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Asymptotic representation of low-frequency dynamic tides in close binaries

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Abstract. An asymptotic representation of low-frequency dynamic tides in close binaries is developed. The dynamic tides are treated as low-frequency, linear, isentropic, forced oscillations of a non-rotating spherically symmetric star. The asymptotic representation is developed to the second order in the forcing frequency. For the sake of simplification, the star is assumed to be everywhere in radiative equilibrium.

As asymptotic approximation of order zero, the divergence-free static tide of which the radial component is solution of Clairaut's equation, is adopted. In the asymptotic approximation of order two, the oscillatory properties of the star play a role. The asymptotic solutions are constructed by means of a two-variable expansion procedure. The regions near the star's centre and surface are treated as boundary layers.

The Eulerian perturbation of the gravitational potential caused by the star's tidal distortion is incorporated in the asymptotic treatment. An expression for that perturbation at the star's surface is derived to the second-order approximation. The expression is determined by the non-oscillatory parts of the asymptotic solutions valid near the star's surface.

Key words: binaries: close – methods: analytical

1. Introduction

In a close binary system of stars, each component is subject to the tidal action of its companion, which is generally time dependent. The linear theory of dynamic tides was first developed by Zahn (1975) and further extended by Polfliet and Smeyers (1990, 1992). Zahn incorporated the effects of radiative damping in the region close to the star's surface where the tidal motions become highly nonadiabatic. On their part, Polfliet and Smeyers restricted themselves to the isentropic approximation. The undisturbed star is considered to be a non-rotating spherically symmetric star.

In accordance with a procedure commonly used in geophysics, the tide-generating potential is expanded successively in a Fourier series in terms of multiples k of the mean motion $n = 2\,\pi/T$, where T is the orbital period, and in a series of spherical harmonics $Y_\ell^m(\theta, \phi)$ of the angular coordinates θ and ϕ with degree ℓ and azimuthal number m. These expansions lead to a resolution of partial tides characterized by the three numbers k, ℓ, m. The tides associated with $k = 0$ are static tides, those associated with $k \neq 0$ are dynamic tides. In this investigation, the partial dynamic tides are treated as forced, linear oscillations of the non-rotating spherically symmetric star.

In the isentropic approximation, partial tides are governed by an inhomogeneous fourth-order system of differential equations in the radial coordinate. The system is made homogeneous by the introduction of the total perturbation of the gravitational potential consisting of the sum of the tide-generating potential and the Eulerian perturbation of the gravitational potential which is due to the star's tidal distortion. The system of equations governing a partial dynamic tide is then formally the same as the system of equations governing a free non-radial oscillation of the star. In the particular cases of static tides, the radial component of the tidal displacement field is determined by the second-order differential equation of Clairaut.

Due to the introduction of the total perturbation of the gravitational potential, the condition resulting from the requirement that the gravitational potential and its gradient be continuous at the star's surface becomes inhomogeneous (Polfliet and Smeyers 1990).

For many realistic orbital periods of close binary systems of stars, the mean motion n and its first multiples are comparable with low eigenfrequencies of free, higher-order g^+-modes so that the use of an asymptotic representation is desirable.

Zahn (1975) first developed an asymptotic approximation for low-frequency dynamic tides in a star composed of a convective core and an envelope in radiative equilibrium. He treated the region near the surface as a boundary layer in order to include the effects of radiative damping. From his asymptotic approximation, he concluded that a partial dynamic tide in a star consists of a static tide and a part sensitive to the star's oscillatory properties.

An asymptotic approximation of dynamic tides was given by Savonije and Papaloizou (1984) for evolved stars containing a semi-convective region between the convective core and the envelope in radiative equilibrium. These authors used the solution of Clairaut's equation as an approximation of order zero but paid no attention to the mobile singularity appearing in the second-order differential equation from which they derived a next approximation [their Eq. (A5)]. This mobile singularity has been taken into account in the asymptotic theory of free, nonradial isentropic oscillations of stars (see, e.g., Tassoul 1980, and Smeyers and Tassoul 1987).

Later, an asymptotic representation of dynamic tides was developed by Polfliet (1991) for a star composed of a convective core and an envelope in radiative equilibrium.

The three asymptotic representations mentioned were derived in the approximation in which the Eulerian perturbation of the gravitational potential caused by the star's tidal distortion is neglected.

In this paper, our aim is to derive an asymptotic representation of low-frequency linear, isentropic dynamic tides in a star that is exact to the second order in the forcing frequency. We incorporate the Eulerian perturbation of the gravitational potential in our treatment so that we are able to determine the perturbation of the external gravitational potential due to the star's tidal distortion. This quantity is of particular interest for studies of dynamical effects resulting from the tidal distortion, as the apsidal motion (Smeyers et al. 1991) and the secular variations of the semi-major axis and the eccentricity (Ruymaekers 1992).

We use a system of two homogeneous second-order differential equations in the divergence and the radial component of the tidal displacement. The equations are formally the same as those established earlier by Pekeris (1938) for free, linear, isentropic oscillations of a star and reintroduced by Tassoul (1990) in her second-order asymptotic treatment of free, high-frequency, linear, isentropic p-modes of a star.

We adopt the dimensionless forced frequency as a small expansion parameter. As asymptotic representation of order zero, we take the divergence-free tidal field whose radial component is solution of Clairaut's equation. This starting point is valid only outside resonances of dynamic tides with free oscillation modes of the star.

For the derivation of higher-order asymptotic approximations, we use procedures described by Kevorkian and Cole (1981) which apply to singular perturbation problems. The procedures were already used for derivations of asymptotic representations of free, linear, isentropic oscillations of stars: to a first asymptotic approximation, for low-frequency g^+-modes by Smeyers et al. (1995), and to a second asymptotic approximation, for high-frequency p-modes by Smeyers et al. (1996).

Because of the singularities appearing in the equations at the star's centre and surface, it is necessary to make a distinction between the region sufficiently far from both singularities, and the two regions adjacent to one of the singularities.

At sufficiently large distances from the star's centre and surface, we consider a second-order differential equation in the divergence of the tidal displacement as a differential equation governing a high-frequency forced oscillator with a small damping, and we construct asymptotic solutions in terms of a fast and a slow variable. We then seek corresponding solutions for the radial component of the tidal displacement.

We treat the regions near the star's centre and surface as boundary layers. We match the boundary-layer solutions with the asymptotic solutions valid in the region sufficiently far from the centre and the surface and construct uniformly valid asymptotic solutions. By imposing the continuity of the gravitational potential and its gradient at the star's surface, we determine the various constants involved in the asymptotic representation.

For the sake of simplification, we assume the whole star to be in radiative equilibrium.

The plan of the paper is as follows. In Sect. 2, we recall useful notions of the theory of dynamic tides in stars and establish the appropriate system of two linear, homogeneous, second-order differential equations in the divergence and the radial component of the tidal displacement. Sect. 3 is devoted to the approximation of order zero in the small expansion parameter.

From Sect. 4 on, we develop asymptotic solutions of the second order in the small expansion parameter. This is done in Sect. 4 for the region sufficiently far from the singular points at $r = 0$ and $r = R_1$, in Sect. 5, for the boundary layer near $r = 0$, and in Sect. 7, for the boundary layer near $r = R_1$. The second-order boundary-layer solutions are matched with the corresponding asymptotic solutions valid at sufficiently large distances from the singular points, in Sects. 6 and 8. In Sect. 9, the continuity of the gravitational potential is imposed at the star's surface. Sect. 10 is devoted to concluding remarks.

2. Basic equations

Consider a star with mass M_1 that is a component of a close binary and is subject to the tidal action of its companion. We treat the companion as a point mass M_2 and assume that it describes a Keplerian orbit with period T and semi-major axis a with respect to the star.

Following Polfliet and Smeyers (1990), we introduce an inertial frame of reference x, y, z with origin at the mass center of the binary. We let the xy-plane coincide with the orbital plane and take the direction of the positive x-axis parallel to the direction from the star's mass center to the periastron in the relative orbit of the companion. Let u and v be the distance and the true anomaly of the companion in its relative orbit.

We pass on from the system of Cartesian coordinates x, y, z to a system of spherical coordinates r, θ, ϕ with origin at the star's mass center by means of the transformation formulae

$$\left.\begin{aligned} x &= r\,\sin\theta\,\cos\phi - \frac{M_2}{M_1 + M_2}\,u\,\cos v, \\ y &= r\,\sin\theta\,\sin\phi - \frac{M_2}{M_1 + M_2}\,u\,\sin v, \\ z &= r\,\cos\theta. \end{aligned}\right\} \tag{1}$$

Let $\varepsilon_T\,W(r, \theta, \phi; t)$ be the tide-generating potential. Here ε_T is a small parameter defined as

$$\varepsilon_T = \left(\frac{R_1}{a}\right)^3 \frac{M_2}{M_1}, \tag{2}$$

where R_1 is the radius of the spherically symmetric star in the absence of any tidal action.

The tide-generating potential is decomposed in a Fourier series in terms of the mean anomaly M of the companion and is expanded in terms of spherical harmonics $Y_\ell^m(\theta,\phi)$ as

$$\varepsilon_T\, W(r,\theta,\phi;t) = \varepsilon_T \sum_{k=-\infty}^{\infty} W_k(r,\theta,\phi)\, \exp(i\,k\,M) \tag{3}$$

with

$$W_k(r,\theta,\phi) = -\frac{G\,M_1}{R_1} \sum_{\ell=2}^{4} \sum_{m=-\ell}^{\ell} c_{\ell,m,k} \left(\frac{r}{R_1}\right)^{\ell} Y_\ell^m(\theta,\phi). \tag{4}$$

The coefficients $c_{\ell,m,k}$ are determined as

$$c_{\ell,m,k} = \frac{(\ell-|m|)!}{(\ell+|m|)!} \left[P_\ell^m(0)\right] \left(\frac{R_1}{a}\right)^{\ell-2} \frac{1}{\pi}\int_0^\pi \left[\frac{u\left(M'\right)}{a}\right]^{-(\ell+1)} \cos\left[m\,v\left(M'\right) + k\,M'\right]\, dM'. \tag{5}$$

We consider tides as forced, linear, isentropic oscillations of a non-rotating spherically symmetric star. Let $\boldsymbol{\xi}$ be the tidal displacement with components ξ_r, ξ_θ, ξ_ϕ with respect to the local orthonormal basis vectors $\partial/\partial r$, $(1/r)\,(\partial/\partial\theta)$, $[1/(r\,\sin\theta)]\,(\partial/\partial\phi)$. Furthermore, let P be the pressure, ρ the mass density, and Φ the gravitational potential. We denote the Eulerian perturbation of a quantity Q by Q' and the Lagrangian perturbation of that quantity by δQ.

We assume that the components of the tidal displacement and the Eulerian perturbations of pressure, mass density, and gravitational potential are decomposed in Fourier series of the form

$$f(r,\theta,\phi;t) = \sum_{k=-\infty}^{\infty} f_k(r,\theta,\phi)\, \exp(i\,k\,M). \tag{6}$$

The governing equations are the perturbed and linearized equation of motion, continuity equation, energy equation for isentropic motions, and Poisson's equation. By the introduction of the total perturbation of the gravitational potential

$$\Psi_k(r,\theta,\phi) = \Phi'_k(r,\theta,\phi) + \varepsilon_T\, W_k(r,\theta,\phi), \tag{7}$$

the equations governing the Fourier component of the tide associated with a forcing frequency $|kn|$ take the same form as the equations governing free, linear, isentropic oscillations that depend on time by a factor $\exp(i\sigma t)$. When we drop the subscript k, the equations can be written as

$$k^2n^2\boldsymbol{\xi} = \nabla\chi + \frac{N^2}{g}\, c^2\,\alpha\,\mathbf{1}_r, \tag{8}$$

$$\frac{\rho'}{\rho} + \frac{1}{\rho}\frac{d\rho}{dr}\,\xi_r + \alpha = 0, \tag{9}$$

$$\frac{P'}{\rho} + \frac{1}{\rho}\frac{dP}{dr}\,\xi_r + c^2\,\alpha = 0, \tag{10}$$

$$\nabla^2\Psi = 4\,\pi\,G\,\rho', \tag{11}$$

where the function χ is defined as

$$\chi = \Psi + \frac{P'}{\rho}, \tag{12}$$

and where

$$\alpha \equiv \nabla\cdot\boldsymbol{\xi} \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,\xi_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\xi_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial\xi_\phi}{\partial\phi} \tag{13}$$

is the divergence of the tidal displacement,

$$\Gamma_1 \equiv \left(\frac{\partial\ln P}{\partial\ln\rho}\right)_S \tag{14}$$

one of the generalized isentropic coefficients,

$$c \equiv \left(\frac{\Gamma_1\,P}{\rho}\right)^{1/2} \tag{15}$$

the isentropic sound velocity, g the gravity,

$$N^2 \equiv -g\left(\frac{g}{c^2} + \frac{1}{\rho}\frac{d\rho}{dr}\right) \tag{16}$$

the square of the Brunt-Väisälä frequency, and $\mathbf{1}_r$ the unit vector in the radial direction.

We sketch the derivation of the appropriate equations in the divergence and the radial component of the tidal displacement, since equations valid in the particular case $k^2n^2 = 0$ are also needed. The divergence of Eq. (8) takes the form

$$k^2n^2\,\alpha = \nabla^2\chi + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,\frac{N^2}{g}\,c^2\,\alpha\right). \tag{17}$$

By deriving an expression for $\nabla^2\left(P'/\rho\right)$ from Eq. (10) and using Eq. (11), one finds

$$\nabla^2\left(c^2\,\alpha - g\,\xi_r\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,\frac{N^2}{g}\,c^2\,\alpha\right) - \left(4\,\pi\,G\,\rho + k^2n^2\right)\,\alpha - 4\,\pi\,G\,\frac{d\rho}{dr}\,\xi_r. \tag{18}$$

Multiplication of Expression (13) for α by k^2n^2 and elimination of $k^2n^2\xi_\theta$ and $k^2n^2\xi_\phi$ by means of the transverse components of Eq. (8) yield

$$k^2n^2\,\alpha = k^2n^2\,\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,\xi_r\right) - \frac{1}{r^2}\,\mathscr{L}^2\chi, \tag{19}$$

where $\mathscr{L}^2$ is the Legendrian defined as

$$\mathscr{L}^2 \equiv -\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]. \tag{20}$$

The radial component of Eq. (8) and Eqs. (18) and (19) form a system of three equations for the radial component of the tidal displacement, ξ_r, the divergence of the tidal displacement, α, and the function χ. By expanding the three functions in terms of spherical harmonics, one derives a system of differential equations for the functions $(\xi_r)_{\ell,m}(r)$, $\alpha_{\ell,m}(r)$, and $\chi_{\ell,m}(r)$ that are associated with the single spherical harmonic $Y_\ell^m(\theta,\phi)$. When one drops the subscript r on the radial component of the tidal displacement and the subscripts ℓ and m on the three functions, the resulting equations take the form

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d}{dr}\left(c^2\alpha-g\,\xi\right)\right]-\frac{\ell(\ell+1)}{r^2}\left(c^2\alpha-g\,\xi\right)$$
$$=\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{N^2}{g}c^2\alpha\right)-\left(4\pi G\rho+k^2n^2\right)\alpha$$
$$-4\pi G\frac{d\rho}{dr}\xi, \tag{21}$$

$$k^2n^2\alpha=k^2n^2\frac{1}{r^2}\frac{d}{dr}\left(r^2\xi\right)-\frac{\ell(\ell+1)}{r^2}\chi, \tag{22}$$

$$k^2n^2\xi=\frac{d\chi}{dr}+\frac{N^2}{g}c^2\alpha. \tag{23}$$

If $k^2n^2\neq 0$, elimination of χ between Eqs. (22) and (23) leads to the equation

$$\frac{d^2\xi}{dr^2}+\frac{4}{r}\frac{d\xi}{dr}-\frac{\ell(\ell+1)-2}{r^2}\xi$$
$$=\frac{d\alpha}{dr}-\left[\frac{c^2}{g}\frac{K_1(r)}{k^2n^2}-\frac{2}{r}\right]\alpha, \tag{24}$$

where

$$K_1(r)=\ell(\ell+1)\frac{N^2}{r^2}. \tag{25}$$

By means of Eq. (24), Eq. (21) can be transformed into the equation

$$\frac{d^2\alpha}{dr^2}+K_2(r)\frac{d\alpha}{dr}+\left[\frac{k^2n^2}{c^2}+K_3(r)+\frac{K_1(r)}{k^2n^2}\right]\alpha$$
$$=-K_4(r)\frac{d\xi}{dr}, \tag{26}$$

where

$$K_2(r)=\frac{2}{r}+\frac{2}{\rho c^2}\frac{d\left(\rho c^2\right)}{dr}-\frac{1}{\rho}\frac{d\rho}{dr}, \tag{27}$$

$$K_3(r)=-\frac{\ell(\ell+1)}{r^2}+\frac{2g}{c^2}\left(\frac{1}{g}\frac{dg}{dr}+\frac{1}{r}\right)$$
$$+\frac{1}{\rho c^2}\frac{d\left(\rho c^2\right)}{dr}\left(\frac{2}{r}-\frac{1}{\rho}\frac{d\rho}{dr}\right)+\frac{1}{\rho c^2}\frac{d^2\left(\rho c^2\right)}{dr^2}, \tag{28}$$

$$K_4(r)=-\frac{2g}{c^2}\left(\frac{1}{g}\frac{dg}{dr}-\frac{1}{r}\right). \tag{29}$$

Eqs. (24) and (26) form a fourth-order system of linear, homogeneous differential equations for the divergence $\alpha(r)$ and the radial component $\xi(r)$ of the tidal displacement in a star associated with a nonzero forcing frequency $|kn|$ and a spherical harmonic $Y_\ell^m(\theta,\phi)$. These equations have the same form as the equations established by Pekeris (1938) for the free, linear, isentropic oscillations of a star.

As forced linear, isentropic oscillations, the partial dynamic tides must satisfy conditions at the star's centre and surface. At $r=0$, the radial component of the tidal displacement must be finite. At $r=R_1$, the Lagrangian perturbation of pressure must be zero, i.e.,

$$(\delta P)_{R_1}=-\left(\Gamma_1 P\,\alpha\right)_{R_1}=0. \tag{30}$$

When $P_{R_1}=0$, the condition requires that the divergence of the tidal displacement be finite at $r=R_1$. Furthermore, at $r=R_1$, the continuity of the gravitational potential and its gradient leads to the condition

$$\left(\frac{d\Phi'}{dr}\right)_{R_1}+\frac{\ell+1}{R_1}\Phi'_{R_1}=-(4\pi G\rho\,\xi)_{R_1}. \tag{31}$$

By using Expansion (4) and Definition (7), the Eulerian perturbation of the gravitational potential can be expressed as

$$\Phi'=\Psi+\varepsilon_T\frac{G\,M_1}{R_1}c_{\ell,m,k}\left(\frac{r}{R_1}\right)^\ell. \tag{32}$$

Homogeneous boundary Condition (31) can then be transformed into an inhomogeneous boundary condition for the total perturbation of the gravitational potential:

$$\left(\frac{d\Psi}{dr}\right)_{R_1}+\frac{\ell+1}{R_1}\Psi_{R_1}+(4\pi G\rho\,\xi)_{R_1}$$
$$=-\varepsilon_T(2\ell+1)\frac{G\,M_1}{R_1^2}c_{\ell,m,k}. \tag{33}$$

We make the equations and boundary conditions dimensionless by expressing the time t, the radial coordinate r, the pressure P, the mass density ρ, the gravity g, the gravitational potential Φ, and the radial component ξ of the tidal displacement in the units $\left[R_1^3/(G\,M_1)\right]^{1/2}$, R_1, $G\,M_1^2/\left(4\pi R_1^4\right)$, $M_1/\left(4\pi R_1^3\right)$, $G\,M_1/R_1^2$, $G\,M_1/R_1$, R_1, respectively. Furthermore, we introduce the small dimensionless expansion parameter

$$\varepsilon=|kn|. \tag{34}$$

For the sake of simplification, we assume N^2 to be positive everywhere in the star.

3. Asymptotic solutions of order ε^0

As approximations of the functions $\alpha(r)$ and $\xi(r)$ at order ε^0, we adopt the solutions $\alpha_0(r)$ and $\xi_0(r)$ derived in the limiting case $\varepsilon=0$. The assumption that the functions $\alpha(r)$ and $\xi(r)$ differ only to a small extent from the functions $\alpha_0(r)$ and $\xi_0(r)$ is unfounded in the cases of resonances of a dynamic tide with a free oscillation mode of the star.

In order to derive the solutions $\alpha_0(r)$ and $\xi_0(r)$, we must return to Eqs. (21)–(23), which, in contrast to Eqs. (24) and (26), are valid in the case $\varepsilon=0$. When $k^2n^2=0$, it follows from Eq. (22) that $\chi_0=0$, and from Eq. (23) that $\alpha_0(r)=0$.

By the use of Poisson's differential equation for the spherically symmetric equilibrium star

$$\frac{dg}{dr} + \frac{2}{r}\, g = \rho, \tag{35}$$

Eq. (21) reduces to

$$\frac{d^2\xi_0}{dr^2} + 2\left(\frac{1}{g}\frac{dg}{dr} + \frac{1}{r}\right)\frac{d\xi_0}{dr} - \frac{\ell(\ell+1)-2}{r^2}\,\xi_0 = 0. \tag{36}$$

By the introduction of the mean mass density interior to the radius r, $\bar{\rho}(r)$, the equation is brought to the form

$$r^2\,\frac{d^2(\xi_0/r)}{dr^2} + 6\,\frac{\rho(r)}{\bar{\rho}(r)}\left[r\,\frac{d(\xi_0/r)}{dr} + \frac{\xi_0}{r}\right] -\ell\,(\ell+1)\,\frac{\xi_0}{r} = 0. \tag{37}$$

This equation is usually derived in the framework of the theory of equilibrium tides, where it is referred to as the equation of Clairaut and is equivalent to the equation of Radau (Sterne 1939, Kopal 1959, Kopal 1960).

Since the tidal displacement is divergence-free, it follows from Eqs. (9) and (10) that

$$\rho_0' = -\frac{d\rho}{dr}\,\xi_0,\ P_0' = \rho\, g\, \xi_0. \tag{38}$$

Furthermore, since $\chi_0 = 0$, it follows that

$$\Psi_0 = -g\,\xi_0. \tag{39}$$

Elimination of Ψ_0 in boundary Condition (33) yields

$$\left(\frac{d\xi_0}{dr}\right)_{R_1} + (\ell-1)\ (\xi_0)_{R_1} = \varepsilon_T\,(2\,\ell+1)\,c_{\ell,m,k}. \tag{40}$$

This condition relates the function $\xi_0(r)$ to its first derivative at $r = 1$.

A general solution of the linear, homogeneous, second-order differential Eq. (36) consists of a linear combination of two independent particular solutions. The point $r = 0$ is a singular point of the equation. Near that point, one particular solution behaves as $r^{\ell-1}$, and the other one as $r^{-(\ell+2)}$. In the particular case of the equilibrium configuration with a uniform mass density, the functions $r^{\ell-1}$ and $r^{-(\ell+2)}$ are exact particular solutions. In any stellar model, the particular solution behaving as $r^{-(\ell+2)}$ as $r \to 0$ must be rejected in order that the radial component of the tidal displacement be finite at $r = 0$. The constant involved in the admissible particular solution, say A_0, is determined by imposing the solution to satisfy inhomogeneous boundary Condition (40).

The divergence-free solution $\xi_0(r)$ corresponds to the equilibrium tide found by Zahn (1975) in his asymptotic treatment and to the first asymptotic approximation used by Savonije and Papaloizou (1984).

According to Relation (32), the Eulerian perturbation of the gravitational potential at $r = 1$ is given by

$$\left(\Phi_0'\right)_{R_1} = \varepsilon_T\, c_{\ell,m,k} - (\xi_0)_{R_1}\,. \tag{41}$$

4. Asymptotic solutions of order ε^2 at distances sufficiently large from $r = 0$ and $r = 1$

At order ε^2, we begin by deriving asymptotic solutions for the divergence and the radial component of the tidal displacement that are valid at distances sufficiently large from the singular points at $r = 0$ and $r = 1$. To this end, we widen the use of a two-variable expansion procedure described by Kevorkian and Cole (1981, Sect. 3.3.3) for Sturm-Liouville differential equations with a large parameter.

We introduce the fast variable $\tau(r)$ as

$$\tau(r) = \frac{1}{\varepsilon}\int_0^r K_1^{1/2}\left(r'\right)\, dr' \tag{42}$$

and consider the radial coordinate r as a slow variable.

We also introduce the following asymptotic expansions for the divergence and the radial component of the tidal displacement in terms of the fast variable τ and the slow variable r:

$$\left.\begin{aligned}
&\alpha^{(o)}(r;\varepsilon) = \varepsilon^2\,\alpha_2^{(o)}(\tau,r) + \varepsilon^3\,\alpha_3^{(o)}(\tau,r)\\
&\quad+\varepsilon^4\,\alpha_4^{(o)}(\tau,r) + \mathcal{O}\left(\varepsilon^5\right),\\
&\xi^{(o)}(r;\varepsilon) = \xi_0(r) + \varepsilon^2\,\xi_2^{(o)}(\tau,r) + \varepsilon^3\,\xi_3^{(o)}(\tau,r)\\
&\quad+\varepsilon^4\,\xi_4^{(o)}(\tau,r) + \mathcal{O}\left(\varepsilon^5\right).
\end{aligned}\right\} \tag{43}$$

These asymptotic expansions differ from the corresponding asymptotic expansions for low-frequency free oscillation modes g^+ by the divergence-free solution $\xi_0(r)$ introduced at order ε^0, although the governing equations are formally the same. The reason of the difference is that the function $\xi_0(r)$ is identically zero for free oscillation modes of a star because of boundary Condition (40). Indeed, since the small parameter ε_T is zero in the absence of any mass M_2, boundary Condition (40) then becomes homogeneous and imposes a relation between the Eulerian perturbation of the gravitational potential and its first derivative at the star's surface which generally cannot be satisfied by a non-zero function $\xi_0(r)$ (Robe 1965).

After transformation of the derivatives with respect to the radial coordinate r according to the chain rule, and substitution of the asymptotic expansions for $\alpha^{(o)}(r;\varepsilon)$ and $\xi^{(o)}(r;\varepsilon)$ into differential Eqs. (24) and (26), the following equations are derived:

at order ε^0,

$$\frac{\partial^2\alpha_2^{(o)}}{\partial\tau^2} + \alpha_2^{(o)} = f^{(o)}(r), \tag{44}$$

$$\frac{\partial^2\xi_2^{(o)}}{\partial\tau^2} = -\frac{1}{K_1}\left[\frac{d^2\xi_0}{dr^2} + \frac{4}{r}\frac{d\xi_0}{dr} - \frac{\ell(\ell+1)-2}{r^2}\,\xi_0\right] -\frac{c^2}{g}\,\alpha_2^{(o)}; \tag{45}$$

at order ε,

$$\frac{\partial^2\alpha_3^{(o)}}{\partial\tau^2} + \alpha_3^{(o)} = -\frac{1}{K_1^{1/2}}\left(K_4\,\frac{\partial\xi_2^{(o)}}{\partial\tau} + 2\,\frac{\partial^2\alpha_2^{(o)}}{\partial r\,\partial\tau}\right.$$

$$+\frac{1}{2\,K_1}\,\frac{dK_1}{dr}\,\frac{\partial\alpha_2^{(o)}}{\partial\tau}+K_2\,\frac{\partial\alpha_2^{(o)}}{\partial\tau}\Bigg), \tag{46}$$

$$\frac{\partial^2\xi_3^{(o)}}{\partial\tau^2}=-\frac{1}{K_1^{1/2}}\Bigg(2\,\frac{\partial\xi_2^{(o)}}{\partial r\,\partial\tau}+\frac{1}{2\,K_1}\,\frac{dK_1}{dr}\,\frac{\partial\xi_2^{(o)}}{\partial\tau}+\frac{4}{r}\,\frac{\partial\xi_2^{(o)}}{\partial\tau}-\frac{\partial\alpha_2^{(o)}}{\partial\tau}\Bigg)-\frac{c^2}{g}\,\alpha_3^{(o)}; \tag{47}$$

and, at order ε^2,

$$\frac{\partial^2\alpha_4^{(o)}}{\partial\tau^2}+\alpha_4^{(o)}=-\frac{1}{K_1^{1/2}}\Bigg(K_4\,\frac{\partial\xi_3^{(o)}}{\partial\tau}+2\,\frac{\partial^2\alpha_3^{(o)}}{\partial r\,\partial\tau}+\frac{1}{2\,K_1}\,\frac{dK_1}{dr}\,\frac{\partial\alpha_3^{(o)}}{\partial\tau}+K_2\,\frac{\partial\alpha_3^{(o)}}{\partial\tau}\Bigg)-\frac{1}{K_1}\Bigg(K_4\,\frac{\partial\xi_2^{(o)}}{\partial r}+\frac{\partial^2\alpha_2^{(o)}}{\partial r^2}+K_2\,\frac{\partial\alpha_2^{(o)}}{\partial r}+K_3\,\alpha_2^{(o)}\Bigg), \tag{48}$$

$$\frac{\partial^2\xi_4^{(o)}}{\partial\tau^2}=-\frac{1}{K_1^{1/2}}\Bigg(2\,\frac{\partial\xi_3^{(o)}}{\partial r\,\partial\tau}+\frac{1}{2\,K_1}\,\frac{dK_1}{dr}\,\frac{\partial\xi_3^{(o)}}{\partial\tau}+\frac{4}{r}\,\frac{\partial\xi_3^{(o)}}{\partial\tau}-\frac{\partial\alpha_3^{(o)}}{\partial\tau}\Bigg)-\frac{1}{K_1}\Bigg[\frac{\partial\xi_2^{(o)}}{\partial r^2}+\frac{4}{r}\,\frac{\partial\xi_2^{(o)}}{\partial r}-\frac{\ell(\ell+1)-2}{r^2}\,\xi_2^{(o)}-\frac{2}{r}\,\alpha_2^{(o)}-\frac{\partial\alpha_2^{(o)}}{\partial r}\Bigg]-\frac{c^2}{g}\,\alpha_4^{(o)}. \tag{49}$$

Eq. (44) is now inhomogeneous by the term $f^{(o)}(r)$, which is defined as

$$f^{(o)}(r)=-\frac{K_4}{K_1}\,\frac{d\xi_0}{dr} \tag{50}$$

and is manifestly due to the approximation at order ε^0 in asymptotic Expansions (43). In solving the equation, the inhomogeneous term can be treated as constant since it does not depend on the fast variable τ. Therefore, a general solution is given by

$$\alpha_2^{(o)}(\tau,r)=A_2^{(o)}(r)\,\cos\tau+B_2^{(o)}(r)\,\sin\tau+f^{(o)}(r), \tag{51}$$

where $A_2^{(o)}(r)$ and $B_2^{(o)}(r)$ are two undetermined functions of the slow variable r.

After two integrations of Eq. (45) with respect to the fast variable τ, we obtain

$$\xi_2^{(o)}(\tau,r)=\frac{c^2}{g}\,\Big[A_2^{(o)}(r)\,\cos\tau+B_2^{(o)}(r)\,\sin\tau\Big]+G_2^{(o)}(r), \tag{52}$$

where $G_2^{(o)}(r)$ is an undetermined function of the slow variable r. In this solution, we have dropped a term which is secular in the fast variable τ.

Substitution of Solutions (51) and (52) into Eq. (46) and removal of the resonant terms lead to

$$A_2^{(o)}(r)=A_2^*\,K_5(r),\ B_2^{(o)}(r)=B_2^*\,K_5(r), \tag{53}$$

where

$$K_5(r)=\frac{g}{\left(N^2\,r^6\,c^8\,\rho^2\right)^{1/4}}, \tag{54}$$

and A_2^* and B_2^* are two undetermined constants. Substitution into Solutions (51) and (52) for the functions $\alpha_2^{(o)}(\tau,r)$ and $\xi_2^{(o)}(\tau,r)$ yields

$$\left.\begin{aligned}\alpha_2^{(o)}(\tau,r)&=K_5(r)\left(A_2^*\,\cos\tau+B_2^*\,\sin\tau\right)+f^{(o)}(r),\\ \xi_2^{(o)}(\tau,r)&=K_6(r)\left(A_2^*\,\cos\tau+B_2^*\,\sin\tau\right)+G_2^{(o)}(r),\end{aligned}\right\} \tag{55}$$

where

$$K_6(r)=\frac{c^2}{g}\,K_5(r)=\left(N^2\,r^6\,\rho^2\right)^{-1/4}. \tag{56}$$

By the removal of the resonant terms, Eq. (46) reduces to a homogeneous differential equation whose general solution can be written as

$$\alpha_3^{(o)}(\tau,r)=A_3^{(o)}(r)\,\cos\tau+B_3^{(o)}(r)\,\sin\tau, \tag{57}$$

where $A_3^{(o)}(r)$ and $B_3^{(o)}(r)$ are undetermined functions of the slow variable r.

For the determination of the function $G_2^{(o)}(r)$, we must consider Eqs. (47), (48), and (49). First, a twofold integration of Eq. (47) with respect to the fast variable yields a solution that is formally the same as Solution (32) in Smeyers et al. (1995).

Secondly, after substitution of the appropriate solutions and removal of the resonant terms in the inhomogeneous part, Eq. (48) reduces to

$$\frac{\partial^2\alpha_4^{(o)}}{\partial\tau^2}+\alpha_4^{(o)}=S_1(r), \tag{58}$$

where

$$S_1(r)=-\frac{1}{K_1}\left(\frac{d^2f^{(o)}}{dr^2}+K_2\,\frac{df^{(o)}}{dr}+K_3\,f^{(o)}+K_4\,\frac{dG_2^{(o)}}{dr}\right). \tag{59}$$

A general solution of the equation is given by

$$\alpha_4^{(o)}(\tau,r)=A_4^{(o)}(r)\,\cos\tau+B_4^{(o)}(r)\,\sin\tau+S_1(r), \tag{60}$$

where $A_4^{(o)}(r)$ and $B_4^{(r)}(r)$ are undetermined functions of the slow variable r.

Thirdly, after substitution of the appropriate solutions, Eq. (49) takes the form

$$\frac{\partial^2\xi_4^{(o)}}{\partial\tau^2}=h_1(r)\,\cos\tau+h_2(r)\,\sin\tau+S_2(r), \tag{61}$$

where $h_1(r)$, $h_2(r)$, and $S_2(r)$ are functions of the slow variable r. The term $S_2(r)$ leads to a term that is secular in the fast variable τ in the solution. Therefore, we remove the term by setting

$$S_2(r)=0. \tag{62}$$

The requirement yields the following inhomogeneous second-order differential equation for the function $G_2^{(o)}(r)$:

$$\begin{aligned}&\frac{d^2G_2^{(o)}}{dr^2}+2\left(\frac{1}{g}\frac{dg}{dr}+\frac{1}{r}\right)\frac{dG_2^{(o)}}{dr}-\frac{\ell(\ell+1)-2}{r^2}\,G_2^{(o)}\\&=\frac{c^2}{g}\frac{d^2f^{(o)}}{dr^2}+\left(\frac{c^2}{g}\,K_2+1\right)\frac{df^{(o)}}{dr}+\left(\frac{c^2}{g}\,K_3+\frac{2}{r}\right)f^{(o)}\\&\equiv H(r).\end{aligned}\tag{63}$$

The homogeneous equation corresponds to Eq. (36). Let $y_1(r)$ be the particular solution of the homogeneous equation that behaves as $r^{\ell-1}$ as $r\to 0$. A second particular solution is obtained as

$$y_2(r)=y_1(r)\int_{r_0}^{r}\left[r'\,g\left(r'\right)y_1\left(r'\right)\right]^{-2}\,dr',\tag{64}$$

where r_0 is an arbitrary value of the radial coordinate. This particular solution behaves as $r^{-(\ell+2)}$ as $r\to 0$. The Wronskian of the particular solutions is

$$W\left[y_1(r),y_2(r)\right]\equiv W(r)=(r\,g)^{-2}.\tag{65}$$

A general solution of Eq. (63) is then given by

$$\begin{aligned}G_2^{(o)}(r)&=C_2^{(o)}\,y_1(r)+D_2^{(o)}\,y_2(r)\\&-y_1(r)\int_0^r\frac{H\left(r'\right)}{W\left(r'\right)}\,y_2\left(r'\right)\,dr'\\&+y_2(r)\int_0^r\frac{H\left(r'\right)}{W\left(r'\right)}\,y_1\left(r'\right)\,dr',\end{aligned}\tag{66}$$

where $C_2^{(o)}$ and $D_2^{(o)}$ are undetermined constants.

For recapitulation, the asymptotic expansions for the divergence and the radial component of the tidal displacement that are valid to order ε^2, at distances sufficiently large from the singular points at $r=0$ and $r=1$, are given by

$$\left.\begin{aligned}\alpha^{(o)}(r;\varepsilon)&=\varepsilon^2\,K_5(r)\Big[\big(A_2^*\,\cos\tau+B_2^*\,\sin\tau\big)\\&\qquad+f(r)\Big],\\\xi^{(o)}(r;\varepsilon)&=\xi_0(r)+\varepsilon^2\,K_6(r)\\&\qquad\left[\big(A_2^*\,\cos\tau+B_2^*\,\sin\tau\big)+\frac{G_2^{(o)}(r)}{K_6(r)}\right],\end{aligned}\right\}\tag{67}$$

where

$$f(r)=\frac{1}{K_5}\,f^{(o)}(r)=-\frac{1}{K_5}\,\frac{K_4}{K_1}\,\frac{d\xi_0}{dr},\tag{68}$$

and A_2^* and B_2^* are two undetermined constants. The functions $K_5(r)$ and $K_6(r)$ are determined by Definitions (54) and (56). The function $G_2^{(o)}(r)$ is given by Solution (66), which involves the two undetermined constants $C_2^{(o)}$ and $D_2^{(o)}$.

In asymptotic Expansions (67), the solutions of order ε^2 contain both an oscillatory and a non-oscillatory part. The asymptotic expansions differ from the corresponding asymptotic expansions for low-frequency free oscillation modes g^+ by the term $f(r)$ appearing in the first expansion and by Solution (66) for the function $G_2^{(o)}(r)$ appearing in the second expansion. Correspondingly, modified boundary-layer solutions must be determined for the boundary layers near the star's centre and surface.

5. Boundary-layer solutions of order ε^2 near the singular point at $r=0$

We treat the regions near $r=0$ and $r=1$ as boundary layers since some coefficients of Eqs. (24) and (26) display a pole at these boundary points. As Smeyers et al. (1995), we pass from the functions $\alpha(r)$ and $\xi(r)$ to functions $v(r)$ and $w(r)$ by means of the transformations

$$\left.\begin{aligned}\alpha(r)&=K_5(r)\,v(r),\\\xi(r)&=\xi_0(r)+K_6(r)\,w(r).\end{aligned}\right\}\tag{69}$$

Eqs. (24) and (26) then become

$$\begin{aligned}&\frac{d^2v}{dr^2}+\left(K_2+\frac{2}{K_5}\frac{dK_5}{dr}\right)\frac{dv}{dr}\\&\quad+\left[\frac{K_1}{\varepsilon^2}+\left(K_3+\frac{1}{K_5}\frac{d^2K_5}{dr^2}+\frac{K_2}{K_5}\frac{dK_5}{dr}\right)\right]v\\&\quad=K_1\,f(r)-K_4\,\frac{K_6}{K_5}\left(\frac{dw}{dr}+\frac{1}{K_6}\frac{dK_6}{dr}w\right),\end{aligned}\tag{70}$$

$$\begin{aligned}&\frac{d^2w}{dr^2}+\left(\frac{4}{r}+\frac{2}{K_6}\frac{dK_6}{dr}\right)\frac{dw}{dr}\\&\quad+\left[\frac{1}{K_6}\frac{d^2K_6}{dr^2}+\frac{4}{r}\frac{1}{K_6}\frac{dK_6}{dr}-\frac{\ell(\ell+1)-2}{r^2}\right]w\Bigg\}\\&\quad=K_1\,f(r)\\&\quad-\left\{\frac{K_1}{\varepsilon^2}\,v-\frac{K_5}{K_6}\left[\frac{dv}{dr}+\left(\frac{2}{r}+\frac{1}{K_5}\frac{dK_5}{dr}\right)v\right]\right\}.\end{aligned}\tag{71}$$

In this section, we derive the dominant boundary-layer solutions valid near $r=0$. We proceed to a large extent in the way followed by Smeyers et al. (1995) for the derivation of the boundary-layer solutions for free low-frequency oscillation modes g^+. We adopt the authors' Taylor series of the pressure P, the mass density ρ, the generalized isentropic coefficient Γ_1, and the functions $K_1(r)$, $K_2(r)$, $K_3(r)$, $K_4(r)$, $K_5(r)$, $K_6(r)$ for small values of r. The Taylor series of the function $f(r)$ for small values of r is given by

$$f(r)=-A_0\,(\ell-1)\,\frac{1}{K_{5,c}}\,\frac{K_{4,c}}{K_{1,c}}\,r^{\ell+1}.\tag{72}$$

We introduce the boundary-layer coordinate

$$\tau_c(r)=\frac{1}{\delta(\varepsilon)}\int_0^r K_1^{1/2}\left(r'\right)dr',\tag{73}$$

where the yet unknown function $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. The Taylor series of the boundary-layer coordinate for small values of r is given by

$$\tau_c(r) = \frac{1}{\delta(\varepsilon)} \left[K_{1,c}^{1/2}\, r + \mathscr{O}\left(r^3\right)\right]. \tag{74}$$

Reversion of this Taylor series yields

$$r = \delta(\varepsilon)\, K_{1,c}^{-1/2}\, \tau_c \left[1 + \mathscr{O}\left(\delta^2\, \tau_c^2\right)\right] \tag{75}$$

[Abramowitz and Stegun 1965, (3.6.25)].

We also introduce boundary-layer expansions of the form

$$\left.\begin{aligned} v^{(c)}\left(\tau_c;\varepsilon\right) &= \mu_2^{(c)}(\varepsilon)\, v_2^{(c)}(\tau_c) + \mu_3^{(c)}(\varepsilon)\, v_3^{(c)}(\tau_c) \\ &\quad + \ldots, \\ w^{(c)}\left(\tau_c;\varepsilon\right) &= \nu_2^{(c)}(\varepsilon)\, w_2^{(c)}(\tau_c) + \nu_3^{(c)}(\varepsilon)\, w_3^{(c)}(\tau_c) \\ &\quad + \ldots. \end{aligned}\right\} \tag{76}$$

In these boundary-layer expansions, $\mu_2^{(c)}(\varepsilon), \mu_3^{(c)}(\varepsilon), \ldots$, and $\nu_2^{(c)}(\varepsilon), \nu_3^{(c)}(\varepsilon), \ldots$ are asymptotic sequences to be determined.

By passing on to derivatives with respect to the boundary-layer coordinate τ_c, substituting boundary-layer Expansions (76), and expanding the coefficients of the various terms for small values of τ_c, we bring Eq. (70) to the form

$$\begin{aligned} &\mu_2^{(c)}(\varepsilon)\left\{\frac{1}{\delta^2(\varepsilon)}\,\frac{d^2 v_2^{(c)}}{d\tau_c^2}\right. \\ &\quad \left. + \left[\frac{1}{\varepsilon^2} - \frac{1}{\delta^2(\varepsilon)}\,\frac{\ell(\ell+1)}{\tau_c^2}\right] v_2^{(c)} + \ldots\right\} + \ldots = f(r) \\ &\quad -\nu_2^{(c)}(\varepsilon)\left[\frac{K_{4,c}}{K_{1,c}}\,\frac{K_{6,c}}{K_{5,c}}\tau_c\left(\frac{dw_2^{(c)}}{d\tau_c} - \frac{2}{\tau_c}\, w_2^{(c)}\right) + \ldots\right] \\ &\quad + \ldots. \end{aligned} \tag{77}$$

In the part of the left-hand member of the equation associated with the function $\mu_2^{(c)}(\varepsilon)$, the term

$$\left[1/\delta^2(\varepsilon)\right]\left[\ell(\ell+1)/\tau_c^2\right] v_2^{(c)}$$

and the term involving the second derivative are of the same order in ε as the term $\left(1/\varepsilon^2\right) v_2^{(c)}$ containing the large parameter, if

$$\delta(\varepsilon) = \varepsilon. \tag{78}$$

Consequently, the boundary-layer coordinate $\tau_c(r)$ corresponds to the fast variable $\tau(r)$ introduced in the region that is situated at distances sufficiently large distances from $r = 0$ and $r = 1$:

$$\tau_c(r) = \tau(r). \tag{79}$$

According to Taylor Series (72), the function $f(r)$ in the right-hand member of Eq. (77) behaves as $\varepsilon^{\ell+1}\,\tau^{\ell+1}$ as $\tau \to 0$. In order to include this term in the dominant boundary-layer equation, we set

$$\mu_2^{(c)}(\varepsilon) = \varepsilon^{\ell+3}. \tag{80}$$

If the ratio $\nu_2^{(c)}(\varepsilon)/\mu_2^{(c)}(\varepsilon)$ is of an order in ε higher than ε^{-2}, the dominant boundary-layer equation takes the form

$$\frac{d^2 v_2^{(c)}}{d\tau^2} + \left[1 - \frac{\ell(\ell+1)}{\tau^2}\right] v_2^{(c)} = \varepsilon^{-(\ell+1)}\, f(r), \tag{81}$$

where the function $\varepsilon^{-(\ell+1)}\, f(r)$ can be regarded as a function of τ of order ε^0, say $g_2^{(c)}(\tau)$. The dominant boundary-layer equation is inhomogeneous in contrast to the corresponding dominant boundary-layer equation that applies to low-frequency free oscillation modes g^+.

The solution of the equation satisfying the requirement that the divergence of the tidal displacement be finite at $r = 0$ can be written as

$$\begin{aligned} \varepsilon^{\ell+3}\, v_2^{(c)}(\tau) = \varepsilon^2 &\left\{\varepsilon^{\ell+1}\, A'_{2,c}\, \tau^{1/2}\, J_{\ell+1/2}(\tau) - \left(\frac{\pi}{2}\right)^{1/2} \tau^{1/2}\right. \\ &\left.\left[I_Y^{(c)}(\tau)\, J_{\ell+1/2}(\tau) - I_J^{(c)}(\tau)\, Y_{\ell+1/2}(\tau)\right]\right\}, \end{aligned} \tag{82}$$

where $A'_{2,c}$ is an undetermined constant, and

$$\left.\begin{aligned} I_J^{(c)}(\tau) &= \left(\frac{\pi}{2}\right)^{1/2} \int_0^{\tau} f\left(r'\right)\, \tau'^{1/2}\, J_{\ell+1/2}\left(\tau'\right)\, d\tau', \\ I_Y^{(c)}(\tau) &= \left(\frac{\pi}{2}\right)^{1/2} \int_0^{\tau} f\left(r'\right)\, \tau'^{1/2}\, Y_{\ell+1/2}\left(\tau'\right)\, d\tau'. \end{aligned}\right\} \tag{83}$$

For convenience, we set

$$A_{2,c} = \varepsilon^{\ell+1}\, A'_{2,c}. \tag{84}$$

Next, we bring Eq. (71) to the form

$$\begin{aligned} &\nu_2^{(c)}(\varepsilon)\left[\frac{1}{\varepsilon^2}\,\frac{d^2 w_2^{(c)}}{d\tau^2} - \frac{1}{\varepsilon^2}\,\frac{\ell(\ell+1)}{\tau^2} w_2^{(c)} + \ldots\right] + \ldots \\ &\quad = f(r) - \varepsilon^2\left(\frac{1}{\varepsilon^2}\, v_2^{(c)} + \ldots\right) + \ldots. \end{aligned} \tag{85}$$

When we set

$$\nu_2^{(c)}(\varepsilon) = \varepsilon^{\ell+3}, \tag{86}$$

the dominant boundary-layer equation is

$$\frac{d^2 w_2^{(c)}}{d\tau^2} - \frac{\ell(\ell+1)}{\tau^2} w_2^{(c)} = \varepsilon^{-(\ell+1)}\, f(r) - v_2^{(c)}(\tau). \tag{87}$$

After subtraction of Eq. (81), it follows that

$$\frac{d^2\left(w_2^{(c)} - v_2^{(c)}\right)}{d\tau^2} - \frac{\ell(\ell+1)}{\tau^2}\left(w_2^{(c)} - v_2^{(c)}\right) = 0. \tag{88}$$

The solution satisfying the requirement that the radial component of the tidal displacement be finite at $r = 0$ can be written as

$$w_2^{(c)}(\tau) - v_2^{(c)}(\tau) = C_{2,c}\, \tau^{\ell+1}, \tag{89}$$

where $C_{2,c}$ is an undetermined constant.

For recapitulation, the asymptotic expansions of the divergence and the radial component of the tidal displacement that are valid in the boundary layer near the singular point at $r = 0$ are given by

$$\left.\begin{aligned} \alpha^{(c)}(r;\varepsilon) &= \varepsilon^{\ell+3}\, K_5(r)\, v_2^{(c)}(\tau), \\ \xi^{(c)}(r;\varepsilon) &= \xi_0(r) + \varepsilon^{\ell+3}\, K_6(r) \\ &\qquad \left[v_2^{(c)}(\tau) + C_{2,c}\, \tau^{\ell+1}\right]. \end{aligned}\right\} \tag{90}$$

The function $\varepsilon^{\ell+3}\, v_2^{(c)}(\tau)$ is determined by Solution (82), which involves the undetermined constant $A_{2,c}$. The constant $C_{2,c}$ appearing in the asymptotic expansion for $\xi^{(c)}(r;\varepsilon)$ is also undetermined.

6. Matching of the second-order boundary-layer solutions valid near r=0

We match the boundary-layer solutions $\varepsilon^{\ell+1}\, v_2^{(c)}(\tau)$ and $\varepsilon^{\ell+1}\, C_{2,c}\, \tau^{\ell+1}$ with the corresponding asymptotic solutions valid at distances sufficiently large from $r = 0$ and $r = 1$. The matching requires the use of the asymptotic approximation of the function $\varepsilon^{\ell+1}\, v_2^{(c)}(\tau)$ for large values of its argument.

Let τ_M be a sufficiently large value of τ so that, for $\tau \geq \tau_M$, the Bessel functions $J_{\ell+1/2}(\tau)$ and $Y_{\ell+1/2}(\tau)$ can be approximated by Hankel's asymptotic expressions. For $\tau \geq \tau_M$, the function $\varepsilon^{\ell+1}\, v_2^{(c)}(\tau)$ can then be approximated as

$$\begin{aligned} \varepsilon^{\ell+1}\, v_2^{(c)}(\tau) &= A_{2,c} \left(\frac{2}{\pi}\right)^{1/2} \sin\left(\tau - \ell\,\frac{\pi}{2}\right) \\ &\quad - I_Y^{(c)}(\tau_M)\, \sin\left(\tau - \ell\,\frac{\pi}{2}\right) - I_J^{(c)}(\tau_M)\, \cos\left(\tau - \ell\,\frac{\pi}{2}\right) \\ &\quad + \left[\int_{\tau_M}^{\tau} f(r')\, \cos\left(\tau' - \ell\,\frac{\pi}{2}\right) d\tau'\right] \sin\left(\tau - \ell\,\frac{\pi}{2}\right) \\ &\quad - \left[\int_{\tau_M}^{\tau} f(r')\, \sin\left(\tau' - \ell\,\frac{\pi}{2}\right) d\tau'\right] \cos\left(\tau - \ell\,\frac{\pi}{2}\right). \end{aligned} \tag{91}$$

Integration by parts in the last two terms leads to

$$\begin{aligned} \varepsilon^{\ell+1}\, v_2^{(c)}(\tau) &= A_{2,c} \left(\frac{2}{\pi}\right)^{1/2} \sin\left(\tau - \ell\,\frac{\pi}{2}\right) \\ &\quad - I_Y^{(c)}(\tau_M)\, \sin\left(\tau - \ell\,\frac{\pi}{2}\right) - I_J^{(c)}(\tau_M)\, \cos\left(\tau - \ell\,\frac{\pi}{2}\right) \\ &\quad - f(r_M)\, \cos(\tau - \tau_M) + f(r) + \mathscr{O}(\varepsilon), \end{aligned} \tag{92}$$

where r_M is the value of r corresponding to the value of τ_M. It follows that, for $\tau \geq \tau_M$, the particular Solution of inhomogeneous Eq. (81) yields both an oscillatory and a nonoscillatory part. The non-oscillatory part contains the function $f(r)$, which appears in asymptotic Expansion (67) for $\alpha^{(o)}(r;\varepsilon)$.

Next, we introduce the intermediate variable

$$\tau_\zeta(r) = \frac{1}{\varepsilon^\zeta} \int_0^r K_1^{1/2}(r')\, dr', \tag{93}$$

where ζ is a constant in the interval $0 < \zeta < 1$. From the definition of $\tau(r)$, it follows that

$$\tau(r) = \varepsilon^{-(1-\zeta)}\, \tau_\zeta(r). \tag{94}$$

Furthermore, from Taylor Series (74) of the boundary-layer coordinate $\tau(r)$ for small values of r, it follows that

$$r = \varepsilon^\zeta\, K_{1,c}^{-1/2}\, \tau_\zeta + \mathscr{O}\left(\varepsilon^{3\zeta}\right). \tag{95}$$

As a first matching condition, we impose, to some order $\gamma_1^{(c)}(\varepsilon)$,

$$\lim_{\substack{\varepsilon \to 0 \\ \tau_\zeta \text{fixed}}} \frac{\varepsilon^2}{\gamma_1^{(c)}(\varepsilon)} \left[A_2^*\, \cos\tau + B_2^*\, \sin\tau + f(r) - \varepsilon^{\ell+1}\, v_2^{(c)}(\tau)\right] = 0, \tag{96}$$

where asymptotic Approximation (92) of the boundary-layer solution $\varepsilon^{\ell+1}\, v_2^{(c)}(\tau)$ for $\tau \geq \tau_M$ must be used.

The matching condition is satisfied to order $\gamma_1^{(c)}(\varepsilon) = \varepsilon^2$ when the coefficients of $\cos\tau$ and $\sin\tau$ are zero. The requirement leads to the following relations between the constants A_2^* and B_2^* and the constant $A_{2,c}$:

$$\left.\begin{aligned} A_2^* &= -\left[A_{2,c}\left(\frac{2}{\pi}\right)^{1/2} - I_Y^{(c)}(\tau_M)\right] \sin\left(\ell\,\frac{\pi}{2}\right) \\ &\quad - I_J^{(c)}(\tau_M)\, \cos\left(\ell\,\frac{\pi}{2}\right) - f(r_M)\, \cos\tau_M, \\ B_2^* &= \left[A_{2,c}\left(\frac{2}{\pi}\right)^{1/2} - I_Y^{(c)}(\tau_M)\right] \cos\left(\ell\,\frac{\pi}{2}\right) \\ &\quad - I_J^{(c)}(\tau_M)\, \sin\left(\ell\,\frac{\pi}{2}\right) - f(r_M)\, \sin\tau_M. \end{aligned}\right\} \tag{97}$$

As a second matching condition, we impose, to some order $\gamma_2^{(c)}(\varepsilon)$,

$$\lim_{\substack{\varepsilon \to 0 \\ \tau_\zeta \text{fixed}}} \frac{\varepsilon^2}{\gamma_2^{(c)}(\varepsilon)} \frac{1}{r^{\ell+1}} \left[\frac{G_2^{(o)}(r)}{K_6(r)} - f(r) - \varepsilon^{\ell+1}\, C_{2,c}\, \tau^{\ell+1}\right] = 0. \tag{98}$$

The factor $r^{-(\ell+1)}$ is incorporated since the sum of functions inside the brackets behaves as $r^{\ell+1}$ as $r \to 0$.

For the derivation of the Taylor series of the function $G_2^{(o)}(r)$ for small values of r, one must examine the behavior of the function $H(r)$ defined by Eq. (63). The terms involving the factor c^2/g behave as $r^{\ell-3}$ as $r \to 0$. Since the sum of these dominant terms is identically zero, the function $H(r)$ behaves as $r^{\ell-1}$. The Wronskian W behaves as r^{-4}. It follows that the third and fourth term in Solution (66) for the function $G_2^{(o)}(r)$ behave as $r^{\ell+1}$ as $r \to 0$. The dominant terms of the first and second term in the solution yield

$$G_2^{(o)}(r) = C_2^{(o)}\, r^{\ell-1} \left[1 + \mathscr{O}\left(r^2\right)\right] + D_2^{(o)}\, r^{-(\ell+2)} \left[1 + \mathscr{O}\left(r^2\right)\right] + \mathscr{O}\left(r^{\ell+1}\right). \tag{99}$$

By using the Taylor series of $K_6(r)$ for small values of r and Taylor Series (95) for r, we then have

$$\frac{1}{r^{\ell+1}} \frac{G_2^{(o)}(r)}{K_6(r)} = \frac{1}{K_{6,c}} \left\{ C_2^{(o)} \left[1 + \mathscr{O}\left(\varepsilon^{2\zeta}\right)\right] + \varepsilon^{-(2\ell+1)\zeta}\, D_2^{(o)}\, K_{1,c}^{(2\ell+1)/2}\, \tau_\zeta^{-(2\ell+1)} \left[1 + \mathscr{O}\left(\varepsilon^{2\zeta}\right)\right] + \mathscr{O}\left(\varepsilon^{2\zeta}\right) \right\}. \tag{100}$$

Similarly, by using Taylor Series (72) for $f(r)$ and Taylor Series (95) for r, we have

$$\frac{f(r)}{r^{\ell+1}} = -A_0\,(\ell-1)\, \frac{1}{K_{5,c}} \frac{K_{4,c}}{K_{1,c}} \left[1 + \mathscr{O}\left(\varepsilon^{2\zeta}\right)\right]. \tag{101}$$

From Relation (94), it follows that

$$\varepsilon^{\ell+1}\, C_{2,c}\, \frac{\tau^{\ell+1}}{r^{\ell+1}} = C_{2,c}\, K_{1,c}^{(\ell+1)/2} \left[1 + \mathscr{O}\left(\varepsilon^{2\zeta}\right)\right]. \tag{102}$$

Consequently, matching Condition (98) is satisfied, to order $\gamma_2^{(c)}(\varepsilon) = \varepsilon^2$, if

$$D_2^{(o)} = 0, \tag{103}$$

$$C_{2,c} = K_{1,c}^{-(\ell+1)/2} \left[A_0\,(\ell-1)\, \frac{1}{K_{5,c}} \frac{K_{4,c}}{K_{1,c}} + \frac{C_2^{(o)}}{K_{6,c}}\right]. \tag{104}$$

Asymptotic expansions of the divergence and the radial component of the tidal displacement that are uniformly valid to order ε^2, from $r = 0$ to a distance sufficiently large from $r = 1$, are obtained by making the sum of the asymptotic Expansions (67) valid at distances sufficiently large from $r = 0$ and $r = 1$ and the boundary-layer Expansions (90), and subtracting their common terms (see Kevorkian and Cole 1981, Sect. 2.2). The uniformly valid asymptotic expansions are then given by

$$\left.\begin{aligned} \alpha^{(c,u)}(r;\varepsilon) &= \varepsilon^{\ell+3}\, K_5(r)\, v_2^{(c)}(\tau), \\ \xi^{(c,u)}(r;\varepsilon) &= \xi_0(r) + \varepsilon^2\, K_6(r) \\ &\quad \left[\varepsilon^{\ell+1}\, v_2^{(c)}(\tau) + \frac{G_2^{(o)}(r)}{K_6(r)} - f(r)\right]. \end{aligned}\right\} \tag{105}$$

The function $\varepsilon^{\ell+1}\, v_2^{(c)}(\tau)$ contains the yet undetermined constant $A_{2,c}$, and the function $G_2^{(o)}(r)$ the yet undetermined constant $C_2^{(o)}$.

7. Boundary-layer solutions of order ε^2 near the singular point at $r = 1$

In the region near $r = 1$, we introduce the independent variable $z = 1 - r$ and assume that the mass density ρ can be expanded in a Taylor series as

$$\rho(r) = \rho_s\, z^{n_e} \left[1 + \frac{\rho_1}{\rho_s}\, z + \mathscr{O}\left(z^2\right)\right]. \tag{106}$$

When the mass contained inside a sphere with radius r is considered to be equal to the star's total mass, the condition of hydrostatic equilibrium reduces to

$$\frac{dP}{dz} = \frac{\rho}{(1-z)^2}. \tag{107}$$

Integration in the supposition that the pressure P vanishes at $z = 0$ yields the Taylor series

$$\begin{aligned} P(r) &= \frac{\rho_s}{n_e+1}\, z^{n_e+1} \left[1 + \frac{n_e+1}{n_e+2} \left(\frac{\rho_1}{\rho_s} + 2\right) z + \mathscr{O}\left(z^2\right)\right] \\ &\equiv P_s\, z^{n_e+1} \left[1 + \frac{P_1}{P_s}\, z + \mathscr{O}\left(z^2\right)\right]. \end{aligned} \tag{108}$$

Hence, the exponent n_e corresponds to the effective polytropic index at the lowest-order approximation of the relation between the pressure P and the mass density ρ.

The Taylor series of the generalized isentropic coefficient Γ_1, the square of the isentropic sound velocity, and the square of the Brunt-Väisälä frequency take the form

$$\Gamma_1 = \Gamma_{1,s} \left[1 + \frac{\Gamma_{1,1}}{\Gamma_{1,s}}\, z + \mathscr{O}\left(z^2\right)\right], \tag{109}$$

$$\begin{aligned} c^2 &= \frac{\Gamma_{1,s}}{n_e+1}\, z \left[1 + \left(\frac{P_1}{P_s} - \frac{\rho_1}{\rho_s} + \frac{\Gamma_{1,1}}{\Gamma_{1,s}}\right) z + \mathscr{O}\left(z^2\right)\right], \\ &\equiv c_s^2\, z \left[1 + \frac{c_1^2}{c_s^2}\, z + \mathscr{O}\left(z^2\right)\right], \end{aligned} \tag{110}$$

$$\begin{aligned} N^2 &= \left(n_e - \frac{n_e+1}{\Gamma_{1,s}}\right) \frac{1}{z} \\ &\quad + \left[2\,n_e + \frac{\rho_1}{\rho_s} + \frac{n_e+1}{\Gamma_{1,s}} \left(\frac{c_1^2}{c_s^2} - 4\right)\right] + \mathscr{O}(z), \\ &\equiv \mathscr{N}_s^2\, \frac{1}{z} \left[1 + \frac{\mathscr{N}_1^2}{\mathscr{N}_s^2}\, z + \mathscr{O}\left(z^2\right)\right]. \end{aligned} \tag{111}$$

Furthermore, the Taylor series of the coefficients $K_1(r)$, $K_2(r)$, $K_3(r)$, $K_4(r)$, $K_5(r)$, $K_6(r)$, and the function $f(r)$ for small values of z, take the form

$$\left.\begin{aligned}
K_1(r) &= \ell(\ell+1)\,\mathcal{N}_s^{\cdot 2}\,\frac{1}{z}\\
&\quad\left[1+\left(\frac{\mathcal{N}_1^{\cdot 2}}{\mathcal{N}_s^{\cdot 2}}+2\right)z+\mathcal{O}\left(z^2\right)\right]\\
&\equiv \frac{K_{1,s}}{z}\left[1+\frac{K_{1,1}}{K_{1,s}}\,z+\mathcal{O}\left(z^2\right)\right],\\
K_2(r) &= -\frac{n_e+2}{z}\,[1+\mathcal{O}(z)]\\
&\equiv \frac{K_{2,s}}{z}\,[1+\mathcal{O}(z)],\\
K_3(r) &= \frac{K_{3,s}}{z}\,[1+\mathcal{O}(z)],\\
K_4(r) &= \frac{6}{c_s^2}\,\frac{1}{z}\,[1+\mathcal{O}(z)]\\
&\equiv \frac{K_{4,s}}{z}\,[1+\mathcal{O}(z)],\\
K_5(r) &= \left(c_s^2\right)^{-1}\,\left(\rho_s^2\,\mathcal{N}_s^{\cdot 2}\right)^{-1/4}\,z^{-(n_e+3/2)/2}\\
&\quad [1+\mathcal{O}(z)]\\
&\equiv K_{s,5}\,z^{-(n_e+3/2)/2}\,[1+\mathcal{O}(z)],\\
K_6(r) &= \left(\rho_s^2\,\mathcal{N}_s^{\cdot 2}\right)^{-1/4}\,z^{-(n_e-1/2)/2}\\
&\quad\left[1+\left(\frac{3}{2}-\frac{1}{4}\,\frac{\mathcal{N}_1^{\cdot 2}}{\mathcal{N}_s^{\cdot 2}}-\frac{1}{2}\,\frac{\rho_1}{\rho_s}\right)z+\mathcal{O}\left(z^2\right)\right]\\
&\equiv K_{6,s}\,z^{-(n_e-1/2)/2}\\
&\quad\left[1+\frac{K_{6,1}}{K_{6,s}}\,z+\mathcal{O}\left(z^2\right)\right],\\
f(r) &= -\frac{1}{K_{5,s}}\,\frac{K_{4,s}}{K_{1,s}}\,\left(\frac{d\xi_0}{dr}\right)_{R_1}\,z^{(n_e+3/2)/2}.
\end{aligned}\right\}\qquad(112)$$

We introduce the boundary-layer coordinate

$$\begin{aligned}
\tau_s(r) &= \frac{1}{\delta'(\varepsilon)}\int_0^z K_1^{1/2}\left(r'\right)dz'\\
&= \frac{1}{\delta'(\varepsilon)}\int_r^1 K_1^{1/2}\left(r'\right)dr',
\end{aligned}\qquad(113)$$

where the yet unknown function $\delta'(\varepsilon)\to 0$ as $\varepsilon\to 0$. For small values of z, the boundary-layer coordinate $\tau_s(r)$ can be expanded in a Taylor series as

$$\tau_s(r)=\frac{1}{\delta'(\varepsilon)}\,2\,K_{1,s}^{1/2}\,z^{1/2}\left[1+\mathcal{O}(z)\right].\qquad(114)$$

Reversion of the Taylor series yields

$$z=\delta'^2(\varepsilon)\,\frac{\tau_s^2}{\left(2\,K_{1,s}^{1/2}\right)^2}\left\{1+\mathcal{O}\left[\delta'^2(\varepsilon)\,\tau_s^2\right]\right\}.\qquad(115)$$

We also introduce boundary-layer expansions of the form

$$\left.\begin{aligned}
v^{(s)}(\tau_s;\varepsilon) &= \mu_2^{(s)}(\varepsilon)\,v_2^{(s)}(\tau_s)+\mu_3^{(s)}(\varepsilon)\,v_3^{(s)}(\tau_s)\\
&\quad+\ldots,\\
w^{(s)}(\tau_s;\varepsilon) &= \nu_2^{(s)}(\varepsilon)\,w_2^{(s)}(\tau_s)+\nu_3^{(s)}(\varepsilon)\,w_3^{(s)}(\tau_s)\\
&\quad+\ldots.
\end{aligned}\right\}\qquad(116)$$

In these boundary-layer expansions, $\mu_2^{(s)}(\varepsilon),\mu_3^{(s)}(\varepsilon),\ldots$, and $\nu_2^{(s)}(\varepsilon),\nu_3^{(s)}(\varepsilon),\ldots$ are asymptotic sequences to be determined.

We then bring Eq. (70) to the form

$$\begin{aligned}
&\mu_2^{(s)}(\varepsilon)\left\{\frac{1}{\delta'^2(\varepsilon)}\,\frac{d^2v_2^{(s)}}{d\tau_s^2}\right.\\
&\quad\left.+\left[\frac{1}{\varepsilon^2}-\frac{1}{\delta'^2(\varepsilon)}\,\frac{(n_e+1)^2-1/4}{\tau_s^2}\right]v_2^{(s)}+\ldots\right\}+\ldots\\
&\quad=f(r)\\
&\quad+\nu_2^{(s)}(\varepsilon)\frac{1}{2}\,\frac{K_{4,s}}{K_{1,s}}\,\frac{K_{6,s}}{K_{5,s}}\,\tau_s\left[\frac{dw_2^{(s)}}{d\tau_s}-\left(n_e-\frac{1}{2}\right)\frac{1}{\tau_s}\,w_2^{(s)}\right]\\
&\quad+\ldots.
\end{aligned}\qquad(117)$$

We first concentrate on the terms of the left-hand member associated with the function $\mu_2^{(s)}(\varepsilon)$. The term

$$\left[1/\delta'^2(\varepsilon)\right]\,\left\{\left[(n_e+1)^2-1/4\right]/\tau_s^2\right\}\,v_2^{(s)}$$

and the term involving the second derivative are of the same order in ε as the term $\left(1/\varepsilon^2\right)\,v_2^{(s)}$ containing the large parameter if

$$\delta'(\varepsilon)=\varepsilon.\qquad(118)$$

From the definitions of the variables $\tau(r)$ and $\tau_s(r)$, it results that, at any point from $r=0$ to $r=1$, the relation holds

$$\tau_s(r)=\tau_{R_1}-\tau(r)\qquad(119)$$

with

$$\tau_{R_1}=\frac{1}{\varepsilon}\int_0^1 K_1^{1/2}\left(r'\right)\,dr'.\qquad(120)$$

According to the last Taylor Series (112), the function $f(r)$ in the right-hand member of Eq. (117) behaves as $\varepsilon^{n_e+3/2}\,\tau_s^{n_e+3/2}$ as $\tau_s\to 0$. In order to include this term in the dominant boundary-layer equation, we set

$$\mu_2^{(s)}(\varepsilon)=\varepsilon^{n_e+7/2}.\qquad(121)$$

If the ratio $\nu_2^{(s)}(\varepsilon)/\mu_2^{(s)}(\varepsilon)$ is of an order in ε higher than ε^{-2}, the dominant boundary-layer equation takes the form

$$\begin{aligned}
&\frac{d^2v_2^{(s)}}{d\tau_s^2}+\left[1-\frac{(n_e+1)^2-1/4}{\tau_s^2}\right]v_2^{(s)}\\
&\quad=\varepsilon^{-(n_e+3/2)}\,f(r),
\end{aligned}\qquad(122)$$

where the function $\varepsilon^{-(n_e+3/2)} f(r)$ can be regarded as a function of τ_s of order ε^0, say $g_2^{(s)}(\tau_s)$. In this case too, the dominant boundary-layer equation is inhomogeneous in contrast with the corresponding dominant boundary-layer equation that applies to low-frequency free oscillation modes g^+.

The solution satisfying the requirement that the divergence of the tidal displacement be finite ar $r = 1$ can be written as

$$\varepsilon^{n_e+7/2} v_2^{(s)}(\tau_s) = \varepsilon^2 \left\{ \varepsilon^{n_e+3/2} A'_{2,s} \tau_s^{1/2} J_{n_e+1}(\tau_s) - \left(\frac{\pi}{2}\right)^{1/2} \tau_s^{1/2} \left[I_Y^{(s)}(\tau_s) J_{n_e+1}(\tau_s) - I_J^{(s)}(\tau_s) Y_{n_e+1}(\tau_s) \right] \right\}, \quad (123)$$

where $A'_{2,s}$ is an undetermined constant, and

$$\left. \begin{aligned} I_J^{(s)}(\tau_s) &= \left(\frac{\pi}{2}\right)^{1/2} \int_0^{\tau_s} f(r') \tau_s'^{1/2} J_{n_e+1}(\tau_s') \, d\tau_s', \\ I_Y^{(s)}(\tau_s) &= \left(\frac{\pi}{2}\right)^{1/2} \int_0^{\tau_s} f(r') \tau_s'^{1/2} Y_{n_e+1}(\tau_s') \, d\tau_s'. \end{aligned} \right\} \quad (124)$$

The functions $I_J^{(s)}(\tau_s)$ and $I_Y^{(s)}(\tau_s)$ behave as $\tau_s^{2n_e+4}$ and τ_s^2, respectively, as $\tau_s \to 0$. For convenience, we set

$$A_{2,s} = \varepsilon^{n_e+3/2} A'_{2,s}. \quad (125)$$

Next, we bring Eq. (71) to the form

$$\nu_2^{(s)}(\varepsilon) \left\{ \frac{1}{\varepsilon^2} \left[\frac{d^2 w_2^{(s)}}{d\tau_s^2} - \frac{2\,n_e}{\tau_s} \frac{dw_2^{(s)}}{d\tau_s} + \frac{(n_e+1/2)^2 - 1}{\tau_s^2} w_2^{(s)} \right] + \ldots \right\} + \ldots = f(r) - \varepsilon^{n_e+7/2} \left\{ \frac{1}{\varepsilon^2} \left[v_2^{(s)}(\tau_s) + 2 \left(c_s^2\right)^{-1} h_2^{(s)}(\tau_s) \right] + \ldots \right\} + \ldots, \quad (126)$$

where

$$h_2^{(s)}(\tau_s) = \frac{1}{\tau_s} \left(\frac{dv_2^{(s)}}{d\tau_s} - \frac{n_e + 3/2}{\tau_s} v_2^{(s)} \right). \quad (127)$$

The function $h_2^{(s)}(\tau_s)$ behaves as $\tau_s^{n_e+3/2}$ as $\tau_s \to 0$.

When we set

$$\nu_2^{(s)}(\varepsilon) = \varepsilon^{n_e+7/2}, \quad (128)$$

the dominant boundary-layer equation is

$$\frac{d^2 w_2^{(s)}}{d\tau_s^2} - \frac{2\,n_e}{\tau_s} \frac{dw_2^{(s)}}{d\tau_s} + \frac{(n_e+1/2)^2 - 1}{\tau_s^2} w_2^{(s)} = \varepsilon^{-(n_e+3/2)} f(r) - \left[v_2^{(s)}(\tau_s) + 2 \left(c_s^2\right)^{-1} h_2^{(s)}(\tau_s) \right]. \quad (129)$$

Subtraction of Eq. (122) and use of the relation $\left(c_s^2\right)^{-1} = n_e - \mathcal{N}_s^{\cdot 2}$ resulting from Taylor Series (110) and (111) yield the equation

$$\frac{d^2 w_2^*}{d\tau_s^2} - \frac{2\,n_e}{\tau_s} \frac{dw_2^*}{d\tau_s} + \frac{(n_e+1/2)^2 - 1}{\tau_s^2} w_2^* - 2\,\mathcal{N}_s^{\cdot 2} h_2^{(s)}(\tau_s) = 0, \quad (130)$$

where

$$w_2^*(\tau_s) = w_2^{(s)}(\tau_s) - v_2^{(s)}(\tau_s). \quad (131)$$

A general solution of Eq. (130) is given by

$$w_2^*(\tau_s) = C_{2,s} \tau_s^{n_e+3/2} + D_{2,s} \tau_s^{n_e-1/2} + \mathcal{N}_s^{\cdot 2} \left[\tau_s^{n_e+3/2} \int_0^{\tau_s} h_2^{(s)}(\tau_s') \tau_s'^{-(n_e+1/2)} \, d\tau_s' - \tau_s^{n_e-1/2} \int_0^{\tau_s} h_2^{(s)}(\tau_s') \tau_s'^{-(n_e-3/2)} \, d\tau_s' \right], \quad (132)$$

where $C_{2,s}$ and $D_{2,s}$ are two undetermined constants.

After transformation of the first integral and part of the second integral, the solution can be rewritten as

$$\varepsilon^{n_e+7/2} w_2^*(\tau_s) = \varepsilon^{n_e+7/2} \left[C_{2,s} \tau_s^{n_e+3/2} + D_{2,s} \tau_s^{n_e-1/2} + 2\,\mathcal{N}_s^{\cdot 2} \tau_s^{n_e-1/2} \int_0^{\tau_s} \tau_s'^{-(n_e+1/2)} v_2^{(s)}(\tau_s') \, d\tau_s' \right]. \quad (133)$$

The remaining integral can be transformed by substitution of Solution (123) for $\varepsilon^{n_e+7/2} v_2^{(s)}(\tau_s)$, application of the recurrence relation for Bessel functions

$$\tau_s^{-n_e} \mathscr{C}_{n_e+1}(\tau_s) = -\frac{d}{d\tau_s} \left[\tau_s^{-n_e} \mathscr{C}_{n_e}(\tau_s) \right], \quad (134)$$

integration by parts, and use of the equality

$$J_{n_e+1}(\tau_s) Y_{n_e}(\tau_s) - J_{n_e}(\tau_s) Y_{n_e+1}(\tau_s) = \frac{2}{\pi \tau_s} \quad (135)$$

[Abramowitz and Stegun 1965, (9.1.16) and (9.1.27)]. Moreover, we set

$$D_{2,s} = \varepsilon^{-2} D'_{2,s}, \quad (136)$$

in order that the term $\varepsilon^{n_e+7/2} D_{2,s} K_6(r) \tau_s^{n_e-1/2}$ be of order ε^2 as $z \to 0$. It follows that

$$\varepsilon^{n_e+7/2} w_2^*(\tau_s) = \varepsilon^{n_e+7/2} C_{2,s} \tau_s^{n_e+3/2} + \varepsilon^{n_e+3/2} D'_{2,s} \tau_s^{n_e-1/2} + \varepsilon^2\, 2\,\mathcal{N}_s^{\cdot 2} \tau_s^{n_e-1/2} \left\{ -A_{2,s} \tau_s^{-n_e} J_{n_e}(\tau_s) + \left(\frac{\pi}{2}\right)^{1/2} \tau_s^{-n_e} \left[I_Y^{(s)}(\tau_s) J_{n_e}(\tau_s) - I_J^{(s)}(\tau_s) Y_{n_e}(\tau_s) \right] + \int_0^{\tau_s} \tau_s'^{-(n_e+1/2)} f(r') \, d\tau_s' \right\}. \quad (137)$$

The last term inside the braces can be transformed by partial integration as

$$\int_0^{\tau_s} \tau_s'^{-(n_e+1/2)} f\left(r'\right)\, d\tau_s' = -\frac{\tau_s^{-(n_e-1/2)}}{n_e - 1/2}\, f(r) + \mathscr{O}(\varepsilon). \tag{138}$$

For recapitulation, the asymptotic expansions of the divergence and the radial component of the tidal displacement that are valid to order ε^2 in the boundary layer near the singular point at $r = 1$ are given by

$$\left.\begin{aligned} \alpha^{(s)}(r;\varepsilon) &= \varepsilon^{n_e+7/2} K_5(r)\, v_2^{(s)}(\tau_s), \\ \xi^{(s)}(r;\varepsilon) &= \xi_0(r) + \varepsilon^{n_e+7/2} K_6(r) \\ &\quad \left[v_2^{(s)}(\tau_s) + w_2^*(\tau_s)\right]. \end{aligned}\right\} \tag{139}$$

The function $\varepsilon^{n_e+7/2}\, v_2^{(s)}(\tau_s)$ is given by Solution (123) and involves the undetermined constant $A_{2,s}$. Furthermore, the function $\varepsilon^{n_e+7/2}\, w_2^*(\tau_s)$ is given by Solution (137) and involves the two undetermined constants $C_{2,s}$ and $D'_{2,s}$.

8. Matching of the second-order boundary-layer solutions valid near r=1

We match the boundary-layer solutions $\varepsilon^{n_e+3/2}\, v_2^{(s)}(\tau_s)$ and $\varepsilon^{n_e+3/2}\, w_2^*(\tau_s)$ with the corresponding asymptotic solutions valid at distances sufficiently large from $r = 0$ and $r = 1$. To this end, asymptotic approximations of the functions $\varepsilon^{n_e+3/2}\, v_2^{(s)}(\tau_s)$ and $\varepsilon^{n_e+3/2}\, w_2^*(\tau_s)$ for large values of their argument are needed.

Let τ_N be a sufficiently large value of τ_s so that, for $\tau_s \ge \tau_N$, the Bessel functions $J_{n_e+1}(\tau_s)$ and $Y_{n_e+1}(\tau_s)$ can be approximated by Hankel's asymptotic expressions. On the analogy of asymptotic Solution (92) for the function $\varepsilon^{\ell+1}\, v_2^{(c)}(\tau)$, the asymptotic approximation for the function $\varepsilon^{n_e+3/2}\, v_2^{(s)}(\tau_s)$ can be written as

$$\begin{aligned} \varepsilon^{n_e+3/2}\, v_2^{(s)}(\tau_s) &= A_{2,s} \left(\frac{2}{\pi}\right)^{1/2} \sin\left[\tau_s - \left(n_e + \frac{1}{2}\right)\frac{\pi}{2}\right] \\ &- I_Y^{(s)}(\tau_N) \sin\left[\tau_s - \left(n_e + \frac{1}{2}\right)\frac{\pi}{2}\right] \\ &- I_J^{(s)}(\tau_N) \cos\left[\tau_s - \left(n_e + \frac{1}{2}\right)\frac{\pi}{2}\right] \\ &- f(r_N)\cos(\tau_s - \tau_N) + f(r) + \mathscr{O}(\varepsilon). \end{aligned} \tag{140}$$

Hence, for $\tau_s \ge \tau_N$, the particular solution of inhomogeneous Eq. (122) leads to both an oscillatory and a non-oscillatory part. In this case too, the non-oscillatory part contains the function $f(r)$ which appears in asymptotic Expansion (67) for $\alpha^{(o)}(r;\varepsilon)$.

Similarly, the asymptotic approximation for the function $\varepsilon^{n_e+3/2}\, w_2^*(\tau_s)$ can be written as

$$\begin{aligned} \varepsilon^{n_e+3/2}\, w_2^*(\tau_s) &= \varepsilon^{n_e+3/2}\, C_{2,s}\, \tau_s^{n_e+3/2} + \varepsilon^{n_e-1/2}\, D'_{2,s}\, \tau_s^{n_e-1/2} \\ &+ \frac{2\mathscr{N}_s^{\cdot 2}}{\tau_s} \left\{ -A_{2,s}\left(\frac{2}{\pi}\right)^{1/2} \cos\left[\tau_s - \left(n_e + \frac{1}{2}\right)\frac{\pi}{2}\right] \right. \\ &+ I_Y^{(s)}(\tau_N) \cos\left[\tau_s - \left(n_e + \frac{1}{2}\right)\frac{\pi}{2}\right] \\ &- I_J^{(s)}(\tau_N) \sin\left[\tau_s - \left(n_e + \frac{1}{2}\right)\frac{\pi}{2}\right] \\ &\left. - f(r_N) \sin(\tau_s - \tau_N) \right\} - \frac{2\mathscr{N}_s^{\cdot 2}}{n_e - 1/2} f(r) + \mathscr{O}(\varepsilon). \end{aligned} \tag{141}$$

Next, we introduce the intermediate variable

$$\tau_\eta(z) = \varepsilon^{-\eta} \int_0^z K_1^{1/2}\left(r'\right)\, dz', \tag{142}$$

where η is a constant in the interval $0 < \eta < 1$. From Definitions (113) and (118), it follows that

$$\tau_s = \varepsilon^{-(1-\eta)}\, \tau_\eta. \tag{143}$$

Furthermore, Taylor Series (115) for small values of z leads to

$$z = \varepsilon^{2\eta} \frac{\tau_\eta^2}{\left(2\, K_{1,s}^{1/2}\right)^2} \left[1 + \mathscr{O}\left(\varepsilon^{2\eta}\right)\right]. \tag{144}$$

As a first matching condition, we impose, to some order $\gamma_1^{(s)}(\varepsilon)$,

$$\lim_{\substack{\varepsilon \to 0 \\ \tau_\eta \text{fixed}}} \frac{\varepsilon^2}{\gamma_1^{(s)}(\varepsilon)} \left\{ \left[A_2^* \cos\left(\tau_{R_1} - \tau_s\right) + B_2^* \sin\left(\tau_{R_1} - \tau_s\right)\right] + f(r) - \varepsilon^{n_e+3/2}\, v_2^{(s)}(\tau_s) \right\} = 0, \tag{145}$$

where asymptotic Approximation (140) of the boundary-layer solution $\varepsilon^{n_e+3/2}\, v_2^{(s)}(\tau_s)$ for $\tau_s \ge \tau_N$ must be used.

The matching condition is satisfied to order $\gamma_1^{(s)}(\varepsilon) = \varepsilon^2$ when the coefficients of $\cos\tau_s$ and $\sin\tau_s$ are zero. The requirement leads to the following relations between the constants A_2^* and B_2^* and the constant $A_{2,s}$:

$$\left.\begin{aligned} A_2^* &= \left[A_{2,s}\left(\frac{2}{\pi}\right)^{1/2} - I_Y^{(s)}(\tau_N)\right] \\ &\quad \sin\left[\tau_{R_1} - \left(n_e + \frac{1}{2}\right)\frac{\pi}{2}\right] \\ &\quad - I_J^{(s)}(\tau_N) \cos\left[\tau_{R_1} - \left(n_e + \frac{1}{2}\right)\frac{\pi}{2}\right] \\ &\quad - f(r_N) \cos\left(\tau_{R_1} - \tau_N\right), \\ B_2^* &= -\left[A_{2,s}\left(\frac{2}{\pi}\right)^{1/2} - I_Y^{(s)}(\tau_N)\right] \\ &\quad \cos\left[\tau_{R_1} - \left(n_e + \frac{1}{2}\right)\frac{\pi}{2}\right] \\ &\quad - I_J^{(s)}(\tau_N) \sin\left[\tau_{R_1} - \left(n_e + \frac{1}{2}\right)\frac{\pi}{2}\right] \\ &\quad - f(r_N) \sin\left(\tau_{R_1} - \tau_N\right). \end{aligned}\right\} \tag{146}$$

Elimination of the constants A_2^* and B_2^* from these equations and Eqs. (97) yields a system of two linear, inhomogeneous algebraic equations in the constants $A_{2,c}$ and $A_{2,s}$ whose determinant is

$$\Delta = \sin\left[\tau_{R_1} - \left(\ell + n_e + \frac{1}{2}\right)\frac{\pi}{2}\right] \equiv \sin\gamma. \qquad (147)$$

When $\sin\gamma \neq 0$, the system admits of the following solutions for the constants $A_{2,c}$ and $A_{2,s}$:

$$\left.\begin{aligned}
A_{2,c} = {} & \frac{1}{\sin\gamma}\left(\frac{2}{\pi}\right)^{1/2}\left\{I_Y^{(c)}(\tau_M)\sin\gamma\right.\\
& + I_J^{(c)}(\tau_M)\cos\gamma - I_J^{(s)}(\tau_N)\\
& + f(r_M)\cos\left[\tau_{R_1} - \tau_M - \left(n_e + \frac{1}{2}\right)\frac{\pi}{2}\right]\\
& \left. - f(r_N)\cos\left[\tau_N - \left(n_e + \frac{1}{2}\right)\frac{\pi}{2}\right]\right\},\\
A_{2,s} = {} & \frac{1}{\sin\gamma}\left(\frac{2}{\pi}\right)^{1/2}\left[I_Y^{(s)}(\tau_N)\sin\gamma\right.\\
& + I_J^{(s)}(r_N)\cos\gamma - I_J^{(c)}(\tau_M)\\
& + f(r_N)\cos\left(\tau_{R_1} - \tau_N - \ell\frac{\pi}{2}\right)\\
& \left. - f(r_M)\cos\left(\tau_M - \ell\frac{\pi}{2}\right)\right].
\end{aligned}\right\} \qquad (148)$$

When $\sin\gamma = 0$, the frequency $|kn|$ of the dynamic tide is equal to the eigenfrequency of a free oscillation mode g^+ [see Smeyers et al. 1995, Eq. (117)]. The constants $A_{2,c}$ and $A_{2,s}$ then become indefinitely large.

As a second matching condition, we impose, to some order $\gamma_2^{(s)}(\varepsilon)$,

$$\lim_{\substack{\varepsilon\to 0\\ \tau_\eta \text{fixed}}} \frac{\varepsilon^2}{\gamma_2^{(s)}(\varepsilon)}\frac{1}{z^{(n_e+3/2)/2}}\left[\frac{G_2^{(o)}(r)}{K_6(r)} - f(r) - \varepsilon^{n_e+3/2}\,w_2^*(\tau_s)\right] = 0, \qquad (149)$$

where asymptotic Approximation (141) of the boundary-layer solution $\varepsilon^{n_e+3/2}\,w_2^*(\tau_s)$ for $\tau_s \geq \tau_N$ must be used. The factor $z^{-(n_e+3/2)/2}$ is incorporated in the matching condition since the sum of the terms inside the brackets behaves as $z^{(n_e+3/2)/2}$ as $z \to 0$.

In terms of the intermediate variable τ_η, we have

$$\frac{1}{z^{(n_e+3/2)/2}}\frac{G_2^{(o)}(r)}{K_6(r)} = \varepsilon^{-2\eta}\left(2\,K_{1,s}^{1/2}\right)^2\frac{G_2^{(o)}(R_1)}{K_{6,s}}\,\tau_\eta^{-2}, \qquad (150)$$

$$\frac{1}{z^{(n_e+3/2)/2}}\,f(r) = -\frac{1}{K_{5,s}}\frac{K_{4,s}}{K_{1,s}}\left(\frac{d\xi_0}{dr}\right)_{R_1}, \qquad (151)$$

$$\varepsilon^{n_e+3/2}\,C_{2,s}\,\frac{\tau_s^{n_e+3/2}}{z^{(n_e+3/2)/2}} = C_{2,s}\left(2\,K_{1,s}^{1/2}\right)^{n_e+3/2}, \qquad (152)$$

$$\varepsilon^{n_e-1/2}\,D'_{2,s}\,\frac{\tau_s^{n_e-1/2}}{z^{(n_e+3/2)/2}} = \varepsilon^{-2\eta}\,D'_{2,s}\left(2\,K_{1,s}^{1/2}\right)^{n_e+3/2}\tau_\eta^{-2}, \qquad (153)$$

A matching to order $\gamma_2^{(s)} = \varepsilon^2$ is possible, if

$$C_{2,s} = -\left(2\,K_{1,s}^{1/2}\right)^{-(n_e+3/2)}\frac{1}{K_{5,s}}\frac{K_{4,s}}{K_{1,s}}\left(\frac{d\xi_0}{dr}\right)_{R_1}\left(\frac{2\,\mathcal{N}_s^{\,2}}{n_e - 1/2} - 1\right), \qquad (154)$$

$$G_2^{(o)}(R_1) = D'_{2,s}\left(2\,K_{1,s}^{1/2}\right)^{n_e-1/2}K_{6,s}. \qquad (155)$$

The constant $C_{2,s}$ is determined by matching Condition (154) and is independent of the forcing frequency of the dynamic tide. The constant $D'_{2,s}$ is fixed subsequently by boundary Condition (33) ensuring the continuity of the gravitational potential and its gradient at $r = 1$.

9. Continuity of the gravitational potential and its gradient at r = 1

In order to impose boundary Condition (33), we derive an expression for the total perturbation of the gravitational potential, Ψ, and its first derivative, $d\Psi/dr$, in terms of the divergence and the radial component of the tidal displacement.

By the use of Definition (12), the relation between the Eulerian and the Lagrangian perturbation of the pressure, and the condition of hydrostatic equilibrium, it follows from Eq. (22) that

$$\Psi = c^2\,\alpha - g\,\xi + \varepsilon^2\,\frac{r^2}{\ell(\ell+1)}\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\,\xi\right) - \alpha\right]. \qquad (156)$$

Differentiation with respect to the radial coordinate r and elimination of the derivatives $d^2\xi/dr^2$ and dg/dr by means of Eq. (24) and Poisson's Eq. (35) yields

$$\frac{d\Psi}{dr} = \frac{d\left(c^2\,\alpha\right)}{dr} - \frac{N^2}{g}\,c^2\,\alpha - g\,\frac{d\xi}{dr} + \left(2\,\frac{g}{r} - \rho\right)\xi + \varepsilon^2\,\xi. \qquad (157)$$

From the first asymptotic Expansion (139) and Solution (123) for $\varepsilon^{n_e+7/2}\,v_2^{(s)}(\tau_s)$, it follows that

$$\alpha_{R_1} = \varepsilon^{-(n_e-1/2)}\,\frac{A_{2,s}}{2^{n_e+1}}\,\frac{\left(c_s^2\right)^{-1}}{\Gamma(n_e+2)}\left(2\,K_{1,s}^{1/2}\right)^{n_e+3/2}K_{6,s}. \qquad (158)$$

Correspondingly, one has

$$\left(c^2\,\alpha\right)_{R_1} = 0, \qquad (159)$$

$$\left[\frac{d\left(c^2\,\alpha\right)}{dr}\right]_{R_1} = -\varepsilon^{-(n_e-1/2)}\,\frac{A_{2,s}}{2^{n_e+1}}\,\frac{1}{\Gamma\left(n_e+2\right)}\left(2\,K_{1,s}^{1/2}\right)^{n_e+3/2}\,K_{6,s}, \tag{160}$$

$$\left(\frac{N^2}{g}\,c^2\,\alpha\right)_{R_1} = \varepsilon^{-(n_e-1/2)}\,\frac{A_{2,s}}{2^{n_e+1}}\,\frac{\mathcal{N}_s^{\cdot 2}}{\Gamma\left(n_e+2\right)}\left(2\,K_{1,s}^{1/2}\right)^{n_e+3/2}\,K_{6,s}. \tag{161}$$

From the second asymptotic Expansion (139), Solution (123) for $\varepsilon^{n_e+7/2}\,v_2^{(s)}(\tau_s)$, and Solution (137) for $\varepsilon^{n_e+7/2}\,w_2^*(\tau_s)$, it follows that

$$\xi_{R_1} = (\xi_0)_{R_1} + \varepsilon^2\,D'_{2,s}\left(2\,K_{1,s}^{1/2}\right)^{n_e-1/2}\,K_{6,s} -\varepsilon^{-(n_e-5/2)}\,\frac{A_{2,s}}{2^{n_e-1}}\,\frac{\mathcal{N}_s^{\cdot 2}}{\Gamma\left(n_e+1\right)}\left(2\,K_{1,s}^{1/2}\right)^{n_e-1/2}\,K_{6,s}, \tag{162}$$

$$\left(\frac{d\xi}{dr}\right)_{R_1} = \left(\frac{d\xi_0}{dr}\right)_{R_1} - \varepsilon^2\,\left(1+\mathcal{N}_s^{\cdot 2}\right)\frac{K_{4,s}}{K_{1,s}}\,\frac{K_{6,s}}{K_{5,s}}\left(\frac{d\xi_0}{dr}\right)_{R_1} -\varepsilon^2\,D'_{2,s}\left(2\,K_{1,s}^{1/2}\right)^{n_e-1/2}\,K_{6,s}\left[\frac{n_e-1/2}{6}\,\frac{K_{1,1}}{K_{1,s}}+\frac{K_{6,1}}{K_{6,s}}\right] -\varepsilon^{-(n_e-1/2)}\,\frac{A_{2,s}}{2^{n_e+1}}\,\frac{1+\mathcal{N}_s^{\cdot 2}}{\Gamma\left(n_e+2\right)}\left(2\,K_{1,s}^{1/2}\right)^{n_e+3/2}\,K_{6,s}. \tag{163}$$

By means of Expressions (158)–(163), one derives that

$$\Psi_{R_1} = -\,(\xi_0)_{R_1} - \varepsilon^2\,D'_{2,s}\,\left(2\,K_{1,s}^{1/2}\right)^{n_e-1/2}\,K_{6,s} +\frac{\varepsilon^2}{\ell(\ell+1)}\,\left(\frac{d\xi_0}{dr}\right)_{R_1}, \tag{164}$$

$$\left(\frac{d\Psi}{dr}\right)_{R_1} = 2\,(\xi_0)_{R_1} - \left(\frac{d\xi_0}{dr}\right)_{R_1} + \varepsilon^2\,(\xi_0)_{R_1} +\varepsilon^2\,D'_{2,s}\,\left(2\,K_{1,s}^{1/2}\right)^{n_e-1/2}\,K_{6,s}\left[2+\frac{n_e-1/2}{6}\,\frac{K_{1,1}}{K_{1,s}}+\frac{K_{6,1}}{K_{6,s}}\right] +\varepsilon^2\,\left(\mathcal{N}_s^{\cdot 2}+1\right)\frac{K_{4,s}}{K_{1,s}}\,\frac{K_{6,s}}{K_{5,s}}\left(\frac{d\xi_0}{dr}\right)_{R_1}. \tag{165}$$

Boundary Condition (33) then yields the following equation for the constant $D'_{2,s}$:

$$\left[(\ell-1)-\frac{n_e-1/2}{6}\,\frac{K_{1,1}}{K_{1,s}}-\frac{K_{6,1}}{K_{6,s}}\right]\left(2\,K_{1,s}^{1/2}\right)^{n_e-1/2}\,K_{6,s}\,D'_{2,s} = (\xi_0)_{R_1} +\frac{1}{\ell(\ell+1)}\left[\ell+1+6\left(1+\frac{1}{\mathcal{N}_s^{\cdot 2}}\right)\right]\left(\frac{d\xi_0}{dr}\right)_{R_1}. \tag{166}$$

Once the constant $D'_{2,s}$ is determined, the constant $C_2^{(o)}$ involved in Solution (66) for $G_2^{(o)}(r)$ is fixed by means of matching Condition (155). Finally, the constant $C_{2,c}$ appearing in Solution (132) is fixed by means of matching Condition (104). All constants involved in the asymptotic treatment are then determined.

From Relation (32) and Solution (164), it follows that the Eulerian perturbation of the gravitational potential at the star's surface that results from the star's tidal distortion is given by

$$\Phi'_{R_1} = \varepsilon_T\,c_{\ell,m,k} - (\xi_0)_{R_1} - \varepsilon^2\,D'_{2,s}\,\left(2\,K_{1,s}^{1/2}\right)^{n_e-1/2}\,K_{6,s} +\frac{\varepsilon^2}{\ell(\ell+1)}\,\left(\frac{d\xi_0}{dr}\right)_{R_1}. \tag{167}$$

The solution for the function $\varepsilon^{n_e+3/2}\,w_2^*(\tau_s)$ that is uniformly valid at order ε^2, from $r=1$ to a distance sufficiently large from $r=0$, can be written as

$$\varepsilon^{n_e+3/2}\,w_2^{*(u)}(\tau_s) = \left[\frac{G_2^{(o)}(r)}{K_6(r)}-f(r)\right] +2\,\mathcal{N}_s^{\cdot 2}\,\tau_s^{n_e-1/2}\left\{A_{2,s}\,\tau_s^{-n_e}\,J_{n_e}(\tau_s)+\left(\frac{\pi}{2}\right)^{1/2}\tau_s^{-n_e}\left[I_Y^{(s)}(\tau_s)\,J_{n_e}(\tau_s)-I_J^{(s)}(\tau_s)\,Y_{n_e}(\tau_s)\right]\right\}. \tag{168}$$

The asymptotic expansions of the divergence and the radial component of the tidal displacement that are uniformly valid to order ε^2, from $r=1$ to a sufficiently large distance from $r=0$, take the form

$$\left.\begin{aligned}\alpha^{(s,u)}(r;\varepsilon) &= \varepsilon^{n_e+7/2}\,K_5(r)\,v_2^{(s)}(\tau_s)\,,\\ \xi^{(s,u)}(r;\varepsilon) &= \xi_0(r)+\varepsilon^{n_e+7/2}\,K_6(r)\left[v_2^{(s)}(\tau_s)+w_2^{*(u)}(\tau_s)\right],\end{aligned}\right\} \tag{169}$$

where the functions $\varepsilon^{n_e+3/2}\,v_2^{(s)}(\tau_s)$ and $\varepsilon^{n_e+3/2}\,w_2^{*(u)}(\tau_s)$ are determined by Solutions (123) and (168), respectively.

10. Concluding remarks

We have developed an asymptotic representation of low-frequency, linear, isentropic dynamic tides in a component of a close binary up to order ε^2. The asymptotic representation is derived in the approximation in which the component is treated as a non-rotating spherically symmetric star.

As asymptotic approximation of order ε^0, we adopted the divergence-free solution $\xi_0(r)$ of Clairaut's Eq. (36) in the supposition that the functions $\alpha(r)$ and $\xi(r)$ differ only to a small extent from the solution valid in the limiting case when $\varepsilon=0$. The amplitude of the function $\xi_0(r)$ is related to the constant $c_{\ell,m,k}$ by boundary Condition (40). Apart from its amplitude, the function $\xi_0(r)$ displays the same behavior from the centre to the surface of the star for all dynamic tides of order k associated with one of the spherical harmonics $Y_\ell^m(\theta,\phi)$ of a given degree ℓ.

Upon the asymptotic approximation of order ε^0, an oscillatory as well as a non-oscillatory asymptotic solution is superposed at order ε^2. This is done by the use of a two-variable expansion procedure in the region of the star situated at distances sufficiently large from the centre and surface. The resulting asymptotic expansions for the divergence and the radial component of the tidal displacement are given by Expansions (67). The superposition of an oscillatory solution with a smaller amplitude upon the non-oscillatory solution $\xi_0(r)$ seems to agree with the variations of the radial component of the tidal displacement that were determined by Polfliet and Smeyers (1990, 1992) by means of numerical integrations of the full system of governing equations for higher-frequency dynamic tides in a 5 $M_\odot$ star.

Boundary-layer solutions at order ε^2 relative to the regions near the star's centre and surface are derived. These solutions are expressed in terms of a single boundary-layer coordinate, which corresponds to the fast coordinate $\tau(r)$ or the fast coordinate $\tau_s(r)$ according to the boundary layer considered. Boundary-layer Solution (137) for $\varepsilon^{n_e+7/2}\, w_s^*(\tau_s)$ has a much more complicated form than the corresponding boundary-layer solution valid near $r = 0$. The origin of the complexity of the solution is connected with the appearance of a mobile singularity for free low-frequency g^+-modes in the region near $r = 1$.

For larger values of the boundary-layer coordinate, the boundary-layer solutions can be decomposed into an oscillatory and a non-oscillatory part. The constants $A_{2,c}$ and $A_{2,s}$ are involved in the oscillatory parts of the boundary-layer solutions.

With regard to the matchings of the asymptotic solutions to order ε^2, it is appropriate to make a distinction between the matchings related to the divergence of the tidal displacement and the matchings related to the radial component of the tidal displacement.

First, the matchings of the asymptotic solutions related to the divergence of the tidal displacement correspond to the matchings of the oscillatory parts of the asymptotic solutions at order ε^2 and lead to Eqs. (148), from which the constants $A_{2,c}$ and $A_{2,s}$ are determined as solutions. Since the various terms in the right-hand members of the equations contain the function $f(r)$, and thus the function $\xi_0(r)$, the constants $A_{2,c}$ and $A_{2,s}$ too are proportional to the constant $c_{\ell,m,k}$ determining the strength of the dynamic tide.

When the frequency of the dynamic tide tends to an eigenfrequency of a free oscillation mode g^+, the constants $A_{2,c}$ and $A_{2,s}$ become indefinitely large. In these limiting cases, the asymptotic expansions differ anymore to a small extent from the divergence-free solution $\xi_0(r)$ adopted at order ε^0, and the validity of the perturbation procedure breaks down.

On the other hand, when $M_2 = 0$ and $\varepsilon_T = 0$, it follows that $\xi_0 = 0$, and $A_{2,c} = 0$ and $A_{2,s} = 0$. This conclusion confirms that, for free oscillations of the star, no oscillatory solutions are possible for frequencies that are not equal to an eigenfrequency.

From a numerical point of view, it may be difficult to determine the constants $A_{2,c}$ and $A_{2,s}$ accurately by means of Eqs. (148), since the values of the constants may be sensitive to the choices of τ_M and τ_N. Therefore, it might be preferable to impose more directly that the uniformly valid asymptotic expansions $\alpha^{(c,u)}(r;\varepsilon)$ and $\alpha^{(s,u)}(r;\varepsilon)$, as given by Expansions (105) and (169), respectively, and their partial derivatives $\partial\alpha^{(c,u)}/\partial\tau$ and $-\partial\alpha^{(s,u)}/\partial\tau_s$ be continuous at any point of their common domain of validity.

Secondly, the matchings of the asymptotic solutions related to the radial component of the tidal displacement correspond to the matchings of the non-oscillatory parts of the asymptotic solutions. The non-oscillatory part of the asymptotic solution for the divergence of the tidal displacement at order ε^2 is determined by the function $f(r)$ given by Definition (68) and depending on the solution $\xi_0(r)$ of Clairaut's equation. The non-oscillatory part of the asymptotic solution for the radial component of the tidal displacement involves in addition the function $G_2^{(o)}(r)$ given by Expression (66).

The divergence and the radial component of the tidal displacement at the star's surface are determined by Expressions (158) and (162). The divergence of the tidal displacement is determined exclusively by the oscillatory part of the solution for $v_2^{(s)}(\tau_s)$ and, in view of Expression (125) for $A_{2,s}$, is of order ε^2. Hence, the non-oscillatory part of a dynamic tide at the star's surface is divergence-free to order ε^2. On the other hand, the radial component of the tidal displacement at order ε^2 is determined by the non-oscillatory part of the solution for $w_2^{(s)}(\tau_s)$. The oscillatory part of the solution, which is of order ε^2 in the boundary layer near $r = 1$, yields a small contribution of order ε^4 at the star's surface.

The Eulerian perturbation of the gravitational potential at the surface caused by the star's tidal distortion is given, to order ε^2, by asymptotic Solution (167). To the order of approximation considered, the perturbation results only from the distortion due to the non-oscillatory parts of the asymptotic solutions. That the perturbation is not influenced by the distortion due to the oscillatory parts of the asymptotic solutions can be understood as follows. The Eulerian perturbation of the gravitational potential is determined by an integral of the varying density perturbations in the star. Undoubtedly, the integral of the highly oscillatory density perturbations must be appreciably smaller than the integral of the non-oscillatory density perturbations.

As a result, the determination of the Eulerian perturbation of the gravitational potential at the star's surface to order ε^2 does not require the derivation of the oscillatory parts of the asymptotic solutions at order ε^2. From Eqs. (166) and (167), it follows that the Eulerian perturbation of the gravitational potential is affected by the values of $(\xi_0)_{R_1}$ and $\left(d\xi_0/dr\right)_{R_1}$, and the constants $\mathcal{N}_s^{-2}$, $K_{1,s}$, $K_{1,1}$, $K_{6,s}$, and $K_{6,1}$, which are related to surface conditions of the equilibrium star.

As the other constants involved in the asymptotic representation, the constants $D'_{2,s}$, $C_2^{(o)}$, and $C_{2,c}$ are proportional to the constant $c_{\ell,m,k}$.

Finally, the asymptotic representation of free low-frequency g^+-modes is recovered in the limiting case when the mass M_2 of the star's companion is set equal to zero so that, according to Definition (2), $\varepsilon_T = 0$. As stated above, it then follows that $\xi_0(r) = 0$ at all points.

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