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Anisotropic diffusion and shear instabilities

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Abstract. We examine the role of anisotropic turbulence on the shear instabilities in a stratified flow. Such turbulence is expected to occur in the radiative interiors of stars, due to their differential rotation and their strong stratification, and the turbulent transport associated with it will be much stronger in the horizontal than in the vertical direction. It will thus weaken the restoring force which is caused by the gradient of mean molecular weight (μ).

We find that the critical shear which is able to overcome the μ-gradient is substantially reduced by this anisotropic turbulence, and we derive an estimate for the resulting turbulent diffusivity in the vertical direction.

Key words: stars: abundances; evolution; interiors; rotation – diffusion.

1. Introduction

The recent observations performed by Herrero et al. (1992) reveal significant abundances anomalies in O and B stars. Even though these observations do not involve a high number of stars, they display a striking correlation between the rotation velocity and the helium and nitrogen overabundances.

Of all instabilities related with the (differential) rotation which have been examined so far, shear instabilities are the most efficient. They operate on a dynamical timescale, unlike the GSF instability (Goldreich & Schubert 1967 and Fricke 1968) or the ABCD instability (Knobloch & Spruit 1983), which both are of double diffusive nature and act on a thermal, hence much longer, timescale. In stellar radiation zones, the shear instability is hindered by the stable stratification, but nothing can prevent it if the flow is sheared in the horizontal direction. For this reason, as was argued by Zahn (1975), one should expect that the differential rotation in latitude induced by the meridian circulation maintains a turbulence which is strongly anisotropic, with a horizontal component of the turbulent diffusivity (D_h) much larger than the vertical one (D_v).

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This turbulence tends to reduce the differential rotation which causes the instability, but it has little effect on the transport in the vertical direction. To obtain such a vertical transport, we have to invoke the instability generated by the vertical shear, although it is impeded by the stratification. Fortunately, the stabilizing effect of the temperature stratification is weakened by radiative losses, as was first pointed out by Townsend (1958), and this mechanism plays an important role in stellar interiors (Zahn 1974).

However in stars the stratification is due not only to the temperature gradient, but also to the variation of the mean molecular weight with depth, in particular outside the regressing convective core of massive stars. These molecular weight gradients could well suppress the shear instability, if it were not for the anisotropic turbulence mentioned above. The horizontal diffusion associated with it will diminish the effect of the vertical molecular weight gradient, much like the thermal diffusivity reduces the effect of temperature stratification. We shall examine in the present paper how this modifies the classical Richardson criterion and we shall give an estimate for the vertical diffusion coefficient.

2. The effect of thermal diffusion on the shear instability

The condition for the shear instability to occur may be obtained by stating that the work done against gravity to exchange two "bubbles" at levels z and $z+\delta z$ with velocities U and $U+\delta U$ must be smaller than the kinetic energy released by equalling their velocities ($cf.$ Chandrasekhar 1961); this yields the familiar Richardson criterion

$$\frac{g}{\rho}\frac{\mathrm{d}\Delta\rho/\mathrm{d}z}{\left(\mathrm{d}U/\mathrm{d}z\right)^2} < \frac{1}{4}, \qquad (1)$$

where z is the vertical coordinate, g the gravity, ρ the density, and $\Delta\rho$ is the density difference between the bubble and the exterior. It remains to evaluate that density gradient $\mathrm{d}\Delta\rho/\mathrm{d}z$.

Quite generally, we may write

$$\begin{aligned}\frac{\mathrm{d}\,\Delta\ln\rho}{\mathrm{d}z} &= \delta\left(\frac{\mathrm{d}\ln T}{\mathrm{d}z} - \frac{\mathrm{d}\ln T'}{\mathrm{d}z}\right) - \varphi\frac{\mathrm{d}\ln\mu}{\mathrm{d}z} \\ &= \frac{1}{H_P}\left[\delta\left(\nabla' - \nabla\right) + \varphi\nabla_\mu\right],\end{aligned}$$

where the $'$ designates the conditions inside of the bubble, the gradients being expressed from now on with respect to the pressure, as is normally done in stellar structure theory:

$$\nabla = \frac{\mathrm{d}\ \ln T}{\mathrm{d}\ \ln P} \quad \text{and} \quad \nabla_\mu = \frac{\mathrm{d}\ \ln \mu}{\mathrm{d}\ \ln P}. \tag{2}$$

We take the usual notations for

$$\delta = -\left(\frac{\partial \ln \rho}{\partial \ln T}\right)_{P,\mu} \quad \text{and} \quad \varphi = \left(\frac{\partial \ln \rho}{\partial \ln \mu}\right)_{P,T} \tag{3}$$

which depend on the equation of state ($\delta = 1, \varphi = 1$ for a perfect gas).

In the absence of thermal diffusion, ∇' is equal to the adiabatic gradient $\nabla_{\rm ad}$ and one retrieves the Richardson criterion relevant for a compressible fluid:

$$\frac{N^2}{\left(\mathrm{d}U/\mathrm{d}z\right)^2} < Ri_c, \quad \text{where} \quad N^2 = N_T^2 + N_\mu^2, \tag{4}$$

having split the Brunt-Väisälä frequency into

$$N_T^2 = \frac{g\delta}{H_P}\left(\nabla_{\rm ad} - \nabla\right) \quad \text{and} \quad N_\mu^2 = \frac{g\varphi}{H_P}\nabla_\mu, \tag{5}$$

with $Ri_c \approx 1/4$ being the critical Richardson number.

The effect of thermal damping has been examined by Dudis (1974), and Zahn (1974) has given an estimate for the turbulent diffusivity in the limit of low Péclet number ($v\ell/K \ll 1$, with K being the thermal diffusivity). Recently, Maeder (1995) derived a new expression for this coefficient, which has the advantage of describing the intermediate regimes. It is inspired by the mixing length treatment of non-adiabatic convection (see Cox & Giuli 1968), where one defines the ratio Γ between the thermal energy transported by an eddy of volume $\mathscr{V}$

$$\rho \mathscr{V} C_p \Delta T = \rho \mathscr{V} C_p T \frac{\ell}{H_P}(\nabla' - \nabla) \tag{6}$$

and the energy which diffuses through its surface of area $\mathscr{A}$ during its lifetime ℓ/v

$$\begin{aligned} \mathscr{A}\chi\left(\frac{\Delta T/2}{\ell/2}\right)\frac{\ell}{v} &= \rho \mathscr{V} T \Delta S \\ &= \rho \mathscr{V} C_P T \frac{\ell}{H_P}(\nabla_{\rm ad} - \nabla') \end{aligned} \tag{7}$$

(ΔT is the maximum temperature difference, ΔS is the entropy excess, and $\chi = \rho C_P K = 16\sigma T^3/3\kappa\rho$ the radiative conductivity, with the usual notations). Assuming that the eddy may be described as a sphere of diameter ℓ (*i.e.* $\mathscr{V}/\mathscr{A} = \ell/6$), one has

$$\Gamma = \frac{v\ell}{6K} = \frac{\nabla' - \nabla}{\nabla_{\rm ad} - \nabla'}. \tag{8}$$

This yields the modified version of the Richardson criterion which was established by Maeder (1995):

$$g\frac{\mathrm{d}\Delta \ln \rho}{\mathrm{d}z} = \left(\frac{\Gamma}{\Gamma + 1}\right) N_T^2 + N_\mu^2 \le Ri_c \left(\frac{\mathrm{d}U}{\mathrm{d}z}\right)^2. \tag{9}$$

According to this criterion, the vertical shear can be unstable only provided

$$\left(\frac{\mathrm{d}U}{\mathrm{d}z}\right)^2 > \frac{1}{Ri_c} N_\mu^2. \tag{10}$$

Then Eq. (9) may be solved for the parameter $\Gamma = v\ell/6K$ which characterizes the largest eddies that satisfy the Richardson criterion, and these produce a turbulent diffusivity $D_v = 2K\Gamma$.

Recently Meynet & Maeder (1996) calculated the evolution of rotating massive stars with this prescription for the turbulent diffusion, assuming local conservation of angular momentum. They found that, even though the model displays a strong differential rotation at the end of the main sequence, no significant mixing occurs because the molecular weight gradient inhibits the shear instability. Since according to the observations such mixing does take place in fast rotators, they conclude that this prescription underestimates the shear mixing in stars, and there must be some way to overcome the μ-barrier.

3. The effect of the horizontal diffusion

As was already mentioned in the Introduction, it seems plausible that stars exhibit an anisotropic turbulence due to the horizontal shear which is induced by the meridian circulation (see also Zahn 1992). This turbulence generates a horizontal transport which is much larger than the vertical one, and which tends to smooth out the horizontal fluctuations of molecular weight. Therefore μ will vary along the trajectory of the bubble considered above (see Fig. 1).

The restoring force per unit mass is then given by

$$g\frac{\mathrm{d}\Delta \ln \rho}{\mathrm{d}z} = \frac{g}{H_P}\left[\delta\left(\nabla' - \nabla\right) - \varphi\left(\nabla'_\mu - \nabla_\mu\right)\right], \tag{11}$$

with the same notations as before.

To estimate $(\nabla'_\mu - \nabla_\mu)$ we proceed as above for the thermal diffusion of heat: it suffices to replace ΔT by $\Delta\mu$ and K by the turbulent diffusivity D_h. The result is

$$\nabla'_\mu - \nabla_\mu = -\frac{\Gamma_\mu}{\Gamma_\mu + 1}\nabla_\mu, \tag{12}$$

where now $\Gamma_\mu = v\ell/6D_h$.

We thereby obtain still another version of the Richardson criterion:

$$\left(\frac{\Gamma}{\Gamma + 1}\right) N_T^2 + \left(\frac{\Gamma_\mu}{\Gamma_\mu + 1}\right) N_\mu^2 \le Ri_c \left(\frac{\mathrm{d}U}{\mathrm{d}z}\right)^2, \tag{13}$$

which describes how the stabilizing effect of the μ-gradient is diminished by the horizontal turbulence. Note that the influence of the temperature gradient will also be modified by this turbulence since the advective transport of heat will be added to the radiative diffusion: $K \to (K + D_h)$.

We see that any vertical shear may become unstable, for small enough $v\ell$. But the Reynolds number characterizing these eddies must be larger than some critical value, otherwise their

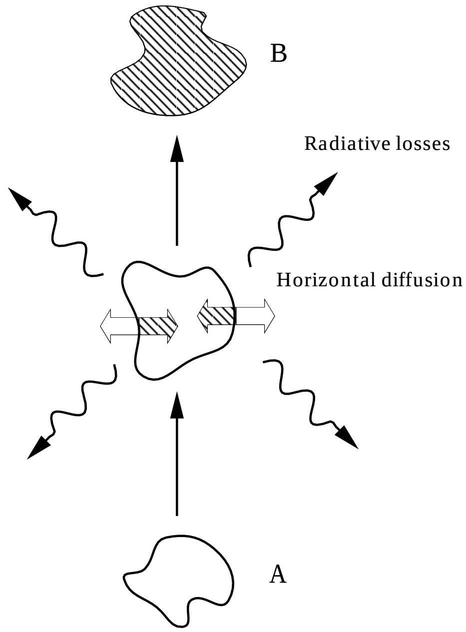

Fig. 1. As a turbulent eddy moves vertically, it is partially mixed with the surrounding medium, and therefore its composition varies.

motion will be damped out through viscous friction. Hence they must obey the condition

$$\frac{v\ell}{\nu} \leq Re_c, \tag{14}$$

where ν is the molecular viscosity, and $Re_c \simeq 10$ is the critical Reynolds number.

Among all turbulent eddies satisfying (13) and (14), those contributing most to the vertical transport are those for which $v\ell$ is the largest. Their size will be given by the equality in Eq. (13), namely

$$\frac{x}{x + K + D_h} N_T^2 + \frac{x}{x + D_h} N_\mu^2 = Ri_c \left(\frac{\mathrm{d}U}{\mathrm{d}z}\right)^2 \tag{15}$$

where $x = v\ell/6 \geq \nu Re_c/6$.

It is an easy matter to solve this second order Eq. (15), and to derive from it the value of vertical diffusivity $D_v \simeq v\ell/3$. Solutions exist only if

$$\left(\frac{\mathrm{d}U}{\mathrm{d}z}\right)^2 \geq \frac{\nu Re_c}{6 Ri_c} \left[\frac{N_T^2}{K + D_h} + \frac{N_\mu^2}{D_h}\right], \tag{16}$$

which replaces the much more stringent condition (10).

Note that the vertical turbulence transports heat and thus modifies the temperature gradient ∇. This was taken into account by Maeder & Meynet (1996) when they evaluated the mixing by shears. However, in most circumstances relevant to radiative interiors, we are in the presence of *mild turbulence*, so that $x \ll K$, and we may ignore this effect. Furthermore, we stated as a working hypothesis the condition $D_v \simeq 2x \ll D_h$. The turbulent diffusivity is then given by the simple expression

$$D_v \simeq \frac{2 Ri_c (\mathrm{d}U/\mathrm{d}z)^2}{N_T^2/(K + D_h) + N_\mu^2/D_h}. \tag{17}$$

One final word about the value of the ratio $\mathscr{V}/\mathscr{A}$ between volume and area of the turbulent eddies. Its choice is rather arbitrary. Following Maeder (1995), we have taken $\mathscr{V}/\mathscr{A} = \ell/6$, whereas Böhm-Vitense (1958) proposed the value $\mathscr{V}/\mathscr{A} = (2/9)\ell$ when she developed her famous mixing length formalism. We could refine this numerical coefficient, for instance by taking our inspiration from Dudis' (1974) work, where $\Gamma = (3/8)v\ell/K$. However, we prefer to hide this uncertainty in the choice of Ri_c and in the prescription we use to evaluate the horizontal diffusion coefficient D_h, which also involves an unknown parameter of order unity (see Zahn 1992).

4. Conclusions

We have shown how the Richardson criterion, which governs the shear instabilities in a stratified fluid, is modified in the presence of a strongly anisotropic turbulence. This turbulent erosion is similar to that which interfers with the advective transport of a large-scale flow, and which has been described by Chaboyer and Zahn (1992). Another example of this phenomenon is given in a numerical simulation performed by Vincent et al. (1996). Here it diminishes the stabilizing action of a molecular weight gradient, to a point which makes it plausible that rapidly rotating stars be partially mixed. The prescription (17) we have derived above for the vertical diffusivity is being implemented in our stellar evolution codes, and the results will be reported in a forthcoming paper.

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