpreted as limiting-cases of instationary waves. The physical meaning of these solutions should form the subject of further investigations.

We have matched an isothermal plane corona to the atmosphere by a discontinuos transition layer. In this case, the range $\omega_c < \omega < \omega_a$, where ω_c and ω_a are the cut-off frequencies of the corona and the atmosphere, is important. The discrete spectrum of the corona-free configuration becomes continuous, and only the discrete frequency of the fundamental mode remains. There are resonances with maxima between the discrete frequencies of the corona-free case. As these resonances are pronounced, they could play a significant role.

Appendix A: Relations for Legendre functions

First, we give relations concerning the symmetry of the pressure perturbation with respect to the mid-plane. As, in general, $P_\nu^{+\mu}(x)$ is not symmetric with respect to x, also the general solution

$$y(x) = C_1 \left[P_\nu^{+\mu}(x) + C_2\, P_\nu^{-\mu}(x) \right] . \tag{A.1}$$

is not symmetric. To construct a symmetric solution, we use the condition $\frac{dy}{dx} = 0$ at $x = 0$. We have (Gradshteyn & Ryzhik 1980, Abramowitz & Stegun 1965):

$$\frac{d}{dx}\, P_\nu^{+\mu}(0) = 2^\mu \frac{2}{\sqrt{\pi}}\, \sin[\frac{\pi}{2}\,(\nu+\mu)]\, \frac{\Gamma(\dfrac{2+\nu+\mu}{2})}{\Gamma(\dfrac{1+\nu-\mu}{2})} . \tag{A.2}$$

Therefore, the condition for symmetry of $y(x)$ is

$$2^\mu\, \sin[\frac{\pi}{2}\,(\nu+\mu)]\, \frac{\Gamma(\dfrac{2+\nu+\mu}{2})}{\Gamma(\dfrac{1+\nu-\mu}{2})} + C_2\, 2^{-\mu}\, \sin[\frac{\pi}{2}\,(\nu-\mu)]\, \frac{\Gamma(\dfrac{2+\nu-\mu}{2})}{\Gamma(\dfrac{1+\nu+\mu}{2})} = 0 . \tag{A.3}$$

We get:

$$C_2 = -\,\frac{2^\mu\, \sin[\frac{\pi}{2}\,(\nu+\mu)]\, \Gamma(\frac{2+\nu+\mu}{2})\, \Gamma(\frac{1+\nu+\mu}{2})}{2^{-\mu}\, \sin[\frac{\pi}{2}\,(\nu-\mu)]\, \Gamma(\frac{2+\nu-\mu}{2})\, \Gamma(\frac{1+\nu-\mu}{2})} . \tag{A.4}$$

Multiplying Eq. (A.1) by the denominator of C_2, we obtain Eq. (28). Eq. (30) follows from

$$P_\nu^{\pm\mu}(-x) = \cos([\nu\pm\mu]\,\pi)\, P_\nu^{\pm\mu}(x) - \frac{2}{\pi}\, \sin([\nu\pm\mu]\,\pi])\, Q_\nu^{\pm\mu}(x) \tag{A.5}$$

(Gradshteyn & Ryzhik 1980) for real or complex ν and μ.

References

Abramowitz M., Stegun I.A., 1965, Handbook of Mathematical Functions. Dover Publ. Inc., New York

Bechert K., 1940, Annalen der Physik 37, 89

Blochinzew D.I., 1957, Grundlagen der Quantenmechanik. Deutscher Verlag der Wissenschaften, Berlin

Carlson M., Stein R.F., 1998, In: Deubner F.-L., Christensen-Dalsgaard J., Kurtz D. (eds.) IAU Symposium 185, New Eyes to see Inside the Sun and Stars. p. 435

Chaplin W.J., Elworth Y., Isaak G.R., et al., 1999, MNRAS 308, 424

Ching E.S.H., Leung P.T., Maassen van den Brink A., et al., 1998, Rev. Mod. Phys. Vol. 70, 1545

Deubner F.-L., 1998, In: Deubner F.-L., Christensen-Dalsgaard J., Kurtz D. (eds.) IAU Symposium 185, New Eyes to see Inside the Sun and Stars. p. 427

Deubner F.-L., Steffens S., 2000, In: Wilson A. (ed.) Magnetic Fields and Solar Processes. 9th European Meeting on Solar Physics, Florence, p. 149

Erdélyi A., 1953, Higher transcendental functions. Vol. 1, Mc Graw Hill Co., New York

Fleck B., Schmitz F., 1993, A&A 273, 671

Gradshteyn I.S., Ryzhik I.M., 1980, Tables of Integrals, Series, and Products. Academic Press, New York

Hindman B.W., Zweibel E.G., 1994, ApJ 436, 629

Holweg J.V., 1982, ApJ 257, 345

Knölker M., 1983, Ph.D. Thesis, Freiburg i. Br.

Landau L.D., Lifshitz E. M., 1959, Quantum Mechanics. Pergamon Press, London

Lazrek M., Baudin F., Bertello L., et al., 1997, Solar Physics 175, 227

Ledoux P., Walraven Th., 1958, In: Flügge S. (ed.) Handbuch der Physik. Vol. 51, Springer, Berlin, Heidelberg

Macke W., 1959, Quanten. Akademische Verlagsgesellschaft, Leipzig

Pekeris C., 1938, ApJ 88, 189

Rabello-Soares M. C., Appourchaux T., 1999, A&A 345, 1027

Simon R., 1965, Ann. d'Ap. 28, 40

Schmitz F., 1986, A&A 166, 368

Schmitz F., Ulmschneider P., Kalkofen W., 1985, A&A 148, 217

Schmitz F., Fleck B., 1993, A&A 279, 499

Schmitz F., Fleck B., 1995, A&A 301, 483

Schmitz F., Steffens S., 1999, A&A 344, 973

Schmitz F., Steffens S., 2000, A&A, 356, 319

Schmitz F., Wolf B.E., 1986, A&A 156, 289

Spitzer L. Jr., 1942, ApJ 95, 329

Thiery S., Boumier P., Gabriel A.H., et al., 2000, A&A 355, 743

Toutain T., Appourchaux T., Fröhlich C., et al., 1998, ApJ 506, L147

Unno W., Osaki Y., Ando H., et al., 1989, Nonradial oscillations of stars. University of Tokyo Press