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Asymptotic representation of nonradial g^+-modes in stars with a convective core

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Abstract. The asymptotic representation of linear, isentropic, low-frequency g^+-modes in spherically symmetric stars developed by Smeyers et al. (1995) is extended to more realistic stellar models composed of a convective core in adiabatic equilibrium and a radiative envelope. By means of a boundary-layer method, it is taken into account that the transition from the convective core to the radiative envelope is a turning point for one of the governing differential equations in the divergence and the radial component of the Lagrangian displacement.

Key words: stars: oscillations – stars: interiors – methods: analytical

1. Introduction

In a recent investigation, Smeyers et al. (1995) derived an asymptotic representation of linear, isentropic, low-frequency nonradial g^+-modes in a spherically symmetric star, without neglecting the Eulerian perturbation of the gravitational potential. They used the system of two second-order differential equations in the divergence and the radial component of the Lagrangian displacement that was established by Pekeris (1938) and reintroduced by Tassoul (1990) for the derivation of a second-order asymptotic representation of linear, isentropic high-frequency p-modes in a star.

For the sake of simplification, Smeyers et al. assumed the star to be everywhere convectively stable. At sufficiently large distances from the star's centre and surface, they adopted a two-variable expansion procedure for the derivation of the asymptotic solutions. They treated the regions near the star's centre and surface as boundary-layers.

In this paper, our aim is to extend the asymptotic representation of Smeyers et al. to more realistic stellar models composed of a convective core in adiabatic equilibrium and a radiative envelope. The transition from the convective core to the radiative envelope in a star is a turning point for one of the two second-order differential equations used. We take the turning point into account by means of a boundary-layer method described by Kevorkian & Cole (1981, Section 3.3.3).

The plan of the paper is as follows. In Sect. 2, we recall the system of two governing linear, homogeneous, second-order differential equations in the divergence and the radial component of the Lagrangian displacement. In Sects. 3 and 4, we construct boundary-layer solutions from the boundary of the adiabatic core into the radiative envelope and match these boundary-layer solutions with the asymptotic solutions valid at larger distances in the radiative envelope. In Sect. 5, we impose the continuity with the solutions that have to be determined numerically in the adiabatic core. In Sect. 6, the boundary-layer solutions constructed from the star's surface are matched with the asymptotic solutions valid at larger distances in the radiative envelope, and the equation determining the eigenfrequencies is derived. Sect. 7 is devoted to concluding remarks and a discussion of results obtained for a 5 $M_\odot$ ZAMS model.

2. Governing equations

Consider a spherically symmetric, static star that is oscillating in a linear, isentropic, low-frequency g^+-mode depending on time by $\exp(i\sigma t)$ and related to a spherical harmonic $Y_\ell^m(\theta,\phi)$. We adopt the notations of Smeyers et al. (1995). The radial parts of the divergence and the radial component of the Lagrangian displacement, $\alpha(r)$ and $\xi(r)$, respectively, are governed by the system of two linear, homogeneous second-order differential equations

$$\frac{d^2\alpha}{dr^2} + K_2(r)\,\frac{d\alpha}{dr} + \left[\frac{K_1(r)}{\sigma^2} + K_3(r) + \frac{\sigma^2}{c^2}\right]\alpha = -K_4(r)\,\frac{d\xi}{dr}, \tag{1}$$

$$\frac{d^2\xi}{dr^2} + \frac{4}{r}\,\frac{d\xi}{dr} - \frac{\ell(\ell+1)-2}{r^2}\,\xi = \frac{d\alpha}{dr} - \left[\frac{c^2}{g}\,\frac{K_1(r)}{\sigma^2} - \frac{2}{r}\right]\alpha, \tag{2}$$

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$$K_1(r) = \ell(\ell+1)\,\frac{N^2}{r^2}, \tag{3}$$

$$K_2(r) = \frac{2}{r} + \frac{2}{\rho\,c^2}\,\frac{d\left(\rho\,c^2\right)}{dr} - \frac{1}{\rho}\,\frac{d\rho}{dr}, \tag{4}$$

$$K_3(r) = -\frac{\ell(\ell+1)}{r^2} + \frac{2\,g}{c^2}\left(\frac{1}{g}\,\frac{dg}{dr} + \frac{1}{r}\right) + \frac{1}{\rho\,c^2}\,\frac{d\left(\rho\,c^2\right)}{dr}\left(\frac{2}{r} - \frac{1}{\rho}\,\frac{d\rho}{dr}\right) + \frac{1}{\rho\,c^2}\,\frac{d^2\left(\rho\,c^2\right)}{dr^2}, \tag{5}$$

$$K_4(r) = -\frac{2\,g}{c^2}\left(\frac{1}{g}\,\frac{dg}{dr} - \frac{1}{r}\right). \tag{6}$$

The solutions of Eqs. (1) and (2) must satisfy the following boundary conditions. At $r = 0$, the radial component of the Lagrangian displacement must be finite. At $r = R$, the divergence of the Lagrangian displacement must be finite, and the continuity of the gravitational potential and its gradient requires that

$$\left(\frac{d\Phi'}{dr}\right)_R + \frac{\ell+1}{R}\,\Phi'_R = -(4\,\pi\,G\,\rho\,\xi)_R. \tag{7}$$

We make differential Eqs. (1) and (2) and boundary Condition (7) dimensionless in the same way as in Smeyers et al. (1995). We also introduce the small dimensionless expansion parameter

$$\varepsilon = |\sigma|. \tag{8}$$

3. Boundary-layer solutions from the boundary of the adiabatic core

Let r_0 be the radius of the convective core in which the square of the Brunt-Väisälä frequency $N^2 = 0$, so that the core is in adiabatic equilibrium. For the asymptotic solutions valid at sufficiently large distances from the boundary of the adiabatic core and the star's surface, we adopt asymptotic solutions of the form of asymptotic Solutions (43) of Smeyers et al. (1995):

$$\left.\begin{aligned}\alpha^{(o)}(r;\varepsilon) &= K_5(r)\left(A_0^*\,\cos\tau + B_0^*\,\sin\tau\right),\\ \xi^{(o)}(r;\varepsilon) &= K_6(r)\left(A_0^*\,\cos\tau + B_0^*\,\sin\tau\right) + G_0^{(o)}(r),\end{aligned}\right\} \tag{9}$$

where

$$\left.\begin{aligned}K_5(r) &= g\left(N^2\,r^6\,c^8\,\rho^2\right)^{-1/4},\\ K_6(r) &= \frac{c^2}{g}\,K_5(r) = \left(N^2\,r^6\,\rho^2\right)^{-1/4},\end{aligned}\right\} \tag{10}$$

and $\tau(r)$ is the fast variable defined as

$$\tau(r) = \frac{1}{\varepsilon}\int_0^y K_1^{1/2}\left(r'\right)\,dy' \tag{11}$$

with

$$y = r - r_0. \tag{12}$$

Furthermore, A_0^* and B_0^* are undetermined constants, and $G_0^{(o)}(r)$ is a general solution of Clairaut's second-order differential equation of the form

$$G_0^{(o)}(r) = C_0^{(o)}\,y_1(r) + D_0^{(o)}\,y_2(r), \tag{13}$$

where $y_1(r)$ and $y_2(r)$ are two linearly independent particular solutions, and $C_0^{(o)}$ and $D_0^{(o)}$ undetermined constants. We consider Solutions (9) to be of order ε^0 in the expansion parameter.

The point $r = r_0$ is a turning point of Eq. (1). For the construction of boundary-layer solutions in the region near the turning point where $r \geq r_0$, we pass on from the functions $\alpha(r)$ and $\xi(r)$ to the functions $v(r)$ and $w(r)$ introduced by Smeyers et al. (1995), by means of the transformations

$$\left.\begin{aligned}\alpha(r) &= K_5(r)\,v(r),\\ \xi(r) &= K_6(r)\,w(r).\end{aligned}\right\} \tag{14}$$

Eqs. (1) and (2) then become

$$\frac{d^2v}{dr^2} + \left(K_2 + \frac{2}{K_5}\,\frac{dK_5}{dr}\right)\frac{dv}{dr} + \left[\frac{K_1}{\varepsilon^2} + \left(K_3 + \frac{1}{K_5}\,\frac{d^2K_5}{dr^2} + \frac{K_2}{K_5}\,\frac{dK_5}{dr}\right) + \frac{\varepsilon^2}{c^2}\right]v = -K_4\,\frac{K_6}{K_5}\left(\frac{dw}{dr} + \frac{1}{K_6}\,\frac{dK_6}{dr}w\right), \tag{15}$$

$$\frac{d^2w}{dr^2} + \left(\frac{4}{r} + \frac{2}{K_6}\,\frac{dK_6}{dr}\right)\frac{dw}{dr} + \left[\frac{1}{K_6}\,\frac{d^2K_6}{dr^2} + \frac{4}{r}\,\frac{1}{K_6}\,\frac{dK_6}{dr} - \frac{\ell(\ell+1)-2}{r^2}\right]w = -\left\{\frac{K_1}{\varepsilon^2}\,v - \frac{K_5}{K_6}\left[\frac{dv}{dr} + \left(\frac{2}{r} + \frac{1}{K_5}\,\frac{dK_5}{dr}\right)v\right]\right\}. \tag{16}$$

We assume that the square of the Brunt-Väisälä frequency can be developed in a Taylor series near $r = r_0$ as

$$N^2(r) = N_{1,t}^2\,y\,[1 + O(y)]. \tag{17}$$

We also develop the mass density ρ, the gravity g, and the square of the isentropic sound velocity c in Taylor series of the form

$$\left.\begin{aligned}\rho(r) &= \rho_0\left[1 + \frac{\rho_1}{\rho_0}\,y + \frac{\rho_2}{\rho_0}\,y^2 + O\left(y^3\right)\right],\\ g(r) &= g_0\left[1 + \frac{g_1}{g_0}\,y + O\left(y^2\right)\right],\\ c^2(r) &= c_0^2\left[1 + \frac{c_1^2}{c_0^2}\,y + \frac{c_2^2}{c_0^2}\,y^2 + O\left(y^3\right)\right].\end{aligned}\right\} \tag{18}$$

Correspondingly, the following Taylor series are derived for the functions $K_1(r)$, $K_2(r)$, $K_3(r)$, $K_4(r)$, $K_5(r)$, $K_6(r)$:

$$\left.\begin{aligned}
K_1(r) &= \frac{\ell(\ell+1)\,N_{1,t}^2}{r_0^2}\,y\,[1+O(y)] \\
&\equiv K_{1,t}\,y\,[1+O(y)], \\
K_2(r) &= \left(\frac{2}{r_0}+\frac{\rho_1}{\rho_0}+2\,\frac{c_1^2}{c_0^2}\right)[1+O(y)] \\
&\equiv K_{2,t}\,[1+O(y)], \\
K_3(r) &= \left[-\frac{\ell(\ell+1)}{r_0^2}+\frac{2\,g_0}{c_0^2}\left(\frac{g_1}{g_0}+\frac{1}{r_0}\right)\right. \\
&\quad+\left(\frac{\rho_1}{\rho_0}+\frac{c_1^2}{c_0^2}\right)\left(\frac{2}{r_0}-\frac{\rho_1}{\rho_0}\right) \\
&\quad\left.+2\left(\frac{\rho_1}{\rho_0}\,\frac{c_1^2}{c_0^2}+\frac{\rho_2}{\rho_0}+\frac{c_2^2}{c_0^2}\right)\right][1+O(y)] \\
&\equiv K_{3,t}\,[1+O(y)], \\
K_4(r) &= -2\,\frac{g_0}{c_0^2}\left(\frac{g_1}{g_0}-\frac{1}{r_0}\right)[1+O(y)] \\
&\equiv K_{4,t}\,[1+O(y)], \\
K_5(r) &= g_0\left(N_{1,t}^2\,r_0^6\,c_0^8\,\rho_0^2\right)^{-1/4}y^{-1/4}\,[1+O(y)] \\
&\equiv K_{5,t}\,y^{-1/4}\,[1+O(y)], \\
K_6(r) &= \left(N_{1,t}^2\,r_0^6\,\rho_0^2\right)^{-1/4}y^{-1/4}\,[1+O(y)] \\
&\equiv K_{6,t}\,y^{-1/4}\,[1+O(y)].
\end{aligned}\right\} \quad (19)$$

Consider the coefficients of the function v in the left-hand member of differential Eq. (15). As $y \to 0$, the coefficient K_1/ε^2 involving the small parameter behaves as y, and the coefficient $(1/K_5)(d^2K_5)/dr^2$ as y^{-2}. Hence, near $y=0$, the latter coefficient may not be neglected in comparison to the first coefficient.

Therefore, we treat the region near $y=0$ as a boundary-layer and introduce the boundary-layer coordinate

$$\tau_t(r) = \frac{1}{\delta_t(\varepsilon)}\int_0^y K_1^{1/2}\left(r'\right)dy', \quad (20)$$

where the yet unknown function $\delta_t(\varepsilon) \to 0$ as $\varepsilon \to 0$. For small values of y, the boundary-layer coordinate can be expanded in a Taylor series as

$$\tau_t(r) = \frac{1}{\delta_t(\varepsilon)}\,\frac{2}{3}\,K_{1,t}^{1/2}\,y^{3/2}\,[1+O(y)]. \quad (21)$$

Reversion yields

$$y = \delta_t^{2/3}(\varepsilon)\left(\frac{3}{2}\right)^{2/3}K_{1,t}^{-1/3}\,\tau_t^{2/3}\left[1+O\left(\delta_t^{2/3}\,\tau_t^{2/3}\right)\right] \quad (22)$$

[Abramowitz & Stegun 1965, (3.6.25)].

We also introduce boundary-layer expansions of the form

$$\left.\begin{aligned}
v^{(t)}(\tau_t;\varepsilon) &= \mu_0^{(t)}(\varepsilon)\,v_0^{(t)}(\tau_t) \\
&\quad+\mu_1^{(t)}(\varepsilon)\,v_1^{(t)}(\tau_t)+\ldots, \\
w^{(t)}(\tau_t;\varepsilon) &= \nu_0^{(t)}(\varepsilon)\,w_0^{(t)}(\tau_t) \\
&\quad+\nu_1^{(t)}(\varepsilon)\,w_1^{(t)}(\tau_t)+\ldots,
\end{aligned}\right\} \quad (23)$$

where $\mu_0^{(t)}(\varepsilon), \mu_1^{(t)}(\varepsilon), \ldots$, and $\nu_0^{(t)}(\varepsilon), \nu_1^{(t)}(\varepsilon), \ldots$ are asymptotic sequences to be determined.

By substituting Taylor Series (19) and boundary-layer Expansions (23) into Eq. (15) and transforming the differentiations with respect to the radial coordinate r into differentiations with respect to the boundary-layer coordinate τ_t, we bring the equation to the form

$$\begin{aligned}
&\mu_0^{(t)}(\varepsilon)\left[\frac{1}{\delta_t^{4/3}(\varepsilon)}\,\frac{d^2v_0^{(t)}}{d\tau_t^2}\right. \\
&\quad\left.+\left(\frac{\delta_t^{2/3}(\varepsilon)}{\varepsilon^2}+\frac{1}{\delta_t^{4/3}(\varepsilon)}\,\frac{5}{36\,\tau_t^2}\right)v_0^{(t)}+\ldots\right]+\ldots \\
&\quad= -\nu_0^{(t)}(\varepsilon)\left\{\frac{K_{4,t}}{K_{1,t}^{1/3}}\,\frac{K_{6,t}}{K_{5,t}}\left[\frac{1}{\delta_t^{2/3}(\varepsilon)}\left(\frac{2}{3}\right)^{1/3}\frac{1}{\tau_t^{1/3}}\,\frac{dw_0^{(t)}}{d\tau_t}\right.\right. \\
&\quad\left.\left.-\frac{1}{\delta_t^{2/3}(\varepsilon)}\left(\frac{2}{3}\right)^{4/3}\frac{1}{4\,\tau_t^{4/3}}\,w_0^{(t)}\right]+\ldots\right\}+\ldots.
\end{aligned} \quad (24)$$

For the derivation of the dominant boundary-layer equation, the part of the left-hand member of the previous equation involving the function $\mu_0^{(t)}(\varepsilon)$ must be considered. The term $\left[1/\delta_t^{4/3}(\varepsilon)\right]\left[5/\left(36\,\tau_t^2\right)\right]v_0^{(t)}(\tau_t)$ and the term containing the second derivative $d^2v_0^{(t)}/d\tau_t^2$ are of the same order in ε as the term $\left[\delta_t^{2/3}(\varepsilon)/\varepsilon^2\right]v_0^{(t)}(\tau_t)$ involving the small parameter if

$$\delta_t(\varepsilon) = \varepsilon. \quad (25)$$

Consequently, the boundary-layer coordinate $\tau_t(r)$ corresponds to the fast variable $\tau(r)$ introduced by Definition (11):

$$\tau_t(r) = \tau(r). \quad (26)$$

From here on, we use the notation $\tau(r)$.

If the ratio $\nu_0^{(t)}(\varepsilon)/\mu_0^{(t)}(\varepsilon)$ is of an order in ε higher than $\varepsilon^{-2/3}$, the dominant boundary-layer equation is homogeneous and takes the form

$$\frac{d^2v_0^{(t)}}{d\tau^2}+\left(1+\frac{5}{36\,\tau^2}\right)v_0^{(t)} = 0. \quad (27)$$

A general solution of this equation is given by

$$v_0^{(t)}(\tau) = \tau^{1/2}\left[A_{0,t}\,J_{1/3}(\tau)+B_{0,t}\,J_{-1/3}(\tau)\right], \quad (28)$$

where $J_{1/3}(\tau)$ and $J_{-1/3}(\tau)$ are Bessel functions of the first kind, and $A_{0,t}$ and $B_{0,t}$ undetermined constants [Abramowitz

& Stegun 1965, (9.1.49)]. As $\tau \to 0$, the first particular solution behaves as $\tau^{5/6}$, the second particular solution as $\tau^{1/6}$.

Proceeding in a similar way as for the derivation of Eq. (24), we bring Eq. (16) to the form

$$\nu_0^{(t)}(\varepsilon)\left[\frac{1}{\varepsilon^{4/3}}\,\frac{d^2 w_0^{(t)}}{d\tau^2}+\frac{1}{\varepsilon^{4/3}}\,\frac{5}{36\,\tau^2}\,w_0^{(t)}+\ldots\right]+\ldots = -\mu_0^{(t)}(\varepsilon)\left[\frac{1}{\varepsilon^{4/3}}\,v_0^{(t)}+\ldots\right]+\ldots. \quad (29)$$

When we set

$$\nu_0^{(t)}(\varepsilon)=\mu_0^{(t)}(\varepsilon), \quad (30)$$

the dominant boundary-layer equation takes the form

$$\frac{d^2 w_0^{(t)}}{d\tau^2}+\frac{5}{36\,\tau^2}\,w_0^{(t)}=-v_0^{(t)}(\tau). \quad (31)$$

After subtraction of Eq. (27), the equation becomes

$$\frac{d^2\left(w_0^{(t)}-v_0^{(t)}\right)}{d\tau^2}+\frac{5}{36\,\tau^2}\left(w_0^{(t)}-v_0^{(t)}\right)=0. \quad (32)$$

A general solution of this homogeneous second-order differential equation is given by

$$w_0^{(t)}(\tau)-v_0^{(t)}(\tau)=C_{0,t}\,\tau^{1/6}+D_{0,t}\,\tau^{5/6}, \quad (33)$$

where $C_{0,t}$ and $D_{0,t}$ are undetermined constants.

The lowest-order boundary-layer solutions of the divergence and the radial component of the Lagrangian displacement that are valid in the boundary-layer located in the star's radiative envelope near the boundary of the adiabatic core, can be written in the more general form

$$\left.\begin{aligned}
&\alpha^{(t)}(r;\varepsilon)=\mu_0^{(t,1)}(\varepsilon)\,K_5(r)\,v_0^{(t)}(\tau),\\
&\xi^{(t)}(r;\varepsilon)=K_6(r)\left[\mu_0^{(t,1)}(\varepsilon)\,v_0^{(t)}(\tau)\right.\\
&\quad\left.+\mu_0^{(t,2)}(\varepsilon)\,C_{0,t}\,\tau^{1/6}+\mu_0^{(t,3)}(\varepsilon)\,D_{0,t}\,\tau^{5/6}\right],
\end{aligned}\right\} \quad (34)$$

where the functions $\mu_0^{(t,1)}(\varepsilon)$, $\mu_0^{(t,2)}(\varepsilon)$, $\mu_0^{(t,3)}(\varepsilon)$ are yet undetermined functions of ε. The functions $\mu_0^{(t,2)}(\varepsilon)$ and $\mu_0^{(t,3)}(\varepsilon)$ are introduced in view of the matching with asymptotic Solutions (9) valid at larger distances in the radiative envelope. The function $v_0^{(t)}(\tau)$ is given by Solution (28) and involves the two yet undetermined constants $A_{0,t}$ and $B_{0,t}$. The constants $C_{0,t}$ and $D_{0,t}$ too are yet undetermined.

4. Matching of the boundary-layer solutions valid from the boundary of the adiabatic core

Boundary-layer Solutions (34) must be matched with asymptotic Solutions (9). To this end, we observe that the asymptotic approximation of the function $v_0^{(t)}(\tau)$ for large values of τ takes the form

$$v_0^{(t)}(\tau)=\left(\frac{2}{\pi}\right)^{1/2}\left[\left(A_{0,t}\,\sin\frac{\pi}{12}+B_{0,t}\,\cos\frac{\pi}{12}\right)\cos\tau+\left(A_{0,t}\,\cos\frac{\pi}{12}+B_{0,t}\,\sin\frac{\pi}{12}\right)\sin\tau\right]. \quad (35)$$

We introduce the intermediate variable

$$\tau_\zeta(r)=\frac{1}{\varepsilon^\zeta}\int_0^y K_1^{1/2}\left(r'\right)dy', \quad (36)$$

where ζ is a constant and satisfies the conditions $0<\zeta<1$. From Definition (20) and Equality (26), it follows that

$$\tau(r)=\varepsilon^{-(1-\zeta)}\,\tau_\zeta(r). \quad (37)$$

Furthermore, from Taylor Series (22), it follows that

$$y=\varepsilon^{2\zeta/3}\left(\frac{3}{2}\right)^{2/3}K_{1,t}^{-1/3}\,\tau_\zeta^{2/3}\left[1+O\left(\varepsilon^{2\zeta/3}\,\tau_\zeta^{2/3}\right)\right]. \quad (38)$$

As a first matching condition, we impose, to some order $\gamma_1^{(t)}(\varepsilon)$,

$$\lim_{\substack{\varepsilon\to 0\\ \tau_\zeta\,\text{fixed}}}\frac{1}{\gamma_1^{(t)}(\varepsilon)}\left\{A_0^*\,\cos\tau+B_0^*\,\sin\tau-\mu_0^{(t,1)}(\varepsilon)\left(\frac{2}{\pi}\right)^{1/2}\left[\left(A_{0,t}\,\sin\frac{\pi}{12}+B_{0,t}\,\cos\frac{\pi}{12}\right)\cos\tau+\left(A_{0,t}\,\cos\frac{\pi}{12}+B_{0,t}\,\sin\frac{\pi}{12}\right)\sin\tau\right]\right\}=0. \quad (39)$$

The matching condition is satisfied, to order $\gamma_1^{(t)}(\varepsilon)=\varepsilon^0$, if

$$\mu_0^{(t,1)}(\varepsilon)=\varepsilon^0 \quad (40)$$

and

$$\left.\begin{aligned}
A_0^*&=\left(\frac{2}{\pi}\right)^{1/2}\left(A_{0,t}\,\sin\frac{\pi}{12}+B_{0,t}\,\cos\frac{\pi}{12}\right),\\
B_0^*&=\left(\frac{2}{\pi}\right)^{1/2}\left(A_{0,t}\,\cos\frac{\pi}{12}+B_{0,t}\,\sin\frac{\pi}{12}\right).
\end{aligned}\right\} \quad (41)$$

As a second matching condition, we impose, to some order $\gamma_2^{(t)}(\varepsilon)$,

$$\lim_{\substack{\varepsilon\to 0\\ \tau_\zeta\,\text{fixed}}}\frac{1}{\gamma_2^{(t)}(\varepsilon)}\,\frac{1}{y^{1/4}}\left\{\frac{G_0^{(o)}(r)}{K_6(r)}-\left[\mu_0^{(t,2)}(\varepsilon)\,C_{0,t}\,\tau^{1/6}+\mu_0^{(t,3)}(\varepsilon)\,D_{0,t}\,\tau^{5/6}\right]\right\}=0. \quad (42)$$

The factor $y^{-1/4}$ is incorporated since the function $G_0^{(o)}(r)/K_6(r)$ behaves as $y^{1/4}$ as $y\to 0$.

By the use of Taylor Series (19) of $K_6(r)$ for small values of y, one has

$$\frac{1}{y^{1/4}}\,\frac{G_0^{(o)}(r)}{K_6(r)} = \frac{G_0^{(o)}(r_0)}{K_{6,t}}\left[1 + O\left(\varepsilon^{2\zeta/3}\,\tau_\zeta^{2/3}\right)\right]. \tag{43}$$

Furthermore, from Relation (37) and Taylor Series (38), it follows that

$$\begin{aligned}&\frac{1}{y^{1/4}}\left[\mu_0^{(t,2)}(\varepsilon)\,C_{0,t}\,\tau^{1/6} + \mu_0^{(t,3)}(\varepsilon)\,D_{0,t}\,\tau^{5/6}\right]\\ &\quad = \left(\frac{3}{2}\right)^{-1/6} K_{1,t}^{1/12}\left[\mu_0^{(t,2)}(\varepsilon)\,\varepsilon^{-1/6}\,C_{0,t}\right.\\ &\quad \left.+\mu_0^{(t,3)}(\varepsilon)\,\varepsilon^{-5/6+2\zeta/3}\,D_{0,t}\,\tau_\zeta^{2/3}\right]\left[1 + O\left(\varepsilon^{2\zeta/3}\,\tau_\zeta^{2/3}\right)\right].\end{aligned} \tag{44}$$

A matching is possible, to order $\gamma_2^{(t)}(\varepsilon) = \varepsilon^0$, if

$$\mu_0^{(t,2)}(\varepsilon) = \varepsilon^{1/6}, \tag{45}$$

and

$$C_{0,t} = \frac{(3/2)^{1/6}}{K_{1,t}^{1/12}\,K_{6,t}}\left[C_0^{(o)}\,y_1(r_0) + D_0^{(o)}\,y_2(r_0)\right], \tag{46}$$

$$D_{0,t} = 0. \tag{47}$$

5. Continuity conditions at the boundary of the adiabatic core

Since $N^2 = 0$ in the adiabatic core, Eqs. (1) and (2) do not contain any term proportional to σ^{-2} in that region.

We introduce asymptotic expansions of the form

$$\left.\begin{aligned}\alpha^{(c)}(r;\varepsilon) &= \mu^{(c)}(\varepsilon)\left[\alpha_0^{(c)}(r) + \varepsilon\,\alpha_1^{(c)}(r) + \ldots\right],\\ \xi^{(c)}(r;\varepsilon) &= \mu^{(c)}(\varepsilon)\left[\xi_0^{(c)}(r) + \varepsilon\,\xi_1^{(c)}(r) + \ldots\right],\end{aligned}\right\} \tag{48}$$

where $\mu^{(c)}(\varepsilon)$ is an undetermined function of ε. The functions $\alpha_0^{(c)}(r)$ and $\xi_0^{(c)}(r)$ are solutions of the equations

$$\left.\begin{aligned}&\frac{d^2\alpha_0^{(c)}}{dr^2} + K_2(r)\,\frac{d\alpha_0^{(c)}}{dr} + K_3(r)\,\alpha_0^{(c)}\\ &\quad = -K_4(r)\,\frac{d\xi_0^{(c)}}{dr},\\ &\frac{d^2\xi_0^{(c)}}{dr^2} + \frac{4}{r}\,\frac{d\xi_0^{(c)}}{dr} - \frac{\ell(\ell+1)-2}{r^2}\,\xi_0^{(c)}\\ &\quad = \frac{d\alpha_0^{(c)}}{dr} + \frac{2}{r}\,\alpha_0^{(c)}.\end{aligned}\right\} \tag{49}$$

The solutions of this fourth-order system of differential equations satisfying the requirement that the Lagrangian displacement be finite at $r = 0$ involve two undetermined constants.

We impose the divergence, the radial component of the Lagrangian displacement and its first derivative to be continuous at the boundary of the adiabatic core. By these conditions, the transverse component $\eta(r)$ of the Lagrangian displacement is also continuous according to the expression for the divergence of the Lagrangian displacement

$$\alpha(r) = \frac{1}{r^2}\,\frac{d}{dr}\left(r^2\,\xi\right) - \frac{\ell(\ell+1)}{r^2}\,\eta. \tag{50}$$

The conditions of continuity require that

$$\left.\begin{aligned}&\mu^{(c)}(\varepsilon)\,\alpha_0^{(c)}(r_0)\\ &\quad = \varepsilon^{-1/6}\,\frac{\sqrt{2}}{3^{1/6}\,\Gamma(2/3)}\,K_{1,t}^{1/12}\,K_{5,t}\,B_{0,t},\\ &\mu^{(c)}(\varepsilon)\,\xi_0^{(c)}(r_0)\\ &\quad = \varepsilon^{-1/6}\,\frac{\sqrt{2}}{3^{1/6}\,\Gamma(2/3)}\,K_{1,t}^{1/12}\,K_{6,t}\,B_{0,t}\\ &\quad + \left(\frac{2}{3}\right)^{1/6} K_{1,t}^{1/12}\,K_{6,t}\,C_{0,t},\\ &\mu^{(c)}(\varepsilon)\left(\frac{d\xi_0^{(c)}}{dr}\right)_{r_0}\\ &\quad = \varepsilon^{-5/6}\,\frac{3^{1/6}\sqrt{2}}{\Gamma(1/3)}\,K_{1,t}^{5/12}\,K_{6,t}\,A_{0,t}.\end{aligned}\right\} \tag{51}$$

where $\Gamma(1/3)$ and $\Gamma(2/3)$ are gamma functions of 1/3 and 2/3.

The first two conditions are satisfied if

$$\mu^{(c)}(\varepsilon) = \varepsilon^{-1/6} \tag{52}$$

and

$$C_{0,t} = 0. \tag{53}$$

By these two conditions, the two constants involved in the solutions for the adiabatic core that are admissible from $r = 0$, are related to the constant $B_{0,t}$ appearing in the boundary-layer solutions. Hence, at $r = r_0$, the relation holds

$$\xi_0^{(c)}(r) = \frac{c^2(r)}{g(r)}\,\alpha_0^{(c)}(r). \tag{54}$$

It implies that the Eulerian perturbation of the pressure, $P'(r)$, is equal to zero at $r = r_0$. In the adiabatic core, however, this perturbation is given by

$$P_0^{\prime(c)}(r) = -\rho\,c^2\left(\alpha_0^{(c)} + \frac{1}{\rho}\,\frac{d\rho}{dr}\,\xi_0^{(c)}\right), \tag{55}$$

so that it is generally different from zero.

The third Condition (51) is satisfied to order $\varepsilon^{-5/6}$ if

$$A_{0,t} = 0. \tag{56}$$

From matching Condition (46) and continuity Condition (53), it follows that

$$G_0^{(o)}(r_0) \equiv C_0^{(o)}\,y_1(r_0) + D_0^{(o)}\,y_2(r_0) = 0,. \tag{57}$$

i.e., the function $G_0^{(o)}(r)$ which appears in asymptotic Solutions (9) vanishes at $r = r_0$.

6. Matching of the boundary-layer solutions valid near the star's surface

Relative to the boundary layer near the star's surface, we adopt boundary-layer Solutions (107) of Smeyers et al. (1995), which take the form

$$\left.\begin{aligned}
&\alpha^{(s)}(r;\varepsilon) = K_5(r)\,\mu_0^{(s,1)}(\varepsilon)\,A_{0,s}\,\tau_s^{1/2}\,J_{n_e+1}(\tau_s)\,,\\
&\xi^{(s)}(r;\varepsilon) = K_6(r)\left\{\mu_0^{(s,3)}(\varepsilon)\,C_{0,s}\,\tau_s^{n_e+3/2}\right.\\
&\quad+\mu_0^{(s,2)}(\varepsilon)\,D_{0,s}\,\tau_s^{n_e-1/2}\\
&\quad+\mu_0^{(s,1)}(\varepsilon)\,A_{0,s}\,\tau_s^{1/2}\,J_{n_e+1}(\tau_s)\\
&\quad\left.\left\{1-\frac{2\,N_s^2}{\tau_s^2}\left[\frac{d\ln J_{n_e+1}(\tau_s)}{d\ln\tau_s}+(n_e+1)\right]\right\}\right\}.
\end{aligned}\right\} \quad (58)$$

Here, $J_{n_e+1}(\tau_s)$ is a Bessel function of the first kind of the boundary-layer coordinate

$$\tau_s(r) = \frac{1}{\varepsilon}\int_r^1 K_1^{1/2}\left(r'\right)dr', \quad (59)$$

n_e is the effective polytropic index, $\mu_0^{(s,1)}(\varepsilon)$, $\mu_0^{(s,2)}(\varepsilon)$, $\mu_0^{(s,3)}(\varepsilon)$ are undetermined functions of ε, and $A_{0,s}$, $C_{0,s}$, $D_{0,s}$ are undetermined constants.

The matching of boundary-layer solution $\alpha^{(s)}(r;\varepsilon)$ with asymptotic solution $\alpha^{(o)}(r;\varepsilon)$ valid at sufficiently large distances from the star's surface and the boundary of the adiabatic core is possible if

$$\mu_0^{(s,1)}(\varepsilon) = \varepsilon^0 \quad (60)$$

and

$$\left.\begin{aligned}
A_0^* &= A_{0,s}\left(\frac{2}{\pi}\right)^{1/2}\sin\left[\tau_R-\left(n_e+\frac{1}{2}\right)\frac{\pi}{2}\right],\\
B_0^* &= -A_{0,s}\left(\frac{2}{\pi}\right)^{1/2}\cos\left[\tau_R-\left(n_e+\frac{1}{2}\right)\frac{\pi}{2}\right]
\end{aligned}\right\} \quad (61)$$

[see Smeyers et al. 1995, Eqs. (114)].

Elimination of the constants A_0^* and B_0^* in Eqs. (41) and (61) leads to the following system of two linear, homogeneous, algebraic equations:

$$\left.\begin{aligned}
&B_{0,t}\,\cos\frac{\pi}{12}-A_{0,s}\,\sin\left[\tau_R-\left(n_e+\frac{1}{2}\right)\frac{\pi}{2}\right]=0,\\
&B_{0,t}\,\sin\frac{\pi}{12}+A_{0,s}\,\cos\left[\tau_R-\left(n_e+\frac{1}{2}\right)\frac{\pi}{2}\right]=0.
\end{aligned}\right\} \quad (62)$$

A necessary and sufficient condition for these equations to admit of a solution for the constants $B_{0,t}$ and $A_{0,s}$ that is different from zero, is

$$\cos\left[\tau_R-\left(n_e+\frac{2}{3}\right)\frac{\pi}{2}\right]=0. \quad (63)$$

This condition leads to the equation determining the eigenfrequencies $|\sigma|$

$$\tau_R \equiv \frac{[\ell(\ell+1)]^{1/2}}{|\sigma|}\int_{r_0}^1\frac{|N|}{r}\,dr = \left(2\,k+n_e-\frac{1}{3}\right)\frac{\pi}{2}, \quad (64)$$

where k is an integer. As in Smeyers et al. (1995), it can be shown that k corresponds to the radial order of the g^{+}-mode. Eigenvalue Eq. (64) corresponds to eigenvalue Eq. (A12) obtained by Tassoul (1980) for similar stellar models, be it in the approximation in which the Eulerian perturbation of the gravitational potential is neglected.

For the matching of boundary-layer solution $\xi^{(s)}(r;\varepsilon)$ with asymptotic solution $\xi^{(o)}(r;\varepsilon)$ valid at sufficiently large distances from the star's surface and the boundary of the adiabatic core, a distinction must be made between the oscillatory parts and the non-oscillatory parts of the solutions. The matching of the oscillatory parts of the solutions is automatically satisfied because of Equalities (61). The matching of the non-oscillatory parts of the solutions is possible if

$$\mu_0^{(s,2)}(\varepsilon) = \varepsilon^{n_e-1/2} \quad (65)$$

and

$$C_{0,s} = 0, \quad (66)$$

$$D_{0,s} = \left(2\,K_{1,s}^{1/2}\right)^{-(n_e-1/2)}K_{6,s}^{-1}\left[C_0^{(o)}\,y_1(R)+D_0^{(o)}\,y_2(R)\right] \quad (67)$$

[see Smeyers et al. 1995, Eqs. (123) and (124)].

Boundary Condition (7) ensuring the continuity of the gravitational potential and its gradient at the star's surface leads to

$$(\ell-1)\,D_{0,s} = 0. \quad (68)$$

Except for $\ell = 1$, it follows that

$$D_{0,s} = 0. \quad (69)$$

By matching Condition (67), we then have

$$C_0^{(o)}\,y_1(R)+D_0^{(o)}\,y_2(R) = 0. \quad (70)$$

The constants $C_0^{(o)}$ and $D_0^{(o)}$ are solutions of the two linear, homogeneous algebraic Eqs. (57) and (70). Since the determinant of the coefficients of the equations is generally different from zero, it results that

$$C_0^{(o)} = 0,\quad D_0^{(o)} = 0, \quad (71)$$

so that

$$G_0^{(o)}(r) = 0. \quad (72)$$

Consequently, in asymptotic Solutions (9), the solution $\xi^{(o)}(r;\varepsilon)$ for the radial component of the Lagrangian displacement is purely oscillatory. From the relation between the solutions for

$\alpha^{(o)}(r;\varepsilon)$ and $\xi^{(o)}(r;\varepsilon)$, it follows that the Eulerian perturbation of the pressure is identically zero in the region considered.

When $\ell = 1$, boundary Condition (68) is satisfied for any value of the constant $D_{0,s}$. Correspondingly, values different from zero can then be used for the constants $C_0^{(o)}$ and $D_0^{(o)}$ in Solution (13) for the function $G_0^{(o)}(r)$. Apparently, in the radiative envelope, a divergence-free non-oscillatory solution of Clairaut's equation can be added to the oscillatory asymptotic solution for the radial component of the Lagrangian displacement. However, it is still possible to adopt a purely oscillatory solution by setting $D_{0,s} = 0$. Equalities (71) and (72) then follow directly.

7. Concluding remarks and numerical results

We have constructed a lowest-order asymptotic representation of low-frequency g^+-modes in stars composed of a convective core in adiabatic equilibrium and a radiative envelope.

Lowest-order asymptotic solutions for the divergence and the radial component of the Lagrangian displacement that are uniformly valid in the radiative envelope, from the boundary of the adiabatic core to a sufficiently large distance from the star's surface, are given by

$$\left.\begin{aligned} \alpha^{(t,u)}(r;\varepsilon) &= B_{0,t}\, K_5(r)\, \tau^{1/2}\, J_{-1/3}(\tau), \\ \xi^{(t,u)}(r;\varepsilon) &= B_{0,t}\, K_6(r)\, \tau^{1/2}\, J_{-1/3}(\tau). \end{aligned}\right\} \qquad (73)$$

In these solutions, $J_{-1/3}(\tau)$ is a Bessel function of the first kind of the fast variable

$$\tau(r) = \frac{1}{\varepsilon}\int_{r_o}^{r} K_1^{1/2}\left(r'\right) dr', \qquad (74)$$

where r_0 is the radius of the adiabatic core. In the region where asymptotic Solutions (73) are uniformly valid, the Eulerian perturbation of the pressure is identically zero.

The solutions $\alpha_0^{(c)}(r)$ and $\xi_0^{(c)}(r)$ valid in the adiabatic core are determined by integration of Eqs. (49). The two constants involved in the solutions that are admissible from $r = 0$, are related to the constant $B_{0,t}$ by means of the first two Conditions (51).

Furthermore, lowest-order asymptotic solutions for the divergence and the radial component of the Lagrangian displacement that are uniformly valid in the radiative envelope, from the star's surface to a sufficiently large distance from the boundary of the adiabatic core, are given by

$$\left.\begin{aligned} &\alpha^{(s,u)}(r;\varepsilon) = A_{0,s}\, K_5(r)\, \tau_s^{1/2}\, J_{n_e+1}\left(\tau_s\right), \\ &\xi^{(s,u)}(r;\varepsilon) = A_{0,s}\, K_6(r)\, \tau_s^{1/2}\, J_{n_e+1}\left(\tau_s\right) \\ &\quad \left\{1 - \frac{2\,N_s^2}{\tau_s^2}\left[\frac{d\ln J_{n_e+1}\left(\tau_s\right)}{d\ln\tau_s} + (n_e+1)\right]\right\}, \end{aligned}\right\} \qquad (75)$$

where $J_{n_e+1}\left(\tau_s\right)$ is a Bessel function of the first kind of the fast variable

$$\tau_s(r) = \frac{1}{\varepsilon}\int_{r}^{1} K_1^{1/2}\left(r'\right) dr'. \qquad (76)$$

The second term in the solution for $\xi^{(s,u)}(r;\varepsilon)$, which is uniformly valid from the star's surface to a sufficiently large distance from the boundary of the adiabatic core, appears here as a term of the lowest-order asymptotic approximation. On the other hand, in the investigations of Tassoul (1980) and Smeyers & Tassoul (1987), the corresponding term does not appear before the second-order asymptotic approximation but becomes a term of the lowest order as $r \to 1$.

The more complicated form of asymptotic Solution (75) for $\xi^{(s,u)}(r;\varepsilon)$ is related to the deviation from the Sturm-Liouville eigenvalue problems the eigenvalue problem of the low-frequency g^+-modes displays in that region. The deviation is seen from the second-order differential equation in the radial component of the Lagrangian displacement that is derived in the Cowling approximation (see Smeyers 1984).

At sufficiently large distances from both the boundary of the adiabatic core and the star's surface, the asymptotic solutions for the divergence and the radial component of the Lagrangian displacement in the radiative envelope reduce to oscillatory functions of the fast variable τ with amplitudes modulated in terms of the slow variable corresponding to the radial coordinate r:

$$\left.\begin{aligned} \alpha^{(o)}(r;\varepsilon) &= K_5(r)\, \left(A_0^*\, \cos\tau + B_0^*\, \sin\tau\right), \\ \xi^{(o)}(r;\varepsilon) &= K_6(r)\, \left(A_0^*\, \cos\tau + B_0^*\, \sin\tau\right). \end{aligned}\right\} \qquad (77)$$

The constants A_0^* and B_0^* are related to the constants $B_{0,t}$ and $A_{0,s}$ by means of matching Conditions (41), in which $A_{0,t} = 0$ according to Equality (56), and matching Conditions (61). Elimination of the constants A_0^* and B_0^* yields the system of two linear, homogeneous algebraic Eqs. (62) in the constants $B_{0,t}$ and $A_{0,s}$. This system admits of non-trivial solutions for the eigenfrequencies $|\sigma|$ determined by Eq. (64).

We have applied the asymptotic theory to g^+-modes of a 5 $M_\odot$ ZAMS model with a radius of 2.67 $R_\odot$ and a solar composition. The model has a convective core extending to a relative radius $r/R = 0.183$, and displays a very thin convective envelope layer nearly at $r/R = 0.995$, which is associated with the partial hydrogen ionization zone.

A propagation diagram for $\ell = 2$, in which the square of the dimensionless angular frequency $\omega = \sigma/(GM/R^3)^{1/2}$ is plot versus the radial distance r of the layer from the centre of the model, is presented in Fig. 1. The boundaries of the regions of propagation in the stellar model are determined by means of the dispersion equation

$$\frac{\sigma^2}{c^2} + \frac{\ell(\ell+1)}{r^2}\left(\frac{N^2}{\sigma^2} - 1\right) - \frac{1}{4}\left(\frac{d\ln\rho}{dr}\right)^2 = 0 \qquad (78)$$

[Smeyers 1984, Eq. (35); see also Bahcall & Ulrich 1988, Eq. (49)].

The application of the asymptotic theory requires a careful determination of the variation of the square of the Brunt-Väisälä frequency as a function of the radial distance r. We artificially removed the negative value of N^2 in the thin outer convective layer by adopting a polytropic relation $P = K\,\rho^{(n_e+1)/n_e}$ between the pressure P and the mass density ρ from $r/R = 0.992$

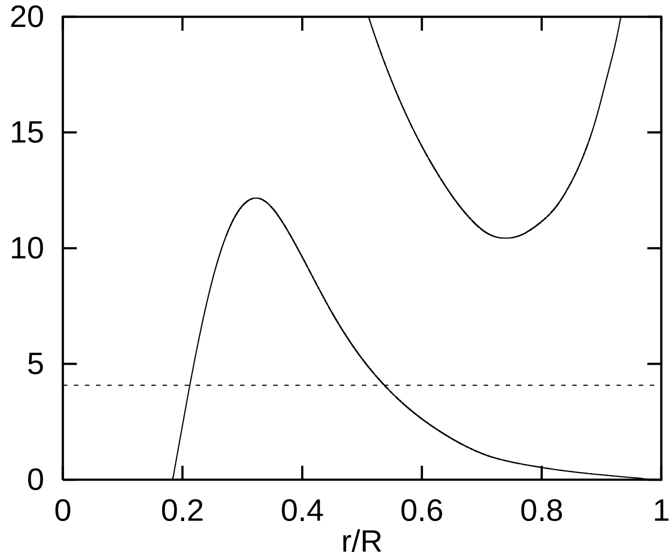

Fig. 1. Propagation diagram of the 5 $M_\odot$ ZAMS model for ℓ = 2. The dashed line is drawn at the frequency of the g_1^+-mode.

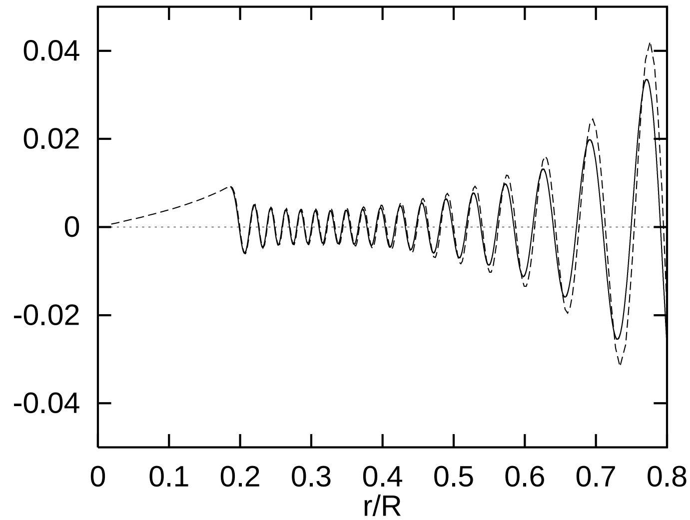

Fig. 3. Variation of the asymptotic (solid line) and exact (dashed line) radial component ξ of the Lagrangian displacement as a function of the radial distance.

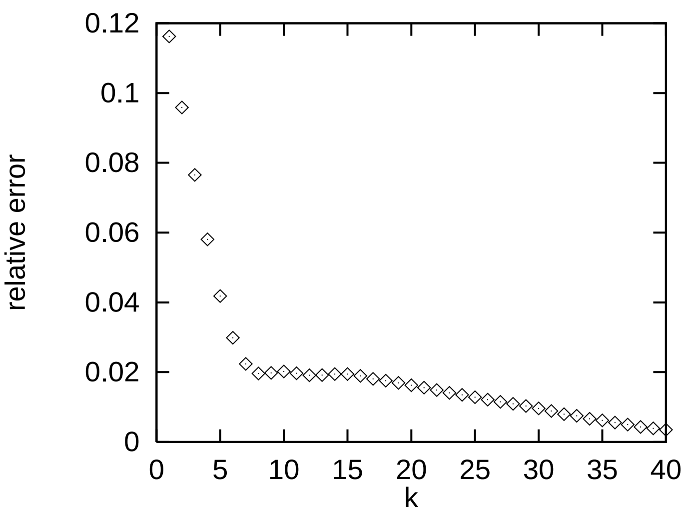

Fig. 2. Relative errors of the asymptotic eigenfrequencies of the g^+-modes belonging to ℓ = 2 for the 5 $M_\odot$ ZAMS model.

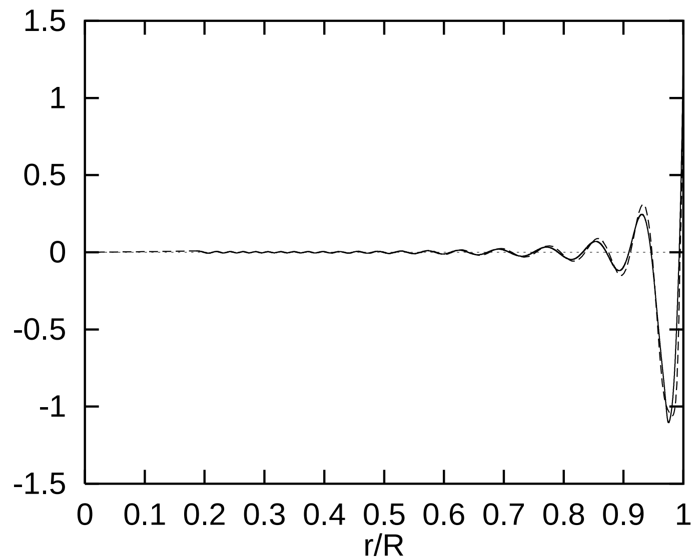

Fig. 4. Variation of the asymptotic (solid line) and exact (dashed line) radial component ξ of the Lagrangian displacement as a function of the radial distance.

to r/R = 1 with a value n_e = 3.18 of the effective polytropic index.

For g^+-modes of various radial orders k belonging to ℓ = 2, we determined the asymptotic approximation of the eigenfrequency by means of Eq. (64). We also determined the exact eigenfrequencies by integrations of the full fourth-order system of governing differential equations. For that purpose, we used an oscillation code of Dr. A. Gautschy based on the Riccati method. The relative errors are displayed in Fig. 2 as a function of the radial order k. They decrease from 0.12 for k = 1 to 0.0035 for k = 40. The relative error is less than 0.02 from k = 8.

The asymptotic approximation and the exact solution for the variation of the radial component of the Lagrangian displacement of the mode g_{40}^+ belonging to ℓ = 2 are presented in Figs. 3 and 4 as functions of the radial distance r. The radial distance extends from r/R = 0 to r/R = 0.8 in Fig. 3, and to r/R = 1 in Fig. 4. The asymptotic approximation is determined by means of Solution (73) for $\xi^{(t,u)}(r;\varepsilon)$ and Solution (75) for $\xi^{(s,u)}(r;\varepsilon)$. As a normalisation, we set the value of $\xi^{(t,u)}(r;\varepsilon)$ at $r = r_0$ equal to the value obtained by the numerical integration. The constant $B_{0,t}$ is then fixed, and the value of the constant $A_{0,s}$ is derived by means of one of the Eqs. (62). For comparison, the exact eigenfunction obtained by numerical integration of the full system of equations is also presented.

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