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Rotational effects in turbulence driven by convection

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Abstract. We analyze rotational effects in turbulence driven by convection in the outer regions of an accretion disk, where opacity is mainly given by ice. These effects are explicitly considered through the introduction of an efficiency factor which takes into account inverse energy cascade processes and through the consideration of a centrifugally supported basic state. By adopting a procedure which assigns some dynamics to the anisotropy factor, we obtain an equation that describes how the turbulent structures behave along the disk. Stationary solutions to that equation are only found if the accretion rate, the efficiency factor, the rotational intensity and the Brunt-Väisälä frequency satisfy a known critical condition. If the rotational intensity is below the critical one, there are two branches of solutions for that equation. If the accretion rate is not very high, the effective Rayleigh number for the onset of the convective instability decreases and, in the upper branch, longitudinal scales are always greater than the horizontal scales; in the lower branch, as we approach the surface of the disk, horizontal scales become greater than longitudinal ones. In both branches, centrifugal effects prevail over the effects due to the Coriolis force. If the accretion rate is high, the effective Rayleigh number for the onset of the convective instability increases. In the lower branch, the size of the turbulent structures increases as $z \to 1$; in the upper branch, the size of the turbulent structures decreases as $z \to 1$. In the lower branch, generation of waves occurs all long the disk. In the upper branch, it is confined to regions close to the point where convection sets in. For those high values of the accretion rate, the effects due to the Coriolis force prevail over the those due to the centrifugal forces.

Key words: accretion – turbulence – convection

1. Introduction

After some 20 years since Shakura & Sunyaev's 1973 proposal for parametrizing an assumed existent but unknown viscosity in accretion disks, it appears that among the possible physical processes capable of generating such viscosity one that succeeded to overcome some basic requisites (e.g. to account for luminosities and time scales) is turbulence driven by convection. This does not mean that there are no unsettled questions. Essentially, there are three main questions faced by convection to be accepted as a model for turbulent viscosity generation in some regions of accretion disks,namely the direction and efficiency of angular momentum transport (Vishniac & Diamond 1992) and the nature of turbulence generated by convection in a very fast rotating medium. The question of the efficiency of angular momentum transport is related to the value of the viscosity parameter and, according to Ruden et al. (1988) $\alpha \ll 1$ and negative. Based on linear normal mode analysis of axisymmetric disturbances, it has been claimed (Korycansky 1992) that angular momentum is transported inwards and, since for general nonaxisymmetric perturbations the radial wavenumber diverges linearly with time, convection will unavoidably transport angular momentum inwards (Goodman & Ryu 1992). For Bernard-Rayleigh convection, in a keplerian thin disk, this drawback is related to the apparent incompability between positive growth rate and positive angular momentum flux, i.e., if $\mathscr{L}_z$, angular momentum, < 0 differential rotation acts as a sink of energy in convective regions, thus inhibiting convective instabilities.The answers we have for the first two questions are dependent on the convective cell dynamics, quasi-linear approach and plane wave solution (Vishniac & Diamond 1992). However, the answer to the third question involves a lot of conceptual aspects. Since energy is injected in the largest structures (eddies), the Heisenberg-Kolmogorov theory does not apply and it is very desirable to have some scheme relating the growth rate of the instability that generates turbulence to a viscosity that transfers energy from the largest eddies to the smallest ones where it is dissipated. This kind of enterprise has been undertaken by Canuto and Goldman (1984) and Canuto et al. (1984) who, using the work of Ledoux et al. (1961), were able to derive an expression for the energy spectral function and, through this, a viscosity related to growth rate of the fastest growing unstable mode. Using a formalism developed by Goldreich & Schubert (1967), neglecting compressibility effects and molecular viscos-

Send offprint requests to: C. Meirelles Filho (at present address) ity, they have obtained two unstable modes, highly dependent on an anisotropy factor, being their contribution to the convective instability taken as a whole. It is worth remarking here, that the whole formalism developed by these authors depends heavily on this anisotropy factor, which is related to the size of the largest eddies and the to size of the convective region (extent in z direction). With rotation, for finite Taylor, Rayleigh and Prandtl numbers, convection sets in the disk only for the anisotropy factor exceeding a certain minimum value. For the Prandtl number equal to zero convection sets in for any value of anisotropy factor, independent of the rotation rate, i.e., independent of the Rossby number.

However, as has been analyzed by Dubrulle & Valdettaro (1992), in turbulent flows with rotation, some new effects come into play and may modify the standard picture we have about turbulence. In that respect, the value of the Rossby number is of crucial importance since it will determine the transition between regimes where rotation is or is not important. With rotation, there will be a tendency to constrain the motion to the plane perpendicular to the rotation axis and, as a consequence, the horizontal scale will increase as compared to the longitudinal one, which means that the turnover time in this direction will increase. The net effect is that the energy cascade down process is hindered by rotation. As a matter of fact, when rotation is present one observes two cascades: an enstrophy (vorticity) cascade from large scales to small scales and an inverse energy cascade from small scales to large scales. Since the first process is not efficient on transporting energy to the dissipation range, what we see is an energy storage in the large structures at the expenses of the small structures. This kind of behavior has been confirmed experimentally by Jacquin et al. (1990), who observed that, with rotation, $L_{hor} \approx {R_o}^{-\gamma} L_z$, where γ is a parameter that depends on the Reynolds number and measures the the influence of rotation on turbulence and R_o is the Rossby number. For γ very large we obtain, in the inertial range, a spectrum that goes like k^{-3} instead of the usual Kolmogorov's $k^{-\frac{5}{3}}$ spectrum. In reality, when rotation is dominant, energy gets stored into inertial, waves which propagate it essentially in the longitudinal direction. In that case, we can no longer assign just one viscosity to the fluid and, what is the most important, the concept of viscosity loses its meaning since we no longer have local dissipation of energy. According to Dubrulle & Valdettaro (1992) $R_o = 1$ is the borderline between these two scenarios: for $R_o > 1$ turbulence is not affected by rotation, for $R_o < 1$ it will be greatly affected. It is worth to mention that compressibility effects will also affect the turbulence through the generation of waves, shocks, etc. Numerical simulations of convection inside spherical shells carried out by Valdettaro & Rieutord (1991) are highly suggestive of the existence of this inverse energy cascade. However, the boundary conditions, the aspect ratio, the value of the Prandtl number and the Froude number they used make quite questionable the application of their results to the accretion disks.

It seems to us that the importance of rotational effects are not transparent in Canuto & Goldman's (1984) and Cabot et al. (1987) treatment of turbulence generated by convective instability, in the sense that no discussion about the behavior of the characteristic scale lengths in the problem under the influence of rotation is made, nor the conditions under which there will be local energy dissipation and an effective viscosity can be assigned to the flow. Also, it is not apparent in their results effects such as inverse energy cascade with consequent diminishing of the angular momentum transport efficiency or,even, how the spectrum in the inertial zone, i.e., Kolmogorov's spectrum, is affected by rotation. Though Dubrulle & Valdettaro's analysis be physically correct, they have neglected the role played by centrifugal acceleration and shear. As has been pointed out by Koschmieder (1967), even at small Froude numbers, effects due to centrifugal acceleration have to be taken into account. Daniels (1980) has studied motions caused by vertical instability coupled to circulation due to centrifugal forces, in a bounded circular layer. Assuming the forces that drive vertical motions are comparable to the centrifugal ones, he discovered that, depending on the parameters of the system, convection may set in at a Rayleigh number below the critical one. It seems, as we intend to show here, that due to this role played by rotation, turbulence is affected by rotation in a very peculiar way and, in some special circumstances, it succeeds forming very anisotropic structures, with horizontal scales much smaller than the longitudinal ones, in such a way as to overcome rotational effects. As far as the efficiency of angular momentum transport is concerned, the value of the viscosity parameter is highly affected, even if the Rossby number is much greater than 1.

In this work, we shall address the questions of the nature of turbulence generated by convective instability as well as the question of the efficiency of angular momentum transport. By no means are we implying that the question of the direction of angular momentum transport is unimportant, we hope to address this problem in future contributions. Our main idea is to improve Canuto & Goldman's (1984) treatment for turbulence generated by convection by taking into account rotational effects as described in Dubrulle & Valdettaro's (1992) paper for turbulent flows with rotation. Since our basic state is a centrifugally supported disk, the effects due to the centrifugal acceleration are implicitly taken into account. This basic state being axisymmetric, doesn't allow for the account of effects due to precession (Knobloch 1993). The physical conditions we shall be interested are similar to those supposedly to exist in the outer portions of the inner disklike region of the primordial solar nebula, where opacity is mainly given by ice. In the treatment we shall be giving here for the large scale turbulence, our starting point is the dispersion relation for the convective unstable perturbations which we manage to transform into an equation relating the Rossby number and the anisotropy factor, in which explicit account for energy going into waves is taken. By matching this expression with that obtained with the solution for the disk equations results in an equation for the anisotropy factor, its solution dependent upon conditions to be satisfied by the rotational rate, accretion rate, Brunt-Väisälä frequency and the efficiency for driving energy into turbulence. Ou main results stem from the analysis of this equation.

A brief outline of the procedure we shall be employing in this work is as follows:
(A.) We present our main arguments to modify Canuto & Goldman's treatment for turbulence and convective instability by taking into account the efficiency of energy going into turbulence. This is done in Sect. 2, in which we present a review of their formalism.
(B.) Under suitable appoximations, the solutions for the disk equations are obtained in Sect. 3.
(C.) In Sect. 4, we match the solution for the Rossby number coming from the solutions of the disk equations with the solution coming from the dispersion relation, obtaining that way an equation for the anisotropy factor. The conditions for the existence of solutions are analyzed.
(D.) In Sect. 5, we obtain the structure of the convective region. The nature of the turbulent structures are analyzed. At the light of the correlation time, a brief discussion of the results is made. In Sect. 7, we present our conclusions.

2. Brief comments about turbulence and convective instability

We shall adopt a theory of turbulence that takes into account the nature of different feeding mechanisms in the energy equation. So, we shall be using the work of Ledoux et al. (1961), in the interpretation given by Canuto et al. (1984) which,essentially, consists in the prescription of a closure relation that yields an expression for the viscosity once we know the growth rate and the wavenumber of the instability that generates turbulence. We, however, shall improve this formulation a little bit by showing, explicitly, the dependence of the Rossby number on the wavenumber interval within which turbulence is fed into the system. To make our point clear, we shall assume that energy is injected into turbulence in the wavenumber interval $k_0 \leq k \leq k_f$ (injection region)and transported by nonlinear processes to the inertial and dissipation regions $k_f \leq k \leq \infty$. Since our main concern is the injection zone, we shall assume that this region smoothly joins the injection zone at k_f, and the flux of energy is constant throughout. In the injection zone, we write a kind of Heisenberg equation

$$\gamma_0 \int_{k_0}^{k_f} F\, dk = \nu_t \int_{k}^{k_0} k^2 F\, dk\,, \tag{1}$$

where F is the energy spectral density and we have assumed that the rate of energy deposition, γ_0, is constant in that interval. We have, also, neglected the contribution from the molecular viscosity and assumed that the rotational efects enter only through the growth rate. The most general expression for the transport coefficient, or turbulent viscosity, ν_t, is

$$\nu_t = B \int_{k}^{\infty} (k^s\, F^n)\, F\, dk\,, \tag{2}$$

where $B\,(k^s\, F^n)$ has the dimension of time, the correlation time. Since we don't have a theory for it, let us assume, for the moment, equality between the turnover time, the correlation time and the time characteristic of the growth of the instability that generates turbulence. Later on, we shall analyze the implications of such an assumption. We, then, set

$$s = 0$$
$$n = 0$$
$$B = \frac{1}{\gamma_0}$$

The solution to Eq. (1) is

$$F = \frac{{\gamma_0}^2}{k_0\, k^2}\,. \tag{3}$$

We now differentiate Eq. (1) respect to k and set $k = k_0$ to obtain

$$\nu_t = \frac{\gamma_0}{{k_0}^2}\,, \tag{4}$$

which is Canuto & Goldmann's (1985) prescription for the turbulent viscosity. Applying Eq. (4) to the usual α prescription we obtain

$$\alpha = \frac{1}{n^2\,(1+x)} \left(\frac{l}{c_s}\right) \gamma(k_0) \left(\frac{L}{H}\right)^2, \tag{5}$$

where c_s is the local sound velocity, H is the scale height of the disk, L is the size of the convective region, n is an integer related to k_z and L and

$$x = \frac{{k_y}^2 + {k_x}^2}{{k_z}^2}\,. \tag{6}$$

If we now adopt the linearization procedure proposed by Goldreich & Schubert (1967), for axisymmetric disturbances, in the version given by Canuto et al. (1984), we obtain the following equation satisfied by the growth rate for the mode that feeds energy into the system,

$$\gamma^3 + \chi\, k^2\, \gamma^2 - \left(\frac{R_a\, x}{T^*\,(1+x)} - \frac{1}{1+x}\right) \Omega^2\, \gamma + \frac{\Omega^2}{1+x}\, \chi\, k^2 = 0\,, \tag{7}$$

where R_a is the Rayleigh number given by

$$R_a = \frac{g_z\, \bar{a}\, \beta\, L^4}{\nu\, \chi}\,, \tag{8}$$

T^* is the effective Taylor number given by

$$T^* = \frac{4\, L^4\, \Omega^2}{\nu^2} \left(1 + \frac{R\, \frac{\partial}{\partial R}\, \Omega}{2\, \Omega}\right) \tag{9}$$

σ being the Prandtl number given by

$$\sigma = \frac{\nu}{\chi}\,, \tag{10}$$

with g_z, $\bar{a}$, β, χ and ν being respectively the z-component of gravitational acceleration, coefficient of thermal expansion, temperature gradient excess over the adiabatic gradient, thermometric conductivity (thermal diffusivity) and kinematic viscosity. Let us now analyze the conditions under which the validity of the assumptions we have made holds. In steady state, the energy injected into turbulence equals the energy dissipated, i.e.,

$$\frac{9}{4}\,\nu_t\,\Omega^2 = \gamma_0 \int_{k_0}^{k_f} F\,dk\,, \tag{11}$$

which results in

$${R_0}^2 = \frac{9}{16}\left(1 - \frac{k_0}{k_f}\right)^{-1}. \tag{12}$$

In the above equation R_0 is the Rossby number defined by

$$R_0 = \frac{\gamma_0}{2\,\Omega}\,. \tag{13}$$

From Eq. (12) we see that $R_0 \geq 0.75$ which, obviously, has to be the range of validity of this formulation. However, as it will be shown, the Rossby number is proportional to the Brunt-Väisälä, frequency which $\rightarrow 0$ as $z \rightarrow 0$. Therefore, in that formulation, there will be a large z extent, above the midplane, for which there will be no solution. A way to remedy this, is to realize that the effects of rotation enter, not only through its implicit effects in the growth rate of the convective instability but also, explicitly, through generation of waves, i.e., wave turbulence. That is to say, not all the energy goes into turbulence, but a part of it goes into waves. We, therefore, rewrite Eq. (1) as

$$Q_0\,\gamma_0 \int_{k_0}^{k_f} F\,dk = \nu_t \int_{k}^{k_0} k^2\,F\,dk\,, \tag{14}$$

which implies

$$\nu_t = Q_0\,\gamma_0\,{k_0}^{-2}\,, \tag{15}$$

where the efficiency for energy deposition into turbulence is Q_0 and $1 - Q_0$ is the efficiency for energy deposition into waves. This kind of procedure is equivalent to take into account the inverse cascade process. It should be remarked that the equality between the turnover and correlation times is no longer valid, i.e.,

$$t_t = \frac{1}{\gamma_0}$$
$$B = \frac{8\,Q_0\,R_0}{9\,\Omega}\left(1 - \frac{k_0}{k_f}\right), \tag{16}$$

where t_t and B are, respectively, turnover and correlation times. Now the energy that goes into turbulence, i.e. the efficiency, depends on the Rossby number. We postpone this discussion. The definition of the viscosity parameter will change, accordingly, to

$$\alpha = \frac{Q_0}{n^2\,(1+x)}\left(\frac{l}{c_s}\right)\gamma_0\left(\frac{L}{H}\right)^2, \tag{17}$$

Our next task will be to solve the disk equations assuming our system can settle down in a stationary state. With the solutions for the physical variables in that state we can have all the similarity numbers and so insert them into Eq. (7) and see if our starting system is convectively unstable ($\gamma > 0$).

3. The solution for the disk equations

To obtain the solutions for the disk equations, we shall adopt a different procedure concerning the mass conservation equation, which in steady state under the assumption of azimuthal symmetry and hydrostatic equilibrium in z direction, in cylindrical coordinates, we write

$$\frac{1}{R}\,\frac{\partial}{\partial R}\,\rho\,R\,V_r = 0\,, \tag{18}$$

ρ, V_r, R being, respectively, mass density, radial velocity and radial distance. Since we shall be solving this equation with null boundary conditions for the density at $z = H$ (the semi-scale height of the disk), and for the radial velocity at the end of convective region, i.e., $z = L$, we have

$$\rho\,V_r = \frac{C(z)}{R}\,, \tag{19}$$

where we'd rather working with volumetric density, instead of column, because we shall be considering the z-structure of the accretion disk and no simplification would result if we do otherwise. Clearly, to the lowest order in z, subject to the null boundary conditions,

$$C(z) = -\frac{3\,\dot{M}}{8\,\pi\,L}\left(1 - \left(\frac{z}{z_t}\right)^2\right), \tag{20}$$

where we have used for the accretion rate

$$\dot{M} = -4\pi\,R\,L \int_0^{z_t} \rho\,V_r\,dz'\,, \tag{21}$$

z is in units of semi-scale height.

Now, from the angular momentum conservation equation, we have

$$\rho\,\nu_T = 9\,\frac{\dot{M}\,S}{16\pi\,L}\left(1 - \left(\frac{z}{z_t}\right)^2\right), \tag{22}$$

ν_T being the turbulent kinematic viscosity and

$$S = 1 - \left(\frac{R}{R_*}\right)^2,$$

R_* is the internal radius of the disk. Null boundary condition for the torque has been used for the torque at R_*. It should be stressed that our dynamical disk is the convective region, extending from the symmetry plane to $z = z_t$. Above this, we have the radiative region. We, finally, write the hydrostatic equilibrium equation,

$$\frac{\partial}{\partial z}\,\rho\,T = -\frac{m_h}{k}\,\Omega^2\,H^2\,\rho\,z\,, \tag{23}$$

and the energy equation, under the radiative diffusion approximation,

$$-4/3\,\frac{\sigma T^3}{\rho\, k_{op}\, H}\,\frac{\partial}{\partial z}\,T$$
$$-\frac{\dot{M}\,S}{4\pi\,L\,n_p}\left(1-\left(\frac{z}{z_t}\right)^2\right)\left(\frac{c_p}{H}\,\frac{\partial}{\partial z}\,T+\Omega^2\,H\,z\right)$$
$$=\frac{9}{16\pi}\,\frac{\dot{M}\,S\,\Omega^2}{z_t}\,z\left(1-1/3\left(\frac{z}{z_t}\right)^2\right), \tag{24}$$

where Ω is the Keplerian angular velocity, k_{op} is the opacity, T is the temperature, m_h is the hydrogen mass, k is the Boltzmann constant, σ is the Stefan-Boltzmann constant, n_p is the turbulent Prandtl number, c_p is the specific heat at constant pressure. We have used a perfect gas law for the equation of the state. The right hand side of the energy equation is obtained integrating the heat generated from $z' = 0$ to $z' = z$.

The solution for the disk equations is highly dependent on the kind of process responsible for the opacity which, usually, depends on the values of the density and temperature. Most often, opacity is expressed in powers of density and temperature. However, these powers and constants entering these expressions are dependent on the region of temperature and density we are. So, to avoid these complexities, we shall specialize to regions with $T \leq 160K$, for which the opacity is mainly given by ice, i.e.,

$$k_{op} = 2\times 10^{-4}\,T^2\,, \tag{25}$$

(Lin & Papaloizou 1980).

Making the substitution

$$T = T_c\,t\,, \tag{26}$$

where T_c is the central temperature, $0 \leq t \leq 1$, in the energy equation, results for the density

$$\rho = -A\,f\,t\,\frac{\partial}{\partial z}\,t\,, \tag{27}$$

with

$$f^{-1}=\frac{1}{n_p\,z_t}\left(1.25\,b\,\frac{\partial}{\partial z}\,t+z\right)$$
$$\times\left(1-\left(\frac{z}{z_t}\right)^2\right)+\frac{9}{4z_t}\left(1-1/3\left(\frac{z}{z_t}\right)^2\right)z\,, \tag{28}$$

$$A=\frac{4.78\,{T_c}^2}{\dot{M}\,S\,H\,\Omega^2}\,, \tag{29}$$

$$T_c = b\,\frac{m_h}{2\,k}\,\Omega^2\,H^2\,. \tag{30}$$

As we shall see later on, b is a kind of eigenvalue that determines the behavior of t along z. Inserting these substitutions into the hydrostatic equilibrium equation yields

$$\frac{\partial}{\partial z}\,f\,t^2\,\frac{\partial}{\partial z}\,t=-2/b\,f\,t\,\frac{\partial}{\partial z}\,t\,z\,. \tag{31}$$

We now rewrite this equation as

$$\frac{\partial}{\partial z}\ln\left(f\,t^2\,\frac{\partial}{\partial z}\,t\right)=-\frac{2}{b\,t}\,z\,, \tag{32}$$

using the behavior of t as $z\to 0$ and keeping terms only to the sixth order in z, we obtain the following approximate solution

$$q\,t^{1/q}=C_0\left(0.5/z_t\left(1/n_p+2.25+\frac{2.5\,b}{n_p\,{z_t}^2}\right)z^2\right.$$
$$-\frac{0.25}{{z_t}^3}\left(\frac{1}{n_p}+0.75-\frac{2.5}{n_p}\,a\,b\right)z^4+2.5/6\,\frac{b\,c}{n_p}\,z^6$$
$$\left.+\frac{1.25}{z_t\,n_p}\,b\,t\left(1-\left(\frac{z}{z_t}\right)^2\right)\right)+C_1\,, \tag{33}$$

a and c being, respectively, the second and the fourth derivative of t, evaluated at $z = 0$. C_0 and C_1 are integration constants to be determined under suitable boundary conditions and

$$q=\frac{a\,b}{3a\,b+1}\,. \tag{34}$$

We now make the assumption that convection develops throughout the whole z extent, except in a very narrow region close to the surface of the disk. We,therefore, set $z = 1$ and impose the following boundary conditions: $t = 1$ at $z = 0$, $t = 0$ at $z = 1$ and $\frac{\partial}{\partial z}\,t = -0.8/b\,e$ at $z = 1$. The boundary condition on the derivative, with $e > 1$, is the only compatible with the previous boundary conditions on the density, radial velocity and laminar regime at the surface. A detailed analysis of Eq. (34) shows that compatibility with the condition on the derivative is only met if $q = 1$ or $1/3 \leq q \leq 0.5$. In the following we analyze both cases separately.

q = 1

Physical solutions are only found for $1.85 \leq n_p$. a, b, c, C_0, C_1, t and $\frac{\partial}{\partial z}\,t$ are given by

$$a=-0.5/b$$
$$c=-0.25\left(\frac{0.75\,n_p-0.25}{2.25\,n_p-0.25}\right)$$
$$C_0=-12\,n_p\left(-0.75+11.25\,n_p+5\,b\,c\right)^{-1}$$
$$C_1=1-1.5/n_p\,b\,C_0\,, \tag{35}$$

$$37.5\,b^2\left(0.75\,n_p-0.25\right)-\Big(5\;e\left(0.75\,n_p-0.25\right)$$
$$+60\left(1.5\,n_p-1.25\right)\left(2.25\,n_p-0.25\right)\Big)\,b$$
$$+e\left(45\,n_p-3\right)\left(2.25\,n_p-.25\right)=0\,, \tag{36}$$

$$t=C_0\Big(0.5\;\left(1/n_p+2.25\,b/n_p\right)\;z^2$$
$$-0.25\left(2.25/n_p+0.75\right)z^4+2.5/6\,n_p\,b\,c\,z^6\Big)$$
$$\times\left(1-1.25/n_p\,b\,C_0\right)^{-1}\,, \tag{37}$$

$$\frac{\partial}{\partial z} t = C_0 \left(\left(1/n_p + 2.25\, b/n_p\right) z - \left(2.25/n_p + 0.75\right) z^3 + 2.5/n_p\, b\, c\, z^5 \right) \times \left(1 - 1.25/n_p\, b\, C_0 \left(1 - z^2\right)\right)^{-1} . \quad (38)$$

It should be mentioned that, since the solutions we have are obtained under the assumption of constant heat generation,i.e.,

$$Q^+ = 9/16\, \dot{M}\, S\, \Omega^2 \left(1 - 1/3 \left(\frac{z}{z_t}\right)^2\right) z\,, \quad (39)$$

which does not depend explicitly on the Prandtl number n_p, the temperature gradient decreases as we increase n_p and so does b.

$$1/3 \leq q \leq 0.5$$

Physical solutions exist only for $0.5 \leq n_p \leq 1.3$. a, b, c, q, C_0, C_1 are given by

$$\begin{aligned} a &= 1/b \left(-0.4 + 0.8\, e - 1.5\, n_p - 2\, b\right) \\ c &= 1/b \left(0.4 - 0.8\, e + 0.9\, n_p + b\right) \\ q &= \frac{a\, b}{3a\, b + 1} \\ C_0 &= -12\, n_p\, q \left(-2 + 4.5\, n_p - 10\, b\right)^{-1} \\ C_1 &= q - 1.25/n_p\, b\, C_0\,, \end{aligned} \quad (40)$$

$$0 = 5\, b^2 + \left(-7\, e + 8.25\, n_p\right) b + 2.5\, e - (1.5 + 2.4\, e)\, n_p + 1.8\, {n_p}^2\,. \quad (41)$$

Since q is not an integer we do not have analytic expressions for both t and its derivative. For the same reason we have already mentioned, b and $\frac{\partial}{\partial z} t$ decrease when n_p increases.

4. The solution for the growth rate

The growth rate is highly dependent on the anisotropy factor. In Cabot, Canuto and Hubickyj it is obtained by imposing it maximizes the growth rate. We shall avoid this procedure and shall look for a different one which assigns some dynamics to the anisotropy factor. From Canuto et al. (1985) prescription for the viscosity, together with our results of the previous section, we write

$$\frac{2\, Q_0\, R_0\, \Omega}{{k_0}^2} = 1/4\, \pi\, \frac{\dot{M}\, S \left(1 - z^2\right)}{H\, \rho}\,. \quad (42)$$

Now, using the expression for the thermometric diffusivity for $T \leq 160$ K, i.e.,

$$\chi = 1.4 \times 10^{-8}\, \frac{T}{\rho^2}\,, \quad (43)$$

we obtain

$$\chi\, {k_0}^2 = -3.5 \times 10^{-7}\, \frac{T_c\, Q_0\, H\, R_0\, \Omega}{A\, \dot{M}\, S \left(1 - z^2\right) f\, \frac{\partial}{\partial z} t}\,. \quad (44)$$

Substituting this last expression in the dispersion relation, Eq. (7), results in

$$4\, {R_0}^2 \left(1 - Z_0\right) = \frac{1}{1 + x} \left(N^2\, x - 1 + 2\, Z_0\right), \quad (45)$$

where

$$Z_0 = 1.75 \times 10^{-7}\, \frac{H\, Q_0\, T_c}{A\, \dot{M}\, S \left(1 - z^2\right) f\, \frac{\partial}{\partial z} t}\,, \quad (46)$$

and N is the Brunt-Väisälä frequency in units of the Keplerian angular velocity. Making now the simplifying assumption that the z extent of the convective region fits only one wave, we use Eq. (41) to write

$$2\, R_0 = -\frac{\pi/4\, \dot{M}\, S \left(1 - z^2\right)(1 + x)}{Q_0\, A\, \Omega\, H^3\, f\, t\, \frac{\partial}{\partial z} t}\,, \quad (47)$$

which substituted into Eq. (45) yields the following equation for the anisotropy factor

$$\left(\frac{\pi}{4}\right)^2 \frac{\left(\dot{M}\, S\right)^2 \left(1 - z^2\right)^2 (1 + x)^3}{{Q_0}^2\, A^2\, f^2\, t^2 \left(\frac{\partial}{\partial z} t\right)^2 \Omega^2\, H^6} \left(1 - Z_0\right) = \left(N^2\, x - 1 + 2\, Z_0\right). \quad (48)$$

The required condition for this equation to have a physical solution for the anisotropy factor is

$$\frac{1.88 \times 10^{32}}{b^4}\, \frac{\left(\dot{M}\, S\right)^4 \left(1 - z^2\right)^2 \left(1 - Z_0\right)}{{Q_0}^2\, \Omega^6\, H^{12}\, f^2\, t^2 \left(\frac{\partial}{\partial z} t\right)^2} \times \left(N^2 - 1 + 2\, Z_0\right)^2 \leq N^6\,. \quad (49)$$

It is apparent from this inequality that the larger is the value of the expression appearing to the left of the expression containing the square of the Brunt-Väisälä on the left hand side, the larger will be the value of the Brunt-Väisälä required for the onset of the turbulent convective regime. However, to solve this inequality and find out some effects of rotation in the establishment of this regime, we should have the knowledge of H which, at this moment, is lacking yet. A reasonable approximation for H, very far away from the central object, such as the present situation, is

$$H \approx 0.1\, R \quad (50)$$

(Cabot et al. 1985). Since the dependence on the radial distance only enters through dependence on Ω, we may write

$$H = 0.1\, (G\, M)^{\frac{1}{3}}\, \Omega^{-\frac{2}{3}}\,. \quad (51)$$

Applying this expression into Eq. (49) results in

$$5.8 \times 10^{13} \left(\frac{\dot{M}\, S\, \Omega^{1/2}}{M}\right)^4 \Psi \left(N^2 - 1 + 2\, Z_0\right)^2 \leq N^6\,, \quad (52)$$

where Ψ, defined by comparison with Eq. (49), does not depend on the radial distance. In this inequality $\dot{M}$ is expressed in units of 3.16×10^{18} g s^{-1} and M in solar masses.

A rapid inspection on this inequality reveals:
(1) the higher is the rotational rate the harder will be the establishment of a turbulent convective region. That is to say, it will require a larger temperature gradient or a higher value of z, which means the extent of a non convective region above the midplane will be larger. Higher values of the rotational rate will imply, for sure, in low efficiency, therefore, decreasing the amount of energy that goes into turbulence,
(2) high values of the accretion rate hinder the establishment of the convective region, requiring larger values of the Brunt-Väisälä frequency, which implies an increase of the non convective region above the midplane,
(3) high values of the mass of the central object will favor the onset of a convective regime as well as the decrease of the non convective region close to the midplane,
(4) the larger is the extent of the convective region, the lesser is the temperature gradient required for its establishment.

For the Brunt-Väisälä frequency satisfying in Eq. (52), Eq. (47) will have two solutions for the anisotropy factor and just one in the critical situation (equality). Let us solve for the critical condition. In that case x is given by

$$x = -1 + \left(\frac{N^2}{3\,D_0}\right)^{1/2}, \tag{53}$$

where

$$D_0 = \frac{\pi^2}{16}\,\frac{\left(\dot{M}\,S\right)^2\left(1-z^2\right)^2(1-Z_0)}{\Omega^2\,{Q_0}^2\,A^2\,H^6\,f^2\,t^2\left(\frac{\partial}{\partial z}\,t\right)^2}\,. \tag{54}$$

Substitution into Eq. (46) for the Rossby number yields

$$R_0 = \frac{0.29}{(1-Z_0)^{1/2}}\,N\,. \tag{55}$$

Since $1 - Z_0 > 1$, and the Brunt-Väisälä frequency goes to zero as $z \rightarrow 0$, $R_0 < 1$ in a large fraction of the convective region. However, what matters is the value of x. If $x < 1$,in that portion of the convective region, through the effects of the Coriolis force, rotation will dominate the dynamics. The inverse cascade process will prevail over the direct one, thus inhibiting dissipation. The turbulent horizontal scale is larger than the vertical scale, i.e., $0 \leq x \leq 1$ and so the concept of viscosity loses its meaning. If the rotational intensity is above the critical one, no stationary solutions are found to the disk equations. If the the rotational intensity is below the critical, Eq. (50) will admit two solutions for the anisotropy factor, one larger and the other smaller than the solution we just obtained. We are now interested in this larger solution, to see if it can change the picture we have drawn. To make our study as analytical as possible, let us write

$$D_0 = \frac{4}{27}\,\xi\,N^6\left(N^2+1-2\,Z_0\right)^{-2}, \tag{56}$$

where $\xi = 1$ corresponds to the critical regime. We want to study the regime $\xi \ll 1$. Inserting into Eq. (49), yields

$$\frac{4}{27}\,\xi\,N^6\,(1+x)^3 = \left(N^2-1+2\,Z_0\right)^2\left(N^2\,x-1+2\,Z_0\right). \tag{57}$$

Let us assume we have a solution $x \gg 1$ for $z \rightarrow 0$. As we know, in that limit, $N \rightarrow 0$. Therefore, in that limit,

$$x \approx \frac{9}{2}\,(1-2\,Z_0)\,N^{-2}\,\xi^{-1/3}\,, \tag{58}$$

which implies

$$R_0 \approx 0.5\,N\,(1-Z_0)^{-1}\,. \tag{59}$$

Solving for $z \rightarrow 1$, $N \rightarrow \infty$, we have

$$(1+x)^3 \approx 4\,\xi^{-1}\,x\,, \tag{60}$$

or

$$x = 2\,\xi^{-1/2}\,, \tag{61}$$

which also implies Eq. (58). However, now the Brunt-Väisälä frequency is much greater than the Keplerian angular velocity and $R_0 > 1$. A cautious analysis of Eq. (57) reveals the absence of solutions with $x < 1$ and $N < 1$.

5. The structure of the convective region

In the last section we have dealt qualitatively with the rotational effects in the convective region of the disk where the opacity is given by ice. In the following we shall work out the actual structure of this region in a more quantitative way, obtaining its limits both in r and z directions. The expression we use for the opacity is valid for $T \leq 160$, therefore this region extends beyond R_i, where R_i is the solution of

$$160 = \frac{b\,m_H}{2\,k}\,\Omega^2\,H^2\,, \tag{62}$$

or

$$R_i = \frac{b\,m_H}{320\,k}\,G\,M\,, \tag{63}$$

which, clearly, depends on the turbulent Prandtl number through b. To find out where the convective region starts in z we insert the criticality condition

$$D_0 = \frac{4}{27}\,N^6\left(N^2+1-2\,Z_0\right)^{-2} \tag{64}$$

into the expression for the anisotropy factor (Eq. (53)) to obtain

$$x = -1 + \frac{3}{2}\left(\frac{N^2+1-2Z_0}{N^2}\right), \tag{65}$$

and since $N \rightarrow 0$ for $z \rightarrow 0$, $x \rightarrow \infty$. To see if this is the correct behavior of x as $z \rightarrow 0$, we have to find out if there is a solution of Eq. (64) close to $z = 0$. In the following we shall specialize to the $q = 1$ solution for the disk equations. For $z \rightarrow 0$, $t \rightarrow 1$ and Eq. (38) yields, in that limit,

$$\frac{\partial}{\partial z}\,t = C_0\,\frac{\left(1/n_p+2.25\,b/n_p\right)}{\left(1-1.25/n_p\,b\,C_0\right)}\,z\,, \tag{66}$$

where b, C_0 are given by Eq. (35).Now, calculating the limit as $z \to 0$ for both sides of Eq. (64)and solving for z_c, we obtain

$$z_c \approx 5.76 \times 10^{-2} \left(\frac{\dot{M}^4}{{Q_0}^2 \, R^3} \right)^{\frac{1}{6}} g_n \, , \quad (67)$$

where $\dot{M}$ is expressed in units of $3.16 \times 10^{18} g s^{-1}$, R in units of 5.16×10^{15} cm and we have assumed 1 solar mass for the central object. g_n, function of the Prandtl number, given by

$$g_n = \left(\left(n_p - 1.25 \, b \, C_0 \right) b \right)^{1/6} {n_p}^{-1/3} \times \frac{\left(\left(2.815 b^2 \, C_0 \right) + 2.25 {n_p}^2 - 2.815 b \; C_0 \, n_p \right)^{1/3}}{\left(-1.25 b \, C_0 \left(1 + 2.25 b \right) - \left(n_p - 1.25 b \, C_0 \right) \right)^{1/2}} \, , \quad (68)$$

is of order of 1. Assuming Q_0 is not much smaller than 1, for reasonable values of $\dot{M}$, say $\dot{M} \leq 1$, our approximation $z_t = 1$ is not so bad.

In the previous section our main concern was to find a solution with $x \gg 1$. As a matter of fact, for D_0 lesser than the value given by Eq. (63), i.e. $z > z_c$, Eq. (48) admits 2 branches of solutions: the upper and the lower branches. In the lower branch, the anisotropy factor decreases monotonically from high values to very low values $\ll 1$, towards the surface of the disk. In the upper branch, x increases monotonically from high values to $x \to \infty$, as we come close to the surface of the disk. Geometrically speaking, the anisotropy factor is a measure of the eddy dimensions, i.e.,

$$\frac{l_z}{l_r} \approx x^{1/2} \, , \quad (69)$$

therefore, we are allowed to say that, close to the symmetry plane, the pattern of the generated turbulence is very anisotropic, the horizontal scales being much smaller than the vertical ones in both branches. As we go farther away from the symmetry plane the level of anisotropy increases in the upper branch and decreases in the lower branch. In other words, to overcome the effects of rotation, turbulence succeeds forming, all long the disk, structures much smaller than those it would form in a medium in which rotation is absent. In the lower branch this kind of behavior only prevails close to the symmetry plane. In that branch, for values of z, such that $N > 1$, $x < 1$, i.e., since turbulence does not affect longitudinal scales, the structures are much greater than those formed in a medium in which rotation is absent. It should be remarked that we have found here that turbulence has a behavior not in complete agreement with the one predicted by Dubrulle & Valdettaro's 1992 paper. That is to say, according to these authors, since as $z \to 0$, $R_0 \to 0$, one would expect $x \ll 1$, i.e., longitudinal scales much smaller than horizontal scales, contrarily to what we have found. It should be stressed that this happens to small values of the accretion rate. Our suspicion is that small values of the accretion rate imply great departure from solid body rotation, i.e., the role played by shear is very important. Under our formulation, let us see how Dubrulle & Valdettaro's results emerge. To do so, let us find what is the least value for the anisotropy factor we can have under critical conditions, assuming $\dot{M} > 1$. Inserting x given by Eq. (53) into Eq. (56), with $\xi = 1$, yields

$$N^4 \left(9 - 4 \left(1 + x \right)^2 \right) + 18 \, N^2 \left(1 - 2 \, Z_0 \right) + 9 \left(1 - 2 \, Z_0 \right)^2 = 0 \; . \quad (70)$$

From that equation we see that under criticality, $x \geq 1/2$. We must realize that to have a value of x close to this, $\dot{M} > 1$, which diminishes the effect of shear. To see where the convective region starts in z, we still may use z_c given by Eq. (67), the dependence of z_t being absorbed into g_n. However, $\dot{M}$ now is larger and z_c will not be much less than 1. Therefore, if the system, now, finds itself in a regime for which the rotational intensity is below the critical one, the disk equations admit two branches of solutions as before. In the upper branch, close to the midplane, $x \approx 1/2$ and increases as we go towards the disk surface. Again, the size of the turbulent structures decreases as we approach the surface. In the lower branch, close to the midplane x has the same value as in the upper branch, but decreases as we come closer to the surface. Here, the size of the turbulent structures increases as we approach the disk surface. The effects of rotation are much more pronounced. Dubrulle & Valdettaro's description now applies.

For a given value of $\dot{M}$, the profile of the turbulence is as follows: crolse to a critical radius, the z-extent of the convective region will be minimum, i.e., z_c will be maximum. In the vicinity of this minimum radius, generation of waves will be intense in the lower branch, all long the z-extent. Compared to a medium in which rotation is absent, turbulent structures will be larger. In the upper branch, generation of waves decreases as z increases. Larger structures are confined to the regions close to z_c. The value of the effective Rayleigh number at which convection sets in is large and the effects of the Coriolis force will prevail over the effects of the centrifugal acceleration.. Going farther away from this critical radius, the z-extent of the convective region increases, i.e., z_c decreases. Waves generation diminishes. The size of the turbulent structures will decrease. The value of the effective Rayleigh number at which convection sets in decreases and the effects of the centrifugal acceleration starts prevailing over the effects of the Coriolis force.

If we interpret correlation time, Eq. (16), as the time required for interaction between different scales structures, we conclude that, close to the symmetry plane, turbulence has a large spectrum, i.e., $k_f \gg k_0$. As we approach the surface of the disk, $k_f \to k_0$ the interaction time is zero and $\nu_t \to 0$.

6. Conclusions

We have treated turbulence with rotation in a thin keplerian disk. Convection is the process assumed to generate turbulence, and we have used Canuto et al. (1984) treatment for the convective instability, whose characteristic growth time we have taken equal to the turnover time. To obtain the turbulent viscosity, we have modified Canuto & Goldman's (1984) procedure to take into account rotational effects by introducing, through an efficiency factor Q_0, the effects of inverse cascade process in the energy equation. This kind of procedure, together with the assumption of inequality between turnover and correlation times, allows us to to extend this treatment to regimes with $R_0 < 0.75$. Rotational effects are also taken into account, explicitly, through an isotropy factor x, simply related to the Rossby number and, since in our basic state the disk is centrifugally supported, centrifugal effects are implicitly considered. Differently from Canuto et al. (1984) paper, we do not obtain the anisotropy factor through a growth rate maximization procedure. We, instead, obtain an expression for the turbulent viscosity, using the solution for the disk equations, assuming an opacity mainly given by ice. By forcing equality between this viscosity and that obtained through modification of Canuto & Goldman's (1984) procedure, an equation for the anisotropy factor is obtained. It is shown that the accretion rate, the amount of energy that goes into waves and the intensity of rotation hinder the establishment of a turbulent convective regime. High values of the mass of the central object and Brunt-Väisälä frequency go in the opposite direction. As a matter of fact, there is a condition to be satisfied for these parameters in order to have a solution for the anisotropy factor. Using that condition, we can define a local critical maximum rotation intensity (angular velocity) above which there is no solution for the anisotropy factor. Below it, there are two branches of solutions: the upper and the lower branches. For $\dot{M} < 1$, the convective occupies the whole z extent of the disk and in the upper branch $x \gg 1$ always, which implies longitudinal turbulent scales much greater than the horizontal ones, even close to symmetry plane, where the Rossby number $\rightarrow 0$. In the lower branch, $x \gg 1$ close to the symmetry plane but, as we approach the surface of the disk, $x \rightarrow 0$. In that branch, close to the surface of the disk, the horizontal scales are much greater than the longitudinal ones. This kind of behavior is somehow intriguing in the sense that it contradicts the behavior we should expect from Dubrulle & Valdettaro's (1992) qualitative analysis. It should be stressed that no solution with both x and $N < 1$ was found.

Another kind of solution is obtained for higher values of the accretion rate. In that case the convective region starts for a value of z_c not much smaller than 1. Above that point, there are two branches of solutions for the disk equations, both starting with $x \approx 1/2$and N not much greater than 1. In the upper branch, x increases as we approach the surface of the disk, decreasing the size of the turbulent structures. In the lower branch an opposite behavior is found. For both these solutions, generation of waves is expected close to the point where the convective region starts. In that case, Dubrulle & Valdettaro's description applies.

We would like to mention that our results are somehow similar to results of Daniels (1980), who has reported that under the the effects of the centrifugal acceleration, depending on the parameters of the system, the onset of convection will occur for smaller values of the critical Rayleigh number. We should stress, however, our basic state is centrifugally supported, quite different from Daniels'(1980) pressure supported. Besides this, we have considered a flow with σ (nonturbulent Prandtl number) $\approx 10^{-8}$ and for Daniels (1980), $\sigma = 1$.

To understand what we mean by that, let us write the equation of force for an element of fluid, as seen by an observer in a frame rotating with Keplerian angular velocity, i.e.,

$$\frac{d}{dt}\bar{v} + 2\bar{\Omega}\wedge\bar{v} + \frac{d}{dt}\bar{\Omega}\wedge\bar{r} - \Omega^2\wedge\bar{r} = -\frac{\nabla P}{\rho} + \nabla\Phi + \bar{F}\,, \qquad (71)$$

where $\bar{v}$ is the relative velocity, Φ is the gravitational potential and $\bar{F}$ is any nonconservative force. To zeroth order, in stationary regime, the relative velocity is zero and all the terms in the lefthand side of Eq. (70) are zero but the centrifugal term. To order superior in H/r, the first term in the lhs is of fourth order, while the 2nd and the 3rd terms are of 2nd order, having comparable magnitudes. However, the centrifugal term is zeroth order and much bigger.Therefore, even not making a detailed analysis of the role played by these new terms due to shear and to centrifugal acceleration, they may justify our claim that Chandrasekhar's (1961) formulation may not apply to accretion disks.

Finally, we would like to mention that our results depend on the assumption of constant efficiency for energy going into waves, as well as on the assumed ratio between radial distance and scale height of the disk. In a future contribution we shall consider rotational effects in accretion disks without resort of those assumptions.

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