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On the stability of the Trojan asteroids*

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Abstract. We reconsider the problem of stability of the triangular Lagrangian equilibria of the restricted problem of three bodies. We consider in particular the Sun–Jupiter model and the Trojan asteroids in the neighbourhood of the point L_4. In the spirit of Nekhoroshev's theory on stability over exponentially large times, we are able to prove that stability over the age of the universe is guaranteed on a region big enough to include a few known asteroids. This significantly improves previous works on the same subject.

Key words: celestial mechanics – instabilities – minor planets, asteroids

1. Introduction

We reconsider the problem of stability of the Lagrangian equilibria of the restricted problem of three bodies in the light of Nekhoroshev's theory of stability over exponentially large time intervals. More precisely, we look for stability for times of the order of the estimated age of the universe.

The problem has been previously investigated in the same spirit by Giorgilli et al. (1989), Simó (1989) and Celletti & Giorgilli (1991) . The underlying idea was to combine analytical and numerical tools in order to prove that if the initial datum of an orbit is sufficiently close to the equilibrium (in phase space), then the orbit is confined in some neighbourhood of the equilibrium for a very long time, fixed in advance. The problem is to produce realistic stability estimates, possibly applicable to real asteroids. Actually, the work of Simó (1989) and of Celletti & Giorgilli (1991) on the Sun–Jupiter–L_4 case has produced realistic estimates: roughly speaking, stability over the age of the universe is proved in a neighbourhood the size of which is of order 10^4 Km. Unfortunately, as discussed by Celletti & Giorgilli (1991), the size of the region where asteroids are actually found turns out to be larger, by a factor 300 (in the best case) to 3000, than the estimated stability region.

The problem is whether or not this kind of estimates can be improved. We attempt to achieve a not negligible improvement by taking into account three possible changes with respect to the quoted work. The first point is concerned with the choice of coordinates; the second point consists in a better choice of the norms used in order to estimate the size of some functions; the third point is connected with the possibility of making expansion up to higher orders with respect to the previous works.

As to the first point, our remark is that numerical investigations show that the projection of the stability region on the plane of Jupiter's orbit is a banana–shaped region which lies close to the circle with center in the sun and radius roughly equal to the Sun–Jupiter distance. Now, all previous works were based on expansions in cartesian coordinates around the Lagrangian equilibrium point. That cartesian coordinates are not suitable to describe regions with a circular shape is evident to everybody. Thus, we used polar coordinates, which are better candidates. This elementary remark is the most important source of improvement.

The second point is rather technical, and will be discussed in detail in Sect. 2.3. Roughly speaking, the problem is how to compute an estimate of the size of a function when we know the coefficients of its Taylor expansion. With respect to previous works we introduce here a better norm.

The third point can be illustrated as follows. According to Nekhoroshev's theory, the series arising from classical perturbation theory have an asymptotic character. This means that at some point one should reach an optimal value for the order of expansion, which gives the best possible result. The exponential stability times typical of Nekhoroshev's theory are actually based on an analytical estimate of such an optimal order. From a practical viewpoint it turns out that the optimal order could be so high that it cannot be reached in explicit expansion using computer algebra. However, it is legitimate to explore how the results improve when the expansion order is increased. For this reason, we decided to study the planar problem instead of the spatial one, as considered in the quoted works. Indeed, reducing the number of degrees of freedom from 3 to 2 allows us to make perturbation expansion up to order 35 instead of 22. However, the most interesting result is that looking for stability results over the age of the universe for the known asteroids we find that

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sometimes the optimal order is less than 34, which means that we are actually close to that limit.

The application of the present method to the known Trojan asteroids in the Sun–Jupiter–L_4 system shows that the overall improvement gives realistic results. Indeed, it turns out that 4 known asteroids are inside the region where stability can be guaranteed for a time as long as the estimated age of the universe. Moreover, the majority of the real asteroids fail to enter this region by just a factor 10. Thus, it is likely that with some further improvement of our method we might succeed in proving the practical stability of the orbit of most Trojan asteroids. A possible suggestion is, for instance, choosing coordinates more adapted to the actual shape of the stability domain, as given by numerical computations.

2. Theoretical framework

We start with the Hamiltonian of the restricted problem of three bodies in the planar circular case. For simplicity, we refer to the Sun–Jupiter case. We introduce a uniformly rotating frame (O, q_1, q_2) as follows: the origin is located at the center of mass of the Jupiter–Sun system; the axes are oriented in such a way that the Sun is always at the point $(\mu, 0)$ and Jupiter at the point $(1-\mu, 0)$; the physical units are chosen so that the mass of Jupiter is μ and the mass of the Sun is $1-\mu$, the distance between Jupiter and the Sun is 1, and the angular velocity of Jupiter is 1. Then the Hamiltonian has the form

$$H = \frac{1}{2}(p_1^2 + p_2^2) + q_2 p_1 - q_1 p_2 - \frac{1-\mu}{\sqrt{(q_1-\mu)^2 + q_2^2}} - \frac{\mu}{\sqrt{(q_1+1-\mu)^2 + q_2^2}} \qquad (1)$$

In this system of coordinates the Lagrangian point L_4 is located at $q_1 = -\frac{1}{2}(1-2\mu)$, $q_2 = \frac{\sqrt{3}}{2}$, $p_1 = -q_2$, $p_2 = q_1$.

2.1. Expansion around the point L_4

On the Hamiltonian (1) we perform a sequence of transformations.

(i) We move the origin to the Sun, thus considering a heliocentric system. The generating function of the corresponding canonical transformation is

$$W_1 = -(Q_1 + \mu)p_1 - Q_2 p_2 + \mu Q_2$$

where Q_1, Q_2, P_2, P_2 denote the heliocentric coordinates.

(ii) We introduce polar coordinates via the canonical transformation generated by

$$W_2 = -\varrho(P_1 \cos\vartheta + P_2 \sin\vartheta) ,$$

where the polar coordinates are denoted by ϱ, ϑ, and the corresponding momenta will be denoted as p_ϱ, p_ϑ.

(iii) Forgetting that ϑ is an angle, we introduce a local reference system in the neighbourhood of the point L_4. Remarking that in polar coordinates the point L_4 is

$$\varrho = 1 , \quad \vartheta = \frac{2\pi}{3} , \quad p_\varrho = 0 , \quad p_\vartheta = 1 ,$$

the canonical transformation is generated by

$$W_3 = p_x(\varrho - 1) + (p_y + 1)\vartheta - \frac{2\pi}{3} p_y .$$

The canonical coordinates will be denoted by x, y, p_x, p_y. After these transformations the Hamiltonian is given the form

$$H = \frac{1}{2}\left[p_x^2 + \frac{(p_y+1)^2}{(x+1)^2}\right] - p_y - \mu(x+1)\cos\left(y + \frac{2\pi}{3}\right) - \frac{1-\mu}{x+1} - \frac{\mu}{\sqrt{(x+1)^2 + 1 + 2(x+1)\cos(y + \frac{2\pi}{3})}} \qquad (2)$$

(iv) We expand the Hamiltonian in Taylor series around the origin, thus giving it the form

$$H = H_2 + H_3 + H_4 + \ldots \qquad (3)$$

where

$$H_2 = \frac{1}{2}(p_x^2 + p_y^2) - 2xp_y + \left(\frac{1}{2} + \frac{9\mu}{8}\right)x^2 - \frac{9\mu}{8}y^2 + \frac{3\sqrt{3}\,\mu}{4}xy \qquad (4)$$

and H_s for $s > 2$ is a homogeneous polynomial of degree s in x, y, p_x, p_y. The explicit expansion of the terms of degree $s > 2$ of the Hamiltonian up to a given order is actually made by computer.

(v) The final transformation gives the quadratic part of the Hamiltonian the diagonal form

$$H_2 = \frac{\omega_1}{2}(x_1^2 + y_1^2) + \frac{\omega_2}{2}(x_2^2 + y_2^2) , \qquad (5)$$

where x_1, x_2, y_1, y_2 are the canonical coordinates, and ω_1 and ω_2 are the frequencies. This is done via the linear symplectic transformation generated by the matrix

$$C = (e_1 m_1^{-1/2}, e_2 m_2^{-1/2}, f_1 m_1^{-1/2}, f_2 m_2^{-1/2}) , \qquad (6)$$

where the real column vectors e_1, e_2, f_1 and f_2 are defined as

$$e_j + i f_j = \begin{pmatrix} \dfrac{8\omega_j^2 + 4\sqrt{3}\alpha + 9}{8} \\ \dfrac{16 i\omega_j + 4\alpha + 3\sqrt{3}}{8} \\ i\omega_j \left(\dfrac{8\omega_j^2 + 4\sqrt{3}\alpha + 9}{8}\right) \\ i\omega_j \dfrac{4\alpha + 3\sqrt{3}}{8} + \dfrac{4\sqrt{3}\alpha + 9}{4} \end{pmatrix} , \qquad (7)$$

the real constants m_j, $(j = 1, 2)$ are given by

$$m_j = \omega_j D_j ,$$

$$D_j = \left(\frac{8\omega_j^2 + 4\sqrt{3}\alpha + 9}{8}\right)^2 - 2\left(\sqrt{3}\alpha + \frac{9}{4}\right) + \left(\frac{4\alpha + 3\sqrt{3}}{8}\right)^2 , \qquad (8)$$

and ω_1^2 , ω_2^2 , α are defined as

$$\omega_1^2 = \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{27}{4} + 4\alpha^2}$$

$$\omega_2^2 = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{27}{4} + 4\alpha^2}\ , \qquad (9)$$
$$\alpha = -\frac{(1-2\mu)3\sqrt{3}}{4}\ .$$

We emphasize that in order to have m_j positive in (8) we must put $\omega_1 > 0$ and $\omega_2 < 0$.

After these transformations the Hamiltonian turns out to be a power series expansion of the form (3), with H_2 in the diagonal form (5). Remark that all the operations above can be explicitly performed on computer, using for instance an algebraic manipulator.

2.2. Construction of Birkhoff's normal form

The form above of the Hamiltonian is specially adapted for application of normal form theory. To this end, following Giorgilli et al. (1989), we use the formalism of Lie transforms. For completeness sake, we recall the main points of the scheme. Considering a generating sequence $\{\chi_s\}_{s\geq 3}$, with $\chi_s(x,y)$ a homogeneous polynomial of degree s, we define the Lie transform operator T_χ as

$$T_\chi = \sum_{s\geq 0} E_s\ , \qquad (10)$$

where the sequence $\{E_s\}_{s\geq 0}$ of operators is recursively defined as

$$E_0 = 1\ ,\quad E_s = \sum_{j=1}^{s} \frac{j}{s} L_{\chi_{j+2}} E_{s-j}\ ; \qquad (11)$$

here, $L_\chi \cdot = \{\chi, \cdot\}$. The operator T_χ is linear and invertible, and preserves products and Poisson brackets. For details and proofs, see the paper of Giorgilli & Galgani (1978).

For a fixed integer $r \geq 3$, we say that the Hamiltonian is in Birkhoff's normal form up to order r if it has the form

$$H^{(r)} = H_2 + Z_3 + \ldots + Z_r + \mathscr{R}^{(r)}\ , \qquad (12)$$

where H_2 has still the form (5), Z_s $(s = 3, \ldots, r)$ depends only on the actions $I_j = (x_j^2 + y_j^2)/2$, and $\mathscr{R}^{(r)}$ is a remainder, actually a power series starting with terms of degree $r+1$. Under suitable nonresonance hypotheses on the unperturbed frequencies ω, the Hamiltonian can be given the normal form above up to order r via a Lie transform generated by a finite generating sequence $\chi^{(r)} = \{\chi_3, \ldots, \chi_r\}$, where χ_s is a homogeneous polynomial of degree s.

Both the generating sequence and the normal form are determined by solving with respect to the unknowns $Z_3, \ldots, Z_r$ and $\chi_3, \ldots, \chi_r$ the equation

$$T_{\chi^{(r)}} H^{(r)} = H\ , \qquad (13)$$

where H is the original Hamiltonian. The explicit algorithm, as well as a computer program doing all necessary algebraic manipulations, is fully described by Giorgilli (1979), so we skip all details.

2.3. Estimate of the stability time

The generating sequence $\chi^{(r)}$ can be used to construct a canonical transformation

$$x' = T_{\chi^{(r)}} x\ ,\quad y' = T_{\chi^{(r)}} y\ , \qquad (14)$$

where x', y' are the new variables, that we shall call "normal coordinates". By construction, the transformed Hamiltonian $H^{(r)}(x',y') = T_{\chi^{(r)}}^{-1} H$ is in normal form up to order r (recall that the operator $T_{\chi^{(r)}}$ is invertible). Thus, $H^{(r)}$ admits approximate first integrals of the form

$$I_j(x',y') = \frac{1}{2}\left({x'_j}^2 + {y'_j}^2\right)\ ,\quad j = 1,2\ , \qquad (15)$$

which are actually action variables for the normalized part of the Hamiltonian. This information is the basis of our estimate of the size of the stability region. Indeed, we have

$$\dot{I}_j = \{I_j, H^{(r)}\} = \{I_j, \mathscr{R}^{(r)}\}\ , \qquad (16)$$

which is a power series starting with terms of degree $r+1$.

We need now a few analytical tools, namely: domains, where stability properties will be investigated, and norms, which will allow us to estimate the size of various functions. We fix positive constants R_1, R_2, and consider a family of domains of the form

$$\Delta_{\varrho R} = \left\{(x,y) \in \mathbf{R}^4\ :\ x_j^2 + y_j^2 \leq \varrho^2 R_j^2\right\}\ , \qquad (17)$$

where ϱ is a positive parameter.

In order to introduce norms we need some considerations. Consider a homogeneous polynomial f of degree s. We are actually interested in estimating the maximum absolute value of f over a domain $\Delta_{\varrho R}$ for fixed values of ϱ and R. In other words, we are interested in estimating a quantity like

$$|f|_{\varrho R} = \sup_{(x,y)\in\Delta_{\varrho R}} |f(x,y)|\ ,$$

namely the supremum norm of f over the domain $\Delta_{\varrho R}$. Actually, computing such a quantity is impractical. So, we do the following. We want to introduce polar coordinates r_j, ϑ_j in each of the coordinate planes x_j, y_j, namely, we want to transform $x_j = r_j \cos\vartheta_j, y_j = r_j \sin\vartheta_j$. Actually, it is more convenient to perform the equivalent transformation to complex variables

$$x_j = \frac{1}{\sqrt{2}}\left(\xi_j + i\eta_j\right)\ ,\quad y_j = \frac{i}{\sqrt{2}}\left(\xi_j - i\eta_j\right)\ ,$$

where

$$\xi_j = \frac{r_j}{\sqrt{2}} e^{-i\vartheta_j}\ ,\quad \eta_j = -\frac{ir_j}{\sqrt{2}} e^{i\vartheta_j}\ .$$

By this, the transformed function $f(\xi,\eta)$ is still a homogeneous polynomial of degree s, that we write as

$$f(\xi,\eta) = \sum_{j_1+j_2+k_1+k_2=s} C_{j_1 j_2 k_1 k_2} \xi_1^{j_1} \xi_2^{j_2} \eta_1^{k_1} \eta_2^{k_2}\ ,$$

where $C_{j_1j_2k_1k_2}$ are complex coefficient which are completely determined by the transformation. From $(x, y) \in \Delta_{\varrho R}$ we clearly have $0 \le r_j < \varrho R_j$. Thus, it is an easy matter to conclude that the supremum norm $|f|_{\varrho R}$ above does not exceed the norm

$$\|f\|_{\varrho R} < \frac{\varrho^s}{2^{s/2}} \sum_{j_1j_2k_1k_2} |C_{j_1j_2k_1k_2}| \, R_1^{j_1+k_1} R_2^{j_2+k_2} \; . \tag{18}$$

The latter norm is easy to compute; thus, we shall use it in the following. Remark that, by definition, the elementary property $\|f\|_{\varrho R} = \varrho^s \|f\|_R$ holds.

Of course, the domains (17) and the norm (18) are properly defined both in the original coordinates (x, y) and in the normal coordinates (x', y').

In order to estimate the stability time we use normal coordinates. We remark that, by definition, one has $(x', y') \in \Delta_{\varrho R}$ if and only if $I_j \le \varrho^2 R_j^2/2$. Suppose that the initial point of an orbit lies in the domain $\Delta_{\varrho_0 R}$ for some positive ϱ_0. We fix a larger domain $\Delta_{\varrho R}$, with $\varrho > \varrho_0$, and ask how long the orbit will be confined in the latter domain. To this end, we use the trivial inequality

$$|I_j(t) - I_j(0)| \le |t| \sup_{\Delta_{\varrho R}} \left|\dot{I}_j\right| \; , \tag{19}$$

which is clearly true until the orbit eventually escapes from $\Delta_{\varrho R}$. The problem is how to estimate $\sup |\dot{I}_j|$. To this end, we proceed as follows. Write the remainder $\mathscr{R}^{(r)}$ as a power series, e.g.,

$$\mathscr{R}^{(r)} = H_{r+1}^{(r)} + H_{r+2}^{(r)} + \ldots \tag{20}$$

It is obviously impossible to determine the whole series, but the first term, namely $H_{r+1}^{(r)}$ can be easily constructed. Thus, we use the approximate estimate

$$\sup_{\Delta_{\varrho R}} \left|\dot{I}_j\right| < 2 \|\{I_j, H_{r+1}^{(r)}\}\|_{\varrho R} \; . \tag{21}$$

This choice is heuristically justified as follows. Standard estimates (see for instance Giorgilli et al. (1989)) allow to prove that the power series above for the remainder is absolutely convergent in the domain $\Delta_{\varrho R}$ provided ϱ is small enough. More precisely, one proves that one has $\|H_s^{(r)}\|_R < C^{s-r-1} D$ for some positive constant C and for $D = \|H_{r+1}^{(r)}\|_R$. Actually, C^{-1} is the convergence radius for ϱ. Now, if we take $\varrho \le C^{-1}/2$, then the supremum of the remainder does not exceed the norm of its first term multiplied by two. This justifies the factor 2 in (21). Of course, we should check that the actual values of ϱ satisfy the restriction above. This technical point will be discussed in Sect. 3.4 below.

Using (21), we estimate the escape time $\tau_r(\varrho_0, \varrho)$ as

$$\tau_r(\varrho_0, \varrho) = \min_{j=1,2} \frac{R_j^2(\varrho^2 - \varrho_0^2)}{4\|\{I_j, H_{r+1}^{(r)}\}\|_{\varrho R}} \; . \tag{22}$$

This quantity still depends on the normalization order r, and on the radii ϱ_0 and ϱ of the initial and the final domain. We now want to remove the dependence on r and ϱ, thus getting an estimated escape time depending only on the initial conditions, namely on ϱ_0. To this end, we optimize the time with respect to r and ϱ. First, keeping r fixed, we write the r.h.s. of (22) as

$$\frac{R_j^2(\varrho^2 - \varrho_0^2)}{4\varrho^{r+1}\|\{I_j, H_{r+1}^{(r)}\}\|_R} \; , \tag{23}$$

here, the mentioned property of the norm has been used. This expression has clearly a maximum for

$$\varrho = \varrho_0 \sqrt{\frac{r+1}{r-1}} \; ; \tag{24}$$

this is the value of ϱ that we shall put in (22), thus getting τ depending only on r and ϱ_0. Secondly, we compute this quantity for r running from 3 to some maximal value $\tilde{r}$, and choose an optimal value, $r_{\rm opt}$ say, which maximizes the estimated escape time. Thus, we produce an estimated escape time depending on ϱ_0 only, namely

$$T(\varrho_0) = \max_{3 \le r \le \tilde{r}} \sup_{\varrho > \varrho_0} \tau_r(\varrho_0, \varrho) \; . \tag{25}$$

The actual value of $\tilde{r}$ clearly depends on the power of the computer and on the efficiency of the program doing all the necessary algebraic manipulations.

3. Results

All our work is based on polynomial expansions. The key remark is that a polynomial in several variables is uniquely represented by an array of coefficients. Thus, performing algebraic manipulations on power series truncated at some order just requires routines for the following operations:

(i) storing and retrieving the coefficient corresponding to a given monomial, which in turn is identified by the exponents of the variables;
(ii) algebraic operations such as sum, products, differentiation and Poisson brackets, linear change of coordinates;
(iii) solution of the homological equation during the process of computation of the normal form, i.e., the equation $\{H_2, \chi_s\} + Z_s = Q_s$ where χ_s is the generating function, Z_s is the normal form and Q_s is a known polynomial of degree s.

A detailed description of the program doing all such manipulations is given by Giorgilli (1979).

All the expansions have been done with a program written *ad hoc*. Use of general purpose symbolic manipulators has been restricted to checking some of the operations.

3.1. Expansion of the Hamiltonian and computation of the normal form

All the algebraic manipulations were done on power expansions truncated at order $\tilde{r} = 35$. The expansion of a function of 4 variables up to degree 35 requires 82 251 coefficients. On the other hand, the process of construction of normal form requires the computation of several functions, with a total of 2 549 782 coefficients.

We start with the Hamiltonian in form (2), and expand it in Taylor series around the origin. To this Hamiltonian we apply the linear transformation with the matrix (6), using as parameter μ the value 9.5387536×10^{-4}, corresponding to the Sun–Jupiter case. The harmonic frequencies turn out to be

$$\omega_1 \simeq 9.9675752552 \times 10^{-1}, \;\; \omega_2 \simeq -8.0463875837 \times 10^{-2}.$$

At the end of this procedure the Hamiltonian has the form of a power expansion $H_2 + H_3 + \ldots + H_{\tilde{r}}$, with H_2 in the diagonal form (5).

The second step consists in determining the generating function which gives the Hamiltonian the normal form (12). At the same time, for $3 \leq r < \tilde{r}$ we compute the coefficients of the first term $H^{(r)}_{r+1}$ of the remainder that we need in order to use (21). This is an easy byproduct of the program computing the normal form.

Having determined the generating sequence, we can determine the canonical transformation (14) as well as its inverse (if we need) by just applying the operator T_χ defined by (10).

The final step consists in estimating the escape time. This is done via a straightforward application of the procedure at the end of the previous section. The results will depend, of course, on the choice of the parameters R_1, R_2 entering the definition (18) of the norm.

3.2. General results concerning the time and the region of stability

For a general discussion we use the values $R_1 = R_2 = 1$ in the definition of the norm. The results are summarized in Fig. 1. In the upper part the graph of the estimated escape time $T(\varrho_0)$ is reported. Recall that this is the minimal time required for an orbit starting on the domain $\Delta_{\varrho_0 R}$ to reach the border of the domain $\Delta_{\varrho R}$, with ϱ given by (24). The value of r to be used in this formula is the optimal one computed according to the procedure at the end of Sect. 3.1, and can be seen in the lower graph of the figure. One will notice that the upper curve in Fig. 1 is composed of straight segments, the slope of which changes in correspondence with a change in the optimal order r. This is easily understood from formula (23). Of course, the actual value of r can not exceed 34, because we stopped the computation of all functions, including the remainder, at order 35.

Let us now consider the problem of stability for an object in the neighbourhood of the point L_4. Precisely, let us look for stability for a time as long as the age of the universe. Since the time unit is $(2\pi)^{-1}$ the period of Jupiter, the estimated age of the universe is about 10^{10} time units. The corresponding value of $\mathrm{Log}\,\varrho_0$ is -1.536, namely, $\varrho_0 \simeq 2.911 \times 10^{-2}$.

In order to find the meaning of this value for the physical system we should perform all transformations back to the polar coordinates. This is a hard task to be performed by hand, of course, because the domains are defined in normal coordinates.

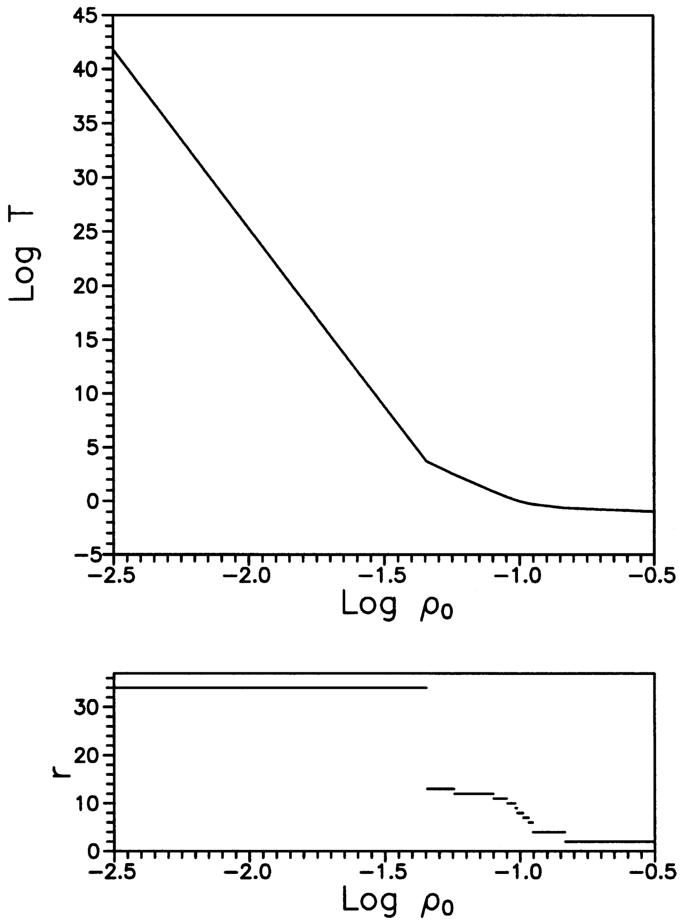

Fig. 1. Upper figure: estimated stability time T as a function of the size ϱ_0 of the initial domain, in Log Log scale. The estimated age of the universe corresponds to $\mathrm{Log}\, T \simeq 10$. Lower figure: optimal order r as a function of ϱ_0

However, a rough evaluation is the following. Accepting that the transformation from the old coordinates to the normal ones

$$x = T^{-1}_{\chi^{(r)}} x' , \quad y = T^{-1}_{\chi^{(r)}} y' , \tag{26}$$

namely the inverse of (14), is well defined on the domain Δ_{ϱ_0}, the image $T^{-1}_{\chi^{(r)}}(\Delta_{\varrho_0 R})$ is close to a polydisk in the old coordinates. An estimate of the size of the latter domain can be computed as follows. Consider the actions $I_j = (x_j^2 + y_j^2)/2$ and $I'_j = (x'^2_j + y'^2_j)/2$ in the old and normal variables, respectively. Consider also the function

$$I_j(x, y)\big|_{x=T^{-1}_{\chi^{(r)}} x' ,\, y=T^{-1}_{\chi^{(r)}} y'} = \big(T^{-1}_{\chi^{(r)}} I_j\big)(x', y')$$

(here, we used the well known property of Lie transforms that the substitution of coordinates can be effected by transforming the functions). The r.h.s. of the latter expression is a power series the lowest order term of which is exactly I'_j; let us write this series as $I'_j + \Phi^{(3)}_j + \Phi^{(4)}_j + \ldots$. Thus, we get the approximate estimate

$$\big|I_j - I'_j\big|_{\varrho_0 R} < \varrho_0^3 \big\|\Phi^{(3)}_j\big\|_R + \ldots + \varrho_0^{\tilde{r}} \big\|\Phi^{(\tilde{r})}_j\big\|_R .$$

Again, this is reasonable provided ϱ_0 is smaller than, say, half of the convergence radius of the series (26). For a further discussion on this point, see Sect. 3.4 below. The r.h.s. of the latter expression can be explicitly computed, since the generating sequence is known, and so the operators $E_1, \ldots, E_{\tilde{r}_{\rm max}}$ are known, too. Using the explicit expressions of the functions and the value above of ϱ_0, we find

$$\left|I_1 - I_1'\right|_{\varrho_0 R} \simeq 5.032 \times 10^{-5} < 0.119\, I_1' \,.$$
$$\left|I_2 - I_2'\right|_{\varrho_0 R} \simeq 1.834 \times 10^{-4} < 0.217\, I_2' \,.$$

We conclude that the stability domain in the old coordinates contains a polydisk of radius $\varrho = \sqrt{\varrho_0^2 - 2|I_2 - I_2'|} \simeq 2.192 \times 10^{-2}$. Now, the coordinates of Jupiter are $(1.307022 \times 10^{-3}, -3.990948 \times 10^{-3}, -4.541613 \times 10^{-3}, 1.717878 \times 10^{-1})$, so that Jupiter is on the border of a polydisk of radius $\varrho_{\rm Jup} \simeq 1.718342 \times 10^{-1}$. Thus, the estimated size of the stability domain is roughly 0.127 times the distance of Jupiter from the point L_4.

3.3. Comparison with the existing asteroids

In order to see how far our estimates can be applied to the existing asteroids we follow Celletti & Giorgilli (1991). Using the 1994 catalog we extract the elements of the Trojan asteroids at a fixed epoch, namely December 14, 1994, J.D.= 2449700.5. Next, we find the elements of Jupiter at the same epoch. Assuming that the orbit of Jupiter is circular, the position of the point L_4 is easily found. Then we transform the elements of the asteroids in cartesian coordinates in a rotating heliocentric system with the z axis orthogonal to the plane of Jupiter's orbit. In order to reduce the problem to a planar one, we project the position of the asteroid in the x, y plane by forcing $z = 0$, and finally we determine the position relative to the point L_4, and the coordinates x_1, x_2, y_1, y_2 which diagonalize the quadratic part of the Hamiltonian. The latter transformation is explicitly described in Sect. 3.1. At this point we adapt the radii R_1, R_2 for the computation of the norm to the initial data of each asteroid by just putting $R_j = \sqrt{x_j^2 + y_j^2}$. Finally, we determine the radius ϱ_0 which ensures stability, in the sense above, for the age of the universe. The results are reported in Table 1. The first column reports the number of the asteroid; the next two columns give the values of the parameters R_1, R_2; the third column gives the estimated value of ϱ_0. By the definition of R_1, R_2 the asteroid is inside the stability region if $\varrho_0 \geq 1$. It is seen that four asteroids fall inside the estimated stability region. On the other hand, in the worst case the estimated size is too small by a factor 30. It is also interesting to remark that for most asteroids an improvement of our estimates by a factor 10 would ensure stability. Thus, the present study constitutes a significant improvement with respect to the previous works. We emphasize that the improvement concerns both the choice of the polar coordinates, which fit better the actual shape of the stability domain with respect to the cartesian ones, and the estimate of the size.

Table 1. Estimated stability region for the known asteroids. The first column gives the catalog number. The second and the third column are the radii R_1, R_2 corresponding to the initial data. The fourth column gives the value of ϱ which ensures stability over the age of the universe; the asteroid is inside if $\varrho > 1$. The fifth column is the optimal order for the wanted time, with a maximum of 34. The table is sorted in decreasing order with respect to the stability parameter ϱ

88181612	$3.130230\,10^{-2}$	$2.101250\,10^{-3}$	1.487790	33
89211605	$3.314960\,10^{-2}$	$1.959370\,10^{-2}$	1.135130	34
41790004	$1.651660\,10^{-2}$	$3.106310\,10^{-2}$	1.100990	34
1870	$3.871410\,10^{-2}$	$1.717610\,10^{-2}$	1.048060	33
2357	$4.234620\,10^{-2}$	$2.850950\,10^{-2}$	$8.470200\,10^{-1}$	34
5257	$3.183610\,10^{-2}$	$4.242410\,10^{-2}$	$7.504500\,10^{-1}$	34
88181912	$7.083260\,10^{-2}$	$6.687100\,10^{-3}$	$6.597200\,10^{-1}$	33
5233	$4.163300\,10^{-2}$	$4.662950\,10^{-2}$	$6.495000\,10^{-1}$	34
4708	$7.099190\,10^{-2}$	$1.894850\,10^{-2}$	$6.275300\,10^{-1}$	32
88181311	$3.914500\,10^{-2}$	$5.262120\,10^{-2}$	$6.063800\,10^{-1}$	34
1871	$5.121390\,10^{-2}$	$4.691570\,10^{-2}$	$6.000700\,10^{-1}$	34
31080004	$7.002890\,10^{-2}$	$2.745100\,10^{-2}$	$5.956600\,10^{-1}$	32
94031908	$1.443780\,10^{-2}$	$6.123500\,10^{-2}$	$5.928600\,10^{-1}$	34
2674	$6.527500\,10^{-2}$	$3.592170\,10^{-2}$	$5.894200\,10^{-1}$	34
88180412	$7.829610\,10^{-2}$	$1.451120\,10^{-2}$	$5.876200\,10^{-1}$	32
88180710	$5.420360\,10^{-2}$	$5.338740\,10^{-2}$	$5.425600\,10^{-1}$	34
88191102	$9.320020\,10^{-2}$	$1.316370\,10^{-2}$	$4.979700\,10^{-1}$	33
88182510	$8.859670\,10^{-2}$	$3.638490\,10^{-2}$	$4.658500\,10^{-1}$	32
2207	$1.747150\,10^{-2}$	$8.093470\,10^{-2}$	$4.487900\,10^{-1}$	34
89201902	$7.247770\,10^{-2}$	$6.844550\,10^{-2}$	$4.163900\,10^{-1}$	34
94031500	$4.552550\,10^{-2}$	$8.358320\,10^{-2}$	$4.075300\,10^{-1}$	34
89212405	$3.008840\,10^{-2}$	$8.992360\,10^{-2}$	$4.005000\,10^{-1}$	34
89211705	$6.369570\,10^{-2}$	$8.261660\,10^{-2}$	$3.826400\,10^{-1}$	34
5907	$9.759570\,10^{-2}$	$6.062860\,10^{-2}$	$3.790100\,10^{-1}$	34
88181411	$9.442780\,10^{-2}$	$6.523500\,10^{-2}$	$3.757900\,10^{-1}$	34
4792	$1.091900\,10^{-1}$	$5.448570\,10^{-2}$	$3.617700\,10^{-1}$	34
88180811	$1.160100\,10^{-1}$	$5.001570\,10^{-2}$	$3.519900\,10^{-1}$	33
3240	$1.362500\,10^{-1}$	$2.751300\,10^{-2}$	$3.359200\,10^{-1}$	32
5638	$1.079900\,10^{-1}$	$8.124580\,10^{-2}$	$3.162000\,10^{-1}$	34
43690004	$1.018300\,10^{-1}$	$9.101430\,10^{-2}$	$3.061600\,10^{-1}$	34
31630002	$1.430300\,10^{-1}$	$4.449490\,10^{-2}$	$3.046700\,10^{-1}$	32
4348	$1.265200\,10^{-1}$	$7.450120\,10^{-2}$	$2.977800\,10^{-1}$	34
4827	$5.760190\,10^{-2}$	$1.213100\,10^{-1}$	$2.868400\,10^{-1}$	34
4722	$1.354100\,10^{-1}$	$8.204770\,10^{-2}$	$2.755600\,10^{-1}$	34
1173	$1.600900\,10^{-1}$	$4.983620\,10^{-2}$	$2.721800\,10^{-1}$	32
10240002	$8.412220\,10^{-2}$	$1.368200\,10^{-1}$	$2.434500\,10^{-1}$	34
2594	$9.109540\,10^{-2}$	$1.393500\,10^{-1}$	$2.360100\,10^{-1}$	34
4829	$6.679660\,10^{-2}$	$1.486500\,10^{-1}$	$2.358500\,10^{-1}$	34
88180812	$1.671900\,10^{-1}$	$9.927530\,10^{-2}$	$2.247200\,10^{-1}$	34
4754	$4.806640\,10^{-2}$	$1.677300\,10^{-1}$	$2.157600\,10^{-1}$	34
4707	$1.470000\,10^{-1}$	$1.294500\,10^{-1}$	$2.138800\,10^{-1}$	34
43170004	$1.345900\,10^{-1}$	$1.403400\,10^{-1}$	$2.106900\,10^{-1}$	34
89210305	$1.881300\,10^{-1}$	$1.057300\,10^{-1}$	$2.032200\,10^{-1}$	34
88182012	$1.910400\,10^{-1}$	$1.094400\,10^{-1}$	$1.989500\,10^{-1}$	34
4805	$1.221800\,10^{-1}$	$1.606700\,10^{-1}$	$1.974600\,10^{-1}$	34
5511	$1.328100\,10^{-1}$	$1.631800\,10^{-1}$	$1.908600\,10^{-1}$	34
89211505	$1.139400\,10^{-1}$	$1.739200\,10^{-1}$	$1.890100\,10^{-1}$	34
20350004	$1.754200\,10^{-1}$	$1.475100\,10^{-1}$	$1.838900\,10^{-1}$	34
884	$1.441100\,10^{-1}$	$1.686700\,10^{-1}$	$1.820300\,10^{-1}$	34
2893	$1.219200\,10^{-1}$	$1.871300\,10^{-1}$	$1.758800\,10^{-1}$	34
1872	$8.983270\,10^{-2}$	$2.039900\,10^{-1}$	$1.723100\,10^{-1}$	34

Table 1. (continued)

88181213	$1.130100\,10^{-1}$	$2.026600\,10^{-1}$	$1.673900\,10^{-1}$	34
51910004	$1.823100\,10^{-1}$	$1.740300\,10^{-1}$	$1.644500\,10^{-1}$	34
4828	$5.293490\,10^{-2}$	$2.216600\,10^{-1}$	$1.637500\,10^{-1}$	34
5130	$5.491180\,10^{-2}$	$2.227200\,10^{-1}$	$1.629200\,10^{-1}$	34
5476	$9.824540\,10^{-2}$	$2.389300\,10^{-1}$	$1.483600\,10^{-1}$	34
88181410	$7.307780\,10^{-2}$	$2.659600\,10^{-1}$	$1.362000\,10^{-1}$	34
88191602	$2.174600\,10^{-1}$	$2.189900\,10^{-1}$	$1.333500\,10^{-1}$	34
2223	$7.393250\,10^{-2}$	$2.835900\,10^{-1}$	$1.278500\,10^{-1}$	34
40350004	$5.551710\,10^{-2}$	$2.853600\,10^{-1}$	$1.272900\,10^{-1}$	34
88180813	$1.353000\,10^{-1}$	$2.759800\,10^{-1}$	$1.255000\,10^{-1}$	34
2241	$6.331600\,10^{-2}$	$2.929700\,10^{-1}$	$1.239700\,10^{-1}$	34
6002	$1.724500\,10^{-1}$	$2.683600\,10^{-1}$	$1.230000\,10^{-1}$	34
88192301	$8.331460\,10^{-2}$	$2.981600\,10^{-1}$	$1.214500\,10^{-1}$	34
3708	$2.174100\,10^{-1}$	$2.654700\,10^{-1}$	$1.171100\,10^{-1}$	34
88190103	$1.383900\,10^{-1}$	$3.006200\,10^{-1}$	$1.162400\,10^{-1}$	34
88181810	$1.248200\,10^{-1}$	$3.104200\,10^{-1}$	$1.144900\,10^{-1}$	34
87171400	$1.661600\,10^{-1}$	$3.084000\,10^{-1}$	$1.106500\,10^{-1}$	34
88191003	$1.184300\,10^{-1}$	$3.296100\,10^{-1}$	$1.089100\,10^{-1}$	34
5119	$1.935700\,10^{-1}$	$3.044200\,10^{-1}$	$1.086700\,10^{-1}$	34
88180512	$2.171600\,10^{-1}$	$2.968600\,10^{-1}$	$1.079300\,10^{-1}$	34
88190703	$1.325300\,10^{-1}$	$3.294100\,10^{-1}$	$1.078800\,10^{-1}$	34
1172	$1.178100\,10^{-1}$	$3.436400\,10^{-1}$	$1.046900\,10^{-1}$	34
31040004	$1.007800\,10^{-1}$	$3.769800\,10^{-1}$	$9.614220\,10^{-2}$	34
4715	$1.572100\,10^{-1}$	$3.739900\,10^{-1}$	$9.453910\,10^{-2}$	34
4832	$2.347900\,10^{-1}$	$3.473200\,10^{-1}$	$9.399910\,10^{-2}$	34
90221206	$2.372900\,10^{-1}$	$4.004800\,10^{-1}$	$8.377690\,10^{-2}$	34
1873	$6.303730\,10^{-2}$	$4.416100\,10^{-1}$	$8.205510\,10^{-2}$	34
88180701	$2.256900\,10^{-1}$	$4.698200\,10^{-1}$	$7.394110\,10^{-2}$	34
41010004	$1.736500\,10^{-1}$	$4.898900\,10^{-1}$	$7.333020\,10^{-2}$	34
617	$2.613400\,10^{-1}$	$4.619700\,10^{-1}$	$7.324830\,10^{-2}$	34
88181510	$1.879700\,10^{-1}$	$5.021700\,10^{-1}$	$7.132770\,10^{-2}$	34
88182511	$2.228500\,10^{-1}$	$5.151500\,10^{-1}$	$6.839150\,10^{-2}$	34
5648	$2.483200\,10^{-1}$	$5.119400\,10^{-1}$	$6.776230\,10^{-2}$	34
88191203	$2.284500\,10^{-1}$	$5.581600\,10^{-1}$	$6.354580\,10^{-2}$	34
5637	$2.749200\,10^{-1}$	$5.709900\,10^{-1}$	$6.081990\,10^{-2}$	34
90202212	$2.078400\,10^{-1}$	$6.032100\,10^{-1}$	$5.963150\,10^{-2}$	34
2895	$1.843700\,10^{-1}$	$6.294600\,10^{-1}$	$5.746530\,10^{-2}$	34
5120	$2.533600\,10^{-1}$	$6.210100\,10^{-1}$	$5.713580\,10^{-2}$	34
3451	$2.285800\,10^{-1}$	$6.288900\,10^{-1}$	$5.705220\,10^{-2}$	34
4791	$1.298900\,10^{-1}$	$6.811600\,10^{-1}$	$5.332690\,10^{-2}$	34
4709	$1.737100\,10^{-1}$	$6.851900\,10^{-1}$	$5.294080\,10^{-2}$	34
3317	$3.085000\,10^{-1}$	$7.051000\,10^{-1}$	$4.989590\,10^{-2}$	34
4867	$2.331200\,10^{-1}$	$7.362600\,10^{-1}$	$4.901550\,10^{-2}$	34
1867	$2.169200\,10^{-1}$	$7.582500\,10^{-1}$	$4.773260\,10^{-2}$	34
88172500	$2.429700\,10^{-1}$	$9.020800\,10^{-1}$	$4.017310\,10^{-2}$	34
1208	$3.619200\,10^{-1}$	$9.975700\,10^{-1}$	$3.597040\,10^{-2}$	34
2363	$2.937300\,10^{-1}$	1.012520	$3.573360\,10^{-2}$	34

3.4. A critical discussion of our approximation method

At some points of our procedure we used approximations of some values by truncating the series at some finite order. We refer in particular to the estimate of the remainder in Sect. 2.3, formula (21) and to the estimate of the deformation in Sect. 3.2. In both cases, the main question is whether the domain of convergence of the series expansions for the transformation of coordinates, the normal form and so on contains or not the polydisk $\Delta_{\varrho_0 R}$ for the values of ϱ_0 that we are considering. For definiteness, let us consider the case $R_1 = R_2 = 1$, and for ϱ_0 the value 2.911×10^{-2}, which, according to our estimates, ensures stability for the age of the universe.

An analytical estimate can be attempted in the following way. We use a result of Giorgilli et al. (1989). It is proven there that if the generating sequence satisfies $\|\chi_s\|_R \le a^{s-3}b$ for all $s \ge 3$ and for some positive a and b, then the coordinate transformation (14) is absolutely convergent in a polydisk $\Delta_{\varrho R}$ with $\varrho = \left(3b + 8a/3\right)^{-1}$. The same is easily proven for the inverse transformation (26), and for the transformation of any other function, for instance the actions I_j or I'_j. Now, since the generating sequence is finite, the constants a and b can be explicitly determined by direct computation of the norms and best fitting. For, it is enough to set

$$b = \|\chi_3\|_R\ , \quad a = \max_{3\le s\le \tilde r}\left(\frac{\|\chi_s\|_R}{\|\chi_3\|_R}\right)^{1/(s-3)}\ .$$

We find $a \simeq 14.164$ and $b \simeq 6.526$, which gives a convergence radius 1.744×10^{-2}, too small by a factor 1.67 with respect to our value of ϱ_0.

However, we stress that the estimates above are purely analytic, and so certainly pessimistic. So, let us proceed heuristically as follows. Consider for instance the series $T_\chi^{-1} I_j$ that we used in Sect. 2.3. Since we know the expansion of that series up to order 35, we can try to evaluate its convergence radius by using some convergence criterion for power series. To this end, having computed the norms of $\Phi_j^{(3)}, \ldots, \Phi_j^{(35)}$, for $3 \le s \le 33$ we fit them with a geometric sequence, i.e., we look for constants c and d such that $\|\Phi_j^{(s)}\| \le cd^s$. We do the same for the coordinate transformations and find the following values:

Function	c	d
χ	6.526	14.164
$T_\chi x_1$	1.0	14.693
$T_\chi x_2$	1.0	14.890
$T_\chi y_1$	1.0	14.782
$T_\chi y_2$	1.0	15.123
$T_\chi^{-1} I_1$	0.5	13.522
$T_\chi^{-1} I_2$	0.5	14.047

The worst case gives $d \simeq 15.123$, which gives an estimated convergence radius $\varrho \simeq 6.612 \times 10^{-2}$. If we accept the latter value as a good indicator of the true convergence radius, we conclude that the value above of ϱ_0 is actually smaller than the convergence radius of the series by a factor 2.2. Thus, our values are safely inside the convergence domain.

Perhaps one might remark that the extra factor 2 that we inserted in the estimate (21) in order to take into account the effect of the rest of the series is actually too small. However, the heuristic argument above shows that replacing 2 by some bigger factor (10 or 100, for instance) should give a correct result. On the other hand, looking at Fig. 1 it is immediately seen that the effect on the estimated value of ϱ_0 is not so relevant: according to formula (22) the curve for T_{ϱ_0} is simply translated, taking slightly smaller values, but the slope of the segment in the region corresponding to $\varrho_0 \simeq 2.911 \times 10^{-2}$ is so high that the value of ϱ giving the age of the universe is changed only a little. For instance, replacing 2 by 100 would change ϱ_0 only by 13 percent.

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