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The linear force-free field in a spherical shell using a new method to determine the coefficients of the eigenfunction expansion

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Abstract. The linear force-free field of a plasma in between spherical shells is found allowing for inhomogeneous boundary conditions. A three-dimensional solution is found by analysis and used as a benchmark to test a solution in terms of an expansion of eigenfunctions where the coefficients are determined by a new method. Alternative methods are also applied in the context of the spherical shell example and used to illustrate some mathematical constraints that can affect their validity. The solution is used to model the solar coronal field in the presence of a large low-latitude coronal hole; SOHO-MDI data provide the inner boundary conditions.

Key words: magnetic fields – Magnetohydrodynamics (MHD) – plasmas – Sun: chromosphere – Sun: corona

1. Introduction

In the strongly magnetised plasma of technology and astrophysical systems in equilibrium, it is often a good approximation to assume a linear force-free magnetic field, that is, the state associated with a magnetic relaxation process (Woltjer 1958; Laurence & Avellaneda 1991). For a slightly dissipative plasma, the notion provides a means to avoid consideration of the detailed plasma dynamics and instead invokes a principle, that of minimum energy with conserved global magnetic helicity (Taylor 1974, 1986). In particular, in technology it is thought that a relaxation process can explain the equilibrium configuration found in several plasma magnetic confinement devices, such as the Spheromak (Dixon et al. 1990), Reversed Field Pinch (Taylor 1974; Ortolani & Schnack 1993), and certain other toroidal configurations (Turner 1984; Taylor & Turner 1989; Browning et al. 1993). It is also common to apply the linear force-free model to a diverse range of problems in solar physics and astrophysics (Nakagawa 1973; Chui & Hilton 1973; Barbosa 1978; Rust & Kumar 1994; Browning 1988; Vekstein et al. 1993; Aly 1993; Kusano et al. 1995; Amari et al. 1997). This paper considers a solar (or stellar) corona modelled between spherical boundaries. In fact the linear force-free field model can only provide a weak abstraction of the complex coronal magnetic field, but the solution itself is of theoretical interest and is used to understand better the different solution forms that arise from the methodology. We are concerned here with some more mathematical aspects of the problem, but we do illustrate the theory with a concrete example of the solar coronal magnetic field.

Although the coronal plasma evolves through a sequence of highly dynamical and non-linear interactions, such as magnetic field reconnections, the relaxation hypothesis is that the final equilibrium state is much simpler, that is, the local plasma current becomes directed along the global magnetic field, except possibly at current sheets, so that the Lorentz force vanishes. In this magnetostatic configuration pressure gradients are ignored because of the low coronal plasma pressure ($\beta \approx 10^{-2}$), and the scale height renders the gravity potential insignificant. In addition no sophistication is added to account for steady flows that could support a stationary equilibrium (Tsinganos 1982). However advections continuously change the boundary conditions at the photosphere and so it is appropriate to interpret any equilibrium obtained as an instantaneous state that is destined to evolve quasi-statically. This is largely true for photospheric motions that are slower than the characteristic Alfven and magnetoacoustic times (Heyvaerts & Priest 1984), but for very fast photospheric motions the equilibrium cannot adjust fast enough and so instead waves are launched. The situation where the photospheric forcing time is similar to the Alfven time is worthy of study but this has not yet been properly investigated.

In considering a solar coronal equilibrium, it is somewhat problematic to define a suitable plasma volume, although even a choice with free boundaries is possible within the theory of relaxed states (Browning 1988). Instead, in the global context, it is usual to establish a fixed inner boundary at the photosphere or lower chromosphere, together with an artificial outer (spherical) boundary. An outer boundary of some kind is required since the force-free approximation (and other modelling assumptions) inevitably break down at some distance from the solar surface. Furthermore, force-free fields in an infinite region in general have unphysical properties such as infinite magnetic energy (Aly 1992, 1993). Thus the coronal plasma may be regarded as residing between two spherical shells. It is then a prerequisite to

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measure or model the flux distribution over those surfaces. In general, comprehensive line-of-sight magnetograms are available for the solar surface (for example the photospheric MDI facility on SOHO, or lower chromospheric magnetograms from Kitt Peak), the problem then being (usually) to transform from the line-of-sight to obtain the normal field component, and to overcome the constraint of access only to the visible disc. Appropriate techniques are described in Clegg et al.(1999a) to extrapolate the boundary condition over the entire solar surface. A more realistic coronal force-free magnetic field model accounts for the variation of current between flux surfaces (see for example Clegg et al. 2000a), representative of the very complex transverse magnetic field gradients at the boundary as recorded on vector magnetograms (Mickey et al. 1996; Sakurai et al. 1995). However, at present vector measurements are only available over strong magnetic field regions where Zeeman triplets can be resolved. Hence, the simplification of a uniform α (current/flux) distribution is still widely studied since it is amenable to analysis, and like the even simpler current-free (potential field) approximation it provides a basis to assess the sensitivity of the field to the assumptions used and to compare with solar observations.

For the potential model it is usual to assume a "source surface" outer boundary condition (Schatten et al. 1969), where the field is wholly radial at some radius $\geq 2.5R_\odot$, but in fact this condition is incompatible with a force-free field in general. Rather it is better to assume that the radial field *component* is uniform in magnitude at some distance from the solar surface, as discovered by the Ulysses mission (Forsyth et al. 1997), and so an estimate of the field strength there is required. However only sparse data is available here and so our approach is to evaluate the field for a family of radial field strengths and then to select the solution that best reproduces other observations. In particular the LASCO coronographs tend to silhouette the field because of the high plasma conductivity. Such evidence also reveals the large radial extent of plumes and streamers and in choosing a spherical outer boundary it could be argued that it is better to place the outer boundary beyond $2.5R_\odot$ where the bounding surface is more isotropic with respect to the plasma β ($\beta(r)$ is different between open and closed field regions). However by definition the force-free model takes no account of pressure gradients and gravity potentials and so here the compromise choice of $2.5R_\odot$ is used, notwithstanding these shortcomings.

An equilibrium field where the Lorentz force vanishes is associated with the well known elliptical problem Eq. (1),

$$\nabla \times \mathbf{B} = \alpha \mathbf{B} \tag{1}$$

in a volume V, solved with appropriate boundary conditions for $\mathbf{B}$ on S (the two spherical shell boundaries). The scalar $\alpha = \mathbf{j} \cdot \mathbf{B}/B^2$ is taken to be uniform in this paper which makes the problem linear. This may be justified partly by relaxation theory (see above). There are several strategies to solve Eq. (1), but two approaches are central to this paper:

First, the equilibrium problem can be formulated in terms of a scalar P, related to the field by

$$\mathbf{B} = \nabla \times \mathbf{A}, \text{ and } \mathbf{A} = \nabla \times (P\hat{\mathbf{r}}) + \alpha P \hat{\mathbf{r}} \tag{2}$$

(Chandrasekhar 1956; Moffatt 1978) where $\mathbf{A}$ is a vector potential and $\hat{\mathbf{r}}$ is a unit radial vector in spherical co-ordinates. The equilibrium problem reduces to a Grad-Shafranov partial differential equation (see Eq. (9)), and a solution can be found in terms of special functions which is used as a benchmark to test the other methods.

Second, a decomposition is often applied where the boundary conditions are accounted for by a potential field $\nabla \times \mathbf{B^V} = \nabla \times (\nabla \times \mathbf{A^V}) = 0$ to allow for the flux across S, superposed with an homogeneous part, i.e. an expansion of eigenfunctions $\mathbf{B_{nmv}} = \nabla \times \mathbf{A_{nmv}}$ (where the mode numbers (n, m, v) refer to the spherical co-ordinates (θ, ϕ, r)) writing

$$\mathbf{B} = \mathbf{B^V} + \sum_{n,m,v} c_{nmv} \mathbf{B_{nmv}} \tag{3}$$

where the eigenfunctions satisfy $\nabla \times \mathbf{B_{nmv}} = \alpha_{nmv} \mathbf{B_{nmv}}$ with eigenvalues α_{nmv} and where $\mathbf{B_{nmv}} \cdot \mathbf{n} = 0$ on S. Such an expansion has been shown to converge by the work of Yoshida and Giga (1990). The problem is to find the coefficients c_{nmv}, but there are now two formulae that purport to determine their values.

In our companion paper Clegg et al.(2000b) it was shown that during the derivation of the coefficients a contentious equation can arise. A similar situation is described in Chu et al.(1999) where, upon taking the curl of Eq. (3) and using the definitions of the potential and eigenfunction fields, there is apparently a relationship for the current that

$$\mathbf{j} = \sum_{n,m,v} c_{nmv} \alpha_{nmv} \mathbf{B_{nmv}} \tag{4}$$

but this is "paradoxical" in the sense that a non-vanishing normal component of current can seemingly be constructed out of a series of terms in Eq. (4) that themselves vanish normal to the boundary. This might be possible in a negative Sobolev space but more usually such an equation cannot converge (given $\mathbf{B_{nmv}}$ are a set of divergence-free vectors) as proved by a theorem described in Clegg et al.(2000b). The problem has arisen because it has been assumed that the curl differential operator can be commuted across an infinite sum of terms to obtain Eq. (4), but it is known that the derivative of a sum of terms does not necessarily equate to the sum of their derivatives. Our interest is not so much with the particular dilemma posed by Eq. (4) but rather because an often used formula for the coefficients in Eq. (3), i.e. Eq. (5) below, can be obtained through a calculation that includes a similar assumption of commutability, creating a similar erroneous equation as Eq. (4), using the usual orthogonality relations (Clegg et al. 2000b). Hence either the formula Eq. (5) is incorrect or there is an alternative means to justify its usage.

The conventional approach then is to find the coefficients for Eq. (3) by a formula developed by Jensen & Chu (1984), i.e.,

$$c_{nmv} = \frac{\alpha}{\alpha_{nmv} - \alpha} \frac{\int \mathbf{A_{nmv}} \cdot \nabla \times \mathbf{A^V} dV}{\int \mathbf{A_{nmv}} \cdot \mathbf{B_{nmv}} dV} \tag{5}$$

and the formula requires that a particular boundary condition is imposed on the vector potential *eigenfunctions* that $\mathbf{A_{nmv}} \times$

$\mathbf{n} = 0$ on S, since this choice affects the numerator of Eq. (5). This extra requirement, and the apparent "paradox" described above has led one of us (PL) to develop a new formula for the coefficients (Clegg et al. 2000b; Laurence et al. in preparation) written as

$$c_{nmv} = \frac{\alpha_{nmv}}{\alpha_{nmv} - \alpha} \frac{\int (\alpha.\mathbf{A^V} - \mathbf{B^V}) \cdot \mathbf{B_{nmv}} dV}{\int B_{nmv}^2 dV} \tag{6}$$

which can be obtained rigorously. It was obtained by only allowing an *inverse* curl operation to be commuted across the infinite series, and this diminishes each term in size to ensure convergence, before applying orthogonality conditions to find c_{nmv} as Eq. (6). The expression is also more convenient than Eq. (5) since any choice can be made for the vector potential constituents within Eq. (6), i.e. there is no *implicit* gauge constraint. In addition, the term $\int \mathbf{B^V} \cdot \mathbf{B_{nmv}} dV$ vanishes for simple geometries.

It turns out that the two formulae Eq. (5) and Eq. (6) are fully compliant because Eq. (5) can also be derived as a special case of the rigorous approach Eq. (6) (Clegg et al. 2000b). The formulae should then be identical in both simply and multiply connected situations but for Eq. (5) the vector potential eigenfunctions must always be constrained by $\mathbf{A_{nmv}} \times \mathbf{n} = 0$ on S. This means that in a torus with net flux only fluxless eigenfunctions are included in the expansion, but these do form a complete set, so that it is the potential field $\mathbf{B^V}$ in Eq. (3) that must account for the flux and boundary conditions. It is the purpose of this paper to apply each coefficient formula Eq. (5) and Eq. (6) to an expansion Eq. (3), to show explicitly that they are identical (when Eq. (5) is properly constrained), and to validate the eigenfunction solution through an independent, non-eigenfunction approach to the problem Eq. (9). The particular situation considered is a force-free field between two spherical shells (see also Cantarella et al. 2000). The solution technique presented can be used for practical extrapolation problems, and an example is presented using SOHO-MDI data to provide boundary data for the solar coronal field.

In Sect. 2 of the paper the benchmark solution is derived from the Grad-Shafranov approach to yield a linear force-free field in a spherical shell with both general and simplified boundary conditions. The simplification is included because the solution P can be expanded into a direct scalar series (Sect. 3.1) and thence Eq. (2) used to obtain a series for $\mathbf{B}$. However this is to *assume* that the differential operator in Eq. (2) can be commuted across an infinite sum of terms, and so for the avoidance of doubt about convergence here, the simplified case is included having only a *finite* number of boundary terms. In retrospect, the instances of commutability encountered in this paper are allowed but the simplified form is helpful because of the removal of special functions. For the remainder of Sect. 3, the series eigenfunction solutions are found using the expansion coefficients derived from both Eq. (5) and Eq. (6). The new formula Eq. (6) is shown to be valid, at least for this example, converging to the independently obtained benchmark solutions. The validity of the formulation Eq. (6) has been demonstrated elsewhere for a cylinder (Clegg et al. 2000b). Thus the kernel of this work is a detailed calculation to clarify the inter-relationships of the scalar and direct vector approaches and the consistent solution is then applied to some particular solar magnetic fields.

2. Bench Mark Solutions

We aim here to solve for the linear force-free field in a spherical shell using the scalar potential approach. This both provides a useful solution in its own right, and also provides a benchmark to test the vector eigenfunction solutions.

2.1. General field in a spherical shell

In spherical co-ordinates, Eq. (2) is explicitly

$$\mathbf{B} = -\frac{\Lambda^2 P}{r^2}\hat{\mathbf{r}} + \left(\frac{1}{r}\frac{\partial^2 P}{\partial r \partial\theta} + \frac{\alpha}{r\sin\theta}\frac{\partial P}{\partial\phi}\right)\hat{\theta} + \left(\frac{1}{r\sin\theta}\frac{\partial^2 P}{\partial r\partial\phi} - \frac{\alpha}{r}\frac{\partial P}{\partial\theta}\right)\hat{\phi} \tag{7}$$

with operator

$$\Lambda^2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{8}$$

The equilibrium problem Eq. (1), with Eq. (7), reduces to solving a Grad-Shafranov equation

$$(L + \alpha^2)P = 0 \text{ where } L = \frac{\partial^2}{\partial r^2} + \frac{\Lambda^2}{r^2} \tag{9}$$

and subject to appropriate boundary contraints on P. The base functions obtained by a separation of variables in spherical coordinates, are given in Appendix A, but they produce a *general* solution written

$$P(r,\mu,\phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} r^{1/2}(A_{nm}J_{n+1/2}(\alpha r) + B_{nm}J_{-n-1/2}(\alpha r))X_{nm}(\mu,\phi) + r^{1/2}(C_{nm}J_{n+1/2}(\alpha r) + D_{nm}J_{-n-1/2}(\alpha r))Y_{nm}(\mu,\phi) \tag{10}$$

where $\mu = \cos\theta$, and $X_{nm}(\mu,\phi)$ and $Y_{nm}(\mu,\phi)$ are surface spherical harmonics (see for example Sneddon (1979), their Eqs. (23.1) and (23.3))

The particular problem of a field between inner and outer spherical shell boundaries at $r = a$ and $r = b$ with given radial field on the boundaries (which is directly proportional to P through Eq. (A.12)) yields conditions $P(a,\mu,\phi)$ and $P(b,\mu,\phi)$ which depend only on (μ,ϕ) and so can be expanded as

$$P\left(\begin{array}{c} a \\ b \end{array},\mu,\phi\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\left[\begin{array}{c} b_{nm} \\ d_{nm}\end{array}\right] P_m^n(\mu)\cos(m\phi) + \sum_{n=1}^{\infty}\sum_{m=1}^{n}\left[\begin{array}{c} a_{nm} \\ c_{nm}\end{array}\right] P_m^n(\mu)\sin(m\phi) \tag{11}$$

and using standard orthogonality relations (see for example Sneddon (1979), their Eqs. (23.2),(23.4),(23.5)) the coefficients are found to be

$$\begin{bmatrix} a_{nm} \\ b_{nm} \end{bmatrix} = N_{nm} \int_0^{2\pi} \int_{-1}^{1} a^2 B_r(a,\mu,\phi) \begin{bmatrix} P_n^m(\mu)\sin m\phi \\ P_n^m(\mu)\cos m\phi \end{bmatrix} d\mu d\phi \quad (12)$$

where $N_{nm} = \frac{C}{2\pi}\frac{2n+1}{n(n+1)}\frac{(n-m)!}{(n+m)!}$ with $C = 1/2$ for $m = 0$, otherwise $C = 1$, and c_{nm}, d_{nm} is found from Eq. (12) by replacing a with b.

It is then simply a matter of comparing Eq. (10) at $r = a, b$ with Eq. (11) to find an expression for the coefficients $A_{nm}, B_{nm}, C_{nm}, D_{nm}$ in terms of $a_{nm}, b_{nm}, c_{nm}, d_{nm}$ and the resulting combinations of the Bessel functions at the boundaries. Following some manipulations, the final solution for P (see also Clegg et al. 1999b) can be written

$$P(r,\mu,\phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} ((a_{nm}, b_{nm})^{\phi} g_b(r) + (c_{nm}, d_{nm})^{\phi} g_a(r)) P_n^m(\mu) \quad (13)$$

where

$$g_a(r) = \frac{r^{1/2}\chi_n(\alpha, r, a)}{b^{1/2}\chi_n(\alpha, b, a)} \quad (14)$$

and $g_b(r)$ is obtained by interchanging b and a in Eq. (14). Here we have used χ_n defined as a radial function of three arguments and we also introduce a similar function ξ_n for use later as

$$\chi_n(\gamma, x, y) = J_{n+1/2}(\gamma x) J_{-n-1/2}(\gamma y) - J_{n+1/2}(\gamma y) J_{-n-1/2}(\gamma x) \quad (15)$$

$$\xi_n(\gamma, x, y) = J_{-n-1/2}(\gamma x) J_{n-1/2}(\gamma y) + J_{-n+1/2}(\gamma y) J_{n+1/2}(\gamma x) \quad (16)$$

and abbreviate various combinations of sinusoidal functions like

$$(a_{nm}, b_{nm})^{\phi} = a_{nm}\sin(m\phi) + b_{nm}\cos(m\phi) \quad (17)$$

Now the field can be obtained directly by transforming Eq. (13) through Eq. (2), although this includes an assumption on commutability, to give

$$B^r(r,\mu,\phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} ((a_{nm}, b_{nm})^{\phi} g_b(r) + (c_{nm}, d_{nm})^{\phi} g_a(r)) r^{-2} f_o(\mu) \quad (18a)$$

$$B^{\theta}(r,\mu,\phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} ((a_{nm}, b_{nm})^{\phi} g_{1b}(r) + (c_{nm}, d_{nm})^{\phi} g_{1a}(r)) f_1(\mu) + ((-b_{nm}, a_{nm})^{\phi} g_{2b}(r) + (-d_{nm}, c_{nm})^{\phi} g_{2a}(r)) f_2(\mu) \quad (18b)$$

$$B^{\phi}(r,\mu,\phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} ((b_{nm}, -a_{nm})^{\phi} g_{1b}(r) + (d_{nm}, -c_{nm})^{\phi} g_{1a}(r)) f_2(\mu) + ((a_{nm}, b_{nm})^{\phi} g_{2b}(r) + (c_{nm}, d_{nm})^{\phi} g_{2a}(r)) f_1(\mu) \quad (18c)$$

where

$$g_{2a}(r) = \frac{\alpha}{r} g_a(r) \quad (19a)$$

$$g_{1a}(r) = \frac{n r^{-3/2}\chi_n(\alpha, r, a) - |\alpha| r^{-1/2}\xi_n(\alpha, a, r)}{b^{1/2}\chi_n(\alpha, b, a)} \quad (19b)$$

and $g_{2b}(r)$ is obtained from Eq. (19a) by replacing $g_a(r)$ with $g_b(r)$, while $g_{1b}(r)$ is obtained from Eq. (19b) by interchanging a with b

$$f_o(\mu) = P_n^m(\mu) n(n+1) \quad (20a)$$

$$f_1(\mu) = \frac{-\mu m}{(1-\mu^2)^{1/2}} P_n^m(\mu) - P_n^{m+1}(\mu) \quad (20b)$$

$$f_2(\mu) = \frac{m}{(1-\mu^2)^{1/2}} P_n^m(\mu) \quad (20c)$$

Field lines obtained from the solution Eqs. (18a,18b,18c) are shown in Fig. 1 for a particular solar magnetic field. In this example, boundary conditions were taken from SOHO-MDI photospheric magnetograms centred on 27 August 1996 when an extended low-latitude coronal hole was present. A uniform radial field strength has been imposed across the boundary at $2.5R_{\odot}$, selected from trial values to ensure that the solution closely resembles SOHO-LASCO observations (as discussed in the introduction). Different magnetic helicities are shown corresponding to different values of α as detailed in the caption. Note that the coronal hole lies alongside the western (right-hand) edge of the N-S aligned arcade which can be seen near the centre of the disc.

2.2. Simplified boundary conditions

The solution Eq. (13) does, with a *broadband* boundary condition spectrum, involve an infinite series in (n, m) and so it is not immediately obvious that, in using Eq. (2), there can be a commutability of the curl across this sum to produce the field Eqs. (18a,18b,18c). However, arbitrary and very simple boundary conditions could be used instead so that the sum can become finite, or even based on a single mode. Such simplifications can properly be described as "exact" and this special case benchmark solution provides an incontravertable test for the eigenfunction expansion scalar and vector field solutions. A suitable set of simplifications is as follows:

(i) take the outer shell boundary, $r = b$, (for now) to be perfectly conducting so that all related coefficients c_{nm}, d_{nm} vanish.
(ii) take only axisymmetric fields $\partial/\partial\phi = 0$, $m = 0$ and so coefficients a_{nm} also vanish,
(iii) admit only an $n = 1$ mode inner boundary condition, $r = a$, specified by $B_r(a) = P_1(\mu) = \mu$.

Now $P(a,\mu) = \mu a^2/2$ (see the r-component of Eq. (A.12) in the Appendix), whereas Eq. (13) reduces to $P(a,\mu) = b_{10}\mu$ so that $b_{10} = a^2/2$, and so the *simplified* version of Eq. (13) and

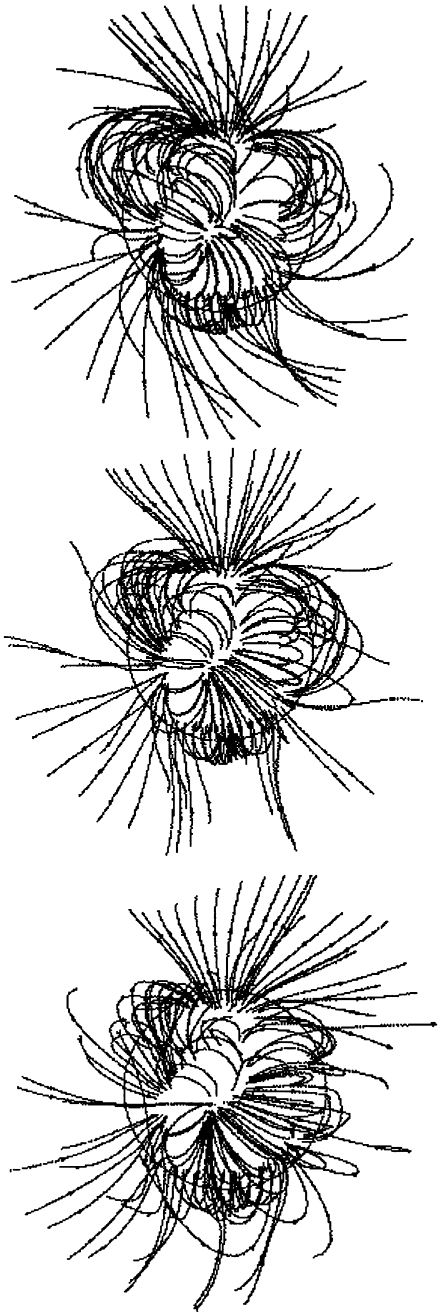

Fig. 1. A sample field solution from Eqs. (18a,18b,18c) modelling the solar corona with boundary conditions from the SOHO-MDI magnetograms for 27 August 1996, a period when a large low-latitude coronal hole developed, extending southward from the north polar hole, to an active region in the southern hemisphere (see Clegg et al. 1999a, 1999b). The solution is shown for $\alpha = -0.2\lambda_{11}, 0, 0.2\lambda_{11}$ (where α is related to the helicity content of the field and λ_{11} is the first eigenvalue discussed in Sect. 3).

Eq. (14), written in terms of sinusoids, with $R = \alpha(r - b)$ and $A = \alpha(a - b)$, is

$$P(r, \theta) = \frac{\mu a^3}{2r} . \frac{(1 + \alpha^2 rb) \sin R - R \cos R}{(1 + \alpha^2 ab) \sin A - A \cos A} \qquad (21)$$

Notice that there is a pole when the denominator vanishes, that is, for values of α where

$$tan A = \frac{A}{1 + \alpha^2 ab} \qquad (22)$$

and for this $n = 1$ model with $a = 1$ and $b = 2.5$, then $\alpha_1 = 2.255589574$ and $\alpha_2 = 4.279689828$.

The $m = 0$, $n = 1$ vector potential, found from $\mathbf{A} = \frac{1}{r}\frac{\partial P}{\partial \theta}\hat{\phi} + \alpha P \hat{\mathbf{r}}$, is explicitly

$$\mathbf{A} = \frac{a^3}{2r}\Big(\frac{\sin\theta}{r}\frac{(1 + \alpha^2 rb) \sin R - R \cos R}{(1 + \alpha^2 ab) \sin A - A \cos A}\hat{\phi} + \alpha \cos\theta \frac{(1 + \alpha^2 rb) \sin R - R \cos R}{(1 + \alpha^2 ab) \sin A - A \cos A}\hat{\mathbf{r}}\Big) \qquad (23)$$

and similarly the simplified model field is obtained from $\mathbf{B} = \frac{2P}{r^2}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial^2 P}{\partial r \partial \theta}\hat{\theta} - \frac{\alpha}{r}\frac{\partial P}{\partial \theta}\hat{\phi}$ which can then be shown directly to satisfy the requirements of Eq. (1) without the need to assume a commutability. A marginally less simplified case can be used to model an idealised corona and is obtained by first inferring the solution with only an inhomogeneous *outer* boundary condition, by exchanging a and b in Eq. (23), and combining this *new* equation with some factor k times the original (inhomogeneous inner boundary) equation Eq. (23). Some examples of such a magnetic field are shown in Fig. 2.

The interesting question of how the field structure changes as α crosses the first eigenvalue, and the topology of fields at higher eigenvalues, is discussed at length in Cantarella et al. (2000) (see their Sect. VII and Figs. 4 & 5) and also in Dixon et al. (1990). Such fields have “islands” of magnetic field detached from the boundary surfaces (such as the photosphere). At an eigenvalue, the field is fully detached but of infinite magnitude. In practice a minimum energy state must always have α below the first eigenvalue because of the infinite magnetic energy barrier there, and it is anyway true that only relatively small (and non-uniform) values of helicity have been measured in the solar corona. It is then better to model the solar corona by assuming a “partially relaxed” magnetic field whereby if enough helicity could be injected it would be a weighted average of $\alpha(\mathbf{B})$ that approaches the first eigenvalue (Kitson & Browning 1990). A final equilibrium (if one exists) then depends on both the initial and boundary conditions and so the model of Fig. 2 is too simplified in practice although it does elucidate the magnetic form. The model does however provide some insight into the paradox, as presented by say Eq. (4), since the curl of this field solution can be compared directly with the partial sums of the right-hand side of Eq. (4) (the coefficients are obtained in Sect. 3.3). It turns out that Eq. (4) is indeed an erroneous expansion but nevertheless the coefficients obtained through Eq. (5), an equation also closely related to the paradox, do provide a correct expansion of the field in Eq. (3) (as discussed in Clegg et al. 2000b and Laurence et al., in preparation).

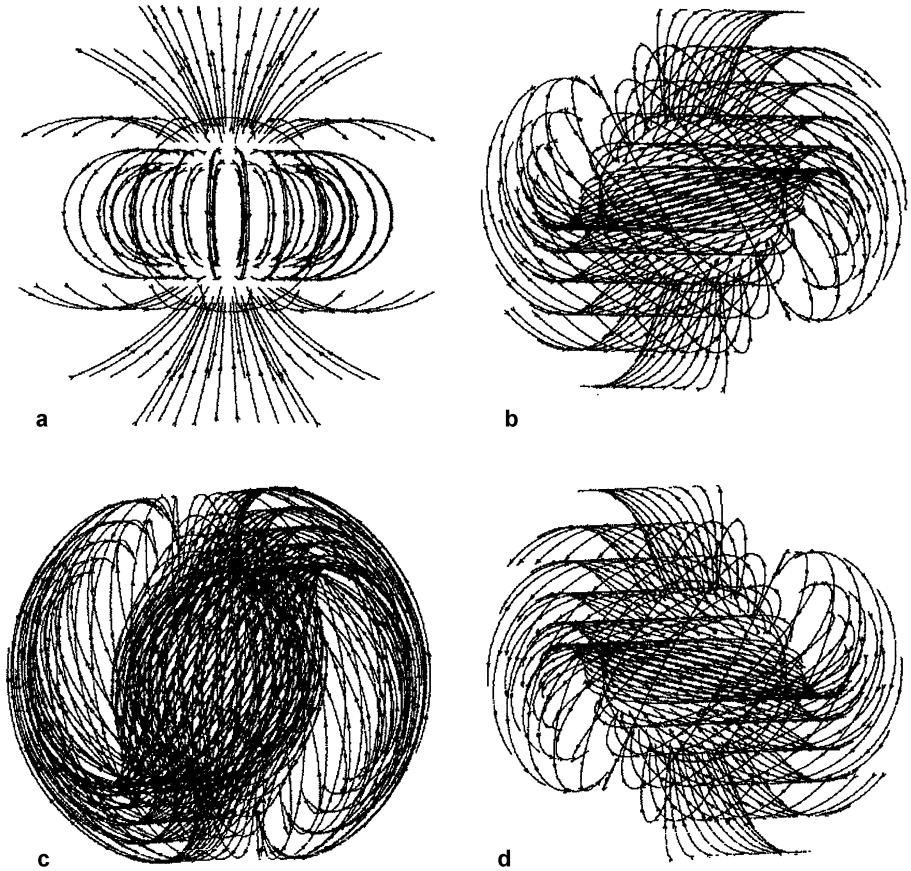

Fig. 2a–d. "Generalised" axisymmetric benchmark field formed by the superposition of two solutions: the curl of Eq. (23) corresponding to an open inner boundary, and the curl of Eq. (23) with a and b exchanged to describe an open outer boundary solution, for **a** $\alpha = 0$, **b** $\alpha = 0.67\lambda_{11}$, **c** $\alpha = \lambda_{11}$, **d** $\alpha = -0.67\lambda_{11}$ relative to the first eigenvalue λ_{11}

3. Eigenfunction expansion for the benchmark fields

3.1. Series solution using a scalar parameter

The calculation now moves on to find some eigenfunction series expressions to compare with the results from the previous section. First note that there is an equivalent (radial) series form for the scalar parameter P written in Eq. (13), and so a series for $\mathbf{B}$ can also be found in this way. To see this we introduce the decomposition

$$P = P^V + \sum_{n,m,v} e_{nmv} P^H_{nmv} \tag{24}$$

The limits on this sum are from $v = 0$ to ∞ in contrast to Eq. (3) which uses $v = -\infty, \infty$. The inhomogeneous boundary condition is carried only through P^V, satisfying

$$L(P^V) = 0, \text{ with, } P^V(S) \neq 0 \tag{25}$$

and the scalar eigenfunctions P^H_{nmv} satisfy

$$(L + \lambda^2_{nv})P^H_{nmv} = 0, \text{ with, } P^H_{nmv}(S) = 0 \tag{26}$$

Here the effective eigenvalues are λ_{nv} (see Eq. (30)), rather than α_{nmv}, being independent of m as explained in Appendix C. The eigenfunctions necessarily form a complete set but one has to be sure that all eigenfunctions have been found (a construction like Chandrasekhar-Kendall (1957) may not yield a full set). It is because the azimuthal components (θ, ϕ) are determined by the boundary conditions that the problem reduces to finding a series for only the radial part of Eq. (13). The detail of the calculation is given in Appendix B, dependent on the orthogonality between combinations of Bessel functions, but the solution then involves replacing $g_a(r)$ and $g_b(r)$ from Eq. (13) by $g^{ser}_a(r)$ and $g^{ser}_b(r)$, respectively, i.e.,

$$P^{ser}(r, \mu, \phi) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} ((a_{nm}, b_{nm})^{\phi} g^{ser}_b(r) + (c_{nm}, d_{nm})^{\phi} g^{ser}_a(r)) P^m_n(\mu) \tag{27}$$

and where the radial functions are structured as

$$g^{ser}_a(r) = g^{V,n}_a(r) + \sum_{v=1}^{\infty} e^{nv}_a g^{*nv}_a(r) \tag{28}$$

Thus the vacuum part of the solution, P^V, corresponds to the terms multiplying $g^{V,n}_a(r)$ and $g^{V,n}_b(r)$ in Eq. (27), using Eq. (28), while the homogeneous part corresponds to the terms in the series over v. Explicitly, the potential parts, radial eigenfunctions and scalar expansion coefficients are respectively

$$g^{V,n}_a(r) = \frac{a^{-2n-1} r^{n+1} - r^{-n}}{a^{-2n-1} b^{n+1} - b^{-n}} \tag{29a}$$

$$g^{*nv}_a(r) = r^{1/2} \chi_n(\lambda_{nv}, r, b) \tag{29b}$$

$$e^{nv}_a = \frac{1}{b^{1/2}} \frac{\alpha^2 \pi (-1)^n}{(\alpha^2 - \lambda^2_{nv})} \frac{J^2_{n+1/2}(\lambda_{nv} a)}{[J^2_{n+1/2}(\lambda_{nv} a) - J^2_{n+1/2}(\lambda_{nv} b)]} \tag{29c}$$

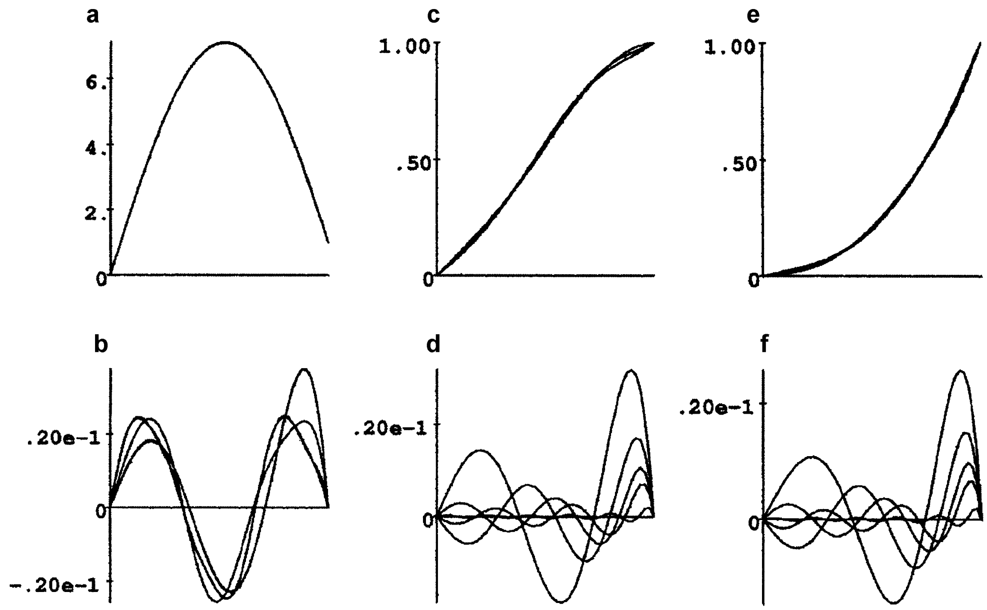

Fig. 3a–f. The convergence of the special function expansion Eq. (31). For **a**, **c** and **e** the exact value of the Bessel function combination, LHS of Eq. (31), is plotted across $a = 1 \leq r \leq b = 2.5$, at $\alpha = 2$, for the modes $n = 0, 3, 5$ respectively; two further lines are added, associated with the RHS of Eq. (31), obtained by summing only the first two terms in the series to $\nu = 2$, or the first three terms to $\nu = 3$, but these are nearly coincident because of the rapid convergence of the eigenfunction expansion. The convergence is shown more clearly in **b**, **d** and **f** by plotting the difference between the exact function (LHS) and the series (RHS) for different partial sums, the sums to $\nu = 2, 3, 4, 5, 10$ for **d** and **f**, where convergence is rapid, and sums to $\nu = 2, 3, 5, 10, 30$ in **b** which again converges but is limited by numerical accuracy. The different forms of **a**, **c** and **e** arise from the proximity of $\alpha = 2$ to the first eigenvalue of each particular mode, i.e. $\lambda_{0,1} = 2.0943951$, $\lambda_{3,1} = 2.9121502$ or $\lambda_{5,1} = 3.7666140$, so that **a** has an amplified, near eigenvalue, character. Although the radial function is zero at $r = a$, the solar magnetic field solution includes a complementary function (interchange a and b in Eq. (31)) which is only zero at $r = b$. Finally, the $n = 0$ mode vanishes if the entire solar surface is considered.

and upon interchanging a and b, the last three equations provide $g_b^{V,n}(r)$, $g_b^{*nv}(r)$ and e_b^{nv} needed to determine $g_b^{ser}(r)$. The radial eigenvalues λ_{nv} correspond to the vth root of the nth order transcendental equation

$$\chi_n(\lambda_{nv}, b, a) = 0 \tag{30}$$

which can be numerically evaluated by a Newton-Raphson method beginning with an approximation from an asymptotic expansion (Abromovitz & Stegun 1970, Eq. 9.5.27). Thus an explicit scalar series Eq. (27) has been found to compare with Eq. (13).

The equivalence of Eq. (28) and Eq. (14) requires that the special functions can be expanded

$$\frac{r^{1/2}\chi_n(\alpha, r, a)}{b^{1/2}\chi_n(\alpha, b, a)} = \frac{a^{-2n-1}r^{n+1} - r^{-n}}{a^{-2n-1}b^{n+1} - b^{-n}} + \sum_{v=1}^{\infty} \frac{r^{1/2}}{b^{1/2}} \frac{\alpha^2\pi(-1)^n}{(\alpha^2 - \lambda_{nv}^2)} \frac{J_{n+1/2}^2(\lambda_{nv}a).\chi_n(\lambda_{nv}, r, b)}{[J_{n+1/2}^2(\lambda_{nv}a) - J_{n+1/2}^2(\lambda_{nv}b)]} \tag{31}$$

with a complementary expansion obtained upon interchanging a and b. The convergence of Eq. (31) is demonstrated numerically in Fig. 3.

3.2. Eigenfunction Field expansion using the new method of Laurence et al.

A vector eigenfunction expansion is now found using the new coefficient formula Eq. (6). We write the sum Eq. (3) explicitly as

$$\mathbf{B} = \sum_{n'=0}^{\infty} \sum_{m'=0}^{n'} \left(\mathbf{B^V_{n'm'}} + \sum_{v=-\infty}^{\infty} c_{n'm'v} \mathbf{B_{n'm'v}} \right) \tag{32}$$

and the formula of Laurence et al. Eq. (6),

$$c_{n'm'v} = \frac{\alpha_{n'm'v}}{(\alpha_{n'm'v} - \alpha)} \frac{\int (\alpha \mathbf{A^V_{nm}} - \mathbf{B^V_{nm}}) \cdot \mathbf{B_{n'm'v}} dV}{\int B^2_{n'm'v} dV} \tag{33}$$

Notice that there is an auxiliary variable structure (see dash marks), that is, our calculation will show *explicitly* that cross-terms in the formula do vanish by orthogonality. In addition, it will be demonstrated that there is no contribution from $\int \mathbf{B^V_{nm}} \cdot \mathbf{B_{n'm'v}} dV$, a result that extends to all *simply-connected* geometries (Clegg et al. 2000b).

By the principle of superposition, the problem Eq. (32), Eq. (33) can be decomposed into the sum of two solutions formed respectively from the boundary condition involving the sine terms and those from the cosine terms. To obtain the field eigenfunctions $\mathbf{B_{n'm'v}}$, we start from the scalar eigenfunctions specified by

$$P^H_{n'm'v} = \begin{bmatrix} E\sin m'\phi \\ F\cos m'\phi \end{bmatrix} P^{m'}_{n'}(\mu) g^{*n'v}_b(r) \tag{34}$$

so that each one of two possible eigenfunctions can be considered, represented by a zero coefficient E or F in Eq. (34) to allow for either a vanishing sine or cosine eigenfunction term in each half of the problem (we will show that crossed terms, for example $\sin m\phi \cos m'\phi$, vanish in the coefficient integral and so the partition does not effect the result). The complementary radial eigenfunction term $g^{*n'v}_a(r)$ does not need to be considered because the two are linearly related through Eq. (B.14) and Eq. (B.17) as

$$g^{*nv}_a(r) = \xi_n(\lambda_{nv}, a, b)\frac{b\pi\lambda_{nv}J^2_{n+1/2}(\lambda_{nv}b)}{2(-1)^n J^2_{n+1/2}(\lambda_{nv}a)} g^{*nv}_b(r) \tag{35}$$

The field eigenfunctions are obtained from a specialisation of Eq. (2),(see Eq. (A.12)), that

$$\mathbf{B_{n'm'v}} = \frac{n(n+1)P^H_{n'm'v}}{r^2}\hat{\mathbf{r}} + \Big(-\frac{(1-\mu^2)^{1/2}}{r}\frac{\partial^2 P^H_{n'm'v}}{\partial r\partial\mu} + \frac{\lambda_{n'v}}{r(1-\mu^2)^{1/2}}\frac{\partial P^H_{n'm'v}}{\partial\phi}\Big)\hat{\theta} + \Big(\frac{1}{r(1-\mu^2)^{1/2}}\frac{\partial^2 P^H_{n'm'v}}{\partial r\partial\phi} + \frac{\lambda_{n'v}(1-\mu^2)^{1/2}}{r}\frac{\partial P^H_{n'm'v}}{\partial\mu}\Big)\hat{\phi} \tag{36}$$

which are explicitly

$$B^r_{n'm'v} = \begin{bmatrix} E\sin m'\phi \\ F\cos m'\phi \end{bmatrix} f^{n'm'}_o(\mu) g^{*n'v}_b(r)/r^2 \tag{37a}$$

$$B^\theta_{n'm'v} = \begin{bmatrix} E\sin m'\phi \\ F\cos m'\phi \end{bmatrix} f^{n'm'}_1(\mu) g^{*n'v}_{1b}(r) + \begin{bmatrix} E\cos m'\phi \\ -F\sin m'\phi \end{bmatrix} f^{n'm'}_2(\mu) g^{*n'v}_{2b}(r) \tag{37b}$$

$$B^\phi_{n'm'v} = \begin{bmatrix} -E\cos m'\phi \\ F\sin m'\phi \end{bmatrix} f^{n'm'}_2(\mu) g^{*n'v}_{1b}(r) + \begin{bmatrix} E\sin m'\phi \\ F\cos m'\phi \end{bmatrix} f^{n'm'}_1(\mu) g^{*n'v}_{2b}(r) \tag{37c}$$

where $g^{*nv}_b(r)$ is found from Eq. (29b), and where

$$g^{*n'v}_{2b}(r) = \frac{\lambda_{n'v}}{r} g^{*n'v}_b(r) \tag{38a}$$

$$g^{*n'v}_{1b}(r) = n'r^{-3/2}\chi_{n'}(\lambda_{n'v}, r, a) - |\lambda_{n'v}| r^{-1/2}\xi_{n'}(\lambda_{n'v}, a, r) \tag{38b}$$

The scalar function P^V_{nm} is the potential part within Eq. (27), Eq. (28) and Eq. (29a), which generates a vector potential $\mathbf{A^V_{nm}}$ and a potential field $\mathbf{B^V_{nm}}$ through

$$\mathbf{A^V_{nm}} = \nabla\times(P^V_{nm}\hat{\mathbf{r}}) = \frac{1}{r\sin\theta}\frac{\partial P^V_{nm}}{\partial\phi}\hat{\theta} - \frac{1}{r}\frac{\partial P^V_{nm}}{\partial\theta}\hat{\phi} \tag{39a}$$

$$\mathbf{B^V_{nm}} = \frac{n(n+1)P^V_{nm}}{r^2}\hat{\mathbf{r}} - \frac{(1-\mu^2)^{1/2}}{r}\frac{\partial^2 P^V_{nm}}{\partial r\partial\mu}\hat{\theta} + \frac{1}{r(1-\mu^2)^{1/2}}\frac{\partial^2 P^V_{nm}}{\partial r\partial\phi}\hat{\phi} \tag{39b}$$

We leave it to the reader to obtain the explicit expression for $A^{V,\theta}_{nm}(r,\mu,\phi)$ and $A^{V,\phi}_{nm}(r,\mu,\phi)$ but recall that any gauge ∇f added to $\mathbf{A^V_{nm}}$ with no effect on the Laurence et al. formula Eq. (33)). The explicit potential field is

$$B^{V,r}(r,\mu,\phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} r^{-2} f^{nm}_0(\mu) ((a_{nm},b_{nm})^\phi g^{V,n}_b(r) + (c_{nm},d_{nm})^\phi g^{V,n}_a(r)) \tag{40a}$$

$$B^{V,\theta}(r,\mu,\phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} f^{nm}_1(\mu) ((a_{nm},b_{nm})^\phi g^{V,n}_{1b}(r) + (c_{nm},d_{nm})^\phi g^{V,n}_{1a}(r)) \tag{40b}$$

$$B^{V,\phi}(r,\mu,\phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} f^{nm}_2(\mu) ((b_{nm},-a_{nm})^\phi g^{V,n}_{1b}(r) + (d_{nm},-c_{nm})^\phi g^{V,n}_{1a}(r)) \tag{40c}$$

where $g^{V,n}_a(r)$ and $g^{V,n}_b(r)$ are given by Eq. (29a) and

$$g^{V,n}_{1a}(r) = \frac{-(n+1)a^{-2n-1}r^{n-1} - nr^{-n-2}}{a^{-2n-1}b^{n+1} - b^{-n}} \tag{41}$$

where $g^{V,n}_{1b}(r)$ follows by interchanging a and b in Eq. (41).

Now the problem is to take Eq. (37a,etc.), Eq. (39a) and Eq. (40a,etc.) and using the standard orthogonality conditions given in Sneddon (1979) (their Eqs. (23.2),(23.4),(23.5)), find the effective coefficients Eq. (33). Details of the integration are given in Appendix D (so as not to further obscure the results), but the final result using the earlier notation, and with $\mathbf{B^V}$ given by Eq. (40a,etc.) is

$$B^r = B^{V,r} + \sum_{n=0}^{\infty}\sum_{m=0}^{n} (a_{nm},b_{nm})^\phi f^{nm}_0(\mu)\sum_{v=-\infty}^{\infty}[\frac{\lambda_{nv}+\alpha}{2\alpha}e^{nv}_b]\frac{g^{*nv}_b(r)}{r^2} + (c_{nm},d_{nm})^\phi f^{nm}_0(\mu)\sum_{v=-\infty}^{\infty}[\frac{\lambda_{nv}+\alpha}{2\alpha}e^{nv}_a]\frac{g^{*nv}_a(r)}{r^2} \tag{42a}$$

$$B^\theta = B^{V,\theta} + \sum_{n=0}^{\infty}\sum_{m=0}^{n}$$

$$(a_{nm}, b_{nm})^\phi f_1^{nm}(\mu) \sum_{v=-\infty}^{\infty} [\frac{\lambda_{nv}+\alpha}{2\alpha} e_b^{nv}] g_{1b}^{*nv}(r)$$
$$+(c_{nm}, d_{nm})^\phi f_1^{nm}(\mu) \sum_{v=-\infty}^{\infty} [\frac{\lambda_{nv}+\alpha}{2\alpha} e_a^{nv}] g_{1a}^{*nv}(r)$$
$$+(-b_{nm}, a_{nm})^\phi f_2^{nm}(\mu) \sum_{v=-\infty}^{\infty} [\frac{\lambda_{nv}+\alpha}{2\alpha} e_b^{nv}] g_{2b}^{*nv}(r)$$
$$+(-d_{nm}, c_{nm})^\phi f_2^{nm}(\mu) \sum_{v=-\infty}^{\infty} [\frac{\lambda_{nv}+\alpha}{2\alpha} e_a^{nv}] g_{2a}^{*nv}(r) \quad (42b)$$

$$B^\phi = B^{V,\phi} + \sum_{n=0}^{\infty}\sum_{m=0}^{n}$$
$$(b_{nm}, -a_{nm})^\phi f_2^{nm}(\mu) \sum_{v=-\infty}^{\infty} [\frac{\lambda_{nv}+\alpha}{2\alpha} e_b^{nv}] g_{1b}^{*nv}(r)$$
$$+(d_{nm}, -c_{nm})^\phi f_2^{nm}(\mu) \sum_{v=-\infty}^{\infty} [\frac{\lambda_{nv}+\alpha}{2\alpha} e_a^{nv}] g_{1a}^{*nv}(r)$$
$$+(a_{nm}, b_{nm})^\phi f_1^{nm}(\mu) \sum_{v=-\infty}^{\infty} [\frac{\lambda_{nv}+\alpha}{2\alpha} e_b^{nv}] g_{2b}^{*nv}(r)$$
$$+(c_{nm}, d_{nm})^\phi f_1^{nm}(\mu) \sum_{v=-\infty}^{\infty} [\frac{\lambda_{nv}+\alpha}{2\alpha} e_a^{nv}] g_{2a}^{*nv}(r) \quad (42c)$$

The form of the expansion Eq. (32) demands identical scalar [coefficients], as seen in Eq. (42a,etc.), i.e. they are isotropic when the limits of the sum are $-\infty \le v \le \infty$.

If instead of the direct vector expansion approach Eq. (32) the problem is approached by transforming the scalar series Eq. (27) into a vector series through Eq. (2), then it becomes evident that, unlike in Eq. (42a,etc), the expansion coefficients differ between the component directions, i.e. they are tensors (Clegg et al. 2000b). This arises naturally because a term $\mathbf{B} = \nabla \times \alpha P^V \hat{\mathbf{r}}$ is generated that has to be recombined into the homogeneous part of the solution (which explains the appearance of tensor coefficients). In fact this tensor coefficient series is simply an alternative representation of Eq. (42a,etc.) noting that the former series is only summed over an infinite *half-space* $0 \le v \le \infty$. To see this, the series Eq. (42a,etc.) can be consolidated into a half-space sum by combining pairs of eigenvalues $+|\lambda_{nv}|$ and $-|\lambda_{nv}|$, noting that e_a^{nv}, e_b^{nv}, $g_a^{*nv}(r)$, $g_b^{*nv}(r)$, $g_{1a}^{*nv}(r)$ and $g_{1b}^{*nv}(r)$ are all invariant under the mapping $\lambda_{nv} \to -\lambda_{nv}$, but $g_{2a}^{*nv}(r)$, and the quotient $(\lambda_{nv}+\alpha)/2\alpha$ are *not* invariant. The result is that

$$\sum_{v=-\infty}^{\infty} [\frac{\lambda_{nv}+\alpha}{2\alpha} e_{a,b}^{nv}] \frac{g_{a,b}^{*nv}(r)}{r^2} = \sum_{v=1}^{\infty} [e_{a,b}^{nv}] \frac{g_{a,b}^{*nv}(r)}{r^2} \quad (43a)$$

$$\sum_{v=-\infty}^{\infty} [\frac{\lambda_{nv}+\alpha}{2\alpha} e_{a,b}^{nv}] g_{1a,1b}^{*nv}(r) = \sum_{v=1}^{\infty} [e_{a,b}^{nv}] g_{1a,1b}^{*nv}(r) \quad (43b)$$

$$\sum_{v=-\infty}^{\infty} [\frac{\lambda_{nv}+\alpha}{2\alpha} e_{a,b}^{nv}] g_{2a,2b}^{*nv}(r) = \sum_{v=1}^{\infty} [\frac{\lambda_{nv}}{\alpha} e_{a,b}^{nv}] g_{2a,2b}^{*nv}(r) \quad (43c)$$

that is the half-space series does have a *tensor* character to its coefficients since some of the terms differ by a factor λ_{nv}/α, except at the (first) eigenvalue $\alpha = \lambda_{nv}$ where this factor is 1. There is also a third way by which we have obtained this result, by exploiting the prior knowledge of the exact solution $\mathbf{B}$ Eq. (18a,etc.), and finding c_{nmv} by a manipulation of Eq. (3), i.e.,

$$c_{nmv} = \int \mathbf{B_{nmv}} \cdot (\mathbf{B} - \mathbf{B^V_{nm}}) dV / \int B_{nmv}^2 dV \quad (44)$$

The field lines for the series solution Eq. (42a,etc.) under the transformation Eq. (43a,etc.) are shown in Fig. 4 to compare with Fig. 1 under the same particular boundary conditions.

3.3. *Simplified boundary conditions: Jensen & Chu formulation*

The consistency of the coefficient Jensen & Chu formula Eq. (5) to the previous results is now shown by looking at the simplified case (as for Sect. 2b). As a prelude to obtaining the Jensen and Chu coefficients a simplification of Eq. (27) can be written as

$$P^{ser} = P^V + \sum_{v=1}^{\infty} e_b^v P_v^H$$
$$= \cos\theta \left(\frac{r^2}{2} \frac{b^{-3} - r^{-3}}{b^{-3} - a^{-3}} + \sum_{v=1}^{\infty} e_b^v \frac{a^2}{2} g_b^{*v}(r) \right) \quad (45)$$

where the superscript $n = 1$ is now implicit. This is related to the corresponding (axisymmetric) vector potential and field by

$$\mathbf{A^V} = \nabla \times P^V \hat{\mathbf{r}} = -\frac{1}{r}\frac{\partial P^V}{\partial\theta}\hat{\phi} = \sin\theta \frac{r}{2} \frac{b^{-3} - r^{-3}}{b^{-3} - a^{-3}} \hat{\phi} \quad (46a)$$

$$\mathbf{B^V} = \nabla \times \mathbf{A^V} = \cos\theta \frac{b^{-3} - r^{-3}}{b^{-3} - a^{-3}} \hat{\mathbf{r}} - \frac{\sin\theta}{2} \frac{2b^{-3} + r^{-3}}{b^{-3} - a^{-3}} \hat{\theta} \quad (46b)$$

The scalar eigenfunctions radial function $g_b^{*v}(r)$ within P_v^H is obtained from Eq. (29b) and Eq. (38b) (upon converting a to b) and converted to sinusoids as

$$g_b^{*v}(r) = -\frac{2}{\pi \lambda_v^3 a^{3/2}}$$
$$(r^{-1} + \lambda_v^2 a) \sin\lambda_v(r-a) - \lambda_v(1 - ar^{-1}) \cos\lambda_v(r-a) \quad (47)$$

Radial eigenvalues λ_v are found from Eq. (30), with $n = 1$ and the zeroes correspond exactly to the poles of α obtained in Eq. (22). The vector potential eigenfunctions and field eigenfunctions are

$$\mathbf{A_v} = \nabla \times P_v^H \hat{\mathbf{r}} + \lambda_v P_v^H \hat{\mathbf{r}} =$$
$$\sin\theta \frac{a^2}{2r} g_b^{*v}(r) \hat{\phi} + \lambda_v \cos\theta \frac{a^2}{2} g_b^{*v}(r) \hat{\mathbf{r}} \quad (48a)$$

$$\mathbf{B_v} = \nabla \times \mathbf{A_v} =$$
$$\cos\theta \frac{a^2}{r^2} g_b^{*v}(r) \hat{\mathbf{r}} + \sin\theta \frac{a^2}{2} g_{1b}^{*v}(r) \hat{\theta} + \lambda_v \sin\theta \frac{a^2}{2r} g_b^{*v}(r) \hat{\phi} \quad (48b)$$

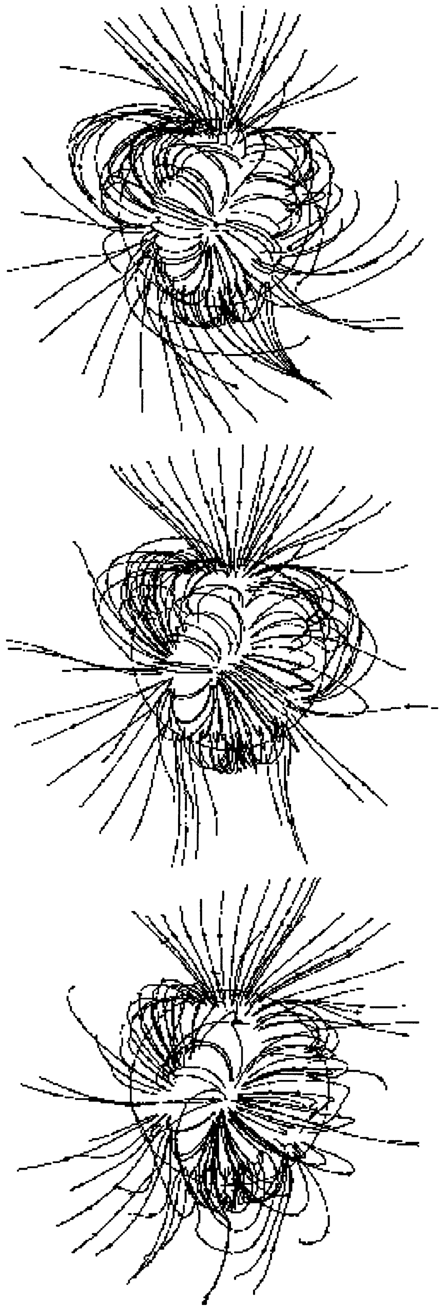

Fig. 4. Field solution based on the eigenfunction expansion Eq. (42a,etc), Eq. (43a,etc) which can be seen to correspond closely to the benchmark field of Fig. 1, with the same values of $\alpha = -0.2\lambda_{11}$, $\alpha = 0$, $\alpha = 0.2\lambda_{11}$. The images here only include the first 7 modes ν in the radial series.

In this case Eq. (48a) already conforms to the requirement that $\mathbf{A_v} \times \mathbf{n} = 0$ on S, but often considerable effort is needed to find an appropriate gauge that fulfills this constraint.

Rewriting Eq. (5), the Jensen and Chu coefficients are

$$c_v = \frac{\alpha}{\lambda_v - \alpha} \int \mathbf{A_v} \cdot \nabla \times \mathbf{A^V} dV / \int \mathbf{A_v} \cdot \mathbf{B_v} dV \tag{49}$$

so that explicitly with an orthogonality condition $\int \mathbf{A_v} \cdot \mathbf{B_{v'}} dV = K\delta_{vv'}$ where K is the normalisation constant and the coefficients are found to be

$$c_v = \frac{1}{2a^{1/2}} \frac{\alpha\pi}{(\lambda_v - \alpha)} \frac{J^2_{3/2}(\lambda_v b)}{[J^2_{3/2}(\lambda_v b) - J^2_{3/2}(\lambda_v a)]} = e_b^v \frac{(\lambda_v + \alpha)}{2\alpha} \tag{50}$$

where the coefficients e_b^v are given by Eq. (29c) (upon converting a to b with $n = 1$), written with sinusoids as

$$e_b^v = -\frac{1}{a^{1/2}} \frac{\alpha^2\pi}{(\alpha^2 - \lambda_v^2)} \tag{51}$$
$$\frac{a^3(\sin\lambda_v b - \lambda_v b\cos\lambda_v b)^2}{a^3(\sin\lambda_v b - \lambda_v b\cos\lambda_v b)^2 - b^3(\sin\lambda_v a - \lambda_v a\cos\lambda_v a)^2}$$

Thus the vector potential series is

$$\mathbf{A} = \sin\theta\frac{r}{2}\left(\frac{b^{-3} - r^{-3}}{b^{-3} - a^{-3}}\right)\hat{\phi} + \sum_{v=-\infty}^{\infty}[e_b^v\frac{(\lambda_v + \alpha)}{2\alpha}] \left(\sin\theta\frac{a^2}{2r}g_b^{*v}(r)\hat{\phi} + \lambda_v\cos\theta\frac{a^2}{2}g_b^{*v}(r)\hat{\mathbf{r}}\right) \tag{52}$$

or written as a *half-space* paired eigenvalue sum, with characteristic *tensor* coefficients,

$$\mathbf{A} = \sin\theta\frac{r}{2}\left(\frac{b^{-3} - r^{-3}}{b^{-3} - a^{-3}}\right)\hat{\phi} + \sum_{v=1}^{\infty}[e_b^v]\sin\theta\frac{a^2}{2r}g_b^{*v}(r)\hat{\phi} + [e_b^v\frac{\lambda_v}{\alpha}]\lambda_v\cos\theta\frac{a^2}{2}g_b^{*v}(r)\hat{\mathbf{r}} \tag{53}$$

The convergence of Eq. (53) to Eq. (23) is shown in Fig. 5

The Jensen & Chu derived series field becomes

$$B^r_{J\&C}(r,\theta) = \cos\theta\left(\frac{b^{-3} - r^{-3}}{b^{-3} - a^{-3}} + \sum_{v=-\infty}^{\infty}[e_b^v\frac{\lambda_v + \alpha}{2\alpha}]\frac{a^2}{r^2}g_b^{*v}(r)\right) \tag{54a}$$

$$B^\theta_{J\&C}(r,\theta) = -\sin\theta\left(\frac{1}{2}\frac{2b^{-3} + r^{-3}}{b^{-3} - a^{-3}} - \sum_{v=-\infty}^{\infty}[e_b^v\frac{\lambda_v + \alpha}{2\alpha}]\frac{a^2}{2}g_{1b}^{*v}(r)\right) \tag{54b}$$

$$B^\phi_{J\&C}(r,\theta) = \sin\theta\sum_{v=-\infty}^{\infty}[e_b^v\frac{\lambda_v + \alpha}{2\alpha}]\frac{a^2}{2}g_{2b}^{*v}(r) \tag{54c}$$

where the square bracketed coefficients of Eq. (54a,etc.) are immediately recognisable from the general field Laurence et al. solution Eq. (42a,etc.).

To summarise, the new Laurence et al. formula for the coefficients Eq. (6), for use in the field eigenfunction expansion Eq. (3), has been used to obtain a solution to the linear force-free field bounded by two spherical shells, Eq. (42a,etc.). The result has been proved correct since it converges to the independently derived, non-eigenfunction, field solution Eq. (18a,etc.), and is

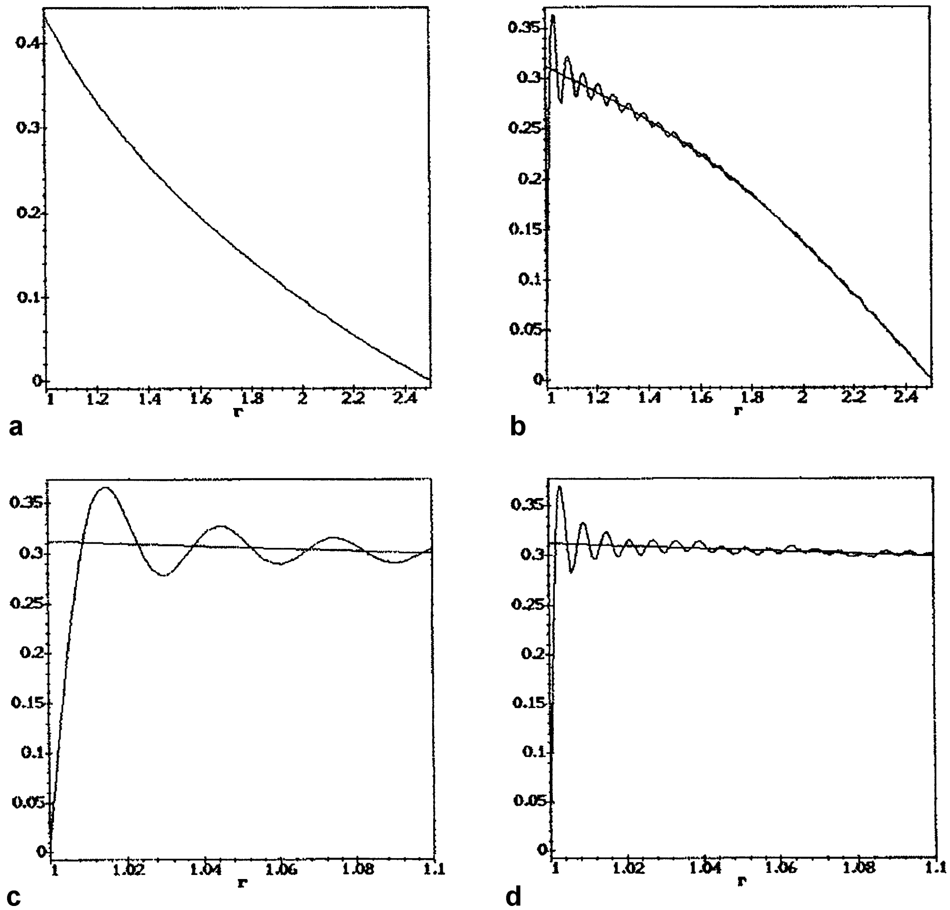

Fig. 5a–d. Evidence that the (axisymmetric $\partial/\partial\phi = 0$) vector potential series Eq. (53) converges, up to a vanishingly thin boundary layer, to the benchmark vector potential Eq. (23) using $\alpha = 0.44\lambda_{11}$ (but see the discussion in Clegg et al. 2000b). In **a** and **b** the components $\mathbf{A}^{\phi}$ and $\mathbf{A}^{\mathbf{r}}$ are plotted against radius (open inner boundary at $r = a = 1$ to the closed outer boundary at $r = b = 2.5$) for the exact and for the series solution including 50 radial modes. In **a** the two lines effectively coincide. In **c** and **d** a zoom view of $\mathbf{A}^{\mathbf{r}}$ is shown $1 \leq r \leq 1.1$, with better convergence using 100 and 500 modes, plotted against the exact solution.

identical to a representation obtained by a field transformation Eq. (2) of the scalar series Eq. (27), although the limits of the resulting series have to be reconciled. It was then encumbent on us to relate these results to those using the usual Jensen and Chu coefficient formula Eq. (5). It is found that the Jensen and Chu approach provides identical results, at least for the case of axisymmetry tested in Eq. (54a,etc.), and provided the constituents of its formula are *properly constrained*. It has been our motivation to demonstrate a formula Eq. (6) that is free from such constraints.

4. Discussion

The paper has concentrated on a detailed mathematical analysis of a linear force-free field in a spherical shell with inhomogeneous conditions over the bounding surfaces. There have been two motivations: first, to obtain a solution that could be used to model the solar or a stellar corona, and second, to obtain and compare different equivalent formulations of the solution. Of particular interest has been the validation of a new formula to obtain the coefficients in an eigenfunction expansion.

The linear force-free field is in fact somewhat inappropriate as a model of the global solar coronal plasma (non-linear force-free models are more realistic but have so far only been used over local domains in the corona, for example Clegg et al. 2000a) but nevertheless it is of interest to compare the overall effects of magnetic helicity with the often used potential field approximation (Clegg et al. 1999a). In this respect, boundary conditions were obtained from SOHO-MDI data centred on 27 August 1996, a time associated with the meridian passage of a large N-S extended coronal hole. This entailed a suitable transformation of the line-of-sight data from an temporal set of magnetograms to obtain a radial field component boundary condition extrapolated over the entire solar surface. Some field results were shown in Fig. 1 and Fig. 4 and further discussion on the boundary conditions and the realism obtained by adding the magnetic helicity can be found in Clegg et al.(1999b). The general solution might also be applied to other areas of astrophysics with spherical domains.

The second objective, to demonstrate and validate a new method, centred on the ability to obtain the solution in several ways. One method involved solving the problem in terms of a scalar then transposing to the magnetic field. A solution in terms of special functions was found, and through deriving an identity this could be converted into a series solution. The series could also be obtained directly by an eigenfunction expansion and we have shown that two different formulae attributed to Jensen and Chu, and Laurence et al. respectively yield the same result, although we feel that the latter formula for the expansion coefficients is the more convenient. The reader is referred to companion papers for the derivation and discussion of the two formulae (see Clegg et al. 2000b; Laurence et al. in preparation).

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Appendix A: separation of variables

Substituting Eq. (7) into Eq. (1), the radial component yields an identity, whilst the θ and ϕ components together imply a scalar Grad-Shafranov equation

$$(L+\alpha^2)P = h(r) = 0 \tag{A.1}$$

where L is given by Eq(9). The arbitrary function $h(r)$ is set to zero because a general solution to Eq. (A.1) comprises a complementary function P^H, the solution to the homogeneous equation, and a "particular solution" $g(r)$ which satisfies

$$\frac{\partial^2 g(r)}{\partial r^2} + \alpha^2 g(r) = h(r) \tag{A.2}$$

which could be solved for $h(r)$, but since the general solution is then $P = P^H + g(r)$ and taking the curl in Eq. (2), it is clear that the components $\mathbf{A}\times\hat{\mathbf{r}}$ arising from $g(r)$ vanish, and although the component $\mathbf{A}\cdot\mathbf{r}$ is initially non-zero for $g(r)$, its effect vanishes upon taking the further curl to generate $\mathbf{B}$. Hence we need only solve the homogeneous problem Eq. (A.1) where $h(r) = 0$. Writing $\mu = \cos\theta$, and separating variables $P = R(r)M(\mu)\Phi(\phi)$

$$(1-\mu^2)\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} + \frac{(1-\mu^2)}{M}\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial M}{\partial\mu}\right) + \frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} + (1-\mu^2)\alpha^2 r^2 = 0 \tag{A.3}$$

but since the ϕ dependence is isolated, this term must be constant, denoted $-m^2$, so that $\partial^2\Phi/\partial\phi^2 + m^2\Phi = 0$ and

$$\Phi = \begin{bmatrix} \sin m\phi \\ \cos m\phi \end{bmatrix} \tag{A.4}$$

In order that Φ be single-valued, m must be an integer since the full azimuthal range is included. The remaining equation has terms separate in r and μ and so equal to a constant C leaving

$$\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial M}{\partial\mu}\right) + \left(C - \frac{m^2}{(1-\mu^2)}\right)M = 0 \tag{A.5}$$

which is simply Legendre's associated differential equation, provided $C = n(n+1)$; n is taken to be a real positive number following Demoulin & Priest (1992), but the resulting series solution would not converge at the poles $\mu = \pm 1$ unless n is an integer. The solution is then simply

$$M = P_n^m(\mu) \tag{A.6}$$

since $Q_n^m(\mu)$ is excluded (being infinite at one pole, $\theta = 0$). Thus the separation can be written $P = R(r)X(\mu,\phi)$ where $X(\mu,\phi)$ are surface spherical harmonics $X_{nm}(\mu,\phi)$ or $Y_{nm}(\mu,\phi)$ since these functions are solutions to

$$\Lambda^2 X(\mu,\phi) + n(n+1)X(\mu,\phi) = 0 \tag{A.7}$$

and then the radial equation is simply

$$\frac{\partial^2 R}{\partial r^2} + \left(\alpha^2 - \frac{n(n+1)}{r^2}\right)R = 0 \tag{A.8}$$

which with a change of variable $R = r^{1/2}\rho$ can be satisfied for $\alpha \neq 0$ by combinations of Bessel functions written

$$R = \begin{bmatrix} r^{1/2}J_{n+1/2}(\alpha r) \\ r^{1/2}J_{-n-1/2}(\alpha r) \end{bmatrix} \tag{A.9}$$

related simply to the second kind of Bessel functions, because n is an integer, by

$$J_{-n-1/2}(\alpha r) = (-1)^{n+1}Y_{n+1/2}(\alpha r) \tag{A.10}$$

Finally using Eq. (A.7), the radial component of Eq. (7) simplifies to

$$B^r = -\frac{R(r)\Lambda^2 X(\mu,\phi)}{r^2} = \frac{n(n+1)P}{r^2} \tag{A.11}$$

and so the field transformation Eq. (7) is explicitly

$$\begin{aligned} \mathbf{B} &= \frac{n(n+1)P}{r^2}\hat{\mathbf{r}} \\ &+ \left(-\frac{(1-\mu^2)^{1/2}}{r}\frac{\partial^2 P}{\partial r\partial\mu} + \frac{\alpha}{r(1-\mu^2)^{1/2}}\frac{\partial P}{\partial\phi}\right)\hat{\theta} \\ &+ \left(\frac{1}{r(1-\mu^2)^{1/2}}\frac{\partial^2 P}{\partial r\partial\phi} + \frac{\alpha(1-\mu^2)^{1/2}}{r}\frac{\partial P}{\partial\mu}\right)\hat{\phi} \end{aligned} \tag{A.12}$$

Appendix B: scalar field eigenfunction analysis

It is required to find a "series" solution to Eq. (9) with operator specified by Eq. (9) and Eq. (8) and subject to the decomposition Eq. (24). The complementary problem Eq. (25) is explicitly

$$L(P^V) = \frac{\partial^2 P^V}{\partial r^2} + \frac{\Lambda^2 P^V}{r^2} = 0 \tag{B.1}$$

where the boundary conditions $P(S) = P^V(S)$, that is $r = a$, $r = b$ on S, can now be directly related to $B^r(S)$ in Eq. (18a) by Eq. (A.11). An *element* of the potential solution is (by inspection) using Eq. (17)

$$\begin{aligned} P_{nm}^V(r,\mu,\phi) &= \\ &(a_{nm},b_{nm})^\phi P_n^m(\mu)\left(\frac{-b^{-2n-1}r^{n+1}+r^{-n}}{-b^{-2n-1}a^{n+1}+a^{-n}}\right) \\ &+(c_{nm},d_{nm})^\phi P_n^m(\mu)\left(\frac{-a^{-2n-1}r^{n+1}+r^{-n}}{-a^{-2n-1}b^{n+1}+b^{-n}}\right) \end{aligned} \tag{B.2}$$

but in general there is a synthesis of such terms indexed by n, m to satisfy the boundary conditions, that is, each coefficient

$a_{nm}, b_{nm}, c_{nm}, d_{nm}$ is found from Eq. (12). Using Eq. (15) we define the radial function

$$g_b^{*nv}(r) = r^{1/2}\chi_n(\lambda_{nv}, r, a) \tag{B.3}$$

the homogeneous equation is Eq. (26) and from Appendix A the eigenfunctions can be written as

$$P^H_{nmv} = g_b^{*nv}(r)\begin{bmatrix}\sin m\phi \\ \cos m\phi\end{bmatrix} P_n^m(\mu) \tag{B.4}$$

or introducing some arbitrary phase coefficients in ϕ these can be written

$$P^H_{nmv} = (E_{nmv}, F_{nmv})^\phi P_n^m(\mu) g_b^{*nv}(r) \tag{B.5}$$

Again there is in general a superposition (n, m) of such functions besides the explicit radial expansion associated with the index v. The eigenvalues are given by Eq. (30). Now Eq. (24) (in generalised notation) can be substituted into Eq. (9) writing

$$(L + \alpha^2)(P^V_{nm} + \sum_{v=1}^{\infty} e_{nmv}P^H_{nmv}) = 0 \tag{B.6}$$

for every term (n, m). Now it turns out that the operator L can be commuted across the sum (see Clegg et al. 2000b) and this is explicitly

$$\left(\frac{\partial^2}{\partial r^2} + \frac{\Lambda^2}{r^2} + \alpha^2\right)P^V_{nm} + \sum_{v=1}^{\infty} e_{nmv}\left(\frac{\partial^2}{\partial r^2} + \frac{\Lambda^2}{r^2} + \alpha^2\right)P^H_{nmv} = 0 \tag{B.7}$$

but substituting Eq. (25) and Eq. (26) into Eq. (B.7) gives an equation as

$$P^V_{nm} = \sum_{v=1}^{\infty} e_{nmv}(-1 + \lambda^2_{nv}/\alpha^2)P^H_{nmv} \tag{B.8}$$

and using Eq. (B.2) and Eq. (B.5)

$$\begin{aligned}&(a_{nm}, b_{nm})^\phi P_n^m(\mu)\left(\frac{-b^{-2n-1}r^{n+1} + r^{-n}}{-b^{-2n-1}a^{n+1} + a^{-n}}\right) \\ &+ (c_{nm}, d_{nm})^\phi P_n^m(\mu)\left(\frac{-a^{-2n-1}r^{n+1} + r^{-n}}{-a^{-2n-1}b^{n+1} + b^{-n}}\right) \\ &= \sum_{v=1}^{\infty} e_{nmv}(-1 + \frac{\lambda^2_{nm}}{\alpha^2})(E_{nmv}, F_{nmv})^\phi P_n^m(\mu) g_b^{*nv}(r)\end{aligned} \tag{B.9}$$

and the unknown coefficients $e_{nmv}E_{nmv}$ and $e_{nmv}F_{nmv}$ are found through some general orthogonality relations in given by Ozisik (1980), Table 3–2.9, that

$$\int_a^b g_b^{*nv}(r) g_b^{*nv'}(r)dr = N(\lambda_{nv})\delta_{vv'} \tag{B.10a}$$

where

$$N(\lambda_{nv}) = -\frac{2[J^2_{n+1/2}(\lambda_{nv}b) - J^2_{n+1/2}(\lambda_{nv}a)]}{\pi^2\lambda^2_{nv}J^2_{n+1/2}(\lambda_{nv}b)} \tag{B.10b}$$

Separating the $\sin m\phi$ and $\cos m\phi$ parts of Eq. (B.9), integrating in r using Eq. (B.10a) gives

$$\begin{bmatrix} e_{nmv}E_{nmv} \\ e_{nmv}F_{nmv}\end{bmatrix} = \frac{-\alpha^2}{(-\lambda^2_{nv} + \alpha^2)N(\lambda_{nv})}\left(\begin{bmatrix}a_{nm} \\ b_{nm}\end{bmatrix} I^1 + \begin{bmatrix}c_{nm} \\ d_{nm}\end{bmatrix} I^2\right) \tag{B.11}$$

where

$$I^1 = \int_a^b \left(\frac{-b^{-2n-1}r^{n+1} + r^{-n}}{-b^{-2n-1}a^{n+1} + a^{-n}}\right) g_b^{*nv}(r)dr \tag{B.12a}$$

$$I^2 = \int_a^b \left(\frac{-a^{-2n-1}r^{n+1} + r^{-n}}{-a^{-2n-1}b^{n+1} + b^{-n}}\right) g_b^{*nv}(r)dr \tag{B.12b}$$

Eq. (B.12a) can be calculated as $I^1 = \xi_n(\lambda_{nv}, a, a).a^{1/2}/\lambda_{nv}$ which is simply

$$I^1 = \frac{2(-1)^n}{\pi\lambda^2_{nv}a^{1/2}} \tag{B.13}$$

whereas Eq. (B.12b) is

$$I^2 = -\xi_n(\lambda_{nv}, a, b)b^{1/2}/\lambda_{nv} \tag{B.14}$$

Ultimately we seek

$$\sum_{v=1}^{\infty} e_{nmv}P^H_{nmv} = \sum_{v=1}^{\infty}(e_{nmv}E_{nmv}, e_{nmv}F_{nmv})^\phi P_n^m(\mu) g_b^{*nv}(r) \tag{B.15}$$

and using Eq. (B.11),

$$\sum_{v=1}^{\infty} e_{nmv}P^H_{nmv} = \sum_{v=1}^{\infty}\frac{-\alpha^2}{(-\lambda^2_{nv} + \alpha^2)N(\lambda_{nv})}P_n^m(\mu) \left((a_{nm}, b_{nm})^\phi I^1 g_b^{*nv}(r) + (c_{nm}, d_{nm})^\phi I^2 g_b^{*nv}(r)\right) \tag{B.16}$$

but using Eq. (B.14), the combination $I^2 g_b^{*nv}(r)$ can be written as

$$I^2 g_b^{*nv}(r) = -\frac{2(-1)^n}{\pi\lambda^2_{nv}b^{1/2}}\frac{J^2_{n+1/2}(\lambda_{nv}a)}{J^2_{n+1/2}(\lambda_{nv}b)} g_a^{*nv}(r) \tag{B.17}$$

(notice the change in the subscript of g). Substituting Eq. (B.13) and Eq. (B.17) into Eq. (B.16) and using Eq. (B.10b)

$$\sum_{v=1}^{\infty} e_{nmv}P^H_{nmv} = \sum_{v=1}^{\infty} P_n^m(\mu) \left((a_{nm}, b_{nm})^\phi e_b^{*nv} g_b^{*nv}(r) + (c_{nm}, d_{nm})^\phi e_a^{*nv} g_a^{*nv}(r)\right) \tag{B.18}$$

where $e^{*nv}_{a,b}$ is specified in Eq. (29c). Finally combining Eq. (B.2) and Eq. (B.18), and superposing all (n, m) elements, the series solution is Eq. (27).

Appendix C: the determination of zeroes

Consider an eigenfunction like Eq. (B.4) but for now with an open specification of the radial function $g(r)$, writing

$$P^H_{nmv} = g(r) \begin{bmatrix} \sin m\phi \\ \cos m\phi \end{bmatrix} P^m_n(\mu) \tag{C.1}$$

This must satisfy a generalised eigenfunction equation written as

$$(L + \alpha^2_{nmv}) P^H_{nmv} = 0 \tag{C.2}$$

where from Eq. (9) and Eq. (8),

$$L = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{C.3}$$

Now $\partial^2 P^H_{nmv}/\partial\phi^2 = -m^2 P^H_{nmv}$ and so substituting into Eq. (C.3) and cancelling

$$r^2\frac{\partial^2 P^H_{nmv}}{\partial r^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P^H_{nmv}}{\partial\theta}\right) + \left(-\frac{m^2}{\sin^2\theta} + \alpha^2_{nmv} r^2\right) P^H_{nmv} = 0 \tag{C.4}$$

also Legendre's associated equation provides that

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P^m_n(\mu)}{\partial\theta}\right) - \frac{m^2}{\sin^2\theta}P^m_n(\mu) = -n(n+1)P^m_n(\mu) \tag{C.5}$$

which is also satisfied by P^H_{nmv}. Substituting into Eq. (C.4) and cancelling the $\begin{bmatrix} \sin m\phi \\ \cos m\phi \end{bmatrix} P^m_n(\mu)$ terms

$$\frac{\partial^2 g(r)}{\partial r^2} + \left(\alpha^2_{nmv} - \frac{n(n+1)}{r^2}\right) g(r) = 0 \tag{C.6}$$

and following the procedure from Eq. (A.8) a solution is

$$g(r) = r^{1/2}\chi_n(\alpha_{nmv}, r, a) \tag{C.7}$$

using Eq. (15) where the linear combination has been imposed to directly satisfy the eigenfunction requirement at $g(a) = 0$. However the eigenfunction must also vanish through $g(b) = 0$, and this zero is independent of m, i.e. the vth radial zero is affected only by the order n of the Bessel function equation Eq. (C.7). Hence it is appropriate to interchange $\alpha_{nmv} \to \lambda_{nv}$ so that $g(r) \to g^{*nv}_b(r)$ (see Eq. (29b)), radial zeroes are found from Eq. (30), and most generally the problem Eq. (C.3) is *di facto* $(L + \lambda^2_{nv})P^H_{nmv} = 0$ as used in the text.

Appendix D: integration and orthogonality

This appendix details the calculation of the field eigenfunction expansion from Sect. 3.2 associated with the Laurence et al. formula Eq. (33). From Eq. (39a) and Eq. (40a,etc.), the difference function is

$$(\alpha\mathbf{A^V_{nm}} - \mathbf{B^V_{nm}})^r = \sum_{n=0}^{\infty}\sum_{m=0}^{n} (-a_{nm}, -b_{nm})^\phi f^{nm}_0(\mu)\frac{g^{V,n}_b(r)}{r^2} + (-c_{nm}, -d_{nm})^\phi f^{nm}_0(\mu)\frac{g^{V,n}_a(r)}{r^2} \tag{D.1a}$$

$$(\alpha\mathbf{A^V_{nm}} - \mathbf{B^V_{nm}})^\theta = \sum_{n=0}^{\infty}\sum_{m=0}^{n} (-b_{nm}, a_{nm})^\phi \alpha f^{nm}_2(\mu)\frac{g^{V,n}_b(r)}{r} + (-d_{nm}, c_{nm})^\phi \alpha f^{nm}_2(\mu)\frac{g^{V,n}_a(r)}{r} + (-a_{nm}, -b_{nm})^\phi f^{nm}_1(\mu) g^{V,n}_{1b}(r) + (-c_{nm}, -d_{nm})^\phi f^{nm}_1(\mu) g^{V,n}_{1a}(r) \tag{D.1b}$$

$$(\alpha\mathbf{A^V_{nm}} - \mathbf{B^V_{nm}})^\phi = \sum_{n=0}^{\infty}\sum_{m=0}^{n} (a_{nm}, b_{nm})^\phi \alpha f^{nm}_1(\mu)\frac{g^{V,n}_b(r)}{r} + (c_{nm}, d_{nm})^\phi \alpha f^{nm}_1(\mu)\frac{g^{V,n}_a(r)}{r} + (-b_{nm}, a_{nm})^\phi f^{nm}_2(\mu) g^{V,n}_{1b}(r) + (-d_{nm}, c_{nm})^\phi f^{nm}_2(\mu) g^{V,n}_{1a}(r) \tag{D.1c}$$

and the eigenfunctions are given by Eq. (37a,etc.). In forming the dot product $\int(\alpha\mathbf{A^V_{nm}} - \mathbf{B^V_{nm}})\cdot\mathbf{B_{n'm'v}}dV$, only the $m = m'$ terms survive the ϕ integration, and also combinations of sines and cosines vanish (justifying the simplification of the decomposition into two distinct eigenfunctions in Sect. 3b and a solution by superposition), because

$$\int_0^{2\pi}\sin m\phi\cos m'\phi d\phi = 0 \quad \text{for any} \quad m, m'$$

$$\int_0^{2\pi}\sin m\phi\sin m'\phi d\phi = \int_0^{2\pi}\cos m\phi\cos m'\phi d\phi = \pi\delta_{mm'}$$

The following integrals of μ occur (given also that $m = m'$ now, but not yet assuming that $n = n'$)

$$\begin{aligned} I_o &= \int_{-1}^{1} f^{nm}_o(\mu) f^{n'm}_o(\mu) d\mu \\ &= nn'(n+1)(n'+1)\int_{-1}^{1} P^m_n(\mu)P^m_{n'}(\mu)d\mu \\ &= \left(n^2(n+1)^2\frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}\right)\delta_{nn'} \end{aligned} \tag{D.2a}$$

$$\begin{aligned} I_1 &= \int_{-1}^{1} f^{nm}_1(\mu) f^{n'm}_1(\mu) d\mu \\ &= \int_{-1}^{1}(1-\mu^2)\frac{\partial P^m_n(\mu)}{\partial\mu}\frac{\partial P^m_{n'}(\mu)}{\partial\mu}d\mu \end{aligned} \tag{D.2b}$$

$$I_2 = \int_{-1}^{1} f_1^{nm}(\mu) f_2^{n'm}(\mu) d\mu = m \int_{-1}^{1} \frac{\partial P_n^m(\mu)}{\partial \mu} P_{n'}^m(\mu) d\mu \tag{D.2c}$$

$$I_3 = \int_{-1}^{1} f_2^{nm}(\mu) f_1^{n'm}(\mu) d\mu = m \int_{-1}^{1} \frac{\partial P_{n'}^m(\mu)}{\partial \mu} P_n^m(\mu) d\mu \tag{D.2d}$$

$$I_4 = \int_{-1}^{1} f_2^{nm}(\mu) f_2^{n'm}(\mu) d\mu = m^2 \int_{-1}^{1} (1-\mu^2)^{-1} P_n^m(\mu) P_{n'}^m(\mu) d\mu \tag{D.2e}$$

Now I_1 can be integrated by parts and simplifies through $P_{n'}^m(1) = 0$ and $P_{n'}^m(-1) = 0$. Next using the Associated Legendre equation on the second integral then
$I_1 + I_4 = n(n+1) \int_{-1}^{1} P_n^m(\mu) P_{n'}^m(\mu) d\mu = \frac{1}{n(n+1)} I_o$
It turns out that I_1 and I_4 only occur in this combination in expanding the dot product and so only $n = n'$ is non-zero. The integrals for I_2 and I_3 are zero for $n = n'$ since they can be written as $I_2^{nn} = I_3^{nn} = \int_{-1}^{1} \frac{\partial}{\partial \mu}(P_n^m(\mu))^2/2 d\mu$ which vanishes at the end-points. For $n \neq n'$ we can integrate by parts to show that $I_2^{nn'} + I_3^{nn'} = m[P_n^m(\mu) P_{n'}^m(\mu)]_{-1}^{1} = 0$ and again this is the only combination encountered in the dot product. Thus only $n = n'$ is involved. Following these reductions, the only *radial integrals* of concern are written below

$$I_A = \int g_b^V g_b^* r^{-2} dr \tag{D.3a}$$

$$I_B = \int g_a^V g_b^* r^{-2} dr \tag{D.3b}$$

$$I_C = \alpha \int g_b^V g_{2b}^* r dr = \alpha \lambda_{n'v} \int g_b^V g_b^* dr \tag{D.3c}$$

$$I_D = \alpha \int g_a^V g_{2b}^* r dr = \alpha \lambda_{n'v} \int g_a^V g_b^* dr \tag{D.3d}$$

$$I_E = \int g_{1b}^V g_{1b}^* r^2 dr = \int \partial g_b^V/\partial r \partial g_b^*/\partial r dr \tag{D.3e}$$

$$= -\int \partial^2 g_b^V/\partial r^2 g_b^* dr \tag{D.3f}$$

$$I_F = \int g_{1a}^V g_{1b}^* r^2 dr = \int \partial g_a^V/\partial r \partial g_b^*/\partial r dr = -\int \partial^2 g_a^V/\partial r^2 g_b^* dr \tag{D.3g}$$

using $[g_b^*]_a^b = 0$ where the superscript n, v is implicit. The dot product integral (after removing $I_2 + I_3 = 0$ terms) becomes

$$\int (\alpha \mathbf{A_{nm}^V} - \mathbf{B_{nm}^V}) \cdot \mathbf{B_{n'm'v}} dV = -I_o \pi \left(\begin{bmatrix} Ea_{nm} \\ Fb_{nm} \end{bmatrix} I_A + \begin{bmatrix} Ec_{nm} \\ Fd_{nm} \end{bmatrix} I_B \right) + (I_1 + I_4)\pi \left(\begin{bmatrix} Ea_{nm} \\ Fb_{nm} \end{bmatrix} (I_C - I_E) + \begin{bmatrix} Ec_{nm} \\ Fd_{nm} \end{bmatrix} (I_D - I_F) \right) \tag{D.4}$$

where the square bracketed terms are associated with the two halves of the problem. Using the results on the μ integrals

$$\int (\alpha \mathbf{A_{nm}^V} - \mathbf{B_{nm}^V}) \cdot \mathbf{B_{n'm'v}} dV = \pi n(n+1) \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} \left(\begin{bmatrix} Ea_{nm} \\ Fb_{nm} \end{bmatrix} (-n(n+1) I_A + I_C - I_E) + \begin{bmatrix} Ec_{nm} \\ Fd_{nm} \end{bmatrix} (-n(n+1) I_B + I_D - I_F) \right) \tag{D.5}$$

The remaining task is to combine and evaluate the radial integrals:
$-n(n+1) I_A + I_C - I_E = \alpha \lambda_{n'v} \int g_b^* g_b^V dr = \alpha \lambda_{nv} I^1$
$-n(n+1) I_B + I_D - I_F = \alpha \lambda_{n'v} \int g_b^* g_a^V dr = \alpha \lambda_{nv} I^2$
where we have used $\partial^2 g_{a,b}^V/\partial r^2 = n(n+1) r^{-2} g_{a,b}^V$ and I^1 and I^2 given by Eq. (B.13), Eq. (B.14). Hence

$$\int (\alpha \mathbf{A_{nm}^V} - \mathbf{B_{nm}^V}) \cdot \mathbf{B_{n'm'v}} dV = \alpha \lambda_{nv} \pi n(n+1) \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} \left(\begin{bmatrix} Ea_{nm} \\ Fb_{nm} \end{bmatrix} I^1 + \begin{bmatrix} Ec_{nm} \\ Fd_{nm} \end{bmatrix} I^2 \right) \tag{D.6}$$

Following similar manipulations the denominator Eq. (33) is

$$\int B_{nmv}^2 dV = \begin{bmatrix} E^2 \\ F^2 \end{bmatrix} \pi n(n+1) \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} \frac{4[J_{n+1/2}^2(\lambda_{nv} a) - J_{n+1/2}^2(\lambda_{nv} b)]}{\pi^2 J_{n+1/2}^2(\lambda_{nv} b)} \tag{D.7}$$

Combining Eq. (D.6) and Eq. (D.7), the expansion becomes

$$\mathbf{B} = \mathbf{B^V} + \sum_{v=-\infty}^{\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{\alpha \lambda_{nv}^2}{(\lambda_{nv} - \alpha)} \frac{\pi^2 J_{n+1/2}^2(\lambda_{nv} b)}{4[J_{n+1/2}^2(\lambda_{nv} a) - J_{n+1/2}^2(\lambda_{nv} b)]} \begin{bmatrix} 1/E^2 \\ 1/F^2 \end{bmatrix} \left(\begin{bmatrix} Ea_{nm} \\ Fb_{nm} \end{bmatrix} I^1 + \begin{bmatrix} Ec_{nm} \\ Fd_{nm} \end{bmatrix} I^2 \right) \mathbf{B_{nmv}} \tag{D.8}$$

Now writing out the field eigenfunctions Eq. (37a,etc.) and using the relationship Eq. (B.17) and the similar conversions
$I^2 g_{1b}^{*nv}(r) = -\frac{2(-1)^n}{\pi \lambda_{nv}^2 b^{1/2}} \frac{J_{n+1/2}^2(\lambda_{nv} a)}{J_{n+1/2}^2(\lambda_{nv} b)} g_{1a}^{*nv}(r)$
$I^2 g_{2b}^{*nv}(r) = -\frac{2(-1)^n}{\pi \lambda_{nv}^2 b^{1/2}} \frac{J_{n+1/2}^2(\lambda_{nv} a)}{J_{n+1/2}^2(\lambda_{nv} b)} g_{2a}^{*nv}(r)$
the field expression eventually simplifies to Eq. (42a,etc), Sect. 3b.

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