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Approximate expressions for polar gap electric field of pulsars

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Abstract. We derive easy-to-handle approximations for the polar gap electric field due to inertial frame dragging as derived by Harding & Muslimov (1998). A formula valid for polar gap height comparable to the polar cap radius is presented.

Key words: acceleration of particles – stars: neutron – stars: pulsars: general

1. Introduction

Accurate models for $E_\parallel$ – the electric field component parallel to the local magnetic field $\boldsymbol{B}$ above polar caps of rotating magnetized neutron stars – are of great theoretical interest in the context of magnetospheric activity of pulsars. In the framework of space charge limited flow model (originally introduced by Arons & Scharlemann (1979)), Harding & Muslimov (1998) (hereafter HM98) derived $E_\parallel$ including the general relativistic frame dragging effect, worked out by Muslimov & Tsygan (1992) (hereafter MT92). HM98 considered the case with $E_\parallel$ screened at both a lower and an upper boundary of acceleration region i.e. at the star surface and at a pair formation front. Since the full expression for $E_\parallel$ is too cumbersome for practical applications, HM98 offered simple analytic expressions valid in various limiting cases. In this paper we revise and extend their results. In particular, we show that in the most frequently considered case, i.e. when the gap accelerator length is comparable to the size of the polar cap radius, the approximation derived by HM98 overestimates $E_\parallel$ by a ratio of the neutron star radius to the polar cap radius.

2. Formulae for $\boldsymbol{E}_\parallel$

In the following, h denotes the altitude above the neutron star surface and h_c is the gap height above which a pair formation front screens $E_\parallel$ (i.e. $E_\parallel = 0$ for $h \ge h_c$). Altitudes $z \equiv h/R_{\rm ns}$, $z_c \equiv h_c/R_{\rm ns}$ and radial distances $\eta \equiv 1+z$, $\eta_c \equiv 1+z_c$ scaled with the star radius $R_{\rm ns}$ will also be used. The magnetic colatitude $\xi \equiv \theta/\theta(\eta)$ is scaled with the half-opening angle of the polar magnetic flux tube $\theta(\eta)$.

A solution of Poisson's equation for the polar gap with $h_c \ll R_{\rm ns}$ as derived by HM98 reads

$$E_\parallel^{(1)} \simeq -E_0\, \theta_0^3\, (1-\epsilon)^{1/2} \left\{ \frac{3}{2}\, \kappa\, S_1 \cos\chi + \right. \\ \left. + \frac{3}{8}\, \theta_0 H(1)\delta(1) S_2 \sin\chi\cos\phi \right\} \quad (1)$$

where,

$$S_1 = \sum_{i=1}^{\infty} \frac{8 J_0(k_i\xi)}{k_i^4 J_1(k_i)} \mathcal{F}(z, z_c, \gamma_i), \\ S_2 = \sum_{i=1}^{\infty} \frac{16 J_1(\tilde{k}_i\xi)}{\tilde{k}_i^4 J_2(\tilde{k}_i)} \mathcal{F}(z, z_c, \tilde{\gamma}_i), \quad (2)$$

$$\mathcal{F}(z, z_c, \gamma) = -[a_1(\gamma\eta - 1)e^{\gamma z} + a_2(\gamma\eta+1)e^{-\gamma z} + \\ +\ a_1(1-\gamma) - a_2(1+\gamma)]/(a_1+a_2), \quad (3)$$

$$a_1 = (\gamma\eta_c + 1)e^{-\gamma z_c} - \gamma - 1, \quad a_2 = \gamma - 1 - (\gamma\eta_c - 1)e^{\gamma z_c} \quad (4)$$

and

$$\gamma_i \approx \frac{k_i}{\theta_0(1-\epsilon)^{1/2}}, \quad \text{and} \quad \tilde{\gamma}_i \approx \frac{\tilde{k}_i}{\theta_0(1-\epsilon)^{1/2}}, \quad (5)$$

where k_i and $\tilde{k}_i$ are the positive roots of the Bessel functions J_0 and J_1, respectively, $H(1)\delta(1) \approx 1$, and χ is a tilt angle between the rotation and the magnetic dipole axes. Values of E_0, θ_0, ϵ, and κ are given by:

$$E_0 \equiv B_{\rm pc} \frac{\Omega R_{\rm ns}}{c}, \qquad \theta_0 = \left(\frac{\Omega R_{\rm ns}}{c f(1)}\right)^{1/2}, \\ \epsilon \equiv \frac{2GM}{R_{\rm ns} c^2}, \qquad \kappa \equiv \frac{\epsilon I}{M R_{\rm ns}^2}, \quad (6)$$

where $B_{\rm pc}$ is the magnetic field strength at the magnetic pole, Ω is the pulsar rotation frequency, and I, M are the moment of inertia, and mass of the neutron star, respectively. The function $f(\eta)$ reads

$$f(\eta) = -3x^3 \left[\ln(1 - x^{-1}) + (1 + (2x)^{-1})/x\right], \quad (7)$$

with $x = \eta/\epsilon$.

The electric field $E_\parallel^{(1)}$ as a function of height h is drawn in Fig. 1 for several values of gap height h_c. It can be seen that $E_\parallel$ saturates in a twofold way: First, for a fixed h_c it assumes a constant value at h exceeding the polar cap radius $r_{\rm pc}$ (after a

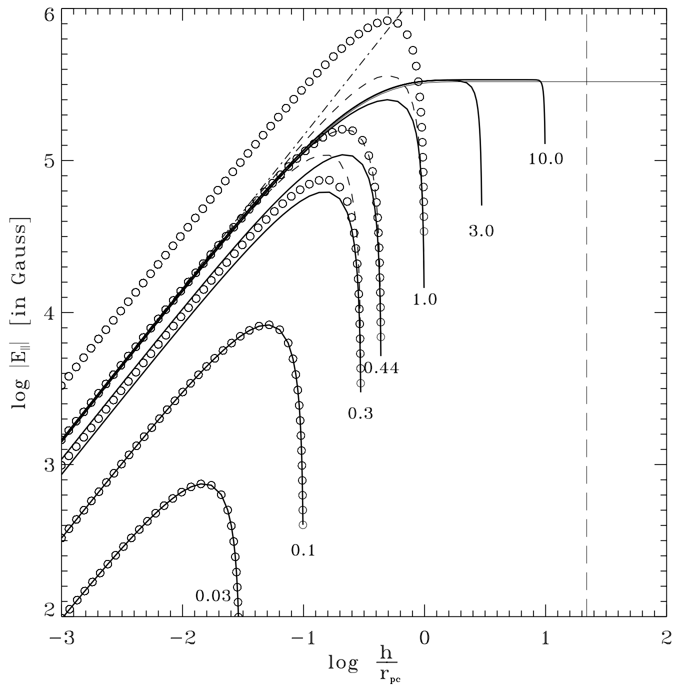

Fig. 1. The electric field $E_\parallel$ in the limit $h_c \ll R_{\rm ns}$ as a function of height h in units of $r_{\rm pc}$ for several values of gap height $h_c = 0.03$, 0.1, 0.3, 0.44, 1.0, 3.0, and 10.0 $r_{\rm pc}$. Thick solid lines present $E_\parallel^{(1)}$ of Eq. (1). Open circles mark approximation $E_\parallel^{(2)}$ (Eq. 9 for the cases 0.03, 0.1, 0.3, 0.44, and 1.0 $r_{\rm pc}$). The dot-dashed line is for $E_\parallel^{(4)}$ (Eq. 13) and dashed lines are for $E_\parallel^{(6)}$ (Eq. 18, the cases $h_c/r_{\rm pc} = 0.3$, 0.44, and 1.0). The thin solid line indicates the case with no upper gap boundary ($E_\parallel^{(3)}$ of Eq. 12). The vertical dashed line marks $h = R_{\rm ns}$. For $h_c = 0.44\ r_{\rm pc}$, approximations $E_\parallel^{(2)}$ and $E_\parallel^{(6)}$ coincide. We assumed $B_{\rm pc} = 10^{12}$ G, $P = 0.1$ s, $\chi = 0.2$ rad, $\xi = 0.7$, $R_{\rm ns} = 10^6$ cm, $M = 1.4 M_\odot$.

linear increase with h). Second, for a fixed $h \ll h_c$ it initially increases linearly with h_c to become constant for $h_c \gtrsim r_{\rm pc}$. As noted by Harding & Muslimov this behaviour can be reproduced with simple approximations of Eq. (1) for different regimes of validity.

Since $k_i \geq 2.4$ it follows form Eq. (5) that $\gamma_i z$ becomes smaller than 1 for $h \lesssim r_{\rm pc}/3$. Thus, in the limit $h \ll r_{\rm pc}/3$ and $h_c \ll r_{\rm pc}/3$ (or $\gamma_i z \ll 1$ and $\gamma_i z_c \ll 1$) the function $\mathcal{F}(z, z_c, \gamma_i)$ may be approximated with

$$\mathcal{F}(z, z_c, \gamma_i) \simeq \frac{1}{2}\gamma_i^3 \left(1 - \frac{z}{z_c}\right) z z_c \tag{8}$$

which leads to

$$E_\parallel^{(2)} \simeq -3\frac{\Omega R_{\rm ns}}{c}\frac{B_{\rm pc}}{1-\epsilon}\left(1 - \frac{z}{z_c}\right) z z_c \left[\kappa\cos\chi \ + \right. \\ \left. + \frac{1}{2}\theta_0 \xi H(1)\delta(1)\sin\chi\cos\phi\right], \tag{9}$$

(HM98), where the relations

$$\sum_{i=1}^{\infty}\frac{2J_0(k_i\xi)}{k_i J_1(k_i)} = 1, \quad \text{and} \quad \sum_{i=1}^{\infty}\frac{2J_1(\tilde{k}_i\xi)}{\tilde{k}_i J_2(\tilde{k}_i)} = \xi \tag{10}$$

have been used (see eg. MT92). As can be seen in Fig. 1, in practice, Eq. (9) (open circles) reproduces Eq. (1) (thick solid line) for $h_c \lesssim r_{\rm pc}/3$ (and $h \leq h_c$).

Approximate behaviour of $E_\parallel$ for $h_c \sim r_{\rm pc}$ can be determined by considering the opposite limiting case for the accelerator height: $h_c \gg r_{\rm pc}/3$. Assuming $\gamma_i z_c \gg 1$, $h < (h_c - r_{\rm pc})$, (and $h_c \ll R_{\rm ns}$) in Eq. (1) one obtains

$$\mathcal{F}(z, z_c, \gamma_i) \simeq \gamma_i\left(1 - e^{-\gamma_i z}\right), \tag{11}$$

which results in the same formula for $E_\parallel$ as the one derived by MT92 for the case with no upper gap boundary (with the condition $E_\parallel = 0$ fulfilled only at the star surface):

$$E_\parallel^{(3)} \simeq -3E_0\theta_0^2\left\{\kappa\cos\chi\sum_{i=1}^{\infty}\frac{4J_0(k_i\xi)}{k_i^3 J_1(k_i)}\left[1 - e^{-\gamma_i z}\right] + \right. \\ \left. + \theta_0 H(1)\delta(1)\sin\chi\cos\phi\sum_{i=1}^{\infty}\frac{2J_1(\tilde{k}_i\xi)}{\tilde{k}_i^3 J_2(\tilde{k}_i)}\left[1 - e^{-\tilde{\gamma}_i z}\right]\right\} \tag{12}$$

Nevertheless, for $r_{\rm pc} \ll h_c \ll R_{\rm ns}$, Eq. (12) reproduces $E_\parallel^{(1)}$ (Eq. 1) almost over the entire acceleration length (see the thin solid line in Fig. 1) except its very final part, where $h \gtrsim (h_c - r_{\rm pc})$.

Taylor expansion of (11) reveals the linear increase of $E_\parallel$ with h for $h \lesssim 0.1 r_{\rm pc}$ and the saturation above $h \simeq r_{\rm pc}$:

$$E_\parallel^{(4)} \simeq -3\frac{\Omega R_{\rm ns}}{c}\frac{B_{\rm pc}}{(1-\epsilon)^{1/2}}\,\theta_0 z\left[\kappa f_1(\xi)\cos\chi \ + \right. \\ \left. + \frac{1}{4}\theta_0 f_2(\xi) H(1)\delta(1)\sin\chi\cos\phi\right], \tag{13}$$

for $h \lesssim 0.1 r_{\rm pc}$ and $0.5 r_{\rm pc} \lesssim h_c \ll R_{\rm ns}$, and

$$E_\parallel^{(5)} \simeq -\frac{3}{2}\frac{\Omega R_{\rm ns}}{c}B_{\rm pc}\theta_0^2(1-\xi^2)\left[\kappa\cos\chi \ + \right. \\ \left. + \frac{1}{4}\theta_0 \xi H(1)\delta(1)\sin\chi\cos\phi\right], \tag{14}$$

for $r_{\rm pc} \lesssim h \lesssim (h_c - r_{\rm pc})$ and $2r_{\rm pc} \lesssim h_c \ll R_{\rm ns}$. The magnetic colatitude profiles

$$f_1(\xi) = \sum_{i=1}^{\infty}\frac{4J_0(k_i\xi)}{k_i^2 J_1(k_i)}, \quad \text{and} \quad f_2(\xi) = \sum_{i=1}^{\infty}\frac{8J_1(\tilde{k}_i\xi)}{\tilde{k}_i^2 J_2(\tilde{k}_i)} \tag{15}$$

are shown in Fig. 2 and the relations

$$\sum_{i=1}^{\infty}\frac{8J_0(k_i\xi)}{k_i^3 J_1(k_i)} = 1 - \xi^2, \quad \sum_{i=1}^{\infty}\frac{16J_1(\tilde{k}_i\xi)}{\tilde{k}_i^3 J_2(\tilde{k}_i)} = \xi(1-\xi^2) \tag{16}$$

have been used in derivation of (14) (eg. MT92). Approximation (13) is shown in Fig. 1 as a dot-dashed line.

For $h_c \ll R_{\rm ns}$ the symmetry

$$E_\parallel^{(1)}(h) \simeq E_\parallel^{(1)}(h_c - h) \tag{17}$$

may easily be proven to hold. Therefore, to extend the validity of Eq. (13) to $h \leq h_c$ it is useful to construct the approximation

$$E_\parallel^{(6)} = E_\parallel^{(4)} \cdot \left(1 - \frac{z}{z_c}\right) \tag{18}$$

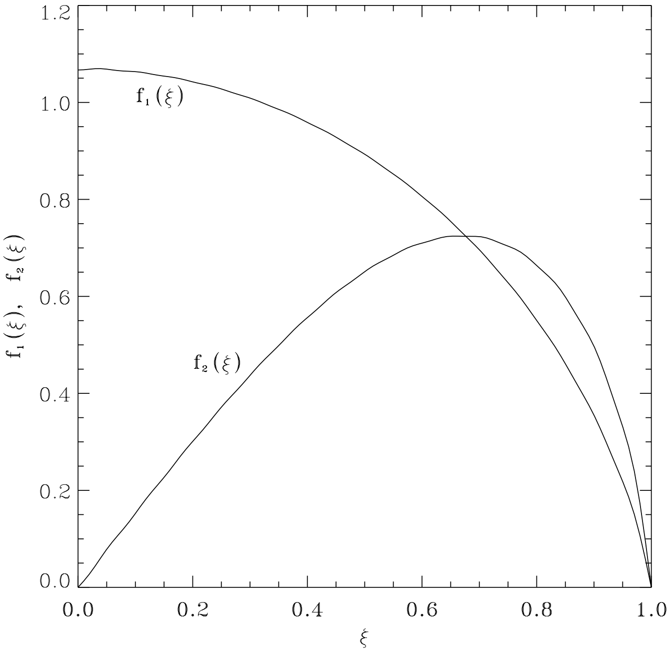

Fig. 2. Magnetic colatitude functions $f_1(\xi)$ and $f_2(\xi)$ (see Eq. (15) for definition).

which works reasonably well for $0.5r_{\rm pc} \lesssim h_c \lesssim r_{\rm pc}$ (dashed lines in Fig. 1).

The expression derived by HM98 for the case $h_c \sim r_{\rm pc}$ clearly overestimates $E_\|$ by a factor $\theta_0^{-1} \approx (r_{\rm pc}/R_{\rm ns})^{-1}$ (cf. Eq. (A3) in HM98, also Eq. (1) in Harding & Muslimov (1998a), or a formula used by Zhang et al. (2000)). Moreover, unlike in Eq. (A3) of HM98, $E_\|^{(4)}$ depends on the magnetic colatitude ξ via $f_1(\xi)$. Our approximations are correct because they reproduce the "exact" formula for $E_\|^{(1)}$ (Eq. 1). Moreover, $E_\|^{(4)}$ of Eq. (13) is identical to the formula of MT92 for the case with no screening at h_c (taken in the limit $\gamma_i z \ll 1$). This identity should hold since for $r_{\rm pc} < h_c \ll R_{\rm ns}$ both gap boundaries influence $E_\|$ only within corresponding adjacent regions of height $\sim r_{\rm pc}$.

Since Eq. (17) holds for $h_c \ll R_{\rm ns}$, $E_\|$ in the upper part of accelerator ($r_{\rm pc} < h \le h_c$) may be approximated with

$$E_\|^{(7)}(h) \simeq E_\|^{(3)}(h_c - h) \simeq \begin{cases} E_\|^{(4)}(h_c - h), \\ \text{for } (h_c - r_{\rm pc}) \lesssim h \le h_c \\ \\ E_\|^{(5)}, \\ \text{for } r_{\rm pc} < h \lesssim (h_c - r_{\rm pc}). \end{cases} \tag{19}$$

In the limit where $h_c \gg r_{\rm pc}$ and $h \gg r_{\rm pc}$, a solution of Poisson's equation for the elongated polar gap reads

$$E_\|^{(8)} \simeq -E_0\theta_0^2 \left\{ \frac{3}{2}\frac{\kappa}{\eta^4}\cos\chi \left[(1-\xi^2) + \right.\right.$$
$$\left. - \left(\frac{\eta_c}{\eta}\right)^4 \sum_{i=1}^{\infty} \frac{8J_0(k_i\xi)}{k_i^3 J_1(k_i)} e^{-\gamma_i(\eta_c)(\eta_c-\eta)} \right] +$$

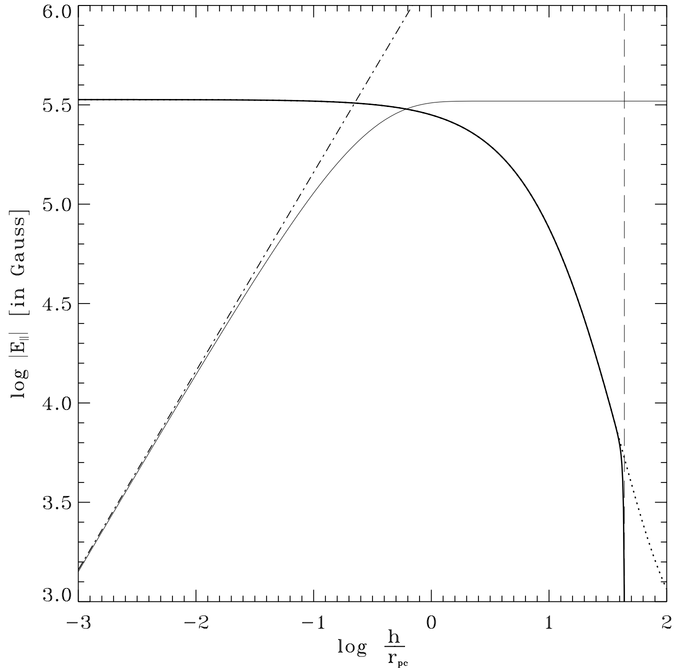

Fig. 3. The run of $E_\|$ as a function of $h/r_{\rm pc}$ in the limit $h_c \gg r_{\rm pc}$. We choose $h_c = 2R_{\rm ns}$ (marked with dashed vertical line); the other parameters are assumed as in Fig. 1. Thick solid line presents $E_\|^{(8)}$ of Eq. (20). It coincides with the case without the upper gap boundary ($E_\|^{(9)}$ of Eq. (22), dotted) except for $h \gtrsim h_c$. The dot-dashed line presents $E_\|^{(4)}$ of Eq. (13). Thin solid line presents the approximation $E_\|^{(3)}$ of Eq. (12).

$$+ \frac{3}{8}\, g(\eta)\sin\chi\cos\phi \left[\xi(1-\xi^2) + \right.$$
$$\left.\left. - \frac{\eta_c}{\eta}\frac{g(\eta_c)}{g(\eta)} \sum_{i=1}^{\infty} \frac{16J_1(\tilde{k}_i\xi)}{\tilde{k}_i^3 J_2(\tilde{k}_i)} e^{-\tilde{\gamma}_i(\eta_c)(\eta_c-\eta)} \right] \right\} \tag{20}$$

(MT92), where now

$$\gamma_i \approx \frac{k_i}{\theta(\eta_c)\eta_c(1-\epsilon/\eta_c)^{1/2}}, \qquad \tilde{\gamma}_i \approx \frac{\tilde{k}_i}{\theta(\eta_c)\eta_c(1-\epsilon/\eta_c)^{1/2}}, \tag{21}$$

$g(\eta) = \theta(\eta)\delta(\eta)H(\eta)$, $\theta(\eta) = \theta_0(\eta f(1)/f(\eta))^{1/2}$, and the functions δ, and H are defined in MT92 or HM98. Since $\gamma_i \gg 1$, the two summation terms which reflect the screening effect of the pair formation front at h_c contribute to $E_\|$ only at h very close to h_c (see Fig. 3) so that $\gamma_i(\eta_c)(\eta_c - \eta) \sim 1$. Thus, for $h_c \gg r_{\rm pc}$ and $r_{\rm pc} \ll h \le h_c$ one may simply use a formula derived in MT92 for the case with no screening at h_c:

$$E_\|^{(9)} \simeq -E_0\theta_0^2 \left\{ \frac{3}{2}\frac{\kappa}{\eta^4}(1-\xi^2)\cos\chi + \right.$$
$$\left. + \frac{3}{8}g(\eta)\xi(1-\xi^2)\sin\chi\cos\phi \right\}. \tag{22}$$

For $P = 0.1$ s the approximation $E_\|^{(9)}$ matches the approximation $E_\|^{(3)}$ (which is valid for low altitudes (Eq. 12)) at

$h \simeq 0.6 r_{\rm pc}$. This altitude is much lower than estimated by Muslimov & Harding (1997). Consequently, below $\sim 0.6 r_{\rm pc}$ either $E_{\|}^{(3)}$ or $E_{\|}^{(4)}$ should be used.

3. Conclusions

We find that for accelerator height h_c approaching the polar cap radius $r_{\rm pc}$ the accelerating electric field (for nonorthogonal rotators) may be approximated according to Eq. (18):

$$E_{\|} = -1.46 \, \frac{B_{12}}{P^{3/2}} \, h \left(1 - \frac{h}{h_c}\right) f_1(\xi) \cos\chi \; \mathrm{Gauss}, \qquad (23)$$

where $B_{12} = B_{\rm pc}/(10^{12} G)$, P is the rotation period in seconds, h is in cm, for a star with $M = 1.4 M_{\odot}$, $R_{\rm ns} = 10^6$ cm. This result differs from the approximation derived in HM98 by a considerable period-dependent factor $r_{\rm pc}/R_{\rm ns}$. Moreover, unlike the HM98' formula it depends on the magnetic colatitude ξ through the factor $f_1(\xi)$. It may be of particular importance for a possibility of limiting acceleration by the resonant inverse compton scattering (see Dyks & Rudak 2000).

We emphasize that the corrections presented in this research note do not affect the results of the numerical calculations presented by Harding & Muslimov (1998) because their numerical procedures include the exact expressions given by Eqs. (1) and (20). However, their analytic estimates of the height and width of the acceleration zone and of the maximum particle energy need to be revised.

Since for an elongated polar gap ($h_c \gg r_{\rm pc}$) the upper gap boundary (pair formation front) influences the electric field only within the negligible ($\ll h_c$) upper part of accelerator, the formulae derived by Muslimov & Tsygan (1992) for no upper boundary at h_c may be used within the entire gap with no significant overestimates of electron energies.

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