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Coordinate transformations and gauges in the relativistic astronomical reference systems

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Abstract. This paper applies a fully post-Newtonian theory (Damour et al. 1991, 1992, 1993, 1994) to the problem of gauge in relativistic reference systems. Gauge fixing is necessary when the precision of time measurement and application reaches 10^{-16} or better. We give a general procedure for fixing the gauges of gravitational potentials in both the global and local coordinate systems, and for determining the gauge functions in all the coordinate transformations. We demonstrate that gauge fixing in a gravitational N-body problem can be solved by fixing the gauge of the self-gravitational potential of each body and the gauge function in the coordinate transformation between the global and local coordinate systems. We also show that these gauge functions can be chosen to make all the coordinate systems harmonic or any as required, no matter what gauge is chosen for the self-gravitational potential of each body.

Key words: relativity – gravitation – astrometry – celestial mechanics, stellar dynamics – reference systems

1. Introduction

Today, it is rather evident that the high precision of astrometric observations and measurements has made it necessary to adopt high-precision astronomical reference systems, and much progress has been made in the past ten years. The so-called high-precision astronomical reference systems are generally based on the 1PN approximation of Einstein's general relativity. For practical purposes, it is common to take some further approximations, for example, some minor 1PN terms may be omitted under certain circumstances.

At present and in the near future, high-precision time measurements need even more precise astronomical reference systems. To fix the gauges of coordinate systems seems to be required. This is one of the problems we have to face and ought to consider seriously at present. We know that hydrogen masers are the most stable atomic clocks, which reach frequency stability of 1×10^{-15} for averaging times of about 1000 seconds. Cesium fountain standards that have been currently developing in some laboratories are expected to provide stability in the low 10^{-16} range. In the longer terms the advent of cooled hydrogen masers and trapped-ion standards may yield stabilities of 10^{-17} and 10^{-18} respectively (Wolf & Petit 1995). Against this background, a relativistic theory for the realization of coordinate time scales at picosecond synchronization and 10^{-18} syntonization accuracy in the vicinity of the Earth has been put forward (Petit & Wolf 1994; Wolf & Petit 1995). Their computations show that the 2PN terms of time may be omitted in the geocentric reference system (GRS) at the present accuracy of time measurements. But this conclusion, in our opinion, should be re-examined. In fact, according to Damour et al. (1991, 1992), the 2PN terms of time are the so-called gauge terms, which do not affect the equations of motion, but they may be considerable in the GRS or BRS because they can take arbitrary functional forms. For instance, it is easy to see that the uncertainty of time coordinate can reach up to 10^{-16} if the gauge of a coordinate system is kept open. The simplest method to solve this problem is to fix the gauges of all the reference systems. It will be discussed in detail in this paper.

The realization of barycentric coordinate times has been discussed previously (Petit & Wolf 1998). But in their paper some viewpoints turn out to be unsatisfactory (Klioner 1998). Especially, Klioner (1998) pointed out that it is necessary to choose a definite gauge for discussing the time scales in the barycentric reference system (BRS). Actually, the GRS also requires its gauge to be fixed. The choice of gauge is one of the main problems listed in Circular No. 2 of IAU WG/RCMA and JCR (Soffel et al. 1998).

In two later Circulars of IAU WG/RCMA, No.3 and No.4 (Soffel & Klioner 1998, 1999), relativistic reference systems and coordinate transformations were discussed, but the choice of time gauges was kept open.[1] Recently the Resolutions of the IAU Colloquium 180 (Washington, March 2000) have recommended that the standard harmonic gauge be employed in relativistic reference systems of the solar system. However, we

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[1] After this paper was submitted, Circular No.5 and its related documents maintain a concrete harmonic gauge in the practice of the solar system.

believe that further analysis and investigation would be useful, and that a general method detailing the steps for fixing the gauges should be given. Which gauge should be preferred, and why? This is the question the present paper intends to answer. Tao and Huang (1998b) has provided a special standard PN gauge for the self-gravitational potential of a body in its local reference system. The choice of the self-potential's gauge is necessary but is only a first step. We will complete a systematic procedure for gauge fixing for all the gravitational potentials, which fully determines all the reference systems and makes time scales reach the 2PN precision.

This paper is arranged as follows: Sect. 2 briefly introduces gauge, gravitational potential, the transformation between two coordinate systems and between gravitational potentials in two different coordinate systems, and demonstrates how to determine the coordinate transformation according to the relation between the gravitational potentials in the framework of DSX theory, which prepares the way for the following section. Sect. 3 is the main part of this paper; it discusses the gauge of self-gravitational potential and its role, and calculates the expressions of inertial gravitational potential, then presents some measures and steps for fixing the gauges of coordinate systems and for determining the coordinate transformation between a local system and the global system. Our discussion covers both harmonic and general gauges. Several practical gauges and the related methods to fix them are presented for astronomical practice. In Sect. 4, our presentation is summarized, and our main conclusions are listed.

2. Gauge, potential and coordinate transformation

Up to now, we still do not know how to deal with the gauge problem. This is because we lack a thorough investigation in gauge, gravitational potential, coordinate transformation and their inter-relations. Let's begin with them in this section.

In order to construct relativistic reference systems, a proper 1PN theory should be employed. Fortunately, two advanced relativistic formalisms have been elaborated to tackle the gravitational N-body problem and astronomical reference systems in the first post-Newtonian approximation of general relativity. One is due to Brumberg and Kopeikin (Brumberg & Kopeikin 1989; Kopeikin 1988; Brumberg 1991). Another is due to Damour, Soffel and Xu (Damour et al. 1991, 1992, 1993, 1994; hereafter cited as the DSX theory). We will apply the DSX formalism to discuss the gauge problem. Its advantages have been gradually recognized. After its appearing, it has been applied in astronomy and other related fields extensively and successfully (e.g. Damour & Vokrouhlicky 1995; Tao et al. 1997; Tao & Huang 1998a, 1998b; Xu et al. 1997). We will adopt its notations and conventions. In the following, most of the materials needed for our further discussion come from the DSX formalism, but some extensions have been made by us.

In the DSX theory, a system of N bodies has $N+1$ coordinate systems, i.e. a global coordinate system and N local ones. There are N independent coordinate transformations between these coordinate systems. We need to choose the gauge for each coordinate system. The gauge of a coordinate system can be represented by a gauge function, so a N-body system has $N+1$ gauge functions.

A coordinate system is represented by all the metric coefficients in it. The DSX scheme introduces a set of gravitational potential (a scalar potential and a 3-dimensional vector potential) to represent the metric coefficients. The Einstein field equations turn into linear differential equations to be satisfied by these gravitational potentials. Therefore the gravitational potentials also satisfy the principle of superposition. This is an important and useful property.

According to what is stated above, a coordinate system can be characterized by the fully 1PN gravitational potentials in it. The field equations are gauge-independent, so the 1PN potentials must contain gauge terms generated by the so-called "gauge function". For example, in the global coordinate system, the field equations are

$$\begin{aligned} &\Box w + 4c^{-2}\partial_t(\partial_t w + \partial_j w_j) = -4\pi G\sigma + O(4) \\ &\Box w_i - \partial_i(\partial_t w + \partial_j w_j) = -4\pi G\sigma^i + O(2) \end{aligned} \tag{1}$$

where d'Alembertian $\Box \equiv -c^{-2}\partial_t^2 + \partial_j\partial_j$. $\sigma^\alpha \equiv (\sigma,\ \sigma^i)$ are the mass density and mass current density. $\sigma \equiv (T^{00} + T^{ii})/c^2$, $\sigma^i \equiv T^{0i}/c$, and $T^{\alpha\beta}$ is the stress-energy tensor of the gravitational source. $O(n)$ denotes $O(c^{-n})$ $(n > 0)$. It is easy to find that Eq. (1) is gauge invariant under a gauge transformation: $w_\mu \to w'_\mu$

$$w' = w - c^{-2}\partial_t\lambda,\ w'_i = w_i + \frac{1}{4}\partial_i\lambda \tag{2}$$

here the gauge function $\lambda = \lambda(t, x^i)$ is an arbitrary differentiable function. Eq. (2) is equivalent to a coordinate transformation $(t, x^i) \to (t', x'^i)$

$$t' = t - \lambda/c^4,\ x'^i = x^i. \tag{3}$$

Eq. (2) also means that the general solution of Eq. (1) is the sum of one of its special solutions and the gauge term. We usually choose this special solution as a definite solution under harmonic gauge (i.e. $\partial_t w + \partial_j w_j = O(2)$), then the general solution is fully determined by the gauge function λ. Its differential gauge condition is

$$\partial_t w + \partial_j w_j = \frac{1}{4}\partial_j\partial_j\lambda + O(2). \tag{4}$$

$\partial_j\partial_j\lambda$ can be viewed as the gauge employed in the field equations, e.g. $\partial_j\partial_j\lambda = O(2)$ represents harmonic gauge, and $\partial_j\partial_j\lambda = \partial_t w + O(2)$ standard PN gauge. But $\partial_j\partial_j\lambda$ can not uniquely determine the coordinate system or the gravitational potentials in it. For our present purpose, it is more suitable to use λ itself directly as the gauge of a coordinate system or the gravitational potentials in it.

Coordinate transformation plays an essential pole in any theory of reference systems, and it must be gauge-dependent. It has a close relation with the transformation of gravitational potentials, which will be stated below.

Let $(ct,\ x^i)$ denote the space-time coordinates of the barycentric reference system, and $(cT,\ X^a)$ denote those attached to some celestial body (e.g. Sun, Moon, Earth or one of other major planets) in the solar system. According to Damour et al. (1991), the coordinate transformation between the global coordinate system $(ct,\ x^i)$ and the local one $(cT,\ X^a)$ of body A has the form of

$$x^\mu(X^\alpha) = z^\mu(T) + e^\mu_a(T)X^a + \xi^\mu(X^\alpha) + O(5,4) \tag{5}$$

where z^μ, e^μ_a and ξ^i ($\sim O(2)$) can be completely determined by a certain procedure (Damour et al. 1991), $\xi^0 \equiv \xi/c^3$, and $\xi(X^\alpha)$ is an arbitrary differentiable function. In fact ξ is the gauge function of the coordinate transformation. It is not negligible, otherwise the time coordinate transformation can't reach the 2PN ($\sim O(4)$) precision. ξ linearly depends on the difference between the gauge functions of these two coordinate systems.

If time coordinates t and T need to reach the precision of $O(4) \sim 10^{-16}$, the gauges of both the global and local coordinate systems must be fixed. Considering $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = G_{\alpha\beta}dX^\alpha dX^\beta$, gauge fixing means that g_{00} and G_{00} can reach the precision of $O(4)$, g_{0i} and G_{0a} the precision of $O(3)$, g_{ij} and G_{ab} the precision of $O(2)$. The metric coefficients $g_{\mu\nu}$ or $G_{\alpha\beta}$ can be expressed in terms of the related gravitational potentials w_μ or W_α (see Eqs. (3.3) and (4.1) in Damour et al. 1991). The gauge of a coordinate system is actually the gauge of the related gravitational potentials. After fixing the gauge, the precision of the scalar gravitational potentials w and W reaches $O(2)$, and the precision of the vector gravitational potentials w_i and W_a is in the Newtonian level.

Theoretically, the choice of gauge is quite arbitrary, and what gauge to be taken is also not important. But in astronomical practice it is convenient to adopt certain gauges. There exist some general rules for us to follow. These are: 1) to make the resulting coordinate transformations as simple as possible; 2) to make the expressions of the gravitational potentials and hence the metric coefficients simple as well. There are two possible ways in choosing a gauge. One is to choose the gauge of gravitational potentials, and then to determine the gauge function $\xi(X^\alpha)$ that finally fixes the coordinate transformation. Another seems to be simpler and more practical. We can directly choose $\xi(X^\alpha)$ in the coordinate transformation (e.g. $\xi = 0$), then determine the expressions of gravitational potentials. This problem will be discussed in detail below.

In accordance with the coordinate transformation (Eq. (5)), the gravitational potential w_μ in the global coordinate system and the gravitational potential W_α in the local coordinate system of body A satisfy the affine transformation as follows (Damour et al. 1991)

$$w_\mu(x) = \mathcal{A}^A_{\mu\alpha}(T)W^A_\alpha(X) + \mathcal{B}^A_\mu(X) + O(4,2) \tag{6}$$

where $w_\mu(x)$ and $W^A_\alpha(X)$ can be expressed as

$$w_\mu = \sum_B w^B_\mu = w^A_\mu + \overline{w}^A_\mu = w^A_\mu + \sum_{B\neq A} w^B_\mu \tag{7}$$

and

$$W^A_\alpha = W^{+A}_\alpha + \overline{W}^A_\alpha = W^{+A}_\alpha + \sum_{B\neq A} W^{B/A}_\alpha + W''^A_\alpha. \tag{8}$$

The transformation matrix $\mathcal{A}^A_{\mu\alpha}$ in Eq. (6) reads

$$\mathcal{A}^A_{\mu\alpha}(T) = \begin{pmatrix} 1 + 2v^2_A/c^2 & 4V^a_A/c^2 \\ v^i_A & R^i_a \end{pmatrix} \tag{9}$$

here $R^i_a = R^i_a(T)$ is an orthogonal matrix, $v^i_A \equiv dz^i_A/dt = R^i_a V^a_A + O(2)$, or $V^a_A = R^i_a v^i_A + O(2)$. In Eq. (6), $\mathcal{B}^A_\mu(X)$ is an inertial term, it is determined by the related coordinate transformation. Its expression is

$$\begin{aligned} \mathcal{B}^A_0(X) &= \tfrac{c^2}{2}\ln(A^0_0A^0_0 - A^0_aA^0_a) + O(4) \\ \mathcal{B}^A_i(X) &= \tfrac{c^3}{4}(A^0_0A^i_0 - A^0_aA^i_a) + O(2) \end{aligned} \tag{10}$$

where $A^\mu_\alpha \equiv \partial x^\mu/\partial X^\alpha$ is the Jacobi matrix of the coordinate transformation. We will see that $\mathcal{B}^A_\mu(X)$ must include the contribution generated by the gauge function $\xi(X)$, which means that ξ can be viewed as a part of the gauge function of the local system.

The gravitational potentials completely describe the related metric in a coordinate system. The transformation of gravitational potentials between two coordinate systems represents in fact the transformation of their metrics. Therefore, it is also equivalent to the related coordinate transformation (Eq. (5)). If w_μ and W^A_α are known, the coordinate transformation can be completely determined, or if we know the coordinate transformation and one of the gravitational potentials (w_μ or W^A_α), then we can also calculate another. In other words, if the gauges of both w_μ and W^A_α are given explicitly, then $\xi(X^\alpha)$ in the coordinate transformation can be figured out straightaway. Or, if we first fix $\xi(X^\alpha)$ and the gauge of w_μ (or W^A_α), then the gauge of W^A_α (or w_μ) is entirely and unambiguously fixed.

In Eq. (5), $z^\mu(T)$ is the coordinate of the origin of the local coordinate system of body A (usually taken as the barycenter of A) in the global coordinate system. It is determined by the translational equations of motion of body A. $\xi^i = c^{-2}R^i_a[A^a\mathbf{X}^2/2 - X^a(\mathbf{A}\cdot\mathbf{X})]$, where A^a is an "acceleration three-vector", $A^a = R^i_a a^i + O(2)$ and $a^i \equiv dv^i/dt \equiv d^2z^i/dt^2$ (see Eq. (2.33) in Damour et al. 1991 for the precise definition of A^a). As for $e^\mu_a(T)$, in order to obtain their explicit expressions, here we have to employ a useful theorem (Damour et al. 1991) as follows

Theorem 1: *In the framework of the DSX theory, W^{+A}_α is the self-gravitational potential generated by body A in its local coordinate system, w^A_μ is its related gravitational potential in the global coordinate system. If W^{+A}_α is under a harmonic gauge, then*

$$w^A_\mu = \mathcal{A}^A_{\mu\alpha}(T)W^{+A}_\alpha + O(4,2) \tag{11}$$

where w^A_μ is also under a harmonic gauge. The converse is also true.

To combine Eq. (6) and Eq. (11), we can easily come to a conclusion (Eq. (4.55) in Damour et al. 1991), that is

$$\overline{w}^A_\mu = \mathcal{A}^A_{\mu\alpha}(T)\overline{W}^A_\alpha(X) + \mathcal{B}^A_\mu(X) + O(4,2). \tag{12}$$

This expression is similar to Eq. (6) in form, and it can replace Eq. (6) to determine $e_a^\mu(T)$ in the coordinate transformation (5). The result (Eq. (5.21) in Damour et al. 1991) is

$$\begin{aligned}
e_0^0 \equiv\ & dz^0/cdT = 1 + \tfrac{1}{c^2}(\tfrac{1}{2}v^2 + \overline{w}) \\
& + \tfrac{1}{c^4}[\tfrac{3}{8}v^4 + \tfrac{1}{2}\overline{w}^2 + \tfrac{5}{2}\overline{w}v^2 - 4\overline{w}_i v^i] + O(6) \\
e_0^i \equiv\ & \tfrac{1}{c}e_0^0 v^i \\
e_a^0 =\ & \tfrac{1}{c}R_a^i\{v^i[1 + \tfrac{1}{c^2}(\tfrac{1}{2}v^2 + 3\overline{w})] - \tfrac{4}{c^2}\overline{w}_i\} + O(5) \\
e_a^i =\ & (1 - \tfrac{1}{c^2}\overline{w})(\delta_{ij} + \tfrac{1}{2c^2}v^i v^j)R_a^j + O(4)
\end{aligned} \tag{13}$$

where we have adopted the so-called "weak effacement condition", $\overline{W}_\alpha^A(T,\mathbf{0}) = 0,\ \forall T$. Besides, we usually choose $R_a^i(T) = \delta_{ia}$, which means that the spatial coordinate axes of both the local and the global coordinate system have the same direction at the origin of the local system, and the local coordinate system is a kinematically non-rotating system other than a quasi-inertial one.

Thus, the coordinate transformation (5) can be entirely determined and explicitly expressed using Eq. (13) except for a gauge function ξ. The determination of ξ directly affects the gauge of the local coordinate system; it will be discussed in the next section.

3. Choice of gauge

According to what is stated above, to fix the gauge of a coordinate system is in fact to fix the gauge function of the related gravitational potential. Note that the field equations are linear. The gravitational potential can be divided into a self-part and an external part. The external gravitational potential can be further split into a sum of terms generated by all the other bodies and an inertial term that vanishes in the global coordinate system as shown in Eqs. (7) and (8). Therefore the gravitational potential has N parts in the global system, and $N+1$ parts in a local system. Every part in a gravitational potential may have a gauge function of its own. Evidently, the gauge function of the total gravitational potential (w_μ or W_α^A) is the sum of the gauge functions of all its parts. It fixes the time coordinates of the related coordinate system within the precision of $O(4)$. To fix the gauge of a gravitational potential needs to choose the gauge of each of its parts, but only the sum of these gauge functions is meaningful. Furthermore, only the expressions of w_μ, W_α^A and ξ under the chosen gauges are relevant to application.

In the framework of the DSX theory, the self-gravitational potential W_α^{+A} is generated by body A in its local coordinate system, where A is a body in the N-body system under consideration. The external gravitational potential $\overline{W}_\alpha^A$ can be expressed theoretically in terms of the self-gravitational potentials of the other bodies, W_α^{+B} ($B \neq A$), while the gravitational potential w_μ in the global coordinate system can be also expressed in terms of W_α^{+A} ($\forall A$). Here the self-gravitational potentials obviously play a fundamental role. Therefore, as a general rule, the choice of gauge should begin with them.

The problem on gauge might be quite complicated. In preparation for it, let us introduce the choice of harmonic gauge that may be the favourite of many astronomers and physicists.

3.1. Harmonic gauge

In the DSX theory, any two gauges are physically equivalent. But there exist some preferable gauges that formally simplify the theory. Harmonic gauge is exactly one of them.

Take the gauge of every self-gravitational potential W_α^{+A} $(A = 1,\ 2,\ \cdots,\ N)$ of N local coordinate systems as harmonic, i.e. their expansions are

$$\begin{aligned}
W^{+A}(T,\mathbf{X}) =\ & G\sum_{l\geq 0}\tfrac{(-)^l}{l!}[M_L^A - \tfrac{1}{2(2l-1)c^2}\ddot{M}_L^A R^2] \\
& \times\partial_L R^{-1} - c^{-2}\partial_T\lambda^{+A} + O(4) \\
W_a^{+A}(T,\mathbf{X}) =\ & G\sum_{l\geq 0}\tfrac{(-)^l}{(l+1)!}[\dot{M}_{aL}^A + l\epsilon_{ab<a_l}S_{L-1>b}^A] \\
& \times\partial_L R^{-1} + \tfrac{1}{4}\partial_a\lambda^{+A} + O(2)
\end{aligned} \tag{14}$$

where $\dot{M}_L^A \equiv dM_L^A/dT$, $\ddot{M}_L^A \equiv d^2M_L^A/dT^2$, and λ^{+A} is a definite function that satisfies the equation $\triangle\lambda^{+A} = O(2)$. Here we see that there are infinite harmonic gauges, so one of them must be explicitly indicated, e.g. "skeletonized-body" harmonic gauge: $\lambda^{+A} = 0$. Then, according to Eq. (11), we can determine w_μ^A ($\forall\ A$) and hence $w_\mu = \sum_{A=1}^N w_\mu^A$. Obviously w_μ and hence the global coordinate system $(ct,\ x^i)$ are fixed under a definite harmonic gauge. The external potential of a local coordinate system consists of two components, i.e. $\overline{W}_\alpha^A = \sum_{B\neq A} W_\alpha^{B/A} + W_\alpha''^A$, here $W_\alpha^{B/A}$ is the contribution from body B to A. It reads

$$W_\alpha^{B/A} = \mathcal{A}_{\alpha\mu}^{(-1)A}(T_A)\mathcal{A}_{\mu\beta}^B(T_B)W_\beta^{+B}(X_B) + O(4,2) \tag{15}$$

where T_A and T_B are the time coordinates of the local coordinate systems of bodies A and B respectively. $T_A = T_B + O(2)$. Clearly, $W_\alpha^{B/A}$ and hence $\sum_{B\neq A} W_\alpha^{B/A}$ are also harmonic. The remainder is $W_\alpha''^A$. It should be harmonic, but its gauge is related with the gauge function ξ in Eq. (5). Let's derive its explicit expressions from Eqs. (10) and (5).

From Eqs. (5) and (13), we have

Lemma 1: *In the framework of the DSX theory, if the "weak effacement condition" is employed, then the Jacobi matrix* $A_\alpha^\mu = \partial x^\mu/\partial X^\alpha$ *has the following expressions*

$$\begin{aligned}
A_0^0 =\ & 1 + \tfrac{1}{c^2}(\tfrac{1}{2}v^2 + \overline{w} + a^i R_a^i X^a) + \tfrac{1}{c^4}\{\tfrac{3}{8}v^4 \\
& + \tfrac{1}{2}\overline{w}^2 + \tfrac{5}{2}\overline{w}v^2 - 4\overline{w}_i v^i + X^a[a^i(v^2 + 4\overline{w}) \\
& + v^i(\mathbf{v}\cdot\mathbf{a} + 3\dot{\overline{w}}) - 4\dot{\overline{w}}^i]R_a^i + c^2v^2\dot{R}_a^i X^a \\
& + \partial_T\xi\}
\end{aligned} \tag{16}$$

$$\begin{aligned}
A_a^0 =\ & \tfrac{1}{c}R_a^i\{v^i[1 + \tfrac{1}{c^2}(\tfrac{1}{2}v^2 + 3\overline{w})] - \tfrac{4}{c^2}\overline{w}_i\} \\
& + \tfrac{1}{c^3}\partial_a\xi
\end{aligned} \tag{17}$$

$$\begin{aligned}
A_0^i =\ & \tfrac{1}{c}v^i[1 + \tfrac{1}{c^2}(\tfrac{1}{2}v^2 + \overline{w})] \\
& + \tfrac{1}{c^3}\{\left[c^2\dot{R}_a^i + \tfrac{1}{2}\left(a^iV^a + v^iA^a\right) - \dot{\overline{w}}\dot{R}_a^i\right]X^a \\
& + R_a^i\left[\tfrac{1}{2}\dot{A}^a\mathbf{X}^2 - X^a\left(\dot{\mathbf{A}}\cdot\mathbf{X}\right)\right]\}
\end{aligned} \tag{18}$$

$$\begin{aligned}
A_a^i =\ & (1 - \tfrac{1}{c^2}\overline{w})(\delta_{ij} + \tfrac{1}{2c^2}v^iv^j)R_a^j \\
& + \tfrac{1}{c^2}R_b^i\left[A^bX^a - A^aX^b - \delta_{ab}(\mathbf{A}\cdot\mathbf{X})\right]
\end{aligned} \tag{19}$$

where $\dot{R}^i_a \equiv dR^i_a/dT$, $\dot{\overline{w}}_\mu \equiv \partial\overline{w}_\mu(t,\mathbf{x})/\partial t|_{\mathbf{x}=\mathbf{z}}$, $\dot{A}^a \equiv dA^a/dT$, *and the expressions of* $e^\mu_\alpha(T)$ *and* $\xi^i(X)$ *have been used.*

Then, $\mathcal{B}^A_\mu$ can be calculated from Eq. (10) and Lemma 1. The result is

Lemma 2: *In the framework of the DSX theory, if* w_μ *and* W^A_α *have the relation expressed by Eq. (6), then* $\mathcal{B}^A_\mu(X)$ *can be written as*

$$\begin{aligned}\mathcal{B}^A_0(X) = & \left(\overline{w} + a^i R^i_a X^a\right) \\ & +\tfrac{1}{c^2}\{(\mathbf{V}\cdot\mathbf{X})\left(\mathbf{A}\cdot\mathbf{V} + 3\dot{\overline{w}}\right) \\ & + c^2\dot{R}^i_a v^i X^a - 4R^i_a X^a \dot{\overline{w}}_i - \tfrac{1}{2}\left(\mathbf{A}\cdot\mathbf{X}\right)^2 \\ & + \tfrac{3}{2}\left(\mathbf{A}\cdot\mathbf{X}\right)\left(v^2 + 2\overline{w}\right) + \partial_t\xi\}\end{aligned} \tag{20}$$

$$\begin{aligned}\mathcal{B}^A_i(X) = & \tfrac{1}{4}\{4\overline{w}_i + \tfrac{5}{2}v^i\left(\mathbf{A}\cdot\mathbf{X}\right) + [c^2\dot{R}^i_a \\ & + \left(\mathbf{A}\cdot\mathbf{V} - \dot{\overline{w}}\right)R^i_a]X^a - \tfrac{1}{2}a^i\left(\mathbf{V}\cdot\mathbf{X}\right) \\ & + R^i_a[\tfrac{1}{2}\dot{A}^a\mathbf{X}^2 - X^a(\dot{\mathbf{A}}\cdot\mathbf{X})] - \partial_i\xi\}.\end{aligned} \tag{21}$$

With the help of Eqs. (8), (12) and (15), W''^A_α can be expressed as

$$W''^A_\alpha\left(X_A\right) = -\mathcal{A}^{A(-1)}_{\alpha\mu}\left(T_A\right)\mathcal{B}^A_\mu\left(X_A\right) + O\left(4,2\right) \tag{22}$$

then we immediately have

Lemma 3: *In the framework of the DSX theory, the inertial gravitational potential* $W''^A_\alpha(X)$ *is expressed as*

$$\begin{aligned}W''^A(X) = & -(\overline{w} + R^i_a a^i X^a)(1 + \tfrac{2}{c^2}V^2) + \tfrac{1}{c^2}\{(\mathbf{A}\cdot\mathbf{X}) \\ & \times(v^2 - 3\overline{w}) + \tfrac{1}{2}[(\mathbf{A}\cdot\mathbf{X})^2 + (\dot{\mathbf{A}}\cdot\mathbf{V})\mathbf{X}^2] \\ & -(\mathbf{V}\cdot\mathbf{X})(4\dot{\overline{w}} + \tfrac{1}{2}\mathbf{A}\cdot\mathbf{V} + \dot{\mathbf{A}}\cdot\mathbf{X}) \\ & +4(\overline{w}_i v^i + \dot{\overline{w}}_i R^i_a X^a) - \partial_T\xi\} + O(4)\end{aligned} \tag{23}$$

$$\begin{aligned}W''^A_a(X) = & \overline{w}V^a - R^i_a\overline{w}_i + \tfrac{1}{4}[\tfrac{3}{2}(\mathbf{A}\cdot\mathbf{X})V^a \\ & + \tfrac{1}{2}(\mathbf{V}\cdot\mathbf{X})A^a + (\dot{\overline{w}} + \dot{\mathbf{A}}\cdot\mathbf{X} - \mathbf{A}\cdot\mathbf{V})X^a \\ & - \tfrac{1}{2}\dot{A}^a\mathbf{X}^2 - c^2 R^i_a\dot{R}^i_b X^b + \partial_a\xi] + O(2).\end{aligned} \tag{24}$$

On the right-hand side of these expressions, the index A that identifies the body is omitted for simplicity. Note that $W''^A_\alpha\left(X_A\right)$ automatically satisfies the "weak effacement condition" (ξ should start with $\mathbf{X}^2$, see Eq. (26)).

We have taken the local coordinate system of body A ($\forall\, A$) to be a harmonic one. Considering that the gauges of W^{+A}_α and $\sum_{B\neq A} W^{B/A}_\alpha$ are harmonic, W''^A_α must be harmonic too. In other words, it will satisfy the equations $\Box_X W''^A_\alpha\left(X_A\right) = O(4,2)$. A straightforward computation results in

Lemma 4: *If* $W''^A_\alpha\left(X_A\right)$ *can be expressed by Lemma 3 and be harmonic, then the gauge function* ξ *in it should satisfy*

$$\begin{aligned}\partial_T\partial_b\partial_b\xi\left(X_A\right) = & \quad \ddot{\overline{w}} + \ddot{\mathbf{A}}\cdot\mathbf{X} + \mathbf{A}^2 + \dot{\mathbf{A}}\cdot\mathbf{V} + O(2). \\ \partial_a\partial_b\partial_b\xi\left(X_A\right) = & \quad \dot{A}^a + O(2)\end{aligned} \tag{25}$$

A simple but useful expression of ξ^A *is*

$$\xi^A\left(X_A\right) = \frac{1}{10}(\dot{\mathbf{A}}\cdot\mathbf{X})\mathbf{X}^2 + \frac{1}{6}\left[\mathbf{A}\cdot\mathbf{V} + \dot{\overline{w}}\right]\mathbf{X}^2. \tag{26}$$

The solution of Eq. (25), i.e. the expression of ξ^A is obviously non-unique. This means that the harmonic gauge condition can't uniquely determine the 1PN gravitational potential. We can choose an expression for the gauge function ξ^A as simple as possible, e.g., the one shown by Eq. (26). In fact, it is the simplest one among all possible expressions of ξ^A. With it we have

Lemma 5: *In the coordinate transformation (5), if the gauge function* ξ ($\equiv c^3\xi^0$) *is expressed by Eq. (26), then the corresponding harmonic* $W''^A_\alpha(X)$ *is*

$$\begin{aligned}W''^A(X) = & -(\overline{w} + R^i_a a^i X^a)(1 + \tfrac{2}{c^2}V^2) + \tfrac{1}{c^2}\{(\mathbf{A}\cdot\mathbf{X}) \\ & \times(v^2 - 3\overline{w}) + [\tfrac{1}{2}(\mathbf{A}\cdot\mathbf{X})^2 + \tfrac{1}{3}(\dot{\mathbf{A}}\cdot\mathbf{V})\mathbf{X}^2] \\ & -(\mathbf{V}\cdot\mathbf{X})(4\dot{\overline{w}} + \tfrac{1}{2}\mathbf{A}\cdot\mathbf{V} + \dot{\mathbf{A}}\cdot\mathbf{X}) \\ & -[\tfrac{1}{10}\ddot{\mathbf{A}}\cdot\mathbf{X} + \tfrac{1}{6}(\mathbf{A}^2 + \ddot{\overline{w}})]\mathbf{X}^2 \\ & +4(\overline{w}_i v^i + \dot{\overline{w}}_i R^i_a X^a)\} + O(4)\end{aligned} \tag{27}$$

$$\begin{aligned}W''^A_a(X) = & \overline{w}V^a - R^i_a\overline{w}_i + \tfrac{1}{4}[\tfrac{3}{2}(\mathbf{A}\cdot\mathbf{X})V^a - \tfrac{2}{5}\dot{A}^a\mathbf{X}^2 \\ & +(\tfrac{4}{3}\dot{\overline{w}} + \tfrac{6}{5}\dot{\mathbf{A}}\cdot\mathbf{X} - \tfrac{2}{3}\mathbf{A}\cdot\mathbf{V})X^a \\ & +\tfrac{1}{2}(\mathbf{V}\cdot\mathbf{X})A^a - c^2 R^i_a\dot{R}^i_b X^b] + O(2).\end{aligned} \tag{28}$$

With this harmonic $W''^A_\alpha(X)$ we have the related harmonic $\overline{W}^A_\alpha$ and hence W^A_α is under a certain harmonic gauge. At the same time, the related coordinate transformation is completely fixed too.

There are two kinds of harmonic gauges adopted in references. The case that

$$\lambda^{+A} = -\Lambda^A = -4G\sum_{l=0}^{\infty}\frac{(-1)^l}{(l+1)!}\,\frac{2l+1}{2l+3}\,P^A_L\,\partial_L|X|^{-1}$$

is called standard harmonic gauge. In this gauge it is inevitable that the "bad moments" P^A_L will appear in W^{+A}_α and hence in W^A_α as well as in w_μ. Its advantage is the existence of a simple relation between the potentials (scalar and vectorial) and the densities (energy and energy current) (Soffel et al. 2000). The case where $\lambda^{+A} = 0$ is called skeletonized harmonic gauge, which involves no bad moments.

By choosing a harmonic gauge and with the expressions of ξ^A and W^{+A}_α in every local coordinate system ($A = 1, 2, \cdots, N$), we have successfully fixed all the coordinate systems and the coordinate transformations between them within the precision stated above, where the time coordinates t and T have reached the precision of 2PN level (less than 10^{-16}). Therefore, it can certainly meet the needs of theory and practice in astronomy at present and in the near future.

3.2. A general procedure for fixing the gauge

Now we may ask the following question: what happens if the gauge of the self-gravitational potential W^{+A}_α ($A = 1, 2, \cdots, N$) is not harmonic? We now present a general way to fix the gauge to meet any kind of requirement.

i) *Fixing the gauges of self-gravitational potentials*

According to Damour et al. (1991), the expansions of self-gravitational potential $W^{+A}_\alpha\left(X_A\right)$ ($A = 1, 2, \cdots, N$) have a general form Eq. (14) under arbitrary gauge, in which the gauge function $\lambda^{+A}\left(X_A\right)$ is an arbitrary differentiable function and can be chosen by personal preference. If $\triangle\lambda^{+A} \neq O(2)$, W^{+A}_α

is not harmonic. $\lambda^{+A}(X_A)$ completely determines the gauge of $W_\alpha^{+A}(X_A)$.

ii) *Calculating the gravitational potentials w_μ^A and $W_\alpha^{B/A}$*

How do we calculate the expressions of w_μ^A and hence w_μ? Apparently, we need a formula like Eq. (11), which can transform W_α^{+A} into w_μ^A. For this purpose, we introduce a theorem and its corollary as follows

Theorem 2: *If $G_\alpha^A(\lambda)$ is a gauge term generated by a gauge function $\lambda(X_A)$, i.e.*

$$G_0^A(\lambda) = -\frac{1}{c^2}\partial_T\lambda, \quad G_a^A(\lambda) = \frac{1}{4}\partial_a\lambda$$

then

$$g_\mu(\lambda) = \mathcal{A}_{\mu\alpha}^A(T_A)G_\alpha^A(\lambda)$$

is certainly a gauge term generated by the same gauge function $\lambda(X_A(x))$, i.e.

$$g_0(\lambda) = -\frac{1}{c^2}\partial_t\lambda, \quad g_i(\lambda) = \frac{1}{4}\partial_i\lambda.$$

The converse is also true.

It is very easy to prove this theorem, since

$$\begin{aligned}\partial_t &= \partial_T - V^a\partial_a, \quad \partial_i = R_a^i\partial_a,\\ \partial_T &= \partial_t + v^i\partial_i, \quad \partial_a = R_a^i\partial_i,\end{aligned}$$

which immediately result in

$$\begin{pmatrix} 1+\frac{2}{c^2}v_A^2 & \frac{4}{c^2}V_A^a \\ v_A^i & R_a^i \end{pmatrix}\begin{pmatrix} -\frac{1}{c^2}\partial_T\lambda \\ \frac{1}{4}\partial_a\lambda \end{pmatrix} = \begin{pmatrix} -\frac{1}{c^2}\partial_t\lambda \\ \frac{1}{4}\partial_i\lambda \end{pmatrix}.$$

That is to say, the matrix $\mathcal{A}_{\mu\alpha}^A(T_A)$ can transform one gauge term in the local system of body A into another in the global system, and both come from the same gauge function. It is exactly what we want to prove.

Combining Theorem 1 in Sect. 2 with Theorem 2, we have

Corollary: *In the framework of the DSX theory, let W_α^{+A} be the self-gravitational potential generated by body A in its local coordinate system, w_μ^A be its related gravitational potential in the global coordinate system. If W_α^{+A} is under a certain gauge, and*

$$w_\mu^A = \mathcal{A}_{\mu\alpha}^A(T)W_\alpha^{+A} + O(4,2) \tag{29}$$

then w_μ^A and W_α^{+A} have the same gauge function. The converse is also true.

Now one can calculate w_μ^A and $W_\alpha^{B/A}$ with the help of Eqs. (29) and (15), respectively.

iii) *Choosing ξ^A, w_μ and W_α^A*

Let $w_\mu = w_\mu^A + \overline{w}_\mu^A$. Then the gauge function of w_μ is $\lambda^+(x)$ where $\lambda^+(x) = \sum_{A=1}^N \lambda^{+A}(X_A(x))$. Sect. 3.1 has pointed out that $\lambda^+(x) = 0$ indicates the skeletonized harmonic gauge. Let $W_\alpha^A = W_\alpha^{+A} + \overline{W}_\alpha^A = W_\alpha^{+A} + \sum_{B\neq A} W_\alpha^{B/A} + W_\alpha^{\prime\prime A}$, where the gauge function of $W_\alpha^{+A} + \sum_{B\neq A} W_\alpha^{B/A}$ is also $\lambda^+ = \lambda^+(x(X_A))$. Then we can choose ξ^A arbitrarily under the restriction that ξ^A should start with square terms of X_A^a, and the expression of $W_\alpha^{\prime\prime A}$ can be written out by Eqs. (23) and (24). A simple way is to choose $\xi^A = 0$.

What is the gauge function of the local system of body A? Actually we do not need to know it, because it does not explicitly appear in the expressions we need to use. But it can be provided if necessary. $W_\alpha^{\prime\prime A}$ includes the terms that are related to $\overline{w}_A$ and $\overline{w}_A^i$, which contain a gauge term generated by $\overline{\lambda}_A = \sum_{B\neq A}\lambda^{+B}$. Those terms in $W_\alpha^{\prime\prime A}$ can change with $\overline{\lambda}_A$ so as not to violate the "weak effacement condition". After some calculation, one can obtain the gauge function of the local system of body A, that is

$$\begin{aligned}&\lambda^+ - (f_0(T_A) + f_1^a(T_A)X_A^a)\\ &-\{\frac{1}{10}(\dot{\mathbf{A}}_A\cdot\mathbf{X}_A)\mathbf{X}_A^2 + \frac{1}{6}\left[\mathbf{A}_A\cdot\mathbf{V}_A + \dot{\overline{w}}\right]\mathbf{X}_A^2\}\end{aligned} \tag{30}$$

where $f_0(T_A)$ and $f_1^a(T_A)$ are the expansion coefficients of $\overline{\lambda}_A$:

$$\overline{\lambda}_A(X_A) = f_0(T_A) + f_1^a(T_A)X_A^a + f_2^{ab}(T_A)X_A^aX_A^b + \cdots$$

Following this procedure to fix the gauges, one will see that the gauges of the global system and the local system are probably no longer harmonic, but the expressions of all the gravitational potentials and the related coordinate transformations are fixed completely.

3.3. A special choice of gauge

As an example to show how to fix the gauge in practice, in this subsection we demonstrate a special choice of gauge. Especially we will show that one can always make the gauges of both the global and local coordinates harmonic, even though the self-gravitational potentials are not harmonic.

Tao & Huang (1998b) showed that there exists a certain standard PN gauge, under which the expressions of the self-gravitational potential $W_\alpha^+(X)$ read

$$\begin{aligned}W^+(T,\mathbf{X}) = \quad & G\sum_{l\geq 0}\tfrac{(-)^l}{l!}M_L\partial_L R^{-1} + O(4)\\ W_a^+(T,\mathbf{X}) = \quad & G\sum_{l\geq 0}\tfrac{(-)^l}{l!}[\tfrac{1}{l+1}(\tfrac{7l+11}{4(2l+3)}\dot{M}_{aL}\\ & +l\epsilon_{ab<a_l}S_{L-1>b}) + \tfrac{l}{8(2l-1)}\\ & \times\delta_{a<a_l}\dot{M}_{L-1>}R^2]\partial_L R^{-1} + O(2).\end{aligned} \tag{31}$$

It shows that $W_\alpha^+(X)$ can be directly expressed in terms of the multipole moments of the related body, M_L, S_L and $\dot{M}_L$ (i.e. dM_L/dT). They argued that the multipole moments M_L, S_L and $\dot{M}_L$ probably change with time and are not suitable observables in astronomical practice. In classical mechanics, we know that the mass multipole moments of a rigid body are constants only in the co-rotating reference system of this body. The astronomical observables should be quantities that are time-independent or at least slowly-changing with time, such as $m_{L'}$, $s_{L'}$ and $m'_{L'}$, or the spherical harmonic coefficients C_{lm}, S_{lm}, C_{lm}^i and S_{lm}^i (Tao & Huang 1998b). Taking Eq. (31) and after a certain operation, the gravitational field parameters of a celestial body can be well defined in terms of several sets of spherical harmonic coefficients, which accord with the demand of astronomical tradition and convention. One advantage of Eq. (31) is that it does not contain the "bad moments". Another advantage lies in the fact that the scalar self-gravitational potential W^+ has the simplest expansion in form, and it is related to the observables C_{lm} and S_{lm} in the same way as in classical mechanics. There are two possible ways to fix the gauge totally.

3.3.1. Method one ($\xi^A = 0$)

Now we will employ the method presented in the previous subsection, and take $W_\alpha^{+A}(X)$ under the standard post-Newtonian gauge expressed by Eq. (31), then its gauge function is

$$\lambda^{+A}(X) = \frac{1}{2}G\sum_{l\geq 0}\frac{(-)^l}{l!}\dot{M}_L^A\partial_L R. \tag{32}$$

We will use Eqs. (29), (15) and (13) to calculate w_μ^A, $W_\alpha^{B/A}$, and $e_a^{A\mu}$, respectively. According to the rules in Sect. 2, we could take Eqs. (23) and (24) as the expressions of $W_\alpha^{\prime\prime A}$, and simply let $\xi^A = 0$. w_μ and W_α^A are still expressed by Eqs. (7) and (8). Thus, all the gravitational potentials in $N+1$ coordinate systems have been completely determined, so fixing all the gauges of both the global and local coordinate systems and hence all the related coordinate transformations.

In order to compare this result with that of the harmonic gauge in Sect. 3.1, we should indicate the gauge functions of w_μ and W_α^A ($A = 1,\ 2,\ \cdots,\ N$). One easily sees that the gauge function of w_μ is $\lambda^+(x) = \sum_{A=1}^N \lambda^{+A}(X_A(x))$, and the gauge function of W_α^A is expressed by (30).

Clearly, this choice leads to a simple result. All the gravitational potentials have fully been determined. Their forms are simple, so is the coordinate transformation. For example, no so-called "bad moment" appears in the expressions of w_μ, W_α^A or in the coordinate transformations. Of course, the gauges of the global coordinate system and the local coordinate systems are not any more harmonic in this case.

3.3.2. Method two (harmonic gauge)

One may wonder if it is possible for us to take all the gauges of both the global and local systems as harmonic even though the gauge of the self-gravitational potential is the standard PN as expressed by Eqs. (31) and (32). Yes, it is indeed. The method and steps are presented as follows: 1) We calculate w_μ, $W_\alpha^{B/A}$ and hence $\sum_{B\neq A} W_\alpha^{B/A}$ according to the procedure described in the preceding subsection, then let $W_\alpha^{\prime\prime A}$ be expressed by Eqs. (27) and (28), i.e. it is harmonic. 2) Now neither w_μ nor W_α^A is harmonic. This problem can be solved with ease, because we know a gauge transformation can change the gauge of gravitational potential. Therefore, we apply gauge transformations on w_μ and W_α^A respectively as follows

$$w_\mu \ \to\ \tilde{w}_\mu = w_\mu - g_\mu\left(\lambda^+\right) \tag{33}$$

$$W_\alpha^A \ \to\ \tilde{W}_\alpha^A = W_\alpha^A - G_\alpha^A\left(\lambda^+\right). \tag{34}$$

It is easy to prove that both $\tilde{w}_\mu$ and $\tilde{W}_\alpha^A$ are harmonic. The coordinate systems undergo a related change: $x^\mu \to \tilde{x}^\mu$, $X_A^\alpha \to \tilde{X}_A^\alpha$, which are

$$\tilde{t} \ =\ t + \lambda^+/c^4, \quad \tilde{x}^i = x^i \quad \text{and}$$
$$\tilde{T} \ =\ T + \lambda^+/c^4, \quad \tilde{X}^a = X^a$$

The related coordinate transformation remains unchanged, i.e. $\xi \to \tilde{\xi} = \xi$. 3) For simplicity, we again use w_μ, W_α^A, x^μ, X_A^α, ξ, $\cdots$ to replace respectively $\tilde{w}_\mu$, $\tilde{W}_\alpha^A$, $\tilde{x}^\mu$, $\tilde{X}_A^\alpha$, $\tilde{\xi}$, $\cdots$, and the last result becomes

$$\begin{aligned} &w_\mu = \textstyle\sum_{A=1}^N w_\mu^A - g_\mu\left(\lambda^+\right) \\ &W_\alpha^A = W_\alpha^{+A} + \textstyle\sum_{B\neq A} W_\alpha^{B/A} + W_\alpha^{\prime\prime A} - G_\alpha^A\left(\lambda^+\right) \\ &\xi^A\left(X_A\right) = \tfrac{1}{10}(\dot{\mathbf{A}}\cdot\mathbf{X})\mathbf{X}^2 + \tfrac{1}{6}\left[\mathbf{A}\cdot\mathbf{V}+\dot{w}\right]\mathbf{X}^2. \end{aligned} \tag{35}$$

Note that no unobservable quantities will appear in the gauge terms $g_\mu\left(\lambda^+\right)$ and $G_\alpha^A\left(\lambda^+\right)$ as long as we adopt Eqs. (31) and (32). Harmonic w_μ and W_α^A are got at the price of introducing $g_\mu\left(\lambda^+\right)$ and $G_\alpha^A\left(\lambda^+\right)$, which naturally make the expressions of w_μ and W_α^A more complicated. This is a disadvantage.

4. Summary and discussion

The preceding section displays methods for fixing the gauge. In this section, we give a brief summary and some discussion.

The choice of gauge has a common procedure. First, we start by choosing the self-gravitational potentials W_α^{+A} ($A = 1,\ 2,\ \cdots,\ N$) with a definite gauge. By so doing, w_μ^A and $W_\alpha^{B/A}$ (and hence w_μ and $\sum_{B\neq A} W_\alpha^{B/A}$) as well as e_α^μ are all determined. Then, we choose the gauge of $W_\alpha^{\prime\prime A}$ and hence determine the gauge function ξ in Eq. (5), or else choose ξ and hence determine $W_\alpha^{\prime\prime A}$ (see Eqs. (23) and (24)). Thus the related coordinate transformation is also fixed.

As to how to choose the gauge of the self-gravitation potential, W_α^{+A}, every choice has its advantages and disadvantages. We are not going to recommend one but just to give a discussion on several possible choices.

1) If W_α^{+A} is chosen as the skeletonized harmonic, i.e. $\lambda^{+A} = 0$, according to Eqs. (11) and (15), w_μ^A and $W_\alpha^{B/A}$ are also harmonic. Moreover, we can choose ξ^A according to Eq. (26) to make both w_μ and W_α^A harmonic. There are no so-called "bad moments" in all the expressions.

2) If W_α^{+A} is chosen to be the standard harmonic, i.e. $\lambda^{+A} = -\Lambda^A = -4G\sum_{l=0}^\infty \frac{(-1)^l}{(l+1)!}\frac{2l+1}{2l+3}P_L^A\,\partial_L|X|^{-1}$, it will bring the so called bad moments P_L^A into the expression of the local potentials. Its advantage is that there exists a simple relation between the potentials and the energy current densities, which would be useful when the Earth interior is concerned.

3) If W_α^{+A} is chosen as the standard post-Newtonian gauge suggested by Tao & Huang (1998b), i.e. $\lambda^{+A}(X) = \frac{1}{2}G\sum\limits_{l\geq 0}\frac{(-)^l}{l!}\dot{M}_L^A\partial_L R$, then the main advantage is to ensure that the scalar self-gravitational potential W^{+A} has a simple form and that it can be legally expanded by spherical harmonic coefficients. Choosing all the $\xi^A = 0$ in this case will simplify all

the expressions. The main disadvantage is that neither the global nor the local coordinate systems are harmonic. In Sect. 3.3 we have shown that one can make all the systems harmonic by adding some gauge terms into the potentials. In this case no bad moments would be brought in.

Harmonic gauges have been widely applied in references. Up to our knowledge its main advantage is that it can greatly simplify the field equation of relativity in the case of linear approximation. A great advantage of the DSX formalism is a simple and linearized field equation in the 1PN N-body problem and leaves the gauge open in a wider set. One more point we would like to make is that both harmonic and standard post-Newtonian gauges can be extended to higher order approximations without difficulty. For the harmonic gauge, the condition is

$$g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0. \tag{36}$$

For the standard PN gauge (called ADM gauge in some literatures), the condition is

$$g^{ij}\Gamma^{0}_{ij} = 0, \qquad g^{\mu\nu}\Gamma^{k}_{\mu\nu} = 0. \tag{37}$$

It can be applied to higher PN orders, too. At present, it is difficult to discuss this problem because we have yet no fully 2PN theory similar to the DSX theory. In such a 2PN theory, what are definitions of σ, σ^i, Σ_A and Σ^a_A? Do the 2PN multipole moments exist or not? What are the 2PN observable quantities? What are the 2PN gravitational potentials? Can they be expanded in terms of the multipole moments? We are not able to answer these questions yet.

There is no ideal gauge choice that has all the advantages and no disadvantage. The choice of gauge is quite a personal preference in the solar system or in other N-body systems.

From the discussions of this paper, we can obtain some conclusions as follows:

a) When the precision of time scales reaches $10^{-16} - 10^{-18}$ in practical measurements or theoretical models of the solar system, a fully 1PN theory should be employed, where the gauges of all the coordinate systems should also be fixed. The time coordinate transformations will include the 2PN terms, in which the gauge terms play a role.

b) There exist definite methods and steps for fixing the gauges of coordinate systems and determining the related coordinate transformations. The gauges can be arbitrary but should be definite.

c) It is possible to choose all the gauges of both the global and local coordinate systems to be harmonic. The gauges of the self-gravitational potentials of celestial bodies can be different from the total potentials. They can be harmonic or standard post-Newtonian.

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