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A fast method for distinguishing between ordered and chaotic orbits

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Abstract. We describe a new method of distinguishing between ordered and chaotic orbits, which is much faster than the methods used up to now, namely (1) the distribution of the Poincaré consequents, (2) the Lyapunov characteristic number and (3) the distribution of the rotation angles. This method is based on the distribution of the helicity angles (the angles of small deviations ξ from a given orbit with a fixed direction), or of the twist angles (the differences of successive helicity angles), and the stretching numbers (the logarithms of the ratios of successive deviations $\mid \xi \mid$, also called 'short time Lyapunov characteristic numbers'). We apply this method to 2-D mappings and 4-D mappings, representing Hamiltonian systems of 2 and 3 degrees of freedom respectively.

Key words: chaos – stellar dynamics – Galaxy: kinematics and dynamics

1. Introduction

Three methods are usually applied in order to distinguish between ordered and chaotic orbits in conservative systems of two degrees of freedom.

1) The distribution of the consequents on a Poincaré surface of section. If the consequents lie on a closed invariant curve we have an ordered orbit, while if the points are scattered we have a chaotic orbit. This method is applicable in two degrees of freedom only. Usually a few tens of points are sufficient to give an idea whether an orbit is ordered or chaotic, but sometimes many hundreds or thousands of points are needed in order to make this distinction (see end of Sect. 3).

2) The Lyapunov characteristic number (LCN), that gives the average exponential deviation of two nearby orbits. The LCN is the limit of the quantity

$$\chi = \frac{\ln \mid \frac{\xi}{\xi_o} \mid}{t}, \tag{1}$$

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where ξ_0 and ξ are small deviations from a given orbit at times $t = 0$ and t, when t tends to infinity. The LCN is independent of the Riemannian metric if the phase space is compact (Benettin et al 1976). In practice we solve the variational equations together with the equation of motion, in order to have ξ infinitesimally small, and see whether χ tends to a finite positive number (chaos) or to zero (order) (Contopoulos et al 1978). Usually χ varies considerably before stabilizing and we need at least some 10^6 periods to find a reliable estimate of the Lyapunov characteristic number. But we get some idea whether an orbit is chaotic or ordered after some thousands of periods. This method can be applied for systems of any number of degrees of freedom.

3) The rotation number is the average rotation angle of successive consequents as seen from a fixed central point on the Poincaré surface of section in a system of two degrees of freedom. In the case of an invariant curve, around the fixed point, the rotation number is well defined, and a good estimate is found after some tens, or hundreds of points. However in the case of chaotic orbits the rotation angles vary considerably and no average value can be defined. The lack of convergence is an indication of chaos.

If we calculate the rotation number as a function of the distance from the fixed point, we find a rotation curve (Fig. 1a and b; Contopoulos 1966, 1971). The rotation number is not defined near the unstable periodic orbits and the boundaries of stable islands, where we have chaotic regions (regions between dashed lines in Fig. 1a and b).

There is one more reason why a rotation angle cannot be defined uniquely in certain cases. Namely in an extended chaotic domain one cannot define unambiguously a 'center' from which to measure the rotation angles. Thus even some periodic orbits, that have a finite number of points on a Poincaré surface of section, may have different rotation numbers depending on the assumed center (Contopoulos 1970, Figs. 8b and c).

The above problems have been bypassed by Laskar et al (1990, 1992, 1993), by considering the basic frequencies of an orbit over a fixed interval of time. In the case of an ordered motion one basic frequency corresponds to the rotation number. It has small oscillations in time and a monotonic variation in space. However in the case of a chaotic motion this number

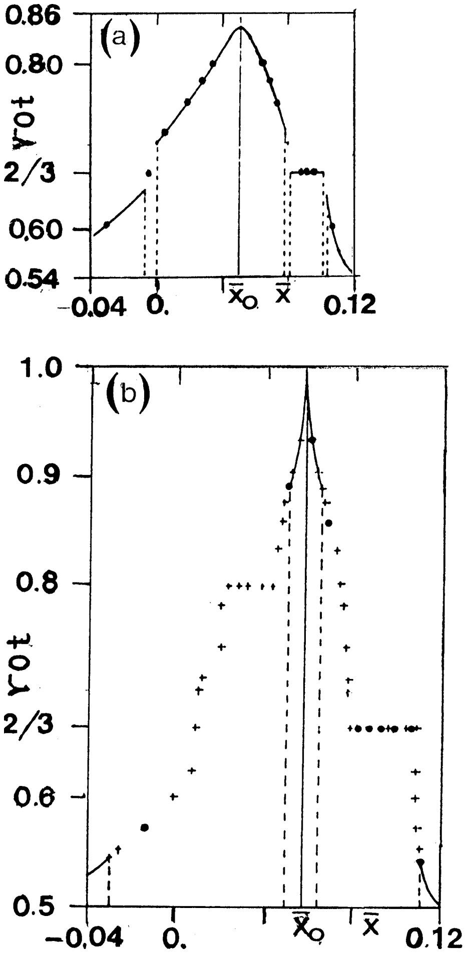

Fig. 1a and b. The rotation curves (rot vs $\bar{x} = A^{1/2}x$) of the Hamiltonian $H = \frac{1}{2}(\dot{x}^2+\dot{y}^2+Ax^2+By^2)-\epsilon xy^2 = 0.00765$ ($A = 1.6$, $B = 0.9$) for **a** $\epsilon = 4$ and **b** $\epsilon = 4.4$. The periodic orbit is at $\bar{x}_0$. The gaps in the rotation curves contain stable (•) and unstable (+) periodic orbits.

has large variations in time along a given orbit, and also large variations between different orbits starting close to each other in space. These variations are used as an indication of chaos. The method of Laskar is much faster than the calculation of the Lyapunov characteristic number.In a particular case (Laskar et al 1992) the chaotic nature of an orbit is established after 10^4 iterations, while the LCN method requires 10^6 iterations. This method can be extended to systems of many degrees of freedom.

We will now introduce a fourth method for distinguishing between order and chaos, which is faster than all the above methods and it can be applied to systems of two or more degrees of freedom. It is the method of helicity angles and stretching numbers.

2. Helicity angles

The "helicity angle" is the angle ϕ formed by the deviation ξ from a given orbit with a given fixed direction, say the x-axis. In the case of a map on a Poincaré surface of section the deviation ξ is on the same surface, but in the full phase space and in systems of more degrees of freedom we define more than one helicity angles.

In a system of two degrees of freedom at every iteration of the map we find one value of the helicity angle ϕ_i on the Poincaré surface of section. As a byproduct we define a "twist angle" $\Delta\phi_i$, which is equal to the difference of two consecutive helicity angles

$$\Delta\phi_i = \phi_{i+1} - \phi_i \tag{2}$$

In the case of ordered orbits the direction of ϕ_i tends to be tangent to the invariant curve. In general the deviation from the tangent decreases very fast and after a number of iterations it is negligible.

The distribution of the helicity angles gives the "spectrum" of the helicity angles. Namely if after N iterations the number of values of ϕ_i between two given values ϕ and $\phi+d\phi$ is $dN(\phi)$, the quantity

$$S(\phi) = \frac{dN(\phi)}{Nd\phi} \tag{3}$$

gives the spectrum. This spectrum is invariant in the same sense as the invariant spectra of stretching numbers (Voglis and Contopoulos, 1994).In the case of chaotic orbits the spectrum is invariant with respect to the initial conditions in the same connected chaotic domain. This is seen in Fig. 2, where we have superimposed the spectra of two orbits in the standard map

$$\begin{aligned} x_{i+1} &= x_i + y_{i+1} \\ & \qquad\qquad (mod 1) \\ y_{i+1} &= y_i + \tfrac{K}{2\pi}\sin 2\pi x_i \end{aligned} \tag{4}$$

for $N = 10^6$ iterations each.

If we take an initial direction ξ_0 different from the previous one it soon becomes essentially tangent to the asymptotic curve after a transient period of about 10 iterations, therefore the spectrum is the same. If we take a different unstable periodic orbit its unstable asymptotic curves cannot intersect the asymptotic curves of the original periodic orbit, thus they are almost parallel to them.

The same happens to any segment of a smooth line in the phase space (x, y); after a few iterations its images become parallel to the asymptotic curve of the main periodic orbit. Therefore if we take any initial point of an orbit in the chaotic domain, and any infinitesimal deviation ξ_0 we find the same spectrum of the helicity angles $S(\phi)$.

In the case of an island of stability the direction ξ tends to the tangent of the invariant curve either in the forward, or in the backward direction. E.g. if the successive consequents move in a clockwise direction the end point of an infinitesimal vector ξ moves also in a clockwise direction but with a slightly different speed. If the initial deviation ξ_0 is inwards from the invariant curve, and the speed increases inwards, the tangent rotates clockwise with slightly larger speed until it becomes tangent. The spectrum does not have a symmetry of 180^o (solid line in Fig. 4). But if we take another ξ_0' outwards (exactly opposite to ξ_0) then this gives successive helicity angles ϕ_i' which

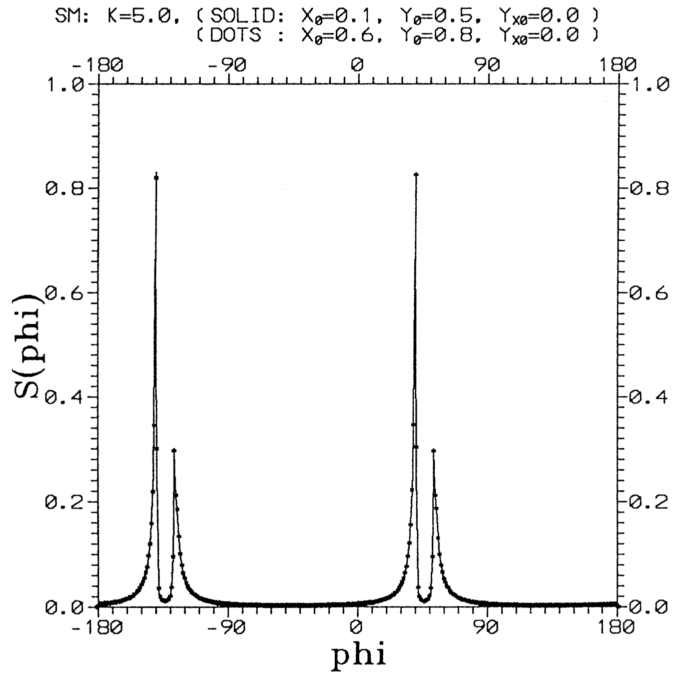

Fig. 2. Two spectra in the chaotic domain of the standard map (4) with K = 5 and initial conditions (x_0 = 0.1, y_0 = 0.5, ξ_{x0} = 1, ξ_{y0} = 0; solid line) and (x_0 = 0.6, y_0 = 0.0, ξ_{x0} = 1, ξ_{y0} = 0; dots). Each orbit is calculated for N = 10^6 iterations.

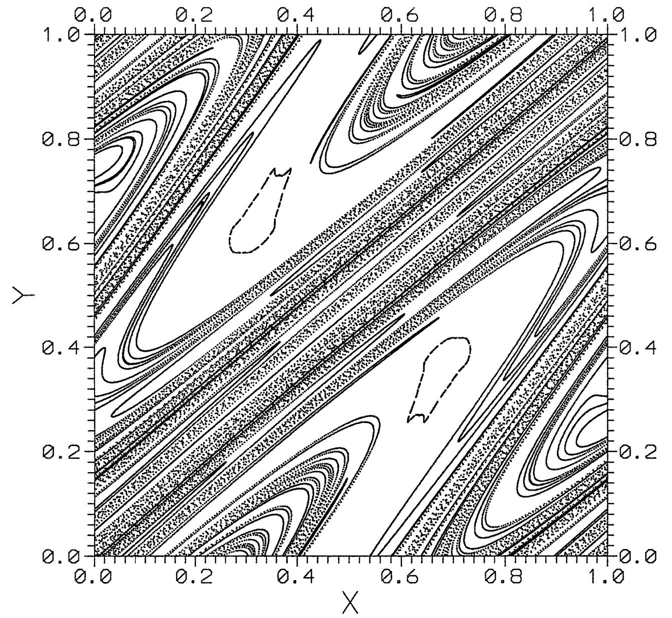

Fig. 3. An unstable asymptotic curve from the basic periodic orbit of period 1 (x_0 = y_0 = 0) in the map (4) with K = 5. This asymptotic curve fills most of the phase space ($0 < x < 1, 0 < y < 1$), but avoids the regions near the two main islands of stability (marked by dashed lines).

are exactly equal to ϕ'_i = ϕ_i + 180^o. Thus the second spectrum is equal to the first in form and size, transposed by 180^o (dashed line in Fig. 4).

The spectra of Fig. 4 refer to one island only. However, if we take all the iterates of the mapping, which alternate from one island to the other, because of the symmetry we have both the original curve of the spectrum and the one transposed by 180^o, thus the total spectrum is the superposition of the two spectra of Fig. 4 and hence it is periodic with period 180^o.

If K is smaller than the critical value K_{crit} = 0.97 there are invariant curves extending all the way from x = 0 to x = 1. If we take an initial deviation ξ_0 from such an invariant curve upwards we find a spectrum, that does not have a period 180^o. But if we take a deviation ξ'_0 directed downwards (say, exactly opposite to the previous one) we find the previous spectrum transposed by 180^o.

The spectra in the ordered domain (islands) vary continuously as the initial point (x, y) moves from one invariant curve to another. Only if the initial point moves along a given invariant curve the spectrum remains the same.

As a consequence if we change the initial conditions along a line (say y=const) that crosses both a chaotic domain and an island we find that the spectrum is constant as long as we are in the chaotic domain, but changes gradually inside the island. This property allows us to distinguish between chaotic and ordered orbits, by taking N much smaller than needed to define the invariant spectrum, as explained in the next section.

3. Distinction between chaotic and ordered orbits

As the spectra are invariant in the chaotic domain, the average values of the helicity angles $< \phi >_n$ after a very large number of iterations are the same for any initial conditions in this domain. This is true even when the spectrum is folded so that the helicity angle is defined in the interval (0,180) only. But even if n is not very large the average values $< \phi >_n$ have relatively small variations. These variations are even smaller if we adopt the folded spectrum. This is seen in Fig. 5 where we have many values of $< \phi >_n$ in the folded spectrum, for n = 10^4 along a line parallel to the x-axis. These values are scattered around the average value $< \phi >$= 38.7 of the spectrum of Fig. 2 (This average value is found by calculating the orbit for 10^8 periods). In fact if we take all the values of ϕ together, with x in the chaotic domain with a step Δx = 10^{-4} we have $\cong$ 10^8 points that form the same spectrum as in Fig. 2. The successive values of $< \phi >_n$ are randomly distributed around the average value $< \phi >$ of the original spectrum.

But as soon as the points x enter into the ordered domain (island) the values of $< \phi >_n$ are quite different from the above average $< \phi >$, and change smoothly from one point to the next. The transition from the chaotic to the ordered domain is quite abrupt in Fig. 5. This allows to separate clearly the ordered from the chaotic domain.

Inside the ordered domain there are secondary islands surrounding higher order stable periodic orbits, and small chaotic regions around unstable periodic orbits. The orbits in the secondary islands of stability give a smooth curve $< \phi >_n$ but different from the original curve $< \phi >_n$ inside the main island,

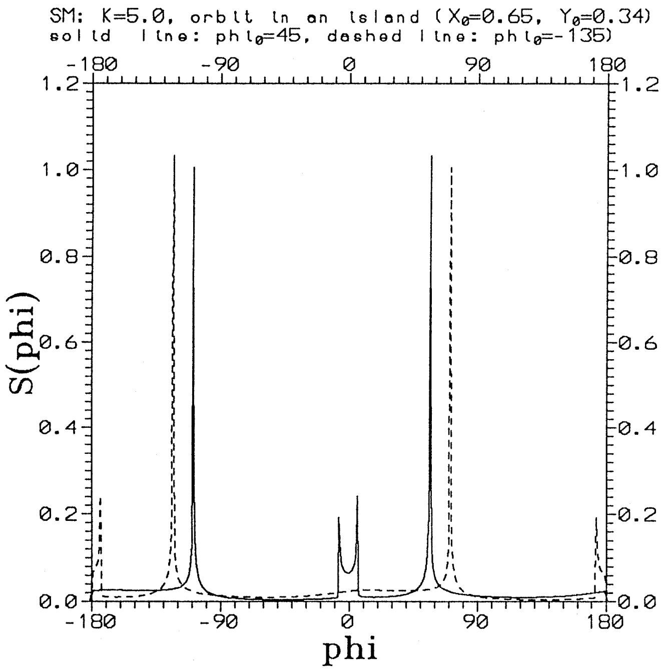

Fig. 4. The spectrum of an orbit in an island of stability in the same model as in Fig. 2, but with initial conditions (x = 0.65, y = 0.34, ξ_x = 1, ξ_y = 1; solid line). The spectrum with the same (x, y) and opposite (ξ_x, ξ_y) is shown as a dotted line. We take only every second iteration of the map, i.e. points on the same closed invariant curve.

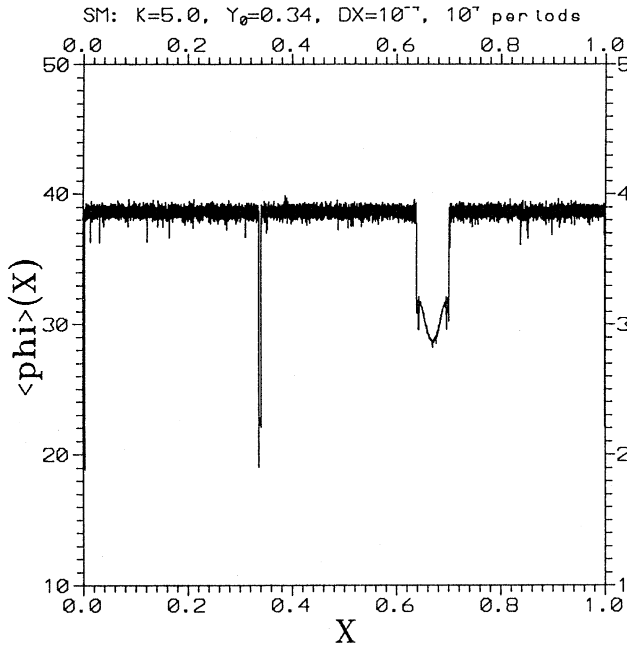

Fig. 5. The average values of the helicity angles $<\phi>_n$ for points along the line y = 0.34 and x from 0 to 1 with a step Δx = 0.0001 (n = 10^4). We see a chaotic domain characterised by values of $<\phi>_n$ randomly distributed around an average value $<\phi>$= 38.7, and two ordered domains. The first refers to the island around the point x = 0.68, and has values of $<\phi>_n$ varying smoothly. The second is a very small island near x = 0.34 that is characterised by a value of $<\phi>_n$ much smaller than in the chaotic domain.

while the secondary chaotic orbits give randomly scattered values of $<\phi>_n$ around an average value which is different from the value $<\phi>$= 38.7 of the large chaotic domain. E.g. in Fig. 6, where we have the values of $<\phi>_n$ along the main island in greater detail, we see two small symmetric chaotic domains at the boundaries of the main island, with $<\phi>\cong$ 31.1, and two more small symmetric chaotic domains, closer to the center of the island, with $<\phi>\cong$ 31.3.Between the two types of chaotic domains there are two symmetric secondary islands with $<\phi>\cong$ 30.9 (and a deeper minimum that is probably due to a higher order island).

In Fig. 5 we see a second very small island near x = 0.34, where the value of $<\phi>_n$ changes very much from the average value $<\phi>$= 38.7 of the chaotic domain. If we reduce the scanning step Δx along the x-axis in Fig. 5 we find further small islands of stability.

If we reduce the number n of iterations the values of $<\phi>_n$ in the chaotic domain have a larger noise but the region of the islands is still clearly distinguished (Fig. 7). The noise is larger when n is smaller.We see now some noise in the distribution of the values of $<\phi>_n$ inside the main island, but this is much smaller than the noise in the chaotic domain. Even the small island of Fig. 5 can be seen as a small gap in the distribution of $<\phi>_n$ in the chaotic domain.

Similar results can be found if we take the spectrum of the "twist angles" $\Delta\phi_i = \phi_{i+1} - \phi_i$. The arguments applicable to the helicity angles apply also to the twist angles, except that the spectrum $S(\Delta\phi)$ has now no periodicity of 180^o even in the chaotic domain. It seems that the noise of the values of $<\Delta\phi>_n$ is smaller than the noise of the values of $<\phi>_n$. Thus we can find a clear distinction between the chaotic and ordered domains. Furthermore $<\Delta\phi>_n$ gives a good estimate of the basic frequency of the orbits in the island. In this respect the use of $<\Delta\phi>_n$ gives similar results to those of Laskar et al (1992).

In Fig. 8 we give the values of $<\Delta\phi>_n$ along the same line as in Fig. 5, but with n = 10 (after the first 10 transient values have been removed).

It is remarkable that even such a small number of iterations, of the order of n = 10, can distinguish between order and chaos. This is due to the fact that the sequences of the values of ϕ or $\Delta\phi$, derived from neighbouring initial conditions, are quite different in a chaotic domain, but close to each other in an island of stablility.

We have checked this method in many cases and it seems that it is always effective. Furthermore our method does not need a Fourier analysis of the data, but uses directly the raw data themselves. Thus we believe that this method is the most promising one for distinguishing between chaotic and ordered domains in mixed systems.

There is only one limitation of this method, which, however, is shared by all other methods. If we are near the outer boundary of an island we have cantori that may keep an orbit close to

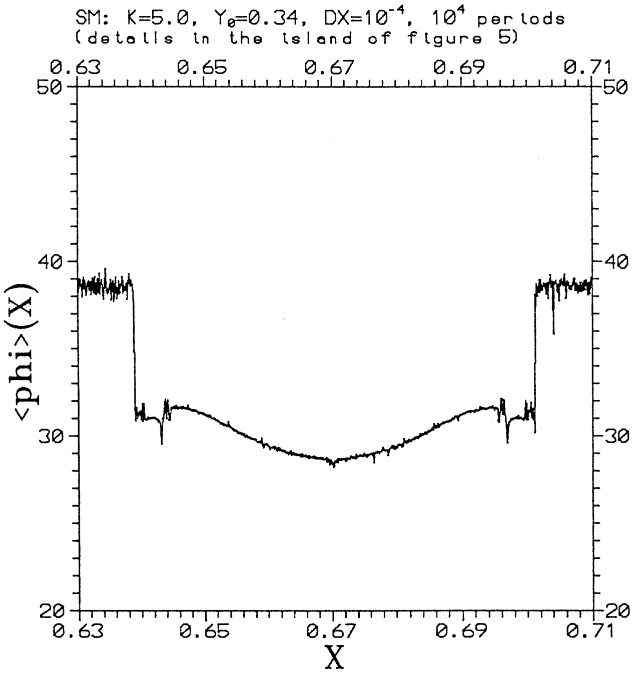

Fig. 6. The values of $< \phi >_n$ in the region of the main island of Fig. 5 in greater detail (Δx = 0.0001).

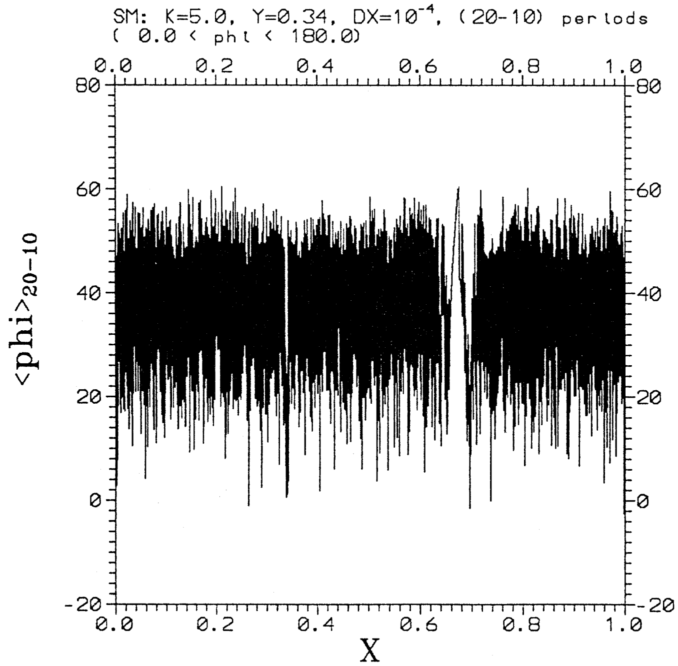

Fig. 7. The same as in Fig. 5, with n = 10, after the first 10 transient values.

the island for a long time before allowing it to escape to the outer chaotic domain. In such a case we have the phenomenon of stickiness (Contopoulos 1971, Shirts and Reinhardt 1982). The orbit looks as ordered for a long time, but later it is clearly chaotic. If one calculates the successive consequents (method 1) they seem to define an invariant curve for a long time. But later the successive consequents are scattered in the whole chaotic domain.

If we calculate the values of χ (Eq.(1)) to define the Lyapunov characteristic number (method 2), they continuously decrease for a long time, giving the impression that χ tends to zero, but later χ increases.

If we calculate the successive rotation angles to find the rotation number (method 3) they give the impression that they converge to an average value (the rotation number), but later on the convergence is worse.

In the same way if we calculate the helicity angles $< \phi >_n$ they seem to give values quite different from those of the chaotic domain, thus indicating an ordered orbit, but later on (for larger n) the value of $< \phi >_n$ is close to the average $< \phi >$ of the chaotic domain.

Thus there is no way to find with a few iterations whether an orbit close to the outer boundary of an island is ordered, and belongs to the island, or it is only a sticky orbit that will eventually merge with the outer chaotic domain. Only longer calculations can decide what is the character of such orbits.

Fortunately, however, the region of sticky orbits, where cantori restrict an orbit for long, but not infinite times, is in general rather small, and not important in many cases of interest.

It is our experience that particular cantori play an important role only for a rather limited range of values of the non-linearity parameter K. If this parameter is smaller than a critical value K_{crit} then the cantorus becomes a torus and seperates completely two regions of phase space. If K is much larger than K_{crit} the holes of the cantorus are so large that they allow a complete and fast interaction of the two regions across the cantorus. Only for a small set of values of K above K_{crit} we have a relatively long, but not absolute, separation of the two regions. In such cases longer calculations are necessary to distinguish the sticky orbits, which are chaotic, but look like ordered for a long time. But in most cases such orbits do not play an important role in the global dynamics of the system. Longer calculations may also be necessary in cases of thin chaotic layers inside islands.

4. Stretching numbers

A "stretching number" is defined by Eq. (1) if we set t=1, i.e.

$$a_i = \ln \mid \frac{\xi_{i+1}}{\xi_i} \mid \tag{5}$$

(Froeschlé et al 1993, Voglis and Contopoulos 1994).

The spectrum of the stretching numbers $S(a)$ is defined in an analogous way to the spectrum of helicity angles

$$S(a) = \frac{dN(a)}{Nda} \tag{6}$$

This spectrum is invariant with respect to the initial conditions in a connected chaotic domain (Voglis and Contopoulos 1994) . As a consequence the Lyapunov characteristic number

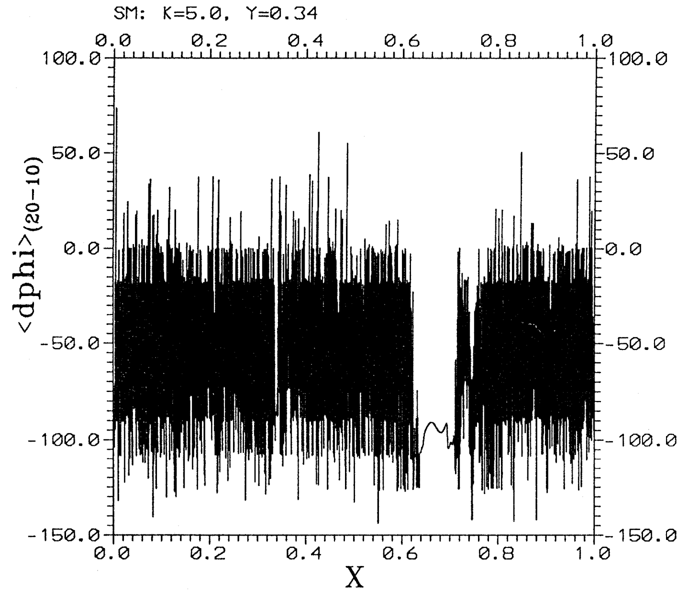

Fig. 8. The average values of the twist angles $< \Delta\phi >_n$ in the same model as in Fig. 2 and initial conditions along the same line as in Fig. 5. (n = 10 after the first 10 transient values).

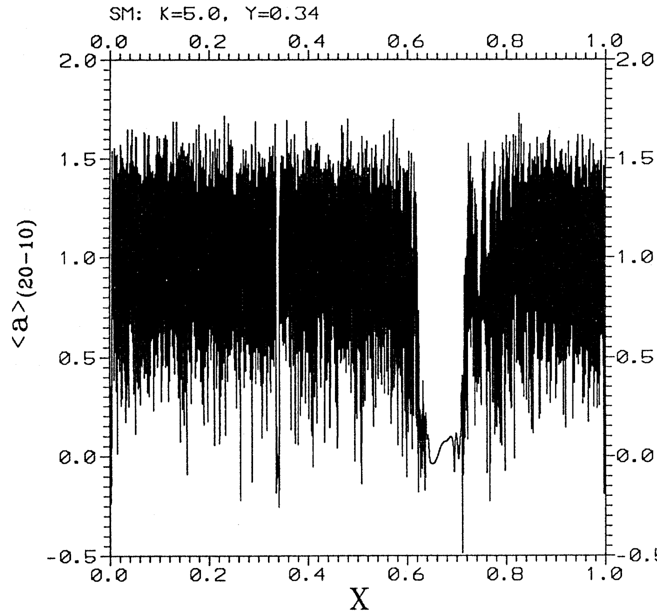

Fig. 9. The average values of the stretching number $< a >_n$ in the same model as in Fig. 2 along the same line as in Fig. 5 (n = 10 after disregarding the first 10 transient values).

(LCN), which is the average value of a when $N \to \infty$, is also invariant.

The spectrum in an ordered domain is invariant only with respect to initial conditions on any particular invariant curve, and with respect to any initial deviation ξ_0. On the other hand the LCN is always zero, although the spectrum may vary if we move from one invariant curve to the next.

We can use the above properties to separate the chaotic and ordered domains, in the same way as with the helicity angles. Namely if we take the average value of a over n = 10 periods, beyond the first 10 periods when the spectrum is transient, we find values that have some scatter around a mean value, which is the LCN of the chaotic domain (LCN = 1.0 in Fig. 9). In the ordered domain we find values that are close to LCN = 0. The change from 1 to 0 is abrupt, as in the case of the helicity angles (Figs. 5-8), and occur at the same values of x as the changes in the average values of the helicity angles.

The method based on the stretching numbers is slightly better than the method based on the helicity angles because the value of LCN is equal to zero in the ordered case (fixed).

5. Four-dimensional maps

Four-dimensional maps (x_1, y_1, x_2, y_2) represent Poincaré surfaces of section of conservative systems of three degrees of freedom. In such cases we define three different helicity angles, e.g. the angles of the projections of ξ on the planes (x_1, x_2), (y_1, x_2), (y_2, x_2) with the axis x_2.

We consider in particular the 4-D map

$$x_1' = x_1 + y_1', \qquad y_1' = y_1 + \frac{K}{2\pi}\sin 2\pi x_1 - \frac{\beta}{\pi}\sin 2\pi(x_2 - x_1),$$

$$x_2' = x_2 + y_2', \qquad y_2' = y_2 + \frac{K}{2\pi}\sin 2\pi x_2 - \frac{\beta}{\pi}\sin 2\pi(x_1 - x_2), \qquad (mod 1)(7)$$

which consists of two coupled standard maps with coupling parameter β.

The spectra of the helicity angles $S(\phi_{x1x2})$, $S(\phi_{y1x2})$, $S(\phi_{y2x2})$ are defined in the same way as the spectrum of the single helicity angle $S(\phi)$ of a 2-D map given by Eq. (3). An example of a spectrum of the helicity angle ϕ_{x1x2} is given in Fig. 10. This spectrum is invariant; the first 10^6 iterations form the spectrum as a solid line, and the next 10^6 iterations form the same spectrum as dots superimposed on the solid line.

In the present case the spectrum does not have a 180^o periodicity as some spectra of 2-D maps.

The other two spectra $S(\phi_{y1x2})$ and $S(\phi_{y2x2})$ have properties similar to those of Fig. 10.

If β is zero the two standard maps are independent. But if β is different from zero the two standard maps interact. In this case one may argue that it is not possible to distinguish between ordered and chaotic domains, because all chaotic domains interact through Arnold diffusion and produce a unique chaotic domain. Even if an orbit is ordered (lying on a KAM torus) any small deviation, due to numerical errors, will make it chaotic. Thus one should expect a unique spectrum, except for transient effects.

However the situation is more complex, because of the extremely long diffusion time of Arnold diffusion. Following the example of other authors (Chirikov 1979, Laskar 1993) we distinguish between Arnold diffusion, which appears for small β, and resonance overlap diffusion, that exists for larger β. We have found (Contopoulos and Voglis 1996) that the transition between Arnold diffusion and resonance overlap diffusion is very

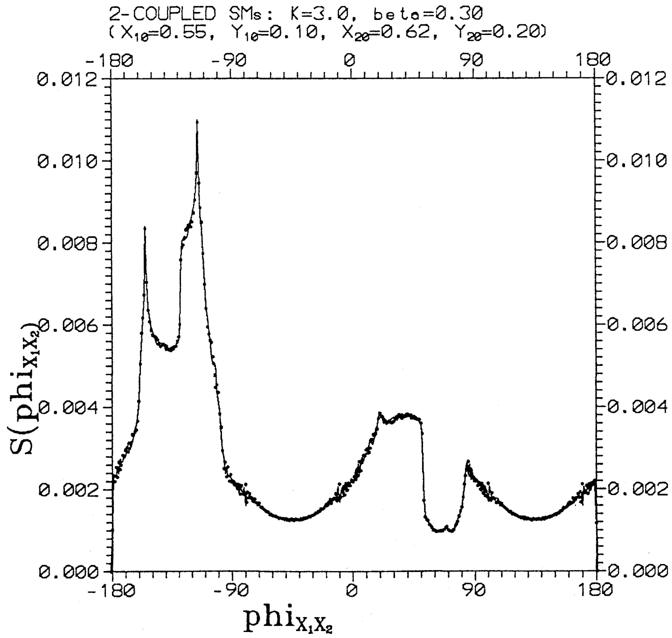

Fig. 10. The spectrum $S(\phi_{x1x2})$ of the helicity angle ϕ_{x1x2} in the model (7) with $K = 3$, $\beta = 0.3$ and initial conditions $x_1 = 0.55$, $y_1 = 0.1$, $x_2 = 0.62$, $y_2 = 0.2$, $\xi_{x1} = \xi_{y1} = \xi_{y2} = 0$, $\xi_{x2} = 1$. The solid line gives the first $N = 10^6$ iterations and the dots the the next 10^6 iterations.

clear and abrupt. Namely the diffusion time T due to resonance overlap depends exponentially on β, for β larger than a certain critical value β_{crit}, but the derivative $\frac{d \ln T}{d\beta}$ changes abruptly as β decreases below β_{crit}. The diffusion time for $\beta < \beta_{crit}$ (this is the regime where Arnold diffusion applies) is so extremely long that in practice no diffusion can be seen over all practically useful times.

Therefore we can distinguish between ordered and chaotic domains in the same way as in systems of 2 degrees of freedom.

In view of this distinction it is of interest to study the variation of the average helicity angles $< \phi_{x1x2} >_n$, $< \phi_{y1x2} >_n$, $< \phi_{y2x2} >_n$ along particular lines in phase space, in order to find the chaotic and the ordered domains.

In Fig. 11 we give the average values of $< \phi_{x1x2} >_n$, $< \phi_{y1x2} >_n$, $< \phi_{y2x2} >_n$ for $K = 3$, $\beta = 0.3$ along a line parallel to the x-axis, with a step $\Delta x = 0.001$ and $n = 10^4$. One can see clearly one main island around $x_1 = 0.55$ and small islands near $x_1 = 0.145$, $x_1 = 0.262$ and $x_1 = 0.432$. The positions of the islands are the same for the two helicity angles.

If we reduce the number n of iterations of each orbit the noise is larger but we still can distinguish the ordered domains (islands) from the chaotic domain. This can be realised even if n is of order $n = 10$ (Fig. 12).

Similar results are found if we use the average values of the twist angles $< \Delta\phi_{x1x2} >_n$ etc, or of the stretching numbers (Fig. 13) $< a >_n$. The islands are found in the same positions as in the case of the helicity angles. Furthermore the value of $< a >_n$ in the ordered domain is almost zero and this helps to identify easily the islands of stability.

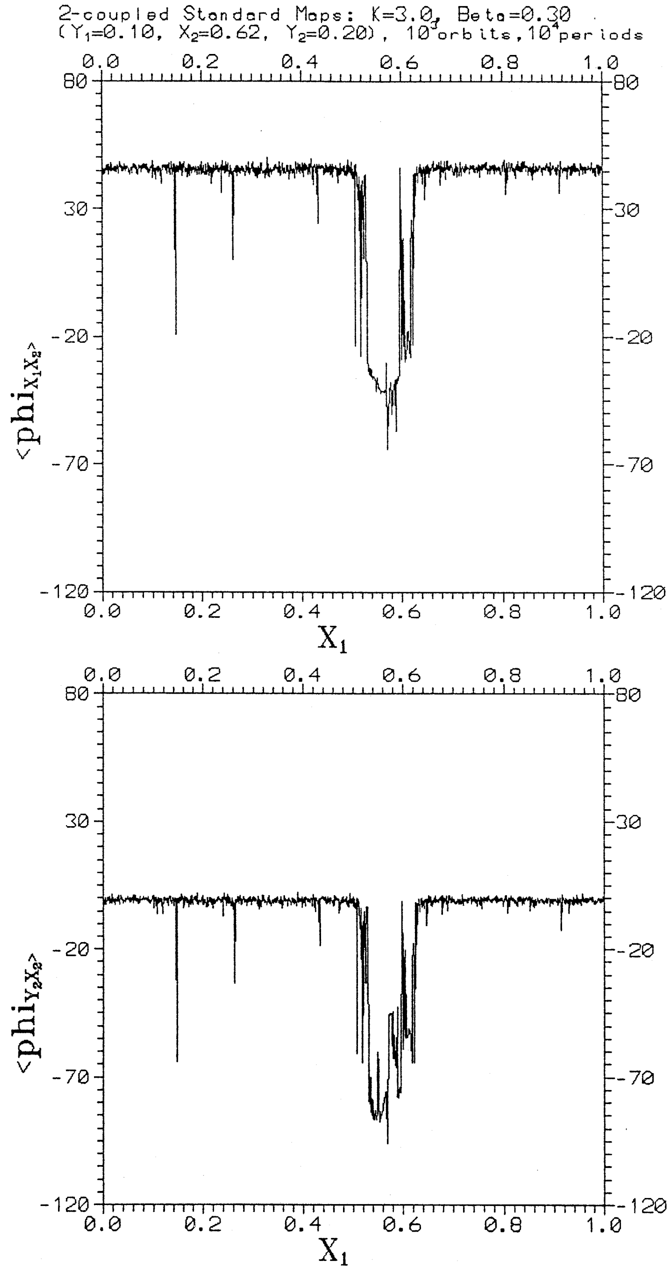

Fig. 11. The average values of the helicity angles $< \phi_{x1x2} >_n$, $< \phi_{y2x2} >_n$, in the model (7) with $K = 3$, $\beta = 0.3$, along the line $y_1 = 0.1$, $x_2 = 0.62$, $y_2 = 0.2$, with x taken from 0 to 1 with a step $\Delta x = 0.001$ ($n = 10^4$).

6. Conclusions

We have introduced a new method for distinguishing between ordered and chaotic orbits, in mixed systems that contain both order and chaos. This is based on the distribution of a relatively small number of 'helicity angles' (or of 'twist angles'), and/or of 'stretching numbers'.

The 'helicity angle' is the angle between a deviation ξ from a given orbit with a fixed direction, say the x-axis. The 'twist angle' is the difference of two successive helicity angles. The

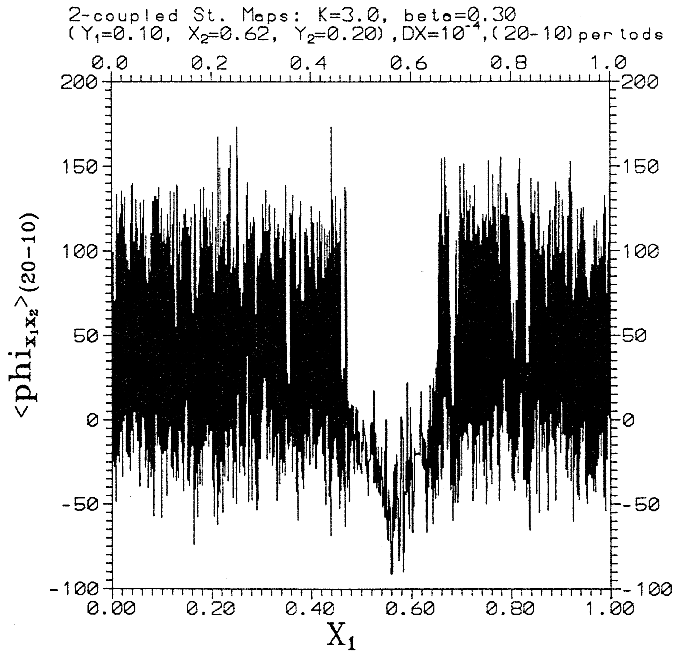

Fig. 12. The average values of the helicity angle $<\phi_{x1x2}>_n$ along the same line as in Fig. 11 with n = 10 (after the first 10 transient values).

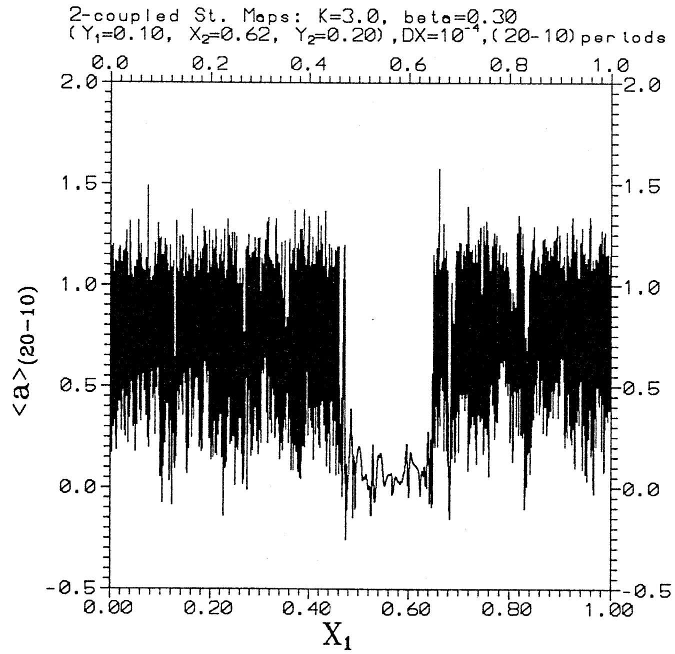

Fig. 13. The average values of the stretching numbers $<a>_n$, along the same line as in Fig. 11 with n = 10 (after the first 10 transient values).

stretching number is the logarithm of the ratio of two successive deviations $\mid \xi_{i+1} \mid$ and $\mid \xi_i \mid$, and is called also a 'short time Lyapunov characteristic number'.

We found the following results:

1) The average value of the helicity angle $<\phi>_n$ (for n iterations) varies in a seemingly chaotic way (like noise) around the average value $<\phi>$ defined over a very large number of iterations. The value of $<\phi>$ is constant in a connected chaotic domain, but it varies smoothly in an ordered domain. If we keep $n = constant$ and calculate $<\phi>_n$ at successive points along a line in phase space we find large variations of $<\phi>_n$ around a fixed $<\phi>$ in the chaotic domain, but small variations of $<\phi>_n$ around a $<\phi>$ varying smoothly in an ordered domain. The distinction between the ordered and chaotic domains is further helped by the fact that the average $<\phi>$ changes abruptly at the boundary between the ordered and the chaotic domains.

2) If the number of iterations n is small we have greater noise, but we can still distinguish ordered and chaotic domains for n as small as 10.

3) Similar results are found if we take the distribution of the twist angles $<\Delta\phi>_n$, or of the stretching numbers $<a>_n$. In the case of regular orbits the values of $<\Delta\phi>_n$ give also the basic frequencies, which are equivalent to the rotation numbers. In the case of the stretching numbers an extra advantage is that even short time average values of $<a>_n$ approximate the Lyapunov characteristic number (LCN) which is known to be zero in the ordered domain.

4) The new method is faster than the 3 known methods of distinguishing between order and chaos, namely (1) the distribution of the consquents on a Poincaré surface of section, (2) the distinction between zero and nonzero LCN, and (3) the method of the distribution of the rotation angles.

5) A further advantage of the helicity angles over the rotation angles, is that they are always defined uniquely, while the rotation angles (angles with respect to a point, normally, representing a periodic orbit at the center of an island) are not uniquely defined if the islands disappear. Furthermore the method can be directly applied, i.e. without a Fourier analysis of the data along an orbit.

6) Similar results can be found in systems of more degrees of freedom. E.g. in a 4-dimensional map we have three helicity angles and one stretching number. In such cases we can again distinguish between ordered and chaotic domains along a fixed line in phase space, by taking short time averages (of the order of 10 periods again) of one helicity angle (or twist angle), or of the stretching number.

In such cases we expect that even if there is a distinction between ordered and chaotic domains over long times nevertheless both types of orbits will merge into a large chaotic domain because of Arnold diffusion (even orbits starting on KAM surfaces will appear in the long run chaotic because of numerical errors). But the time scale of Arnold diffusion is so extremely long (in general much longer than any physically relevant time interval) that we can neglect it. Thus a distinction between ordered and chaotic domain is still possible.

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