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Viscosity-alpha and dynamo-alpha for magnetically driven compressible turbulence in Kepler disks

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Abstract. For a given isotropic and homogeneous field of *magnetic* fluctuations both the viscosity-α as well as the dynamo-α have been computed for accretion disks on the basis of a quasi-linear approximation with shear flow and density fluctuations (i.e. magnetic buoyancy) included. The resulting viscosity-α proves to be positive for sufficiently strong shear (i.e. the angular momentum transport is *outwards*) while the sign of the dynamo-α depends on the hemisphere. Again, for sufficiently strong shear it changes its sign, it is now *negative* for the upper disk plane and positive for the lower one.

The current helicity $\langle \boldsymbol{j}' \cdot \boldsymbol{B}' \rangle$ also changes its sign with increasing shear. For a Kepler flow in the upper (lower) disk plane, the sign is positive (negative). In our turbulence model the current helicity of the fluctuations and the α-effect of dynamo theory are almost always out of phase; the signs of all the quantities are in perfect correspondence to the numerical simulations of Brandenburg (1998, 2000). The kinetic helicity has the *same sign* as the α-effect – not, as often assumed, the opposite one.

The resulting ratio between the dynamo-α and the viscosity-α reveals the dynamo-α amplitude as rather small compared with the turbulence intensity. This is in contrast to earlier results on the basis of a quantitative approximation but again is in agreement with recent results of numerical simulations.

Key words: accretion, accretion disks – instabilities – turbulence – Magnetohydrodynamics (MHD)

1. Introduction

There is now evidence that the accretion disk dynamo works with an α-effect with negative sign in the upper disk plane and positive sign in the lower disk plane. This is important because in $\alpha\Omega$-dynamos the sign of the α-effect directs the resulting geometry. The most easily excited mode has quadrupolar geometry for positive α-effect[1] and has dipolar geometry for negative α-effect (Torkelsson & Brandenburg 1994). Rüdiger et al. (1995), working with positive α-effect, found only solutions with the quadrupolar symmetry dominating (Fig. 1).

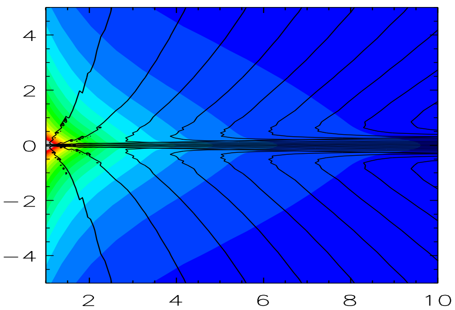

Fig. 1. The magnetic geometry for accretion-disk dynamos with positive α-effect[1] is quadrupolar, i.e. even with respect to the equator. The vertical axis at the left gives the rotation axis. Note the poloidal field lines *not* supporting jets and outflows. Figure taken from Rekowski et al. (2000).

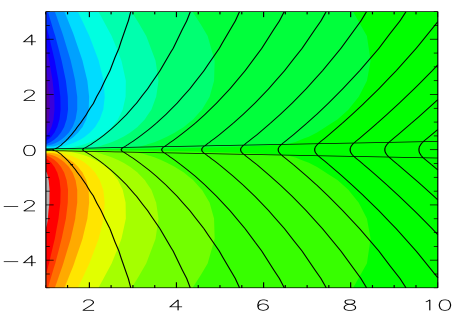

Fig. 2. The same as in Fig. 1 but for negative α-effect. The magnetic geometry is dipolar, i.e. odd with respect to the equator. Note the poloidal field lines supporting jets and outflows. Maxima of the toroidal fields are located in the halo.

In Rekowski et al. (2000) the different geometries of the dynamo-generated magnetic fields are demonstrated. For *negative* dynamo-α, however, a stationary dipolar structure of the magnetic field results (Fig. 2). The additional magnetic torque at the disk surface significantly changes the profile of the effective temperature to a profile which is more flat. The magnetic torque becomes of the same order as the radial viscous torque. The inclination angle of the poloidal field exceeds 30° even for a magnetic Prandtl number of order unity, and also the criterion

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[1] In order to avoid confusion, the α-effect of the dynamo theory – which is always antisymmetric with respect to the equator – is represented in this article by a characteristic value for the *upper* disk plane.

for poloidal collimation after Spruit et al. (1997) is fulfilled. The dynamo-generated magnetic field configuration thus supports the magnetic wind launching concept for accretion disks not only for unrealisticly high turbulent magnetic Prandtl numbers.

On the other hand, an accretion disk can only exist if there is an instability which transports the angular momentum outwards, or, in other words, the 'viscosity-α' is positive. This is not a trivial constraint as we know from several hydrodynamical simulations (Ryu & Goodman 1992; Cabot & Pollack 1992; Kley et al. 1993; Goldman & Wandel 1995; Stone & Balbus 1996, see also Balbus et al. 1996). The situation drastically changes for electrically conducting media, however, if (weak) magnetic fields are allowed to play their own role and, in particular, to feedback onto the momentum transport via the Lorentz force (Balbus & Hawley 1991; Hawley et al. 1996, Brandenburg et al. 1995; Ziegler & Rüdiger 2000). On the other hand, Brandenburg (1998) proposes an interesting argument for magnetic shear flows that for positive viscosity-α the dynamo-α must be negative in the upper disk plane.

There is much discussion about the existence of *negative* α-effect which is also needed in order to reproduce the observed butterfly diagram of solar activity with an $\alpha\Omega$-dynamo and the helioseismologically-derived profile of internal rotation[2]. Within the frame of the anelastic approximation, i.e. if the mass conservation can be described with $\mathrm{div}\,\rho\boldsymbol{u} = 0$ for density-stratified fluids the kinetic helicity is always negative (positive) on the northern (southern) hemisphere. As there is a *minus* between the α-effect and the helicity, the resulting α-effect is positive. Also a strong differential rotation does not change this situation (Pipin et al. 2000). The only possibility for negative α-effect is given if the turbulence intensity behaves in opposition to the density stratification – as it is realized in the solar tachocline layer (Krivodubskij & Schultz 1993).

In a previous paper (Rüdiger et al. 2000) we have considered quite another turbulence model ignoring the density stratification in the continuity equation

$$\frac{\partial \rho'}{\partial t} + \bar{\rho}\,\mathrm{div}\,\boldsymbol{u}' = 0. \tag{1}$$

Note that here we do *not* apply the anelastic approximation. All the resulting effects are thus vanishingly small for a very high speed of sound, c_{ac}. The turbulence may be driven by Lorentz force fluctuations due to a field $\boldsymbol{B}'$ of magnetic field fluctuations ('flux tubes') and density fluctuations, i.e. buoyancy is included. A quasilinear second-order correlation-approximation provides the surprising result that the famous *minus* between kinetic helicity and α-effect disappears but nevertheless the α-effect proves to be positive again (see Table 1 below). As the only possibility to find *negative* α-effects, we must consider differential rotation, i.e. the inclusion of a shear.

For rigid rotation the magnetically driven turbulence model yields inward transport of angular momentum. Only for shear flows, however, we can compute the total angular momentum transport in accretion disks as only in this case does the dominating eddy viscosity appear in the expression for the angular momentum transport.

In the present paper, therefore, for a magnetically driven turbulence field subject to a large-scale shear flow the dynamo-α, the two helicities and the angular momentum transport (which must be outwards!) are simultaneously derived. Drastic differences of the results for rigid rotation and Kepler rotation are found. Indeed, for a sufficiently high shear rate the dynamo-α changes its sign and even takes the desired negative values for the case of Kepler rotation.

2. Mean-field electrodynamics

The equations are close to those in Rüdiger et al. (2000). The momentum equation for non-rigid rotation in the inertial system with buoyancy included is

$$\frac{\partial \boldsymbol{u}'}{\partial t} + \boldsymbol{u}'\cdot\nabla\bar{\boldsymbol{u}} + \bar{\boldsymbol{u}}\cdot\nabla\boldsymbol{u}' = -\frac{1}{\bar{\rho}}\,\mathrm{grad}\left(p' + \frac{\boldsymbol{B}'\cdot\bar{\boldsymbol{B}}}{\mu_0}\right) + + \frac{\rho'}{\bar{\rho}}\boldsymbol{g} + \frac{1}{\mu_0\bar{\rho}}(\bar{\boldsymbol{B}}\cdot\nabla)\boldsymbol{B}' + \nu\Delta\boldsymbol{u}'. \tag{2}$$

Overbars indicate prescribed mean quantities such as the homogeneous magnetic field, large-scale flow and density. $\boldsymbol{g}$ denotes the acceleration due to gravity.

As an energy equation for the turbulence the adiabacity relation

$$p' = c_{\mathrm{ac}}^2 \rho' \tag{3}$$

is used with c_{ac} as the isothermal speed of sound. Eqs. (2) and (3) lead to a turbulence field $\boldsymbol{u}'$, driven by the Lorentz force on the RHS of (2). The original, prescribed magnetic field fluctuations may be denoted by $\boldsymbol{B}^{(0)}$. Their correlation tensor is assumed to form a homogeneous, isotropic and stationary field of magnetic turbulence. The resulting kinetic turbulence is subject to a basic rotation and subject to shear or – in other words – to differential rotation.

After some algebra one can find the correlation tensor of the turbulence and, in particular, its covariance $\langle u'_s u'_\phi\rangle$, s here being the distance from the rotation axis. This quantity is part of the angular momentum transport. The total angular momentum transport is given by

$$T_{s\phi} = \langle u'_s u'_\phi\rangle - \frac{1}{\mu_0\bar{\rho}}\langle B'_s B'_\phi\rangle, \tag{4}$$

taking into account also the Maxwell stress. The latter results from the magnetic fluctuations, $\boldsymbol{B}'$, driven by the turbulence field considered. The corresponding equation is the induction equation in its linearised version, i.e.

$$\frac{\partial \boldsymbol{B}'}{\partial t} - \mathrm{rot}(\bar{\boldsymbol{u}}\times\boldsymbol{B}') - \eta\Delta\boldsymbol{B}' = \mathrm{rot}(\boldsymbol{u}'\times\bar{\boldsymbol{B}}). \tag{5}$$

Here again both the influences of the basic rotation (only on non-axisymmetric field components) as well as differential rotation can be isolated.

[2] If meridional circulation is neglected.

The resulting (rather complex) magnetic fluctuations must be used to compute the Maxwell stress in (4), or, as the next interesting quantity, to compute the current helicity

$$\mathcal{H}_{\rm curr} = \langle \boldsymbol{j}' \cdot \boldsymbol{B}' \rangle = \frac{1}{\mu_0} \langle {\rm rot}\, \boldsymbol{B}' \cdot \boldsymbol{B}' \rangle, \tag{6}$$

which has the same kind of equatorial (anti-)symmetry as the dynamo-α. For *homogeneous* global magnetic fields the dynamo-α is directly related to the turbulent electromotive force (EMF) according to $\boldsymbol{\mathcal{E}} = \langle \boldsymbol{u}' \times \boldsymbol{B}' \rangle = \alpha \circ \bar{\boldsymbol{B}}$, so that $\alpha_{ij} \bar{B}_i \bar{B}_j = \boldsymbol{\mathcal{E}} \cdot \bar{\boldsymbol{B}}$. Rädler & Seehafer (1990) propose to apply this equation to $\alpha\Omega$-dynamos with dominating azimuthal field belts so that

$$\alpha_{\phi\phi} = \boldsymbol{\mathcal{E}} \cdot \bar{\boldsymbol{B}} / \bar{B}_\phi^2, \tag{7}$$

where $\alpha_{\phi\phi}$ is the azimuthal component of the α-tensor. We are, in particular, interested to check their and Keinigs' (1983) antiphase relation,

$$\alpha_{\phi\phi} \mathcal{H}_{\rm curr} < 0, \tag{8}$$

between α-effect and current helicity. There is an increasing number of papers presenting observations of the current helicity of the solar surface always with the result that it is *negative* at the northern hemisphere and *positive* at the southern hemisphere (Seehafer 1990; Pevtsov et al. 1995; Abramenko et al. 1996; Bao & Zhang 1998). If (8) is correct then there is a strong empirical evidence for an α-effect that is *positive* (*negative*) in the northern (southern) hemisphere of the Sun.

Here we start to find the relation between α-effect and current helicity for shear-flow disks. We shall see that there are exceptions, indeed, to the simple relation (8). This is not a surprise. Blackman & Field (1999) argue that Keinigs' result,

$$\frac{\alpha_{\phi\phi} \bar{B}_\phi^2}{\mu_0 \mathcal{H}_{\rm curr}} = -\eta, \tag{9}$$

strongly depends on the assumed stationarity and homogeneity of the magnetic fields and flows which are, however, not realistic for dynamo problems.

3. The current helicity and the α-effect

The complete relation for the current helicity is

$$\mathcal{H}_{\rm curr} = \frac{2}{15} \epsilon_{inj} \Big(g_j \bar{B}^2 \bar{u}_{n,i} + 2 g_j \bar{B}_n \bar{B}_p \bar{u}_{p,i} - g_n \bar{B}_j \bar{B}_p \bar{u}_{i,p} - (\boldsymbol{g} \cdot \bar{\boldsymbol{B}}) \bar{B}_j \bar{u}_{i,n} \Big) \frac{I_1}{\mu_0^2 \bar{\rho} c_{\rm ac}^2} \tag{10}$$

with

$$I_1 = \iint_0^\infty \frac{\eta k^2 (\nu^2 k^4 + \omega^2) - 2\omega^2 \nu k^2}{(\omega^2 + \nu^2 k^4)^2 (\omega^2 + \eta^2 k^4)} k^2 \mathcal{B}(k,\omega)\, dk d\omega \tag{11}$$

and the spectral function $\mathcal{B}$ in the definition $\langle B^{(0)2} \rangle = \iint \mathcal{B} dk d\omega$.

In disk geometry the deformation tensor is simply

$$\bar{u}_{i,j} = -\epsilon_{ijp} \Omega_p + e_i^\phi e_j^s \frac{\partial \Omega}{\partial \log s}, \tag{12}$$

with $\boldsymbol{e}^s$ and $\boldsymbol{e}^\phi$ as the unit vectors in radial and azimuthal directions. Insertion of (12) into (10) gives

$$\mathcal{H}_{\rm curr} = \frac{2}{5} \bar{B}_\phi^2 (\boldsymbol{g} \cdot \boldsymbol{\Omega}) \left(1 + \frac{\partial \log \Omega}{\partial \log s} \right) \frac{I_1}{\mu_0^2 \bar{\rho} c_{\rm ac}^2}. \tag{13}$$

The sign of the I_1 determines the sign of the current helicity which we have to discuss for various turbulence models. The integral (11) does not prove to be definite in sign. It is negative-definite for very large magnetic Prandtl numbers ($\eta = 0$) but it is positive for the more realistic case of moderate magnetic Prandtl number and spectral functions $calB$ decreasing for increasing frequency ω. In the sense of the 'τ-approximation' the spectrum of the given field of magnetic fluctuations has been approximated by $\mathcal{B} \propto \delta(k - \ell_{\rm corr}^{-1})\, \delta(\omega)$ with $\nu \simeq \ell_{\rm corr}^2 / \tau_{\rm corr}$ (Kitchatinov 1991) and the I_1 becomes positive-definite. For the current helicity (6) of the shear flow we then find

$$\mathcal{H}_{\rm curr} = \frac{2}{5} \frac{\tau_{\rm corr}^3}{\ell_{\rm corr}^2} V_{\rm A}^2 \frac{\langle B^{(0)2} \rangle}{\mu_0 \bar{\rho} c_{\rm ac}^2} \left(1 + \frac{\partial \log \Omega}{\partial \log s} \right) g_z \Omega. \tag{14}$$

$V_{\rm A} = \bar{B}_\phi / \sqrt{\mu_0 \bar{\rho}}$ is the Alfvén velocity. Indeed, for $c_{\rm ac} \to \infty$ the current helicity disappears. It is negative on the northern hemisphere for weak differential rotation but changes its sign for sufficiently large shear. For a Kepler disk with its vertical gravity, $g_z = -\Omega^2 z$, the current helicity becomes

$$\mathcal{H}_{\rm curr} = \frac{z \Omega^3}{5} \frac{\tau_{\rm corr}^3}{\ell_{\rm corr}^2} \frac{\bar{B}_\phi^2}{\mu_0} \frac{\langle B^{(0)2} \rangle}{\mu_0 \bar{\rho} c_{\rm ac}^2}, \tag{15}$$

which is *positive* in the upper disk plane and *negative* in the lower disk plane. This is exactly the numerical result of Brandenburg (1999) for the current helicity of magnetic field fluctuations in Kepler disks.

The next step concerns the α-effect defined by the above relation. It results from the general expression

$$\alpha_{fm} = \frac{\epsilon_{fij}}{5} \Bigg(\Big(\frac{2}{3} g_m \bar{u}_{i,j} - g_i \bar{u}_{m,j} + \frac{2}{3} g_l \delta_{mj} \bar{u}_{i,l} \Big) \frac{I_2}{\mu_0 \bar{\rho} c_{\rm ac}^2} - \Big(g_i (\bar{u}_{m,j} + \bar{u}_{j,m}) + \delta_{mj} g_l (\bar{u}_{i,l} + \bar{u}_{l,i}) \Big) \frac{I_3}{\mu_0 \bar{\rho} c_{\rm ac}^2} \Bigg) \tag{16}$$

with

$$I_2 = \iint_0^\infty \frac{\nu^2 k^4 - \omega^2}{(\omega^2 + \nu^2 k^4)^2} \mathcal{B}(k,\omega) dk d\omega, \tag{17}$$

$$I_3 = \frac{1}{3} \iint_0^\infty \frac{\nu^2 k^4 (\nu^2 k^4 - 3\omega^2)}{(\omega^2 + \nu^2 k^4)^3} \mathcal{B}(k,\omega) dk d\omega. \tag{18}$$

Again the total effect vanishes for $c_{\rm ac} \to \infty$. Only the most important component $\alpha_{\phi\phi}$ need be discussed. We obtain

$$\alpha_{\phi\phi} = -\frac{(\boldsymbol{g} \cdot \boldsymbol{\Omega})}{5} \left(\left(1 + \frac{\partial \log \Omega}{\partial \log s} \right) \frac{I_2}{\mu_0 \bar{\rho} c_{\rm ac}^2} + \frac{\partial \log \Omega}{\partial \log s} \frac{I_3}{\mu_0 \bar{\rho} c_{\rm ac}^2} \right). \tag{19}$$

Our magnetic flux tube model yields

$$\alpha_{\phi\phi} = -\frac{4}{15}\frac{\tau_{\rm corr}^2}{c_{\rm ac}^2}(\boldsymbol{g}\cdot\boldsymbol{\Omega})\left(\frac{3}{4}+\frac{\partial\log\Omega}{\partial\log s}\right)\frac{\langle B^{(0)2}\rangle}{\mu_0\bar\rho}, \tag{20}$$

so that for Kepler disks

$$\alpha_{\phi\phi} = -\frac{z\Omega^3}{5}\ \frac{\tau_{\rm corr}^2}{c_{\rm ac}^2}\frac{\langle B^{(0)2}\rangle}{\mu_0\bar\rho} \tag{21}$$

results. For rigid rotation the α-effect proves to be positive in the upper disk plane and negative in the lower disk plane (Rüdiger et al. 2000). The opposite is true for Kepler flows. The dynamo-α becomes negative in the upper disk plane and positive in the lower disk plane. Again the results comply with the results of the numerical simulations by Brandenburg (1999). After (20) the dynamo-α completely vanishes for $\Omega\propto s^{-0.75}$ rather than for $\Omega\propto s^{-1}$ for which the current helicity vanishes after (14). So a small interval exists with exponents between 0.75 and 1 where the α-effect and the current helicity have the same sign. Box simulations should be used to test the relevance of this surprising result.

The ratio of the α-effect and current helicity here follows to

$$\frac{\alpha_{\phi\phi}\bar B_\phi^2}{\mu_0\mathcal{H}_{\rm curr}} = -\frac{\ell_{\rm corr}^2}{\tau_{\rm corr}} \tag{22}$$

– very close to (9). For rigid rotation the factor sinks to 1/2 (see paper I). The small differences to Keinigs' result certainly result from the fact that we are not using the anelastic approximation.

A similar question arises concerning the kinetic helicity

$$\mathcal{H}_{\rm kin} = \langle\boldsymbol{u}'\cdot{\rm rot}\,\boldsymbol{u}'\rangle, \tag{23}$$

which is often believed to be in antiphase to the α-effect (Moffatt 1978). We obtain

$$\mathcal{H}_{\rm kin} = -\frac{4}{15}\epsilon_{inj}g_j\bar B_n\bar B_p(\bar u_{p,i}-\bar u_{i,p})\frac{I_4}{\mu_0^2\bar\rho^2c_{\rm ac}^2}+ \\ +\frac{4}{15}\epsilon_{pij}g_i\bar B_n\bar B_j\left(\bar u_{p,n}+\bar u_{n,p}\right)\frac{I_7}{\mu_0^2\bar\rho^2c_{\rm ac}^2} \tag{24}$$

resulting in

$$\mathcal{H}_{\rm kin} = -\frac{4}{15}\frac{\bar B^2}{\mu_0^2\bar\rho^2c_{\rm ac}^2}(\boldsymbol{g}\cdot\boldsymbol{\Omega})\left(\left(2+\frac{\partial\log\Omega}{\partial\log s}\right)I_4+ \right. \\ \left. +\frac{\partial\log\Omega}{\partial\log s}I_7\right) \tag{25}$$

with the positive quantity

$$I_4 = \iint\limits_0^\infty \frac{\nu k^4\mathcal{B}(k,\omega)}{(\omega^2+\nu^2k^4)^2}dkd\omega, \tag{26}$$

and with

$$I_7 = \iint \frac{\nu k^4(\nu^2k^4-\omega^2)}{(\omega^2+\nu^2k^4)^3}\mathcal{B}(k,\omega)dkd\omega. \tag{27}$$

Hence, for rigid rotation the kinetic helicity is positive in the upper disk plane, and it is negative in the lower disk plane. For the one-mode flux tube model we find for the amplitude the value

$$\mathcal{H}_{\rm kin} = -\frac{8}{15}\frac{\tau_{\rm corr}^3(\boldsymbol{g}\cdot\boldsymbol{\Omega})}{\ell_{\rm corr}^2}\left(1+\frac{\partial\log\Omega}{\partial\log s}\right)V_{\rm A}^2\frac{\langle B^{(0)2}\rangle}{\mu_0\bar\rho c_{\rm ac}^2}. \tag{28}$$

Again the sign of the (pseudo-)scalar changes with increasing shear in the same way as it happens for the current helicity, see Eq. (14). For Kepler rotation the kinetic helicity,

$$\mathcal{H}_{\rm kin} = -\frac{4}{15}z\Omega^3\frac{\tau_{\rm corr}^3}{\ell_{\rm corr}^2}V_{\rm A}^2\frac{\langle B^{(0)2}\rangle}{\mu_0\bar\rho c_{\rm ac}^2}, \tag{29}$$

in the upper hemisphere proves to be negative – as it does in the simulations by Brandenburg (1999).

The amplitude ratio of the dynamo-α and the kinetic helicity is

$$\frac{\alpha_{\phi\phi}}{\mathcal{H}_{\rm kin}} \simeq \tau_{\rm corr}\frac{{\rm Ma}^2}{{\rm Mm}^2}, \tag{30}$$

with the turnover velocity $u_{\rm T} = \ell_{\rm corr}/\tau_{\rm corr}$, the turbulence Mach number ${\rm Ma} = u_{\rm T}/c_{\rm ac}$ and the magnetic Mach number ${\rm Mm} = V_{\rm A}/c_{\rm ac}$. For equipartition of the magnetic energy with the thermal energy (Mm$\simeq 1$) we find the α-effect to be much smaller than the traditional value ('helicity times correlation time') if the turbulence is subsonic. Even the sign is opposite. However, as the kinetic helicity can not be observed at the disk surface we are not able to estimate the amplitude of the dynamo-α, $\alpha_{\phi\phi}$, from the given expressions. To this end we need the comparison with a quantity representing, e.g., the angular momentum transport in accretion disks which can directly be observed via the radiation or the temporal behavior of the real disks. The quantity describing these effects has been introduced by Shakura & Sunyaev (1973) and will be computed in the following section.

4. Angular momentum transport

Our turbulence can only model the situation in accretion disks if it transports the angular momentum outwards, i.e. if the stress $T_{s\phi}$ is positive. Additionally, we know from observations the value of the normalized angular momentum transport,

$$\alpha_{\rm SS} = \frac{T_{s\phi}}{c_{\rm ac}^2}, \tag{31}$$

being of order $10^{-3}...1$ so that – if we find a relation between both the alphas – the dynamo-α can be estimated. For historical reasons the quantity (31) is called the viscosity-α. The notation arises from the Boussinesq relation postulating a direct correspondence between stress and strain. In paper I we have shown that even for rigid rotation a finite (negative) value for the angular momentum (31) exists which clearly can not be due to an 'eddy viscosity'.

For the correlation tensor of the magnetic-forced turbulence the complex expression

$$\langle u_i'u_j'\rangle = -\frac{23}{105}\left(\bar u_{i,j}+\bar u_{j,i}\right)\frac{\bar B^2I_4}{\mu_0^2\bar\rho^2}+$$

$$
\begin{aligned}
&+\Big\{-\left(\bar{B}_i g_j+\bar{B}_j g_i\right)\left(\bar{u}_{m,n}+\bar{u}_{n,m}\right)\bar{B}_n g_m-\\
&-5\left(\bar{B}_i g_j+\bar{B}_j g_i\right)\bar{B}_m g_n\bar{u}_{m,n}+\\
&+3g^2\left(\bar{B}_i\bar{u}_{j,n}+\bar{B}_j\bar{u}_{i,n}\right)\bar{B}_n-\frac{\bar{B}^2}{9}\left(g_i\bar{u}_{j,n}+g_j\bar{u}_{i,n}\right)g_n\\
&-\frac{5}{3}\bar{B}^2\left(\bar{u}_{n,i}g_j+g_i\bar{u}_{n,j}\right)g_n-\\
&-\frac{13}{3}g^2\bar{B}^2\left(\bar{u}_{i,j}+\bar{u}_{j,i}\right)\Big\}\frac{I_5}{\mu_0^2\bar{\rho}^2c_{\rm ac}^4}-\\
&-\Big\{4\left(g_i\bar{B}_j+g_j\bar{B}_i\right)\left(\bar{u}_{m,n}+\bar{u}_{n,m}\right)\bar{B}_n g_m-\\
&-\frac{76}{9}\bar{B}^2\left(g_i\left(\bar{u}_{j,n}+\bar{u}_{n,j}\right)+g_j\left(\bar{u}_{i,n}+\bar{u}_{n,i}\right)\right)g_n+\\
&+\frac{16}{9}\bar{B}^2g^2\left(\bar{u}_{i,j}+\bar{u}_{j,i}\right)\Big\}\frac{I_6}{\mu_0^2\bar{\rho}^2c_{\rm ac}^4}
\end{aligned}
\qquad (32)
$$

is obtained with

$$I_5=\frac{1}{105}\int\!\!\int_0^\infty\frac{\nu k^2\mathcal{B}(k,\omega)}{(\omega^2+\nu^2k^4)^2}dkd\omega \qquad (33)$$

and

$$I_6=\frac{1}{105}\int\!\!\int_0^\infty\frac{\nu k^2(\nu^2k^4-\omega^2)\mathcal{B}(k,\omega)}{(\omega^2+\nu^2k^4)^3}dkd\omega. \qquad (34)$$

It follows

$$\langle u_s'u_\phi'\rangle=-\left(\frac{3g^2I_5}{c_{\rm ac}^4}+\left(\frac{23}{105}I_4+\frac{13}{3}\frac{g^2I_5}{c_{\rm ac}^4}+\frac{16}{9}\frac{g^2I_6}{c_{\rm ac}^4}\right)\frac{\partial\log\Omega}{\partial\log s}\right)\frac{\bar{B}^2\Omega}{\mu_0^2\bar{\rho}^2}. \qquad (35)$$

Note that in (35) a basic viscosity term[3] exists which does *not* vanish for $c_{\rm ac}\to\infty$.

One could discuss the sign and the value of (35) in full generality for a series of turbulence models which, however, is not in the scope of the present paper. All we need is to find whether the 'viscosity' term in (35) may dominate the non-viscosity term so that the sign of the angular momentum transport could be changed from minus to plus.

For our magnetic-driven turbulence and for a magnetic field with dominating toroidal component $(|\bar{B}_\phi|\gg|\bar{B}_s|,|\bar{B}_z|)$ we find

$$\langle u_s'u_\phi'\rangle=-\frac{\tau_{\rm corr}^3}{105}\left(3\frac{g^2}{c_{\rm ac}^4}+\frac{1}{\ell_{\rm corr}^2}\left(23+\frac{55}{9}\frac{g^2\ell_{\rm corr}^2}{c_{\rm ac}^4}\right)\frac{\partial\log\Omega}{\partial\log s}\right)\Omega V_{\rm A}^2\frac{\langle B^{(0)2}\rangle}{\mu_0\bar{\rho}} \qquad (36)$$

for the Reynolds stress at the equator. It does not vanish for rigid rotation so that the term 'viscosity' here indeed makes no real sense. The rigid-rotation term in (36) reflects the Λ-effect of rotating turbulence fields which is responsible for the maintenance of differential rotation in stellar convection zones (see paper I). Here it is negative. The total angular momentum transport, however, for a Kepler flow is positive as then

$$\langle u_s'u_\phi'\rangle\approx\frac{1}{105}\frac{\tau_{\rm corr}^3}{\ell_{\rm corr}^2}\left(\frac{69}{2}+\frac{37}{6}\left(\frac{\ell_{\rm corr}}{H_p}\right)^2\right)\Omega V_{\rm A}^2\frac{\langle B^{(0)2}\rangle}{\mu_0\bar{\rho}} \qquad (37)$$

results if $gH_p\simeq c_{\rm ac}^2$, H_p as the radial pressure scale. In Kepler flows the Reynolds stress is positive, hence the angular momentum is always transported outwards.

For the Maxwell stress we simply obtain

$$\langle B_i'B_j'\rangle=\frac{23}{14}V_{\rm A}^2(\bar{u}_{i,j}+\bar{u}_{j,i})I_1, \qquad (38)$$

which leads to the magnetic-induced angular momentum transport

$$\langle B_s'B_\phi'\rangle=\frac{23}{105}\frac{\tau_{\rm corr}^3}{\ell_{\rm corr}^2}\frac{\partial\log\Omega}{\partial\log s}\Omega V_{\rm A}^2\langle B^{(0)2}\rangle. \qquad (39)$$

It is a viscosity-term and it survives the limit $c_{\rm ac}\to\infty$. For (31) we then arrive at the positive value

$$\alpha_{\rm SS}=\left(\frac{23}{35}+\frac{2}{35}\frac{\ell_{\rm corr}^2}{H_p^2}\right)\frac{\tau_{\rm corr}^3}{\ell_{\rm corr}^2}\frac{V_{\rm A}^2}{c_{\rm ac}^2}\frac{\langle B^{(0)2}\rangle}{\mu_0\bar{\rho}}\Omega \qquad (40)$$

with the amplitude

$$\alpha_{\rm SS}\simeq\frac{\Omega^*}{2}{\rm Mm}^2\frac{\langle B^{(0)2}\rangle/\mu_0\bar{\rho}}{u_{\rm T}^2}, \qquad (41)$$

where the Coriolis number is $\Omega^*=2\tau_{\rm corr}\Omega$. If the magnetic fields are in equipartition with the turbulence then we find

$$\alpha_{\rm SS}\lesssim{\rm Mm}^2\ \Omega^* \qquad (42)$$

as an estimate. The maximal value of (42) might be Ω^*, which in accretion disks itself hardly exceeds the order of unity. The viscosity-α, therefore, proves to be *smaller* than unity for subsonic turbulence. On the other hand, for equipartition of the magnetic energy with the thermal energy (${\rm Mm}\simeq 1$) the value of (42) should not be much smaller than unity. The desired order of magnitude of 10^{-3}...1 for the viscosity-α might be well reproduced by the presented theory.

[3] Typically, 'viscosity' is the coefficient of $\nabla\Omega$ in the angular momentum transport relations

Table 1. The signs of the MHD coefficients for rigid rotation and Kepler rotation.

rotation	location	$\alpha_{\rm SS}$	$\alpha_{\phi\phi}$	$\mathcal{H}_{\rm curr}$	$\mathcal{H}_{\rm kin}$
rigid	north	−	+	−	+
	south	−	−	+	−
Kepler	north	+	−	+	−
	south	+	+	−	+

5. Discussion: dynamo-α and viscosity-α

In Table 1 a summary is given for the signs of the resulting MHD mean-field coefficients obtained by our turbulence model for the two cases of rigid rotation and Kepler rotation. The first line with positive α-effect, with positive kinetic helicity and negative current helicity (all in the upper disk plane) is just the same as given by Brandenburg & Schmitt (1998) for a simulation of the solar north pole. For a MHD shear flow simulated by Brandenburg (1999) the case 'Kepler' in Table 1 is valid and there is also not even one exception from the general agreement. The kinetic helicity in the upper disk plane for Brandenburg's simulation is negative and the same is true in our flux tube model. It is interesting to formulate for Kepler disks the relation between both alphas. With (21) and (40) follows

$$\frac{\alpha_{\phi\phi}}{\alpha_{\rm SS}} \lesssim -\frac{1}{5}\ \frac{z\Omega^2\tau_{\rm corr}u_{\rm T}^2}{V_{\rm A}^2}, \qquad (43)$$

so that the amplitude of the dynamo-α becomes

$$|\alpha_{\phi\phi}| \lesssim \frac{\alpha_{\rm SS}}{5}\ \frac{\ell_{\rm corr}}{H}\ \frac{u_{\rm T}}{\rm Mm^2}. \qquad (44)$$

We have also used the relation $H\Omega \simeq c_{\rm ac}$ between the disk thickness and the temperature of a thin accretion disk. The magnetic Mach number Mm can be assumed to be of order unity.

We find the dynamo-α to be a rather small fraction of the turbulent velocity $u_{\rm T}$. Ziegler & Rüdiger (2000) find with a box simulation that the dynamo-α is of order $5 \cdot 10^{-3}$ in units of the sound velocity. The turbulent velocity is of the order of the sound velocity (of the midplane) so that the α-effect approaches $5 \cdot 10^{-3}$ in units of the eddy velocity. This is indeed smaller than the viscosity-α which in the simulations was of order 10^{-2}.

The dynamo-α proves to be negative in the upper disk plane and positive in the lower one. We can thus expect a dipolar symmetry with respect to the equator for the dynamo-maintained large-scale magnetic fields. In order to ensure self-excitation for the magnetic fields with such a small α-effect, the eddy diffusivity of the turbulence must be sufficiently small. As it works with uniform magnetic fields we can not compute this coefficient with our model. The same also holds for almost all numerical simulations so that here it must remain open whether a magnetic dynamo really works.

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