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ASTRONOMY AND ASTROPHYSICS

Linear, isentropic oscillations of the compressible MacLaurin spheroids

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Abstract. Smeyers' procedure (1986) for the determination of linear, isentropic oscillations of the incompressible MacLaurin spheroids is extended to the compressible MacLaurin spheroids. It is shown that the solutions can be constructed by a direct integration of a finite set of differential equations written in spherical coordinates. Oblate spheroidal coordinates are used with regard to the boundary conditions that must be satisfied at the surface of the MacLaurin spheroid.

For compressible MacLaurin spheroids with eccentricities e varying from zero to unity, the modes are determined that stem from the fundamental radial mode and the second-harmonic Kelvin modes in the non-rotating equilibrium sphere with uniform mass density. The modes obtained agree with the modes determined earlier by Chandrasekhar and Lebovitz (1962a, 1962b) by means of the second-order tensor virial equations.

Next, four axisymmetric modes are determined that stem from the first radial overtone, the second-harmonic p_1- and g_1^--mode, and the fourth-harmonic Kelvin mode in the non-rotating equilibrium sphere with uniform mass density. The g_1^--mode becomes dynamically stable at the eccentricity e = 0.7724 and again dynamically unstable at e = 0.9952.

Key words: stars: oscillations – rotation

1. Introduction

The linear, isentropic oscillations of the incompressible MacLaurin spheroids were determined by Bryan more than a century ago (1889). Bryan adopted a procedure used by Poincaré for the study of self-gravitating, uniformly rotating ellipsoids. By a projection of the MacLaurin spheroid on an auxiliary spheroid, the problem reduces to the resolution of Laplace's equation over the volume of the auxiliary spheroid. Bryan simplified the procedure by expressing the solutions in terms of spheroidal harmonics instead of Lamé functions.

Later, Chandrasekhar and Lebovitz (1962a, 1962b, 1969) used the second-order tensor virial equations to solve the eigenvalue problem of the linear, isentropic, second-harmonic oscillations of both the incompressible and the compressible MacLaurin spheroids. From the third- and fourth-order virial equations, they determined the linear and isentropic, third- and fourth-harmonic modes of the incompressible MacLaurin spheroids. They observed that some modes of the incompressible MacLaurin spheroids remain solutions for the compressible spheroids.

Smeyers (1986) presented a method to solve the eigenvalue problem of the linear, isentropic oscillations of the incompressible MacLaurin spheroids by a direct integration of a finite set of governing equations written in spherical coordinates. The set of the equations consists of the equations belonging to all, even or uneven, degrees ℓ of the spherical harmonic not larger than a fixed value L. Solutions are found in terms of finite power series in the radial coordinate and in terms of a finite number of spherical harmonic functions. In order to impose them to satisfy the boundary conditions at the spheroidal surface of the equilibrium configuration, one expresses the solutions in terms of oblate spheroidal coordinates.

In this paper, we extend Smeyers' procedure to the determination of linear, isentropic oscillations of the *compressible* MacLaurin spheroids. We show that the solutions can be determined by a direct integration of a finite set of differential equations written in spherical coordinates. Here too, we use oblate spheroidal coordinates in order to impose the boundary conditions at the surface of the MacLaurin spheroid.

We consider two applications. By setting L = 2, we determine the modes of the compressible MacLaurin spheroids that stem from the fundamental radial mode and the second-harmonic Kelvin modes in the non-rotating equilibrium sphere with uniform mass density. Comparison of our results with those of Chandrasekhar and Lebovitz (1962a, 1962b) confirms the validity of the method.

Next, by setting L = 4, we determine the axisymmetric modes of the compressible MacLaurin spheroids that stem from the following four axisymmetric modes in the non-rotating equilibrium sphere with uniform mass density: the first radial overtone, the second-harmonic p_1- and g_1^--mode, and the fourth-

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harmonic Kelvin mode. The g_1^--mode becomes dynamically stable at the eccentricity $e = 0.7724$ and again dynamically unstable at $e = 0.9952$.

The plan of the paper is as follows. In Sect. 2, we recall briefly the structure of the MacLaurin spheroids. In Sect. 3, we present the governing equations in spherical coordinates, and the boundary conditions. Sect. 4 is devoted to the separation of the angular variables. In Sect. 5, the equations are brought in an appropriate form. Their solution is described in Sect. 6. In Sects. 7 and 8, we derive the various modes of the compressible MacLaurin spheroids mentioned above, for different values of the eccentricity.

2. Equilibrium configurations

Consider a uniformly rotating MacLaurin spheroid with a uniform mass density ρ and an angular velocity $\boldsymbol{\Omega}$. We let coincide the origin and the z-axis of a corotating frame of reference with the centre of the spheroid, and the rotation axis, respectively.

With respect to that corotating frame of reference, we introduce a system of spherical coordinates r, θ, φ by means of the following transformation from a system of Cartesian coordinates x, y, z defined in the inertial frame of reference with the same origin and the same z-axis:

$$\left.\begin{aligned} x &= r\,\sin\theta\,\cos(\varphi+\Omega\,t),\\ y &= r\,\sin\theta\,\sin(\varphi+\Omega\,t),\\ z &= r\,\cos\theta. \end{aligned}\right\} \tag{1}$$

We also introduce a system of oblate spheroidal coordinates ξ, η, φ in the corotating frame of reference by means of the transformation formulae

$$\left.\begin{aligned} x &= a\left[\left(1+\xi^2\right)\left(1-\eta^2\right)\right]^{1/2}\cos\left(\varphi+\Omega\,t\right),\\ y &= a\left[\left(1+\xi^2\right)\left(1-\eta^2\right)\right]^{1/2}\sin\left(\varphi+\Omega\,t\right),\\ z &= a\,\xi\,\eta, \end{aligned}\right\} \tag{2}$$

where a is the radius of the focal circle of the confocal spheroidal and hyperboloidal surfaces (see, e.g., Morse and Feshbach 1953, Sect. 10.3).

We let coincide the spheroidal coordinate surface $\xi = \xi_S$ with the surface of the MacLaurin spheroid. The eccentricity e of the surface and the equatorial radius $R_{\rm e}$ of the MacLaurin spheroid are related to the coordinate ξ_S and the radius a by

$$e = \frac{1}{\left(\xi_S^2+1\right)^{1/2}}, \tag{3}$$

$$R_{\rm e} = a\left(\xi_S^2+1\right)^{1/2}. \tag{4}$$

The distributions of the internal gravitational potential Φ_i, the external gravitational potential Φ_e, and the pressure P are given by

$$\Phi_i\left(\xi,\eta\right) = -\pi\,G\,\rho\,a^2\left\{\frac{4}{3}\,\mathrm{i}\,\xi_S\left(\xi_S^2+1\right)Q_0\left(\mathrm{i}\,\xi_S\right)\right.$$
$$\left.+\frac{2}{3}\left(\xi_S^2-\xi^2\right)+\frac{2}{3}\left[\left(A_1-A_3\right)\xi^2+A_1\right]P_2\left(\eta\right)\right\}, \tag{5}$$

$$\Phi_e\left(\xi,\eta\right) = -\frac{4}{3}\,\pi\,G\,\rho\,a^2\,\mathrm{i}\,\xi_S\left(\xi_S^2+1\right)$$
$$\left[Q_0\left(\mathrm{i}\,\xi\right)-Q_2\left(\mathrm{i}\,\xi\right)\,P_2\left(\eta\right)\right], \tag{6}$$

$$P = \pi\,G\,\rho^2\,a^2\,A_3\,\frac{\xi_S^2+\eta^2}{\xi_S^2+1}\left(\xi_S^2-\xi^2\right), \tag{7}$$

where $P_2\left(\eta\right)$ is a Legendre function of the first kind, $Q_0\left(\mathrm{i}\,\xi_S\right)$ and $Q_2\left(\mathrm{i}\,\xi_S\right)$ are Legendre functions of the second kind, and A_1 and A_3 are constants defined as

$$A_1 \equiv \mathrm{i}\,\xi_S\left(\xi_S^2+1\right)Q_0\left(\mathrm{i}\,\xi_S\right)-\xi_S^2, \tag{8}$$

$$A_3 \equiv 2\left(\xi_S^2+1\right)\left[1-\mathrm{i}\,\xi_S\,Q_0\left(\mathrm{i}\,\xi_S\right)\right] \tag{9}$$

(Chandrasekhar 1969, Sect. 17; Smeyers 1986).

The pressure P can be expressed in terms of the spherical coordinates as

$$P = K\,\rho\left[\left(\xi_S^2+\cos^2\theta\right)r^2-a^2\,\xi_S^2\left(\xi_S^2+1\right)\right], \tag{10}$$

where K is a constant defined as

$$K = 2\,\pi\,G\,\rho\,Q_1\left(\mathrm{i}\,\xi_S\right). \tag{11}$$

The angular velocity Ω of the rotation is related to the coordinate ξ_S as

$$\frac{\Omega^2}{2\,\pi\,G\,\rho} = A_1 - A_3\,\frac{\xi_S^2}{\xi_S^2+1}. \tag{12}$$

The angular velocity Ω as a function of the eccentricity of the MacLaurin spheroid increases from zero to a maximum value $\Omega^2_{\rm max}/(\pi\,G\,\rho) = 0.44933$ at $e = 0.92995$ and then decreases to zero. The angular momentum as a function of the eccentricity e increases monotonically.

In the limit for $e \to 0$, $\xi_S \to \infty$ according to Relation (3). When $R_{\rm e}$ is kept constant, a must tend to zero as $\left(\xi_S^2+1\right)^{-1/2}$ according to Relation (4). The angular velocity Ω is then zero, so that the equilibrium configuration is a non-rotating sphere of radius $R_{\rm e}$. In the limit for $e \to 1$, it follows from Relation (3) that $\xi_S \to 0$. When $R_{\rm e}$ is kept constant, a tends to $R_{\rm e}$, and the angular velocity Ω tends to zero so that the equilibrium configuration becomes a non-rotating flat disk.

3. Governing equations and boundary conditions

Let $\boldsymbol{\xi}$ be the Lagrangian displacement of a mass element. A prime on a quantity denotes the Eulerian perturbation of that quantity.

In terms of the spherical coordinates, the equations governing free, linear, isentropic oscillations of a compressible, rotating MacLaurin spheroid that depend on time t by $\exp(i\sigma t)$ can be written as

$$\sigma^2\,\xi_r + 2\,\mathrm{i}\,\sigma\,\Omega\,\sin\theta\,\xi_\varphi = \frac{\partial\chi}{\partial r} - A_r\,c^2\,\alpha, \tag{13}$$

$$\sigma^2\,\xi_\theta + 2\,\mathrm{i}\,\sigma\,\Omega\,\cos\theta\,\xi_\varphi = \frac{1}{r}\,\frac{\partial\chi}{\partial\theta} - \frac{A_\theta}{r}\,c^2\,\alpha\,, \tag{14}$$

$$\sigma^2\,\xi_\varphi - 2\,\mathrm{i}\,\sigma\,\Omega\,(\sin\theta\,\xi_r + \cos\theta\,\xi_\theta) = \frac{1}{r\,\sin\theta}\,\frac{\partial\chi}{\partial\varphi}\,, \tag{15}$$

$$\rho' = -\rho\,\alpha\,, \tag{16}$$

$$\chi - \Phi' = -c^2\,\alpha - \frac{\nabla P}{\rho}\cdot\boldsymbol{\xi}\,, \tag{17}$$

$$\nabla^2\Phi' = 4\,\pi\,G\,\rho'\,, \tag{18}$$

where α is the divergence of the Lagrangian displacement, $c^2 \equiv \Gamma_1\,P/\rho$ the square of the isentropic sound velocity, and

$$A_r = -\frac{1}{\rho\,c^2}\,\frac{\partial P}{\partial r}, \quad A_\theta = -\frac{1}{\rho\,c^2}\,\frac{\partial P}{\partial\theta}, \tag{19}$$

$$\chi = \Phi' + \frac{P'}{\rho}. \tag{20}$$

Instead of Eqs. (14) and (15), we use the r-component of the vorticity equation and an equation for the divergence of the horizontal component of the Lagrangian displacement

$$\begin{aligned}
&\sigma^2\left[\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\xi_\varphi\right) - \frac{1}{\sin\theta}\,\frac{\partial\,\xi_\theta}{\partial\varphi}\right] \\
&\quad - 2\,\mathrm{i}\,\sigma\,\Omega\left\{\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin^2\theta\,\xi_r\right) - \sin\theta\,\xi_\theta\right. \\
&\quad \left. + \cos\theta\left[\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\,(\sin\theta\,\xi_\theta) + \frac{1}{\sin\theta}\,\frac{\partial\,\xi_\varphi}{\partial\varphi}\right]\right\} \\
&\quad = 2\,K\,r\,\cos\theta\,\frac{\partial\,\alpha}{\partial\varphi}\,,
\end{aligned} \tag{21}$$

$$\begin{aligned}
&\sigma^2\left[\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\,(\sin\theta\,\xi_\theta) + \frac{1}{\sin\theta}\,\frac{\partial\,\xi_\varphi}{\partial\varphi}\right] \\
&\quad + 2\,\mathrm{i}\,\sigma\,\Omega\left\{-\sin\theta\,\xi_\varphi - \frac{\partial\,\xi_r}{\partial\varphi}\right. \\
&\quad \left. + \cos\theta\left[\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\xi_\varphi\right) - \frac{1}{\sin\theta}\,\frac{\partial\,\xi_\theta}{\partial\varphi}\right]\right\} \\
&\quad = \frac{1}{r}\left[\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\,\chi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\,\frac{\partial^2\,\chi}{\partial\varphi^2}\right] \\
&\quad - \frac{2\,K}{\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin^2\theta\,\cos\theta\,r\,\alpha\right)\,,
\end{aligned} \tag{22}$$

where we have inserted Expression (10) for the distribution of the pressure P in the MacLaurin spheroid.

The solutions must satisfy boundary conditions at the centre and the surface of the MacLaurin spheroid. At $r = 0$, the Lagrangian displacement must be finite. At the surface, the Lagrangian perturbation of the pressure must vanish:

$$(\delta P)_{\xi=\xi_S} = 0. \tag{23}$$

Since the equilibrium pressure P vanishes there, and

$$\delta P = -\Gamma_1\,P\,\alpha, \tag{24}$$

the boundary condition requires that α be finite at $\xi = \xi_S$. Furthermore, at the surface, the continuity of the gravitational potential and its gradient requires that the Eulerian perturbation of the internal gravitational potential, Φ'_i, and the Eulerian perturbation of the external gravitational potential, Φ'_e, satisfy the conditions

$$\left(\Phi'_i\right)_{\xi=\xi_S} = \left(\Phi'_e\right)_{\xi=\xi_S}\,, \tag{25}$$

$$\left(\frac{\partial\Phi'_i}{\partial\xi}\right)_{\xi=\xi_S} = \left(\frac{\partial\Phi'_e}{\partial\xi}\right)_{\xi=\xi_S} - 4\,\pi\,G\,\rho\,a^2\,\frac{\xi_S^2+\eta^2}{\xi_S^2+1}\,(\delta\xi)_{\xi=\xi_S}\,, \tag{26}$$

$$\left(\frac{\partial\Phi'_i}{\partial\eta}\right)_{\xi=\xi_S} = \left(\frac{\partial\Phi'_e}{\partial\eta}\right)_{\xi=\xi_S}\,. \tag{27}$$

The Eulerian perturbation of the external gravitational potential is given by

$$\Phi'_e(\xi,\eta,\varphi) = \sum_{\lambda=0}^{\infty} c_{\lambda,m}\,Q_\lambda^m(i\xi)\,P_\lambda^m(\eta)\,e^{i\,m\,\varphi}, \tag{28}$$

where the $c_{\lambda,m}$ are undetermined constants, and $P_\lambda^m(\eta)$ and $Q_\lambda^m(i\xi)$ associated Legendre functions of the first and the second kind, respectively (Smeyers 1986).

4. Separation of the angular variables

Since the equilibrium configuration is axially symmetric, we may search for solutions proportional to $\exp(im\varphi)$.

The scalar perturbations $\chi(r,\theta,\varphi)$, $\Phi'(r,\theta,\varphi)$, $\rho'(r,\theta,\varphi)$ can be expanded in terms of spherical harmonics $Y_\lambda^m(\theta,\varphi)$ with $\lambda \geq |m|$. In order to derive the appropriate expansions for the components of the Lagrangian displacement, we decompose the displacement field into a longitudinal, a toroidal, and a poloidal part as

$$\boldsymbol{\xi} = \nabla\Psi + \nabla\times(T\,\mathbf{1}_r) + \nabla\times\nabla\times(S\,\mathbf{1}_r)\,, \tag{29}$$

and expand the scalar potentials $\Psi(r,\theta,\varphi)$, $T(r,\theta,\varphi)$, $S(r,\theta,\varphi)$ in terms of spherical harmonics. It follows that

$$\left.\begin{aligned}
\xi_r &= \sum_{\lambda=|m|}^{\infty} a_{\lambda,m}(r)\,Y_\lambda^m(\theta,\varphi), \\
\xi_\theta &= \sum_{\lambda=|m|}^{\infty}\left[b_{\lambda,m}(r)\,\frac{\partial Y_\lambda^m(\theta,\varphi)}{\partial\theta}\right. \\
&\qquad \left. + \frac{1}{r\,\sin\theta}\,T_{\lambda,m}(r)\,\frac{\partial Y_\lambda^m(\theta,\varphi)}{\partial\varphi}\right], \\
\xi_\varphi &= \sum_{\lambda=|m|}^{\infty}\left[b_{\lambda,m}(r)\,\frac{1}{\sin\theta}\,\frac{\partial Y_\lambda^m(\theta,\varphi)}{\partial\varphi}\right. \\
&\qquad \left. - \frac{1}{r}\,T_{\lambda,m}(r)\,\frac{\partial Y_\lambda^m(\theta,\varphi)}{\partial\theta}\right],
\end{aligned}\right\} \tag{30}$$

where

$$\left.\begin{aligned}
a_{\lambda,m}(r) &= \frac{d\Psi_{\lambda,m}(r)}{dr} + \frac{\lambda(\lambda+1)}{r^2}\,S_{\lambda,m}(r), \\
b_{\lambda,m}(r) &= \frac{1}{r}\left(\Psi_{\lambda,m}(r) + \frac{dS_{\lambda,m}(r)}{dr}\right).
\end{aligned}\right\} \tag{31}$$

By inserting the various expansions into Eqs. (13), (18), (21)–(22), and (16), multiplying by $\overline{Y_\ell^m(\theta,\varphi)}$, where the bar denotes the complex conjugate, and integrating over a spherical surface, we derive the following equations:

$$\frac{\mathrm{d}\chi_{\ell,m}}{\mathrm{d}r} = \sigma^2 a_{\ell,m} - 2\,m\,\sigma\,\Omega\,b_{\ell,m} - 2\,\sigma\,\Omega\Big[(\ell-1)\;K^-(\ell-1,m)\;\frac{\mathrm{i}}{r}\,T_{\ell-1,m} - (\ell+2)\;K^+(\ell+1,m)\;\frac{\mathrm{i}}{r}\,T_{\ell+1,m}\Big] - 2\,K\,\xi_S^2\,r\,\alpha_{\ell,m} - 2\,K\left[L^-(\ell-2,m)\;r\,\alpha_{\ell-2,m} + L(\ell,m)\;r\,\alpha_{\ell,m} + L^+(\ell+2,m)\;r\,\alpha_{\ell+2,m}\right], \tag{32}$$

$$\frac{1}{r^2}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\,a_{\ell,m}\right) - \frac{\ell\,(\ell+1)}{r}\,b_{\ell,m} = \alpha_{\ell,m}\,, \tag{33}$$

$$\frac{1}{r^2}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\,\frac{\mathrm{d}\Phi'_{\ell,m}}{\mathrm{d}r}\right) - \frac{\ell\,(\ell+1)}{r^2}\,\Phi'_{\ell,m} = -\,4\,\pi\,G\,\rho\,\alpha_{\ell,m}\,, \tag{34}$$

$$\left[\ell\,(\ell+1)\;\sigma^2 - 2\,m\,\sigma\,\Omega\right]\frac{\mathrm{i}}{r}\,T_{\ell,m} + 2\,K\,m\left[K^-(\ell-1,m)\;r\,\alpha_{\ell-1,m} + K^+(\ell+1,m)\;r\,\alpha_{\ell+1,m}\right] + 2\,\sigma\,\Omega\left[(\ell+1)\;K^-(\ell-1,m)\;a_{\ell-1,m} - \ell\,K^+(\ell+1,m)\,a_{\ell+1,m}\right] - 2\,\sigma\,\Omega\left[(\ell-1)\,(\ell+1)\,K^-(\ell-1,m)\;b_{\ell-1,m} + \ell\,(\ell+2)\,K^+(\ell+1,m)\;b_{\ell+1,m}\right] = 0\,, \tag{35}$$

$$\left[\ell\,(\ell+1)\;\sigma^2 - 2\,m\,\sigma\,\Omega\right]b_{\ell,m} - 2\,m\,\sigma\,\Omega\,a_{\ell,m} - 2\,\sigma\,\Omega\left[(\ell-1)\,(\ell+1)\,K^-(\ell-1,m)\;\frac{\mathrm{i}}{r}\,T_{\ell-1,m} + \ell\;(\ell+2)\;K^+(\ell+1,m)\;\frac{\mathrm{i}}{r}\,T_{\ell+1,m}\right] - \frac{\ell\;(\ell+1)}{r}\,\chi_{\ell,m} - 2\,K\left[(\ell+1)\;L^-(\ell-2,m)\;r\,\alpha_{\ell-2,m} + M(\ell,m)\;r\,\alpha_{\ell,m} - \ell\,L^+(\ell+2,m)\;r\,\alpha_{\ell+2,m}\right] = 0\,, \tag{36}$$

$$\frac{1}{r}\,\chi_{\ell,m} - \frac{1}{r}\,\Phi'_{\ell,m} + 2\,K\,\xi_S^2\,a_{\ell,m} + 2\,K\left[L^-(\ell-2,m)\;a_{\ell-2,m} + L(\ell,m)\,a_{\ell,m} + L^+(\ell+2,m)\;a_{\ell+2,m}\right] + 2\,K\left[M^-(\ell-2,m)\;b_{\ell-2,m} + M(\ell,m)\;b_{\ell,m} + M^+(\ell+2,m)\;b_{\ell+2,m}\right] - 2\,K\,m\left[K^-(\ell-1,m)\;\frac{\mathrm{i}}{r}\,T_{\ell-1,m} + K^+(\ell+1,m)\;\frac{\mathrm{i}}{r}\,T_{\ell+1,m}\right] + \Gamma_1\,K\,\xi_S^2\left[r^2 - a^2\left(\xi_S^2+1\right)\right]\frac{1}{r}\,\alpha_{\ell,m} + \Gamma_1\,K\left[L^-(\ell-2,m)\;r\,\alpha_{\ell-2,m} + L(\ell,m)\;r\,\alpha_{\ell,m} + L^+(\ell+2,m)\;r\,\alpha_{\ell+2,m}\right] = 0\,. \tag{37}$$

In these equations, the following definitions are used:

$$K^-(\ell,m) = \frac{\ell-|m|+1}{2\,\ell+1}\,, \tag{38}$$

$$K^+(\ell,m) = \frac{\ell+|m|}{2\,\ell+1}\,, \tag{39}$$

$$L^-(\ell,m) = K^-(\ell,m)\;K^-(\ell+1,m)\,, \tag{40}$$

$$L(\ell,m) = K^-(\ell,m)\;K^+(\ell+1,m) + K^+(\ell,m)\;K^-(\ell-1,m)\,, \tag{41}$$

$$L^+(\ell,m) = K^+(\ell,m)\;K^+(\ell-1,m)\,, \tag{42}$$

$$M^-(\ell,m) = -\ell\,L^-(\ell,m)\,, \tag{43}$$

$$M(\ell,m) = -\ell\,K^-(\ell,m)\;K^+(\ell+1,m) + (\ell+1)\;K^+(\ell,m)\;K^-(\ell-1,m) = \frac{3}{2}\,L(\ell,m) - \frac{1}{2}\,, \tag{44}$$

$$M^+(\ell,m) = (\ell+1)\;L^+(\ell,m)\,. \tag{45}$$

5. Transformation of the equations

We consider only modes with frequencies different from zero. In view of the construction of solutions, we transform Eqs. (32)–(37) in the following way.

First, in Eq. (35), we replace ℓ by $\ell+1$. The resulting equation is valid for $\ell \geq |m|-1$ and takes the form

$$\ell\,b_{\ell,m} - a_{\ell,m} = -\frac{(\ell+1)\;K^+(\ell+2,m)}{(\ell+2)\;K^-(\ell,m)}\left[a_{\ell+2,m} + (\ell+3)\;b_{\ell+2,m}\right] + \frac{(\ell+1)\,(\ell+2)\;\sigma - 2\,m\,\Omega}{2\,\Omega\;(\ell+2)\;K^-(\ell,m)}\,\frac{\mathrm{i}}{r}\,T_{\ell+1,m} + \frac{m\,K}{\sigma\,\Omega\;(\ell+2)\;K^-(\ell,m)}\left[K^-(\ell,m)\;r\,\alpha_{\ell,m} + K^+(\ell+2,m)\,r\,\alpha_{\ell+2,m}\right]. \tag{46}$$

We use this equation in order to eliminate the product $\ell\,b_{\ell,m}$ in Eq. (33). The latter equation then becomes

$$\frac{\mathrm{d}a_{\ell,m}}{\mathrm{d}r} - \frac{(\ell-1)}{r}\,a_{\ell,m} = \alpha_{\ell,m} + \frac{m\,K\;(\ell+1)}{\sigma\,\Omega\;(\ell+2)\;K^-(\ell,m)}\left[K^-(\ell,m)\,\alpha_{\ell,m} + K^+(\ell+2,m)\,\alpha_{\ell+2,m}\right] - \frac{(\ell+1)^2\;K^+(\ell+2,m)}{(\ell+2)\;K^-(\ell,m)}\,\frac{1}{r}\left[a_{\ell+2,m} + (\ell+3)\,b_{\ell+2,m}\right] + \frac{(\ell+1)\,[(\ell+1)\;(\ell+2)\;\sigma - 2\,m\,\Omega]}{2\,\Omega\;(\ell+2)\;K^-(\ell,m)}\,\frac{\mathrm{i}}{r^2}\,T_{\ell+1,m}\,. \tag{47}$$

Secondly, we use Eq. (36) in order to eliminate the terms involving $T_{\ell-1,m}$ and $\alpha_{\ell-2,m}$ in Eq. (32). In the resulting equation, we eliminate the difference $a_{\ell,m} - \ell\,b_{\ell,m}$ by means of Eq. (46). We then find the equation

$$\frac{\mathrm{d}\chi_{\ell,m}}{\mathrm{d}r} - \frac{\ell}{r}\,\chi_{\ell,m}$$

$$= \left(\sigma^2 + \frac{2\,\sigma\, m\, \Omega}{\ell+1}\right) \frac{(\ell+1)\; K^+(\ell+2,m)}{(\ell+2)\; K^-(\ell,m)} \left[a_{\ell+2,m} + (\ell+3)\; b_{\ell+2,m}\right]$$
$$+ \left[-\left(\sigma^2 + \frac{2\,\sigma\, m\,\Omega}{\ell+1}\right) \frac{[(\ell+1)\;(\ell+2)\;\sigma - 2\, m\,\Omega]}{2\,\Omega\;(\ell+2)\; K^-(\ell,m)} + 2\,\sigma\,\Omega\, \frac{(2\,\ell+1)\;(\ell+2)}{\ell+1}\; K^+(\ell+1,m)\right] \frac{\mathrm{i}}{r}\; T_{\ell+1,m}$$
$$- \left[\sigma^2 + \frac{2\, m\,\sigma\,\Omega}{\ell+1}\right] \frac{K\, m}{\sigma\,\Omega\;(\ell+2)\; K^-(\ell,m)} \left[K^-(\ell,m)\; r\,\alpha_{\ell,m} + K^+(\ell+2,m)\; r\,\alpha_{\ell+2,m}\right]$$
$$- 2\,K\,\xi_S^2\, r\,\alpha_{\ell,m}$$
$$+ \frac{2\,K}{\ell+1}\left[M(\ell,m) - (\ell+1)\; L(\ell,m)\right] r\,\alpha_{\ell,m}$$
$$- \frac{2\,K\,(2\,\ell+1)}{\ell+1} L^+(\ell+2,m)\; r\,\alpha_{\ell+2,m}\,. \qquad (48)$$

Thirdly, for values of ℓ higher than $|m|+1$, we eliminate $(\ell-2)\, b_{\ell-2,m} - a_{\ell-2,m}$ in Eq. (37) by using Eq. (46) for $\ell-2$. It follows that

$$\frac{\ell-1}{\ell\,\Omega}\,(\ell\,\sigma + 2\, m\,\Omega)\, K\, K^-(\ell-1,m)\; \frac{\mathrm{i}}{r}\; T_{\ell-1,m} + \frac{(2\, m\, K - \sigma\,\Omega\,\ell\,\Gamma_1)\, K}{\sigma\,\Omega\,\ell}\; L^-(\ell-2,m)\; r\,\alpha_{\ell-2,m}$$
$$= \frac{1}{r}\,\chi_{\ell,m} - \frac{1}{r}\,\Phi'_{\ell,m}$$
$$- 2\,K\, m\, K^+(\ell+1,m)\; \frac{\mathrm{i}}{r}\; T_{\ell+1,m} + 2\,K\,\xi_S^2\, a_{\ell,m}$$
$$+ 2\,K\left[L(\ell,m)\, a_{\ell,m} + L^+(\ell+2,m)\; a_{\ell+2,m}\right]$$
$$+ 2\,K\left[M(\ell,m)\; b_{\ell,m} + M^+(\ell+2,m)\; b_{\ell+2,m}\right]$$
$$+ \frac{2\,K\,(\ell-1)}{\ell}\; K^-(\ell-1,m)\; K^+(\ell,m)\; \left[a_{\ell,m} + (\ell+1)\; b_{\ell,m}\right]$$
$$+ \Gamma_1\, K\,\xi_S^2\left[r^2 - a^2\;\left(\xi_S^2+1\right)\right]\; \frac{1}{r}\;\alpha_{\ell,m}$$
$$+ \Gamma_1\, K\, r\left[L(\ell,m)\;\alpha_{\ell,m} + L^+(\ell+2,m)\;\alpha_{\ell+2,m}\right]$$
$$- \frac{2\, m\, K^2\, K^-(\ell-1,m)\; K^+(\ell,m)}{\sigma\,\Omega\,\ell}\; r\,\alpha_{\ell,m}\,. \qquad (49)$$

6. Solution of the equations

Eqs. (47), (48), and (34) can be considered as three differential equations determining the functions $a_{\ell,m}$, $\chi_{\ell,m}$, and $\Phi'_{\ell,m}$ when the right-hand members of these equations are known. The right-hand members contain the functions $\alpha_{\ell,m}$, $a_{\ell+2,m}$, $b_{\ell+2,m}$, $\alpha_{\ell+2,m}$, $T_{\ell+1,m}$.

To the three differential equations, algebraic Eqs. (46), (36), and (37) or (49) are added, which relate the functions $b_{\ell,m}$, $T_{\ell-1,m}$, and $\alpha_{\ell-2,m}$ to the three functions $a_{\ell,m}$, $\chi_{\ell,m}$, and $\Phi'_{\ell,m}$. In the algebraic equations, the functions $a_{\ell+2,m}$, $b_{\ell+2,m}$, $\alpha_{\ell,m}$, $\alpha_{\ell+2,m}$, $T_{\ell+1,m}$ also appear.

Hence, the terms in the functions a, b, χ, Φ', and α with an even/uneven degree are coupled to the terms in the function T with an uneven/even degree so that the infinite system of coupled equations can be decomposed into two subsystems.

Both coupled systems of equations can be satisfied exactly when all functions $a_{\ell,m}$, $\chi_{\ell,m}$, $\Phi'_{\ell,m}$, $b_{\ell,m}$, $T_{\ell-1,m}$, and $\alpha_{\ell-2,m}$ that are associated with a degree ℓ larger than a given value L, are set equal to zero, and the *finite* number of coupled systems of equations associated with degrees ℓ less than or equal to L are solved.

In this procedure, the systems of equations associated with degrees ℓ larger than the given value L are identically satisfied.

For $\ell = L$, differential Eqs. (47), (48), and (34) are homogeneous and admit of the following solutions that remain finite as $r \to 0$:

$$a_{L,m} = A\, r^{L-1}\,, \qquad \chi_{L,m} = B\, r^L\,, \qquad \Phi'_{L,m} = C\, r^L\,, \qquad (50)$$

where A, B, and C are yet undetermined constants.

Once the solutions for $a_{L,m}$, $\chi_{L,m}$, and $\Phi'_{L,m}$ are determined, it is possible to derive the corresponding solutions for $b_{L,m}$, $T_{L-1,m}$, and $\alpha_{L-2,m}$ by means of Eqs. (46), (36), and (49) for $\ell = L$.

Then, the inhomogeneous system of differential Eqs. (47), (48), and (34) associated with $\ell = L-2$ can be solved for the functions $a_{L-2,m}$, $\chi_{L-2,m}$, and $\Phi'_{L-2,m}$. The solution of the system of algebraic Eqs. (46), (36), and (49) associated with $\ell = L-2$ yields the functions $b_{L-2,m}$, $T_{L-3,m}$, and $\alpha_{L-4,m}$. It turns out that both the differential equations and the algebraic equations can be solved analytically, and that the solutions are finite power series in r.

The procedure is continued until the lowest admissible value of ℓ is reached. The equations associated with the lowest admissible value of ℓ impose conditions on the yet undetermined constants. The constants that remain undetermined are fixed by boundary Conditions (25)–(27). Boundary Condition (23) is automatically satisfied since the expansion of the divergence of the Lagrangian displacement in terms of spherical harmonics consists of a finite number of terms in powers of r.

7. Modes for $L = 2$

In this section, we consider $L = 2$. For $|m| = 1$ and $|m| = 2$, it was already observed by Chandrasekhar and Lebovitz (1962a) and Carini and Ruffini (1991) that the modes of the compressible MacLaurin spheroids are identical to those of the incompressible spheroids. In accordance with this identity, we recovered the results of Smeyers (1986) for the incompressible MacLaurin spheroids for both values of $|m|$.

For $|m| = 0$, the modes of the compressible MacLaurin spheroids differ from those of the incompressible spheroids and are determined as follows.

The solutions of the differential Eqs. (47), (48), and (34) for $\ell = 2$ are given by

$$a_{2,0} = A_1\, r\,,\; \chi_{2,0} = B_1\, r^2\,,\; \Phi'_{2,0} = C_1\, r^2\,. \qquad (51)$$

From Eqs. (46), (36) and (49) for $\ell = 2$, it follows that

$$\left.\begin{array}{lcl} b_{2,0} & = & D_1\, r\,, \\ \sigma\,\mathrm{i}\,T_{1,0}/\left(\Omega\, r\right) & = & E_1\, r\,, \\ \alpha_{0,0} & = & F_1\,, \end{array}\right\} \tag{52}$$

where the constants D_1, E_1, and F_1 are related to the constants A_1, B_1, C_1 by

$$A_1 - 2\,D_1 = 0\,, \tag{53}$$

$$\sigma^2\,A_1 - 2\,B_1 - \frac{4}{3}\,\Omega^2\,E_1 - \frac{4}{3}\,K\,F_1 = 0\,, \tag{54}$$

$$2\,K\left(\xi_S^2 + 1\right)A_1 + B_1 - C_1 - \frac{2}{3}\,K\,E_1 + \frac{2}{3}\,\Gamma_1\,K\,F_1 = 0\,. \tag{55}$$

By solving differential Eqs. (47), (48), and (34) for $\ell = 0$, we obtain the following solutions that remain finite as $r \to 0$:

$$\left.\begin{array}{lcl} a_{0,0} & = & A_2^2\, r\,, \\ \chi_{0,0} & = & B_2^1 + B_2^2\, r^2\,, \\ \Phi'_{0,0} & = & C_2^1 + C_2^2\, r^2\,, \end{array}\right\} \tag{56}$$

where the constants A_2^2, B_2^2, and C_2^2 are defined as

$$A_2^2 = -\frac{1}{4}\,A_1 + \frac{1}{4}\,E_1 + \frac{1}{2}\,F_1\,, \tag{57}$$

$$B_2^2 = \frac{\sigma^2}{4}\,A_1 - \left(\frac{\sigma^2}{4} - \frac{2}{3}\,\Omega^2\right)E_1 - K\left(\xi_S^2 + \frac{1}{3}\right)F_1\,, \tag{58}$$

$$C_2^2 = -\frac{2}{3}\,\pi\,G\,\rho\,F_1\,. \tag{59}$$

Eq. (36) is identically satisfied for $\ell = 0$. Eq. (46) yields the following relation between constants:

$$-\frac{1}{2}\,A_1 + \frac{1}{2}\,E_1 + A_2^2 = 0\,. \tag{60}$$

From Eq. (37) for $\ell = 0$, two more relations between constants are derived:

$$B_2^2 - C_2^2 + 2\,K\left(\xi_S^2 + \frac{1}{3}\right)A_2^2 + \frac{2}{3}\,K\,A_1 + \Gamma_1\,K\left(\xi_S^2 + \frac{1}{3}\right)F_1 = 0\,, \tag{61}$$

$$B_2^1 - C_2^1 - \Gamma_1\,K\,a^2\,\xi_S^2\left(\xi_S^2 + 1\right)F_1 = 0\,. \tag{62}$$

The Eulerian perturbation of the internal gravitational potential Φ'_i can be expressed in terms of the spheroidal coordinates as

$$\Phi'_i = C_2^1 + \left(\xi^2 + \frac{2}{3}\right)a^2\,C_2^2 - \frac{a^2}{3}\,C_1 + \frac{a^2}{3}\left[\left(3\,\xi^2 + 1\right)C_1 - 2\,C_2^2\right]P_2\left(\eta\right)\,. \tag{63}$$

From Expansion (28) and boundary Condition (25), it follows that the Eulerian perturbation of the external gravitational potential Φ'_e is given by

$$\Phi'_e = \left[C_2^1 + \left(\xi_S^2 + \frac{2}{3}\right)a^2\,C_2^2 - \frac{a^2}{3}\,C_1\right]\frac{Q_0\left(\mathrm{i}\,\xi\right)}{Q_0\left(\mathrm{i}\,\xi_S\right)} + \frac{a^2}{3}\left[\left(3\,\xi_S^2 + 1\right)C_1 - 2\,C_2^2\right]\frac{Q_2\left(\mathrm{i}\,\xi\right)}{Q_2\left(\mathrm{i}\,\xi_S\right)}\,P_2\left(\eta\right)\,. \tag{64}$$

Boundary Condition (27) is then automatically satisfied. In terms of the spheroidal coordinates, the component $\delta\xi$ of the Lagrangian displacement takes the form

$$\delta\xi = \frac{\xi\,\left(1 + \xi^2\right)}{\left(\xi^2 + \eta^2\right)}\left[A_2^2 + A_1\,P_2\left(\eta\right)\right]\,. \tag{65}$$

By inserting Expressions (63), (64), and (65) for the Eulerian perturbations of the internal and the external gravitational potential and for the component $\delta\xi$ of the Lagrangian displacement into Condition (26), we obtain the relations

$$C_1 = 6\,\pi\,G\,\rho\,A_1\,\frac{\mathrm{i}\,\xi_S\,Q_2\left(\mathrm{i}\,\xi_S\right)}{W_2\left(\mathrm{i}\,\xi_S\right)} - \frac{Q_2'\left(\mathrm{i}\,\xi_S\right)}{W_2\left(\mathrm{i}\,\xi_S\right)}\,C_2^2\,, \tag{66}$$

$$\frac{1}{3}\,a^2\,C_1\,Q_0'\left(\mathrm{i}\,\xi_S\right) - 4\,\pi\,G\,\rho\,a^2\,\mathrm{i}\,\xi_S\,Q_0\left(\mathrm{i}\,\xi_S\right)\,A_2^2 - \left[\left(\xi_S^2 + \frac{2}{3}\right)Q_0'\left(\mathrm{i}\,\xi_S\right) + 2\,\mathrm{i}\,\xi_S\,Q_0\left(\mathrm{i}\,\xi_S\right)\right]a^2\,C_2^2 - Q_0'\left(\mathrm{i}\,\xi_S\right)\,C_2^1 = 0\,, \tag{67}$$

where $W_\ell^m\left(\mathrm{i}\,\xi_S\right)$ is the Wronskian $W\left[P_\ell^m\left(\mathrm{i}\,\xi_S\right), Q_\ell^m\left(\mathrm{i}\,\xi_S\right)\right]$.

Eqs. (53)–(55) and (57)–(62) and boundary Conditions (66) and (67) yield a homogeneous system of eleven linear and algebraic equations for the eleven unknown constants A_1, B_1, C_1, D_1, E_1, F_1, A_2^2, B_2^2, C_2^2, B_2^1, and C_2^1.

By solving Eqs. (57) and (60) for E_1 and A_2^2 in terms of A_1 and F_1, we find

$$E_1 = A_1 - \frac{2}{3}\,F_1\,, \tag{68}$$

$$A_2^2 = \frac{1}{3}\,F_1\,. \tag{69}$$

Using the previous relations, Eqs. (54), (55), and (58) yield

$$B_1 = \left(\frac{\sigma^2}{2} - \frac{2}{3}\,\Omega^2\right)A_1 + \frac{2}{9}\left(2\,\Omega^2 - 3\,K\right)F_1\,, \tag{70}$$

$$C_1 = \left[\frac{\sigma^2}{2} - \frac{2}{3}\,\Omega^2 + 2\,K\left(\xi_S^2 + \frac{2}{3}\right)\right]A_1 + \frac{2}{9}\left[2\,\Omega^2 + K\left(3\,\Gamma_1 - 1\right)\right]F_1\,, \tag{71}$$

$$B_2^2 = \frac{2}{3}\,\Omega^2\,A_1 + \left[\frac{\sigma^2}{6} - \frac{4}{9}\,\Omega^2 - K\left(\xi_S^2 + \frac{1}{3}\right)\right]F_1\,. \tag{72}$$

By eliminating the constants B_1, C_1, E_1, A_2^2, B_2^2, and C_2^2 in Eqs. (61) and (66) by means of Eqs. (68)–(72) and Eq. (59), we derive the following two equations containing only the constants A_1 and F_1:

$$\left(K+\Omega^2\right) A_1 + \left[\frac{\sigma^2}{4} - \frac{2}{3}\,\Omega^2 - \frac{1}{3}\,K\,(3\,\Gamma_1-1)\,P_2\,(\mathrm{i}\,\xi_S) + \pi\,G\,\rho\right] F_1 = 0\,, \tag{73}$$

$$\left[\frac{\sigma^2}{2} - \frac{2}{3}\,\Omega^2 + 2\,K\left(\xi_S^2+\frac{2}{3}\right) - 6\,\pi\,G\,\rho\,\frac{\mathrm{i}\,\xi_S\,Q_2\,(\mathrm{i}\,\xi_S)}{W_2\,(\mathrm{i}\,\xi_S)}\right] A_1 + \left[\frac{4}{9}\,\Omega^2 + \frac{2}{9}\,K\,(3\,\Gamma_1-1) - \frac{2}{3}\,\pi\,G\,\rho\,\frac{Q_2'\,(\mathrm{i}\,\xi_S)}{W_2\,(\mathrm{i}\,\xi_S)}\right] F_1 = 0\,. \tag{74}$$

The eigenfrequencies are determined by the condition that the determinant of this homogeneous system of equations be zero. The condition leads to the quadratic equation for σ^2

$$\left[\sigma^2 - \frac{8}{3}\,\Omega^2 - \frac{4}{3}\,K\,(3\,\Gamma_1-1)\,P_2\,(\mathrm{i}\,\xi_S) + 4\,\pi\,G\,\rho\right] \left[\sigma^2 - \frac{4}{3}\,\Omega^2 + 4\,K\left(\xi_S^2+\frac{2}{3}\right) - 12\,\pi\,G\,\rho\,\frac{\mathrm{i}\,\xi_S\,Q_2\,(\mathrm{i}\,\xi_S)}{W_2\,(\mathrm{i}\,\xi_S)}\right] - \frac{16}{9}\left(K+\Omega^2\right)\left[2\,\Omega^2 + K\,(3\,\Gamma_1-1) - 3\,\pi\,G\,\rho\,\frac{Q_2'\,(\mathrm{i}\,\xi_S)}{W_2\,(\mathrm{i}\,\xi_S)}\right] = 0\,. \tag{75}$$

We have solved the quadratic equation as a function of the eccentricity e for $\Gamma_1 = 5/3$ and have found two positive values of σ^2 for all values of e. Our results agree with those found by means of the tensor virial method (Chandrasekhar and Lebovitz 1962a, Carini and Ruffini 1991).

In the limiting case for $e \to 0$, we find, for the largest root,

$$\lim_{\xi_S\to\infty} \frac{\sigma^2}{\pi\,G\,\rho} = 4\left(\Gamma_1 - \frac{4}{3}\right)\,, \tag{76}$$

and, for the smallest root,

$$\lim_{\xi_S\to\infty} \frac{\sigma^2}{\pi\,G\,\rho} = \frac{16}{15}\,. \tag{77}$$

Hence, the mode associated with the largest value of σ^2 stems from the fundamental radial mode in the non-rotating equilibrium sphere with uniform mass density, and the mode associated with the smallest value of σ^2 stems from the second-harmonic Kelvin mode of this configuration (Smeyers 1966). For the sake of simplicity, we refer to these modes as the pseudo-radial mode $ps\,r_0$ and the Kelvin mode of $\ell = 2$.

In the limiting case for $e \to 1$, we find, for the largest root,

$$\lim_{\xi_S\to 0} \frac{\sigma^2}{\pi\,G\,\rho} = 4\,\Gamma_1\,, \tag{78}$$

while the smallest root becomes zero.

According to Eq. (76), the square of the eigenfrequency of the fundamental radial mode in the equilibrium sphere with uniform mass density is negative when $\Gamma_1 < 4/3$. From Eq. (78), it appears that this dynamical instability is removed by a sufficient amount of angular momentum for any positive value of Γ_1, whatever small it may be (Chandrasekhar and Lebovitz 1962b, Lebovitz 1967).

The variation of the eigenfrequencies σ^2 as a function of the eccentricity e is represented in Fig. 1. The values of the eigenfrequencies remain nearly constant up to $e = 0.5$.

For a given eigenfrequency, Eqs. (73)–(74) can be solved for A_1 and F_1. The solutions for the divergence and the radial component of the Lagrangian displacement take the form

$$\left.\begin{aligned} \alpha &= F_1\,, \\ \xi_r &= \left[\frac{F_1}{3} + A_1\,P_2\,(\cos\theta)\right] r\,. \end{aligned}\right\} \tag{79}$$

For both modes, the divergence of the Lagrangian displacement is constant over the whole rotating MacLaurin spheroid. The radial component of the Lagrangian displacement is zero at the center and depends on the colatitude. At a fixed colatitude, it increases linearly with the radial distance.

In the limiting case of the non-rotating equilibrium sphere with uniform mass density, it follows from Eq. (73) that the constant F_1 is zero for the Kelvin mode of $\ell = 2$ so that the mode is divergence-free. For the fundamental radial mode, it follows from Eq. (74) that A_1 is zero so that the radial displacement is independent of the colatitude θ. The known relation between the divergence and the radial component of the Lagrangian displacement

$$\xi_r = \frac{\alpha}{3}\,r\,, \tag{80}$$

is then recovered (Sauvenier-Goffin 1951).

8. Modes for $L = 4$.

For $L = 4$, we must solve the systems of equations belonging to the degrees $\ell = 4$, $\ell = 2$, and $\ell = 0$.

The homogeneous differential Eqs. (47), (48), and (34) associated with $\ell = 4$, yield the solutions

$$\left.\begin{aligned} a_{4,0} &= A_1\,r^3\,, \\ \chi_{4,0} &= B_1\,r^4\,, \\ \Phi'_{4,0} &= C_1\,r^4\,. \end{aligned}\right\} \tag{81}$$

From algebraic Eqs. (46), (36), and (49), it follows that

$$\left.\begin{aligned} b_{4,0} &= D_1\,r^3\,, \\ \sigma\,\mathrm{i}\,T_{1,0}/(\Omega\,r) &= E_1\,r^3\,, \\ \alpha_{2,0} &= F_1\,r^2\,, \end{aligned}\right\} \tag{82}$$

where the constants D_1, E_1, and F_1 are related to the constants A_1, B_1, and C_1 by

$$A_1 - 4\,D_1 = 0\,, \tag{83}$$

$$\sigma^2\,A_1 - 4\,B_1 - \frac{24}{7}\,\Omega^2\,E_1 - \frac{24}{35}\,K\,F_1 = 0\,, \tag{84}$$

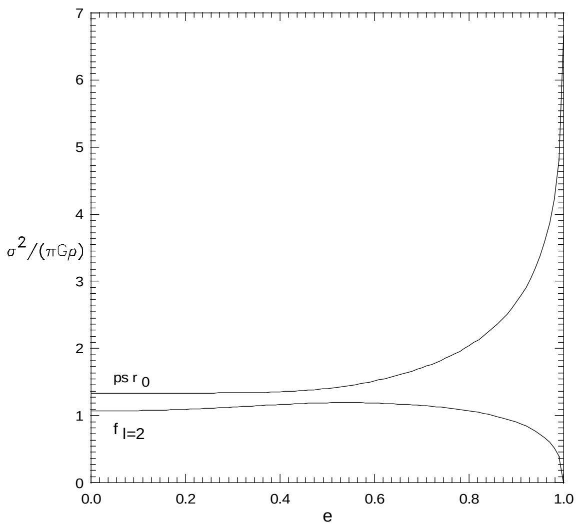

Fig. 1. Frequencies of the modes for $L = 2$ as a function of the eccentricity e. In the limit for $e \longrightarrow 0$, the largest frequency tends to that of the fundamental radial mode, and the smallest frequency to that of the Kelvin mode or f-mode of $\ell = 2$ of the non-rotating sphere. The mode with the largest frequency is denoted as the pseudo-radial mode $ps\ r_0$.

$$2\,K\left(\xi_S^2+1\right)A_1+B_1-C_1-\frac{12}{7}\,K\,E_1 + \frac{12}{35}\,\Gamma_1\,K\,F_1=0\,. \tag{85}$$

Next, by solving the inhomogeneous differential Eqs. (47), (48), and (34) associated with $\ell = 2$, we obtain

$$\left.\begin{aligned} a_{2,0} &= A_2^1\,r+A_2^2\,r^3\,,\\ \chi_{2,0} &= B_2^1\,r^2+B_2^2\,r^4\,,\\ \Phi'_{2,0} &= C_2^1\,r^2+C_2^2\,r^4\,, \end{aligned}\right\} \tag{86}$$

where

$$A_2^2 = -\frac{15}{8}\,A_1+\frac{15}{4}\,E_1+\frac{1}{2}\,F_1\,, \tag{87}$$

$$B_2^2 = \frac{5}{8}\,\sigma^2\,A_1+\left(-\frac{5}{4}\,\sigma^2+\frac{20}{7}\,\Omega^2\right)E_1 - K\left(\xi_S^2+\frac{3}{7}\right)F_1\,, \tag{88}$$

$$C_2^2 = -\frac{2}{7}\,\pi\,G\,\rho\,F_1\,. \tag{89}$$

The solutions of the algebraic equations associated with $\ell = 2$ are then given by

$$\left.\begin{aligned} b_{2,0} &= D_2^1\,r+D_2^2\,r^3\,,\\ \sigma\,\mathrm{i}\,T_{1,0}/\left(\Omega\,r\right) &= E_2^1\,r+E_2^2\,r^3\,,\\ \alpha_{0,0} &= F_2^1+F_2^2\,r^2\,. \end{aligned}\right\} \tag{90}$$

The constants with superscript 2 are related to the constants A_1, B_1, C_1, D_1, E_1, and F_1 by

$$\frac{5}{8}\,A_1-\frac{5}{4}\,E_1-\frac{1}{2}\,A_2^2+D_2^2=0\,, \tag{91}$$

$$\sigma^2\,A_2^2-2\,B_2^2-\frac{4}{3}\,\Omega^2\,E_2^2-\frac{4}{3}\,K\,F_2^2 - \frac{5}{4}\,\sigma^2\,A_1+\left(\frac{5}{2}\,\sigma^2-\frac{16}{7}\,\Omega^2\right)E_1-\frac{4}{21}\,K\,F_1=0\,, \tag{92}$$

$$2\,K\left(\xi_S^2+1\right)A_2^2+B_2^2-C_2^2-\frac{2}{3}\,K\,E_2^2+\frac{2}{3}\,\Gamma_1\,K\,F_2^2 + \frac{12}{7}\,K\,E_1+\Gamma_1\,K\left(\xi_S^2+\frac{11}{21}\right)F_1=0\,. \tag{93}$$

The constants with superscript 1 are related to the constants A_2^1, B_2^1, and C_2^1 by

$$A_2^1-2\,D_2^1=0\,, \tag{94}$$

$$\sigma^2\,A_2^1-2\,B_2^1-\frac{4}{3}\,\Omega^2\,E_2^1-\frac{4}{3}\,K\,F_2^1=0\,, \tag{95}$$

$$2\,K\left(\xi_S^2+1\right)A_2^1+B_2^1-C_2^1 - \frac{2}{3}\,K\,E_2^1+\frac{2}{3}\,\Gamma_1\,K\,F_2^1 - a^2\,\xi_S^2\left(\xi_S^2+1\right)\Gamma_1\,K\,F_1=0\,. \tag{96}$$

Thirdly, the inhomogeneous differential Eqs. (47), (48), and (34) associated with $\ell = 0$ admit of the solutions

$$\left.\begin{aligned} a_{0,0} &= A_3^2\,r+A_3^3\,r^3\,,\\ \chi_{0,0} &= B_3^1+B_3^2\,r^2+B_3^3\,r^4\,,\\ \Phi'_{0,0} &= C_3^1+C_3^2\,r^2+C_3^3\,r^4\,, \end{aligned}\right\} \tag{97}$$

where

$$A_3^3 = \frac{3}{32}\,A_1-\frac{3}{16}\,E_1-\frac{1}{8}\,A_2^2+\frac{1}{8}\,E_2^2+\frac{1}{4}\,F_2^2\,, \tag{98}$$

$$B_3^3 = -\frac{3}{32}\,\sigma^2\,A_1+\frac{3}{16}\,\sigma^2\,E_1-\frac{K}{15}\,F_1+\frac{\sigma^2}{8}\,A_2^2 - \left(\frac{\sigma^2}{8}-\frac{\Omega^2}{3}\right)E_2^2-\frac{K}{2}\left(\xi_S^2+\frac{1}{3}\right)F_2^2\,, \tag{99}$$

$$C_3^3 = -\frac{1}{5}\,\pi\,G\,\rho\,F_2^2\,, \tag{100}$$

and

$$A_3^2=-\frac{1}{4}\,A_2^1+\frac{1}{4}\,E_2^1+\frac{1}{2}\,F_2^1\,, \tag{101}$$

$$B_3^2=\frac{\sigma^2}{4}\,A_2^1-\left(\frac{\sigma^2}{4}-\frac{2}{3}\,\Omega^2\right)E_2^1-K\left(\xi_S^2+\frac{1}{3}\right)F_2^1\,, \tag{102}$$

$$C_3^2=-\frac{2}{3}\,\pi\,G\,\rho\,F_2^1\,. \tag{103}$$

Eq. (36) is identically satisfied for $\ell = 0$. By inserting the solutions into Eqs. (46) and (37) for $\ell = 0$, one obtains the following relations between constants:

$$\frac{3}{8}A_1 - \frac{3}{4}E_1 - \frac{1}{2}A_2^2 + \frac{1}{2}E_2^2 + A_3^3 = 0\,, \tag{104}$$

$$\begin{aligned} &B_3^3 - C_3^3 + 2K\left(\xi_S^2 + \frac{1}{3}\right)A_3^3 + \frac{2}{3}K A_2^2 \\ &\quad + \Gamma_1 K\left(\xi_S^2 + \frac{1}{3}\right)F_2^2 - \frac{K}{2}A_1 + K E_1 \\ &\quad + \frac{2}{15}\Gamma_1 K F_1 = 0\,, \end{aligned} \tag{105}$$

$$-\frac{1}{2}A_2^1 + \frac{1}{2}E_2^1 + A_3^2 = 0\,, \tag{106}$$

$$\begin{aligned} &B_3^2 - C_3^2 + 2K\left(\xi_S^2 + \frac{1}{3}\right)A_3^2 + \frac{2}{3}K A_2^1 \\ &\quad + \Gamma_1 K\left(\xi_S^2 + \frac{1}{3}\right)F_2^1 \\ &\quad - \Gamma_1 K a^2 \xi_S^2\left(\xi_S^2 + 1\right)F_2^2 = 0\,, \end{aligned} \tag{107}$$

$$B_3^1 - C_3^1 - \Gamma_1 K a^2 \xi_S^2\left(\xi_S^2 + 1\right)F_2^1 = 0\,. \tag{108}$$

The solution for the Eulerian perturbation of the internal gravitational potential can be expressed in terms of the spheroidal coordinates as

$$\begin{aligned} \Phi_i' = {} & \frac{8}{35}a^4\left[C_3^3 - \left(\frac{3}{2}\xi^2 + \frac{1}{2}\right)C_2^2\right. \\ & \left. + \left(\frac{35}{8}\xi^4 + \frac{30}{8}\xi^2 + \frac{3}{8}\right)C_1\right]P_4(\eta) \\ & + \left[-\frac{4}{3}\left(\xi^2 + \frac{4}{7}\right)a^4 C_3^3 + \left(\xi^4 + \frac{17}{21}\xi^2 + \frac{8}{21}\right)a^4 C_2^2\right. \\ & - \frac{2}{7}\left(3\xi^2 + 1\right)a^4 C_1 - \frac{2}{3}a^2 C_3^2 \\ & \left. + \left(\xi^2 + \frac{1}{3}\right)a^2 C_2^1\right]P_2(\eta) \\ & + \left[\left(\xi^4 + \frac{4}{3}\xi^2 + \frac{8}{15}\right)a^4 C_3^3 - \left(\frac{7}{15}\xi^2 + \frac{4}{15}\right)a^4 C_2^2\right. \\ & + \frac{1}{5}a^4 C_1 + \left(\xi^2 + \frac{2}{3}\right)a^2 C_3^2 \\ & \left. - \frac{1}{3}a^2 C_2^1 + C_3^1\right]P_0(\eta)\,. \end{aligned} \tag{109}$$

From boundary Conditions (25) and (27), it follows that only the constants $c_{0,0}$, $c_{2,0}$, and $c_{4,0}$ in Expansion (28) for the Eulerian perturbation of the external gravitational potential Φ_e' are different from zero and are given by

$$\begin{aligned} c_{0,0} = {} & \frac{1}{Q_0(\mathrm{i}\,\xi_S)}\left[\left(\xi_S^4 + \frac{4}{3}\xi_S^2 + \frac{8}{15}\right)a^4 C_3^3\right. \\ & - \left(\frac{7}{15}\xi_S^2 + \frac{4}{15}\right)a^4 C_2^2 + \frac{1}{5}a^4 C_1 \\ & \left. + \left(\xi_S^2 + \frac{2}{3}\right)a^2 C_3^2 - \frac{1}{3}a^2 C_2^1 + C_3^1\right]\,, \end{aligned} \tag{110}$$

$$\begin{aligned} c_{2,0} = {} & \frac{1}{Q_2(\mathrm{i}\,\xi_S)}\left[-\frac{4}{3}\left(\xi_S^2 + \frac{4}{7}\right)a^4 C_3^3\right. \\ & + \left(\xi_S^4 + \frac{17}{21}\xi_S^2 + \frac{8}{21}\right)a^4 C_2^2 - \frac{2}{7}\left(3\xi_S^2 + 1\right)a^4 C_1 \\ & \left. - \frac{2}{3}a^2 C_3^2 + \left(\xi_S^2 + \frac{1}{3}\right)a^2 C_2^1\right]\,, \end{aligned} \tag{111}$$

$$\begin{aligned} c_{4,0} = {} & \frac{1}{Q_4(\mathrm{i}\,\xi_S)}\frac{8}{35}a^4\left[C_3^3 - \left(\frac{3}{2}\xi_S^2 + \frac{1}{2}\right)C_2^2\right. \\ & \left. + \left(\frac{35}{8}\xi_S^4 + \frac{30}{8}\xi_S^2 + \frac{3}{8}\right)C_1\right]\,. \end{aligned} \tag{112}$$

In terms of the spheroidal coordinates, the solution for the component $\delta\,\xi$ of the Lagrangian displacement can be written as

$$\begin{aligned} \delta\,\xi = {} & \frac{\xi\left(\xi^2 + 1\right)}{\xi^2 + \eta^2}\left\{\frac{8}{35}a^2\left[-\frac{3}{2}A_2^2 + \left(\frac{35}{8}\xi^2 + \frac{15}{4}\right)A_1\right.\right. \\ & \left. - \frac{15}{4}E_1\right]P_4(\eta) + \left[-\frac{2}{3}a^2 A_3^3 + \left(\xi^2 + \frac{10}{21}\right)a^2 A_2^2\right. \\ & \left. - \frac{17}{28}a^2 A_1 + \frac{5}{14}a^2 E_1 + A_2^1\right]P_2(\eta) \\ & + \left[\left(\xi^2 + \frac{2}{3}\right)a^2 A_3^3 - \frac{2}{15}a^2 A_2^2\right. \\ & \left.\left. - \frac{1}{4}a^2 A_1 + \frac{1}{2}a^2 E_1 + A_3^2\right]P_0(\eta)\right\}\,. \end{aligned} \tag{113}$$

By inserting the expressions for the Eulerian perturbations of the internal and the external gravitational potential and for the component $\delta\,\xi$ of the Lagrangian displacement into boundary Condition (26), we derive the following relations between constants:

$$\begin{aligned} & -\frac{5}{2}\pi G\rho\xi_S\left(7\xi_S^2 + 6\right)A_1 - \frac{W_4(\mathrm{i}\,\xi_S)}{\mathrm{i}\,Q_4(\mathrm{i}\,\xi_S)}C_1 \\ & + 15\pi G\rho\xi_S E_1 + 6\pi G\rho\xi_S A_2^2 \\ & - \frac{W\left(P_2(\mathrm{i}\,\xi_S), Q_4(\mathrm{i}\,\xi_S)\right)}{\mathrm{i}\,Q_4(\mathrm{i}\,\xi_S)}C_2^2 - \frac{Q_4'(\mathrm{i}\,\xi_S)}{\mathrm{i}\,Q_4(\mathrm{i}\,\xi_S)}C_3^3 = 0\,, \end{aligned} \tag{114}$$

$$\begin{aligned} & \frac{17}{7}\pi G\rho\xi_S a^2 A_1 + \frac{1}{\mathrm{i}\,Q_2(\mathrm{i}\,\xi_S)}\left[\frac{12}{7}\mathrm{i}\,\xi_S Q_2(\mathrm{i}\,\xi_S)\right. \\ & \left. + \frac{2}{7}\left(3\xi_S^2 + 1\right)Q_2'(\mathrm{i}\,\xi_S)\right]a^2 C_1 - \frac{10}{7}\pi G\rho\xi_S a^2 E_1 \\ & - 4\pi G\rho\xi_S\left(\xi_S^2 + \frac{10}{21}\right)a^2 A_2^2 \\ & + \frac{1}{\mathrm{i}\,Q_2(\mathrm{i}\,\xi_S)}\left[-\mathrm{i}\,\xi_S\left(4\xi_S^2 + \frac{34}{21}\right)Q_2(\mathrm{i}\,\xi_S)\right. \\ & \left. - \left(\xi_S^4 + \frac{17}{21}\xi_S^2 + \frac{8}{21}\right)Q_2'(\mathrm{i}\,\xi_S)\right]a^2 C_2^2 \\ & + \frac{8}{3}\pi G\rho\xi_S a^2 A_3^3 + \frac{1}{\mathrm{i}\,Q_2(\mathrm{i}\,\xi_S)}\left[\frac{8}{3}\mathrm{i}\,\xi_S Q_2(\mathrm{i}\,\xi_S)\right. \\ & \left. + \frac{4}{3}\left(\xi_S^2 + \frac{4}{7}\right)Q_2'(\mathrm{i}\,\xi_S)\right]a^2 C_3^3 - 4\pi G\rho\xi_S A_2^1 \\ & + \frac{2}{3}\frac{W_2(\mathrm{i}\,\xi_S)}{\mathrm{i}\,Q_2(\mathrm{i}\,\xi_S)}C_2^1 + \frac{2}{3}\frac{Q_2'(\mathrm{i}\,\xi_S)}{\mathrm{i}\,Q_2(\mathrm{i}\,\xi_S)}C_3^2 = 0\,, \end{aligned} \tag{115}$$

$$
\begin{aligned}
&\pi G \rho \xi_S a^4 A_1 - \frac{1}{5} \frac{Q_0'(\mathrm{i}\,\xi_S)}{\mathrm{i}\,Q_0(\mathrm{i}\,\xi_S)} a^4 C_1 - 2\pi G \rho \xi_S a^4 E_1 \\
&+ \frac{8}{15} \pi G \rho \xi_S a^4 A_2^2 + \frac{1}{\mathrm{i}\,Q_0(\mathrm{i}\,\xi_S)} \left[\frac{14}{15} \mathrm{i}\,\xi_S Q_0(\mathrm{i}\,\xi_S) \right. \\
&\left. + \frac{1}{15} \left(7\xi_S^2 + 4\right) Q_0'(\mathrm{i}\,\xi_S) \right] a^4 C_2^2 \\
&- 4\pi G \rho \xi_S \left(\xi_S^2 + \frac{2}{3}\right) a^4 A_3^3 \\
&+ \frac{1}{\mathrm{i}\,Q_0(\mathrm{i}\,\xi_S)} \left[-4\,\mathrm{i}\,\xi_S \left(\xi_S^2 + \frac{2}{3}\right) Q_0(\mathrm{i}\,\xi_S) \right. \\
&\left. - \left(\xi_S^4 + \frac{4}{3}\xi_S^2 + \frac{8}{15}\right) Q_0'(\mathrm{i}\,\xi_S) \right] a^4 C_3^3 \\
&+ \frac{1}{3} \frac{Q_0'(\mathrm{i}\,\xi_S)}{\mathrm{i}\,Q_0(\mathrm{i}\,\xi_S)} a^2 C_2^1 - 4\pi G \rho \xi_S a^2 A_3^2 \\
&+ \frac{1}{\mathrm{i}\,Q_0(\mathrm{i}\,\xi_S)} \left[-2\,\mathrm{i}\,\xi_S Q_0(\mathrm{i}\,\xi_S) \right. \\
&\left. - \left(\xi_S^2 + \frac{2}{3}\right) Q_0'(\mathrm{i}\,\xi_S) \right] a^2 C_3^2 - \frac{Q_0'(\mathrm{i}\,\xi_S)}{\mathrm{i}\,Q_0(\mathrm{i}\,\xi_S)} C_3^1 = 0\,. \qquad (116)
\end{aligned}
$$

Boundary Conditions (114)–(116) together with Eqs. (83)–(85), (87)–(89), (91)–(96), (98)–(108) yield a homogeneous system of 26 linear and algebraic equations in the 26 unknown constants A_1, B_1, C_1, D_1, E_1, F_1, A_2^2, B_2^2, C_2^2, D_2^2, E_2^2, F_2^2, A_3^3, B_3^3, C_3^3, A_2^1, B_2^1, C_2^1, D_2^1, E_2^1, F_2^1, A_3^2, B_3^2, C_3^2, B_3^1, C_3^1.

The eigenfrequencies corresponding to the values of σ^2 for which this system of equations has a non-trivial solution can be determined by eliminating twenty constants in terms of the six constants A_1, F_1, A_2^2, F_2^2, A_2^1, and F_2^1. First, solving Eqs. (87), (98), and (104) for E_1, E_2^2, and A_3^3, yields

$$E_1 = \frac{1}{2} A_1 + \frac{4}{15} A_2^2 - \frac{2}{15} F_1\,, \qquad (117)$$

$$E_2^2 = -\frac{1}{5} F_1 + \frac{7}{5} A_2^2 - \frac{2}{5} F_2^2\,, \qquad (118)$$

$$A_3^3 = \frac{1}{5} F_2^2\,. \qquad (119)$$

By inserting the expressions for E_1 and E_2^2 into Eqs. (84), (85), (88), and (99), we obtain

$$
\begin{aligned}
B_1 = {} & \left(\frac{\sigma^2}{4} - \frac{3}{7}\Omega^2\right) A_1 - \frac{8}{35}\Omega^2 A_2^2 \\
& + \frac{2}{35}\left(2\Omega^2 - 3K\right) F_1\,, \qquad (120)
\end{aligned}
$$

$$
\begin{aligned}
C_1 = {} & \left[\frac{\sigma^2}{4} - \frac{3}{7}\Omega^2 + 2K\left(\xi_S^2 + \frac{4}{7}\right)\right] A_1 + \frac{2}{35}\left[2\Omega^2\right. \\
& \left. + (1 + 6\Gamma_1) K\right] F_1 - \frac{8}{35}\left(\Omega^2 + 2K\right) A_2^2\,, \qquad (121)
\end{aligned}
$$

$$
\begin{aligned}
B_2^2 = {} & \frac{10}{7}\Omega^2 A_1 + \left[\frac{\sigma^2}{6} - \frac{8}{21}\Omega^2 - K\left(\xi_S^2 + \frac{3}{7}\right)\right] F_1 \\
& + \left(-\frac{\sigma^2}{3} + \frac{16}{21}\Omega^2\right) A_2^2\,, \qquad (122)
\end{aligned}
$$

$$
\begin{aligned}
B_3^3 = {} & -\frac{1}{15}\left(K + \Omega^2\right) F_1 + \frac{7}{15}\Omega^2 A_2^2 \\
& + \left[\frac{\sigma^2}{20} - \frac{2}{15}\Omega^2 - \frac{K}{2}\left(\xi_S^2 + \frac{1}{3}\right)\right] F_2^2\,. \qquad (123)
\end{aligned}
$$

By eliminating the constants A_3^3, B_3^3, B_2^2, E_2^2, E_1, C_1, C_2^2, and C_3^3 in Eqs. (92)–(93), (105), and (114) by means of the previous expressions and Eqs. (89) and (100), one derives the following four equations containing only the four constants A_1, F_1, A_2^2, and F_2^2:

$$
\begin{aligned}
& -2\Omega^2 A_1 + \left[-\frac{\sigma^2}{3} + \frac{2}{3}\Omega^2 + K\left(\xi_S^2 + \frac{1}{3}\right)\right] F_1 \\
& \quad + \left(\frac{7}{6}\sigma^2 - 2\Omega^2\right) A_2^2 + \frac{2}{15}\left(2\Omega^2 - 5K\right) F_2^2 = 0\,, \qquad (124)
\end{aligned}
$$

$$
\begin{aligned}
& \frac{2}{7}\left(5\Omega^2 + 3K\right) A_1 + \left[\frac{2}{7}\pi G \rho + \frac{\sigma^2}{6} - \frac{8}{21}\Omega^2\right. \\
& \quad \left. + K\left(\xi_S^2 + \frac{11}{21}\right)(\Gamma_1 - 1)\right] F_1 \\
& \quad + \left[-\frac{\sigma^2}{3} + \frac{16}{21}\Omega^2 + 2K\left(\xi_S^2 + \frac{16}{21}\right)\right] A_2^2 \\
& \quad + \frac{2}{15} K (2 + 5\Gamma_1) F_2^2 = 0\,, \qquad (125)
\end{aligned}
$$

$$
\begin{aligned}
& \frac{1}{15}\left[(2\Gamma_1 - 3) K - \Omega^2\right] F_1 + \frac{7}{15}\left(\Omega^2 + 2K\right) A_2^2 \\
& \quad + \left[\frac{\sigma^2}{20} - \frac{2}{15}\Omega^2 + K\left(\Gamma_1 - \frac{1}{10}\right)\left(\xi_S^2 + \frac{1}{3}\right)\right. \\
& \quad \left. + \frac{1}{5}\pi G \rho\right] F_2^2 = 0\,, \qquad (126)
\end{aligned}
$$

$$
\begin{aligned}
& \left[\frac{\sigma^2}{4} - \frac{3}{7}\Omega^2 + 2K\left(\xi_S^2 + \frac{4}{7}\right)\right. \\
& \quad \left. + \frac{5}{2}\pi G \rho \left(7\xi_S^2 + 3\right) \frac{\mathrm{i}\,\xi_S Q_4(\mathrm{i}\,\xi_S)}{W_4(\mathrm{i}\,\xi_S)}\right] A_1 \\
& \quad + \left[\frac{4}{35}\Omega^2 + \frac{2}{35}(1 + 6\Gamma_1) K + 2\pi G \rho \frac{\mathrm{i}\,\xi_S Q_4(\mathrm{i}\,\xi_S)}{W_4(\mathrm{i}\,\xi_S)}\right. \\
& \quad \left. - \frac{2}{7}\pi G \rho \frac{W\left(P_2(\mathrm{i}\,\xi_S), Q_4(\mathrm{i}\,\xi_S)\right)}{W_4(\mathrm{i}\,\xi_S)}\right] F_1 \\
& \quad + \left[-\frac{8}{35}\left(\Omega^2 + 2K\right) - 10\pi G \rho \frac{\mathrm{i}\,\xi_S Q_4(\mathrm{i}\,\xi_S)}{W_4(\mathrm{i}\,\xi_S)}\right] A_2^2 \\
& \quad - \frac{1}{5}\pi G \rho \frac{Q_4'(\mathrm{i}\,\xi_S)}{W_4(\mathrm{i}\,\xi_S)} F_2^2 = 0\,. \qquad (127)
\end{aligned}
$$

Eqs. (101) and (106) can be solved as

$$A_3^2 = \frac{1}{3} F_2^1\,, \qquad (128)$$

$$E_2^1 = A_2^1 - \frac{2}{3} F_2^1\,. \qquad (129)$$

By the use of the previous relations, it follows from Eqs. (95), (96), and (102) that

$$B_2^1 = \left(\frac{\sigma^2}{2} - \frac{2}{3}\Omega^2\right) A_2^1 + \frac{2}{9}\left(2\Omega^2 - 3K\right) F_2^1\,, \qquad (130)$$

$$C_2^1 = \left[\frac{\sigma^2}{2} - \frac{2}{3}\,\Omega^2 + 2\,K\left(\xi_S^2 + \frac{2}{3}\right)\right] A_2^1 + \frac{2}{9}\left[2\,\Omega^2 + K\,(3\,\Gamma_1 - 1)\right] F_2^1 - a^2\,\xi_S^2\left(\xi_S^2 + 1\right)\Gamma_1\,K\,F_1\,, \tag{131}$$

$$B_3^2 = \frac{2}{3}\,\Omega^2\,A_2^1 + \left[\frac{\sigma^2}{6} - \frac{4}{9}\,\Omega^2 - K\left(\xi_S^2 + \frac{1}{3}\right)\right] F_2^1\,. \tag{132}$$

By eliminating the constants A_3^2, B_3^2, C_3^2, B_2^1, and C_2^1 by means of the previous expressions and Eq. (103), and C_1, E_1, C_2^2, A_3^3, and C_3^3 in Eqs. (107) and (115) by means of Relations (89), (100), (117), (119), and (121), one derives two more relations between the constants A_1, F_1, A_2^2, F_2^2, A_2^1, and F_2^1:

$$\left(K + \Omega^2\right) A_2^1 + \left[\frac{\sigma^2}{4} - \frac{2}{3}\,\Omega^2 - \frac{1}{3}\,K\,(3\,\Gamma_1 - 1)\,P_2\,(\mathrm{i}\,\xi_S) + \pi\,G\,\rho\right] F_2^1 = \frac{3}{2}\,\Gamma_1\,K\,a^2\,\xi_S^2\,\left(\xi_S^2 + 1\right) F_2^2\,, \tag{133}$$

$$\begin{aligned}
&\left[\frac{\sigma^2}{2} - \frac{2}{3}\,\Omega^2 + 2\,K\left(\xi_S^2 + \frac{2}{3}\right) - 6\,\pi\,G\,\rho\,\frac{\mathrm{i}\,\xi_S\,Q_2\,(\mathrm{i}\,\xi_S)}{W_2\,(\mathrm{i}\,\xi_S)}\right] A_2^1 \\
&+ \left[\frac{4}{9}\,\Omega^2 + \frac{2}{9}\,K\,(3\,\Gamma_1 - 1) - \frac{2}{3}\,\pi\,G\,\rho\,\frac{Q_2'\,(\mathrm{i}\,\xi_S)}{W_2\,(\mathrm{i}\,\xi_S)}\right] F_2^1 \\
&= -a^2\left\{\frac{6}{7}\left[-\frac{\sigma^2}{4} + \frac{3}{7}\,\Omega^2 - 2\,K\left(\xi_S^2 + \frac{4}{7}\right) + 3\,\pi\,G\,\rho\,\frac{\mathrm{i}\,\xi_S\,Q_2\,(\mathrm{i}\,\xi_S)}{W_2\,(\mathrm{i}\,\xi_S)}\right] A_1\right. \\
&+ \left\{-\xi_S^2\,\left(\xi_S^2 + 1\right)\Gamma_1\,K - \frac{12}{245}\left[2\,\Omega^2 + (1 + 6\,\Gamma_1)\,K\right]\right. \\
&+ \frac{2}{7}\,\frac{\pi\,G\,\rho}{W_2\,(\mathrm{i}\,\xi_S)}\left[6\,\mathrm{i}\,\xi_S\left(\xi_S^2 + \frac{4}{7}\right) Q_2\,(\mathrm{i}\,\xi_S) + \left.\left(\frac{3}{2}\,\xi_S^4 + \frac{17}{14}\,\xi_S^2 + \frac{4}{7}\right) Q_2'\,(\mathrm{i}\,\xi_S)\right]\right\} F_1 \\
&+ \left[\frac{48}{245}\left(\Omega^2 + 2\,K\right) - 6\,\pi\,G\,\rho\left(\xi_S^2 + \frac{4}{7}\right)\frac{\mathrm{i}\,\xi_S\,Q_2\,(\mathrm{i}\,\xi_S)}{W_2\,(\mathrm{i}\,\xi_S)}\right] A_2^2 \\
&\left. - \frac{2}{5}\,\pi\,G\,\rho\left(\xi_S^2 + \frac{4}{7}\right)\frac{Q_2'\,(\mathrm{i}\,\xi_S)}{W_2\,(\mathrm{i}\,\xi_S)} F_2^2\right\}\,.
\end{aligned} \tag{134}$$

The eigenfrequencies of the modes for $L = 4$ are determined by the zeros of the determinant of the homogeneous system of the six Eqs. (124)–(127) and (133)–(134) in the six unknown constants A_1, F_1, A_2^2, F_2^2, A_2^1, and F_2^1. The determinant is of the sixth degree in σ^2.

The first four equations contain only the four constants A_1, F_1, A_2^2, and F_2^2. Consequently, the determinant Δ of the coefficients of the full system of equations can be written as the product of two determinants: a (4×4)-determinant Δ_1 composed of the coefficients of the first four equations and a (2×2)-determinant Δ_2 composed of the coefficients of the terms in the last two equations that involve the constants A_2^1 and F_2^1. The determinant Δ_1 is of the fourth degree in σ^2, the determinant Δ_2 of the second degree in σ^2.

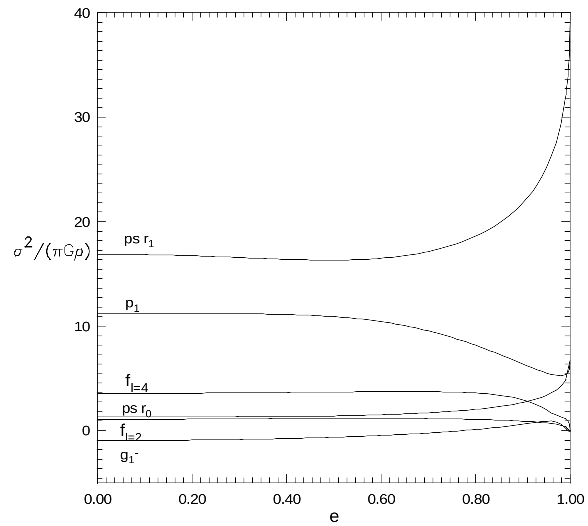

Fig. 2. The eigenfrequencies σ^2 of the modes found for $L = 4$ as a function of the eccentricity e. In the limit for $e \to 0$, the largest frequency corresponds to that of the first harmonic radial mode ($ps\ r_1$), the second largest frequency to that of the p_1-mode of $\ell = 2$ (p_1), the third largest frequency to that of the Kelvin mode of $\ell = 4$ ($f_{\ell=4}$), the fourth largest frequency to that of the fundamental radial mode ($ps\ r_0$), the fifth largest frequency to that of the Kelvin mode of $\ell = 2$ ($f_{\ell=2}$), and the smallest frequency to that of the g_1^--mode of $\ell = 2$ (g_1^-).

The requirement that the determinant Δ be zero can be satisfied by the conditions $\Delta_1 = 0$ or/and $\Delta_2 = 0$.

For $\Delta_2 = 0$, two eigenfrequencies are found. These eigenfrequencies correspond to those obtained in the case $L = 2$. The constants A_1, F_1, A_2^2, and F_2^2 are equal to zero, so that one recovers the eigenfunctions determined above.

The equation $\Delta_1 = 0$ admits of four eigenfrequencies σ^2. In the limiting case of a non-rotating spherically symmetric equilibrium configuration, these eigenfrequencies are given by

$$\lim_{\xi_S \to \infty} \frac{\sigma_1^2}{\pi\,G\,\rho} = \frac{32}{9}\,, \tag{135}$$

$$\lim_{\xi_S \to \infty} \frac{\sigma_{2,3}^2}{\pi\,G\,\rho} = -\frac{2}{3}\,(4 - 7\,\Gamma_1) \pm \left[\frac{4}{9}\,(4 - 7\,\Gamma_1)^2 + \frac{32}{3}\right]^{1/2}\,, \tag{136}$$

$$\lim_{\xi_S \to \infty} \frac{\sigma_4^2}{\pi\,G\,\rho} = \frac{8}{3}\,(5\,\Gamma_1 - 2)\,. \tag{137}$$

They correspond to the eigenfrequency of the Kelvin mode associated with $\ell = 4$, the p_1- and the g_1^--mode associated with $\ell = 2$, and the first harmonic radial mode, respectively (Smeyers 1966). Again, for the sake of brevity, we refer to these modes as the Kelvin mode of $\ell = 4$, the p_1- and the g_1^- mode of $\ell = 2$, and the pseudo-radial mode $ps\ r_1$.

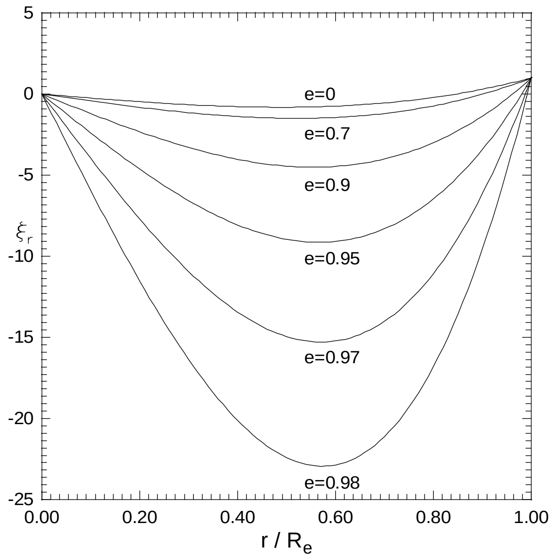

Fig. 3. The variation of the radial component of the Lagrangian displacement in the equatorial plane for the mode $ps\ r_1$ stemming from the first harmonic radial mode in the non-rotating equilibrium sphere. The variation is presented for various values of the eccentricity.

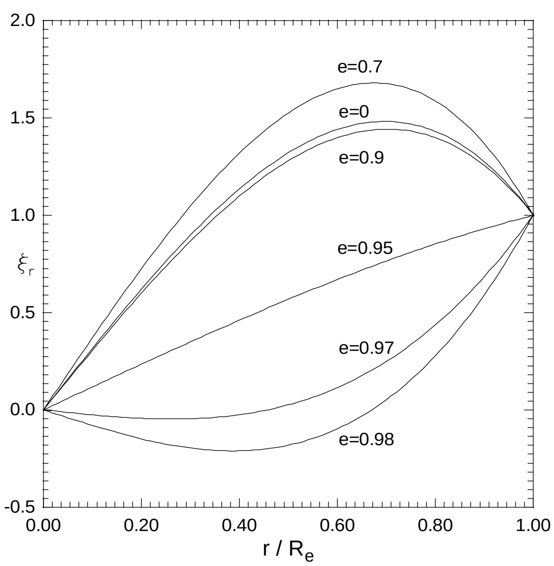

Fig. 4. The variation of the radial component of the Lagrangian displacement in the equatorial plane for the mode stemming from the g_1^--mode of ℓ = 2 in the non-rotating equilibrium sphere. The variation is presented for various values of the eccentricity.

In the limiting case for $e \to 1$, two of the four frequencies determined by the (4 × 4)-determinant become zero, while, for the other two frequencies, we find

$$\lim_{\xi_S \to 0} \frac{\sigma_3^2}{\pi\, G\, \rho} \;=\; 4\,\Gamma_1\,, \tag{138}$$

$$\lim_{\xi_S \to 0} \frac{\sigma_4^2}{\pi\, G\, \rho} \;=\; 24\,\Gamma_1\,. \tag{139}$$

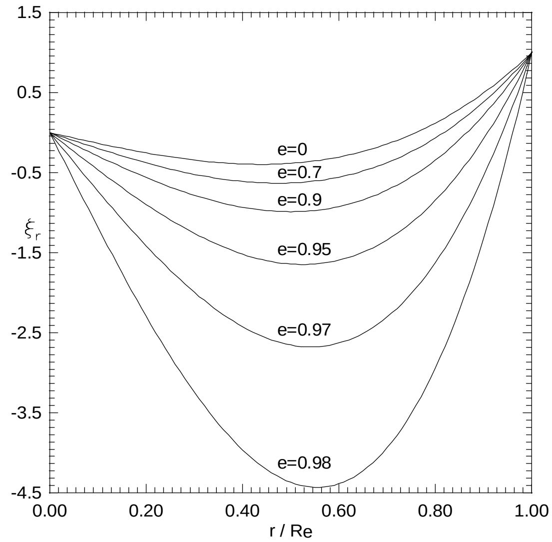

Fig. 5. The variation of the radial component of the Lagrangian displacement in the equatorial plane for the mode stemming from the p_1-mode of ℓ = 2 in the non-rotating equilibrium sphere. The variation is presented for various values of the eccentricity.

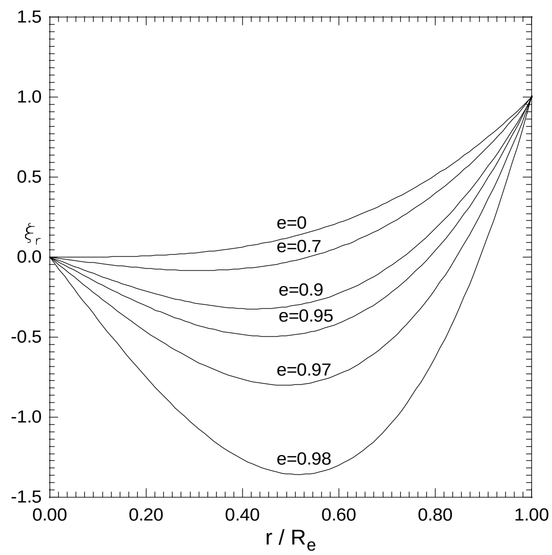

Fig. 6. The variation of the radial component of the Lagrangian displacement in the equatorial plane for the mode stemming from the Kelvin mode of ℓ = 4 in the non-rotating equilibrium sphere. The variation is presented for various values of the eccentricity.

The variations of the six eigenfrequencies σ^2 of the axisymmetrix modes that are found for L = 4 are displayed in Fig. 2, as a function of the eccentricity e of the MacLaurin spheroid. The value $\Gamma_1 = 5/3$ is used. Some of the lines representing the evolution of eigenfrequencies σ^2 determined by the condition Δ_1 = 0 cross lines representing the evolution of eigenfrequencies σ^2 determined by the condition Δ_2 = 0. At the crossings, both determinants Δ_1 and Δ_2 are equal to zero. The following crossings occur: at e = 0.9103, between the curve for the Kelvin mode of ℓ = 4, and the curve for the fundamental pseudo-radial mode $ps\ r_0$, at e = 0.9321 and e = 0.9839, between the curve for the g_1^--mode of $\ell = 2$ and the curve for the Kelvin mode of $\ell = 2$.

The g_1^--mode of $\ell = 2$ becomes dynamically stable at $e = 0.7724$ and again dynamically unstable at $e = 0.9952$. The eigenfrequencies σ^2 of the five other modes are positive for all values of e.

For each eigenfrequency σ^2, the solutions for the radial component and the divergence of the Lagrangian displacement are given by expressions of the form

$$\left.\begin{aligned} \xi_r &= A_1\, r^3\, P_4(\cos\theta) + \left(A_2^2\, r^3 + A_2^1\, r\right) P_2(\cos\theta) \\ &\quad + \left(A_3^3\, r^3 + A_3^2\, r\right) . \\ \alpha &= F_1\, r^2\, P_2(\cos\theta) + \left(F_2^2\, r^2 + F_2^1\right) , \end{aligned}\right\} \qquad (140)$$

where A_1, F_1, A_2^2, F_2^2, A_2^1, and F_2^1 are non-trivial solutions of the system of six Eqs. (124)–(127) and (133)–(134), and A_3^3 and A_3^2 are given by Expressions (119) and (128).

In the limit for $e \to 0$, the solutions for the radial component and the divergence of the Lagrangian displacement reduce to the known solutions for the p_1- and the g_1^--mode of $\ell = 2$

$$\left.\begin{aligned} \xi_r &= \left[\frac{1}{7}\left(2 + \frac{4}{\sigma^2}\right) r^3 \right. \\ &\quad \left. - \frac{R_e^2}{7}\left(1 + \frac{4}{\sigma^2}\right) r\right] F_1\, P_2(\cos\theta) , \\ \alpha &= F_1\, r^2\, P_2(\cos\theta) , \end{aligned}\right\} \qquad (141)$$

the known solution for the first harmonic radial mode

$$\left.\begin{aligned} \xi_r &= \frac{F_2^2}{5}\left(r^3 - \frac{5}{7} R_e^2\, r\right) , \\ \alpha &= F_2^2\left(r^2 - \frac{3}{7} R_e^2\right) , \end{aligned}\right\} \qquad (142)$$

and the known solution for the Kelvin mode of $\ell = 4$

$$\left.\begin{aligned} \xi_r &= A_1\, r^3\, P_4(\cos\theta) , \\ \alpha &= 0 , \end{aligned}\right\} \qquad (143)$$

(Sauvenier-Goffin 1951, Smeyers 1966). The g_1^--mode of $\ell = 2$ and the Kelvin mode of $\ell = 4$ display no nodes in their radial displacement, the p_1-mode of $\ell = 2$ and the pseudo-radial mode $ps\ r_1$ have a single node.

In Figs. (3)–(6), we present the variation of the radial component of the Lagrangian displacement in the equatorial plane of the MacLaurin spheroid as a function of the eccentricity e. The eigenfunctions are normalised so that the radial component is equal to unity at the equator.

For the g_1^--mode of $\ell = 2$, the curve representing the radial displacement in the equatorial plane is convex for $e < 0.9527$, it is almost a straight line for $e = 0.9527$, and it becomes concave for larger values of e. A node appears near $e = 0.9663$, and moves away from the centre for increasing values of e.

For any value of the eccentricity e different from zero, the radial displacement in the equatorial plane of the Kelvin mode of $\ell = 4$ displays one node. For $e = 0$, the node is situated at the centre. For increasing values of e, it moves towards the surface.

The number of nodes in the radial components of the p_1-mode of $\ell = 2$ and the pseudo-radial mode $ps\ r_1$ remains equal to one. As for the other two modes, the node moves towards the surface for increasing values of e.

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