Astron. Astrophys. 318, 657–661 (1997)

ASTRONOMY AND ASTROPHYSICS

Analytic solution of the two-body problem with slowly decreasing mass

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Received 3 May 1996 / Accepted 3 July 1996

Abstract. We intend to present an approximate analytic solution of the two-body problem with slowly decreasing mass which is obtained through the integration of the Hamilton equations using A. Deprit's method of perturbations. The solution, obtained through the law of mass variation $\dot{m} = -\alpha m^n$ is put into practice in a specific case and compared with F. Mestschersky's exact equation (n = 2) and with that which results from numerically integrating the equations.

Key words: stars – mass loss – binaries: close – celestial mechanics – methods: analytical

1. Introduction

M. Ch. Dufour (Dufour 1866) seems to have been the first to examine the astronomical phenomenon of variable mass by relating the secular variation of lunar acceleration with the increase of the Earth's mass due to the impact of meteorites.

Years later, H. Gylden (Gylden 1884) set out the solution to the system of differential equations which describes the two-body motion when the masses are subject to variations. However it was F. Mestschersky, (Mestschersky 1893, 1902, 1949), basing his studies on the latter's work, who was the first to point out a specific case of this problem which is integrable. To be more precise, he proved that, taking into consideration a law of mass variation

$$\mu(t) = m(t) = \frac{1}{a + \alpha t} \tag{1}$$

the transformation

$$\mathbf{R} = \frac{\mathbf{r}}{a + \alpha t}; \quad \tau = \frac{-1}{a(a + \alpha t)} \tag{2}$$

converts the equations of motion

$$\ddot{\mathbf{r}} = -\frac{\mu(t)}{r^3}\mathbf{r} \tag{3}$$

into other integrable ones, given by

$$\mathbf{R}'' = -\frac{\mathbf{R}}{\mathbf{R}^3} \tag{4}$$

a and α being constant and ($'$) denotes derivative with respect to the variable τ.

If the initial mass m_0 begins to vary from a certain instant t_0 = 0, then it is obvious that the value of the constant a is $1/m_0$, and therefore if we take m_0 = 1, without losing generality, the expression (1) can be written

$$\mu = m(t) = \frac{1}{1 + \alpha t} \tag{5}$$

Under these conditions it can be proven that together with the transformation

$$\mathbf{R} = \frac{\mathbf{r}}{1 + \alpha t} \tag{6}$$

any function of the form

$$\tau = \frac{\pm 1}{\alpha(1 + \alpha t)} + C \qquad C = \text{constant} \tag{7}$$

can be used as a relation between t and τ to transform the Eqs. (3) into (4).

In particular, it is more convenient to take $C = \frac{1}{\alpha}$ and minus on numerator because (7) becomes

$$\tau = \frac{t}{1 + \alpha t} \tag{8}$$

which has the advantage that when $\alpha \rightarrow 0$, $\tau \rightarrow$t, that is, it tends towards identity transformation.

It is of vital importance to have this exact solution available when studying the nature of other solutions both numerical as well as analytic, since they can be compared with it.

Since F. Mestschersky's initial study a great many researchers have dedicated much of their time to this problem, one of the classics of Celestial Mechanics, which has become known as the Gylden-Mestschersky.

An ample bibliography can be found in the published works of E. N. Polyakova (Polyakova 1994) and C. Prieto (Prieto 1995).

The specific case which results in a slow isotropic mass loss has also been the focus of exhaustive studies carried out by researchers like, for instance, J. D. Hadjidemetriou (Hadjidemetriou 1963, 1966), L. Van Der Laan (Van Der Laan and Verhulst 1972), F. Verhulst (Verhulst 1969(1), 1969(2)), W. Eckhaus (Verhulst and Eckhaus 1972) to name but a few. The vast majority of these, in search of the stellar application, have taken the so-called Eddington-Jeans law (Jeans 1924, 1925) as a law of the variation of mass.

$$\dot{m} = -\alpha m^n \tag{9}$$

where α and n are real numbers, the first of them positive proximate to zero and n varying between 1.4 and 4.4.

So, for example, J. D. Hadjidemetriou used this to integrate the Lagrange equations, first numerically and then analytically, considering the problem as one of two-body perturbation.

In the present article and using the same law of mass variation, the equations of motion are written in the Hamiltonian form and analytically integrated until the second-order of perturbation.

The solution obtained is presented by an example and compared with the Mestschersky exact equation and with that resulting from the integration of the equations using a numerical method.

2. Development of Hamiltonian Function

Since we are dealing with planar motion, the polar coordinates and their conjugated moments coincide with the system of polar-nodal variables in which the Hamiltonian function can be written in the form:

$$F = \frac{1}{2}\left(R^2 + \frac{U^2}{r^2}\right) - \frac{\mu(t)}{r} \tag{10}$$

Using the Delaunay variables depending on time:

$L^2 = \mu a \quad ; \quad \ell =$ mean anomaly

$G^2 = \mu a(1-e^2) \quad ; \quad g =$ argument of the periastron

A. Deprit (Deprit 1983) demonstrated that the function (10) becomes:

$$F = F(L, G, \ell; t) = -\frac{1}{2}\frac{m^2}{L^2} + \frac{\dot{m}}{m} Le \sin E \tag{11}$$

where $m = \mu$. According to this, the canonical equations which govern the motion of the two-body problem with slowly decreasing mass are:

$$\begin{aligned}
\frac{d\ell}{dt} &= \frac{\partial F}{\partial L} = \frac{m^2}{L^3} + \frac{\dot{m}}{m}\left(e + \frac{G^2}{L^2 e}\right)\sin E \\
\frac{dg}{dt} &= \frac{\partial F}{\partial G} = -\frac{\dot{m}}{m}\frac{G}{eL}\sin E \\
\frac{dL}{dt} &= -\frac{\partial F}{\partial \ell} = -\frac{\dot{m}}{m}\frac{eL\cos E}{(1-e\cos E)} \\
\frac{dG}{dt} &= -\frac{\partial F}{\partial g} = 0
\end{aligned} \tag{12}$$

The fact that the G momentum is constant was to be expected because in the general two-body problem with variable mass the angular momentum remains constant.

It is clear that if in (12) $\dot{m} = 0$, the equations become those of the classic two-body problem, therefore it could be considered, as other authors already have, that the slow loss of isotropic mass acts like a perturbation and the problem could be dealt with using the typical methods of the theory of perturbations.

To do so would first of all require the development of the Hamilton function (11) in a small parameter power ϵ so that for $\epsilon = 0$ the Keplerian case would result. So it seems reasonable that this parameter should be related to the coefficient α which appears in the law of mass variation (9).

We will simply chose ϵ as the nondimensioned value of α.

Let us now consider for the function $m(t)$ a development in the Taylor series of the form:

$$m = m_0 + \sum_{p=1}^{\infty} \frac{1}{p!} m_0^{(p)} (t - t_0)^p$$

m_0 being the value of the mass in a certain initial instant t_0 and where $m_0^{(p)}$ represents the p^{th} derivative of the function m with respect to t evaluated as $t = t_0$. When $\dot{m} = -\alpha m^n$, we have

$$m_0^{(p)} = (-1)^p \alpha^p m_0^{pn-p+1} \times \prod_{s=1}^{\infty} \frac{sn - s + 1}{(s-1+p)n - (s+p-2)} \quad / \quad s \in N$$

and up to second-order, we have

$$m = m_0 - \alpha m_0^n (t - t_0) + \alpha^2 \frac{m_0^{2n-1}}{2} n(t-t_0)^2 + \cdots \tag{13}$$

whose convergence is assured for $|t - t_0| < 1$.

If this expression is substituted in the Hamiltonian function and certain operations effected, the result is

$$F = F_0 + \frac{\epsilon}{1!}F_1 + \frac{\epsilon^2}{2!}F_2 + \cdots$$

where the F_i terms are, up to the second order:

$$F_0 = -\frac{m_0^2}{2L^2}$$

$$F_1 = \frac{m_0^{n+1}}{L^2}(t - t_0) - eLm_0^{n-1}\sin E$$

$$F_2 = 2eLm_0^{2n-2}(n-1)\sin E(t-t_0) - \frac{m_0^{2n}(n+1)}{L^2}(t-t_0)^2$$

3. Integration of the equations of motion.

Once the devolpment of Hamiltonian function has been achieved in powers of the small parameter, the equations of motion can be analytically integrated until a certain order of perturbation using methods based on the application of the Lie transformation to the initial canonical system with the aim of obtaining other systems which are simpler to integrate.

To this end we will use the method suggested by A. Deprit (Deprit 1969). Taking for granted that this method is widely know, since it has been used in diverse research studies over the past few years, we will proceed directly to putting it into prectice without any further comment.

So let $(L, G, \ell, g) \rightarrow (L^*, G^*, \ell^*, g^*)$, an infinitesimal canonical transformation, so that the new Hamiltonian be independent from ℓ^*.

Selecting F^* of the form:

$$F^* = F_0^* + \frac{\epsilon}{1!}F_1^* + \frac{\epsilon^2}{2!}F_2^*$$

with

$$F_0^* = F_0$$

$$F_1^* = \frac{m_0^{n+1}}{L^{*2}}(t - t_0)$$

$$F_2^* = -\frac{m_0^{2n}(n+1)}{L^{*2}}(t - t_0)^2$$

from the equations:

$$F_0^* = F_0$$

$$F_1^* = F_1 + \{F_0; W_1\} - \frac{\partial W_1}{\partial t} \quad (14)$$

$$F_2^* = F_2 + \{F_1; W_1\} + \{F_1^*; W_1\} + \{F_0; W_2\} - \frac{\partial W_2}{\partial t} \quad (15)$$

we can calculate the terms W_1 and W_2 of the generating function of the canonical transformation which relates the old and the new variables obtaining

$$W_1 = -e^* L^{*4} m_0^{n-3}\left[\frac{e^*}{2} + \cos E^* - \frac{e^*}{2}\cos(2E^*)\right]$$

$$\begin{aligned} W_2 = {} & L^{*4} m_0^{2(n-3)}\Big[\tfrac{E^* L^*}{2}\left[e^* L^{*2}(2n-3) - 2G^{*2}\right] + \tfrac{e^* L^*}{4} \\ & \left[4G^{*2} - 8L^{*2}(n+1) - 2e^{*2}L^{*2}(n-3)\right]\sin E^* \\ & -e^{*2}\left(n\tfrac{L^{*3}+1}{4} + L^{*3} + 1\right)\sin(2E^*) + \tfrac{L^{*3}e^{*3}(n+1)}{6} \\ & \sin(3E^*) + 2e^*(n+1)m_0^2(t-t_0)\cos E^* - \tfrac{e^{*2}}{2}(n+1) \\ & m_0^2(t-t_0)\cos(2E^*) - \tfrac{e^{*3}}{2}(n+1)\sin E^*\cos(2E^*)\Big] \end{aligned}$$

Hence the change of variables will be given by:

$$\begin{aligned} & L = L^* - \tfrac{\partial W}{\partial \ell^*} \;;\; G = G^* - \tfrac{\partial W}{\partial g^*} = G^* \\ & \ell = \ell^* + \tfrac{\partial W}{\partial L^*} \;;\; g = g^* + \tfrac{\partial W}{\partial G^*} \end{aligned} \quad (16)$$

where

$$W = \epsilon W_1 + \frac{\epsilon^2}{2!}W_2$$

is the generating function until the second-order.

On the other hand, since the new Hamilton function:

$$\begin{aligned} F^* = {} & F_0^* + \epsilon F_1^* + \tfrac{\epsilon^2}{2!}F_2^* = -\tfrac{m_0^2}{2L^{*2}} - \epsilon\tfrac{m_0^{n+1}}{L^{*2}}(t-t_0) - \\ & -\tfrac{\epsilon^2}{2!}\tfrac{m_0^{2n}(n+1)}{L^{*2}}(t-t_0)^2 \end{aligned}$$

depends only on L^* and time, the canonical equations can be integrated into the form:

$$\frac{dG^*}{dt} = -\frac{\partial F^*}{\partial g^*} = 0 \Rightarrow G^* = cte$$

$$\frac{dL^*}{dt} = -\frac{\partial F^*}{\partial \ell^*} = 0 \Rightarrow L^* = cte$$

$$\frac{dg^*}{dt} = \frac{\partial F^*}{\partial G^*} = 0 \Rightarrow g^* = cte$$

$$\frac{d\ell^*}{dt} = \frac{\partial F^*}{\partial L^*} = \frac{m_0^2}{L^{*3}} - \frac{2\epsilon m_0^{n+1}}{L^{*3}}(t-t_0) + \frac{\epsilon^2 m_0^{2n}(n+1)}{L^{*3}}(t-t_0)^2 \quad (17)$$

$$\ell^* = A^*_0 + A^*_1 t + A^*_2 t^2 + A^*_3 t^3$$

where

$$\begin{aligned} & A^*_1 = \tfrac{m_0^2 + 2\epsilon m_0^{n+1}t_0 + \epsilon^2 m_0^{2n} t_0^2(n+1)}{L^{*3}} \\ & A^*_2 = -\tfrac{\epsilon m_0^{n-1}}{L^{*3}}\left(m_0^2 + \epsilon m_0^{n+1}t_0(n+1)\right) \\ & A^*_3 = \tfrac{\epsilon^2 m_0^{2n}(n+1)}{3L^{*3}} \end{aligned} \quad (18)$$

and A^*_0 an integration constant.

4. Modus Operandi.

Starting off with the following initial values for the orbital elements: T_0, a_0, e_0, ω_0 coresponding to an instant t_0 and m_0 being the initial value of the mass, we will proceed first to calculate the corresponding Delaunay variables for the initial time.

$$\begin{aligned} & L_0 = \sqrt{m_0 a_0}; \quad \ell_0 = 2\pi\sqrt{\tfrac{a_0^3}{m_0}}(t_0 - T_0) \\ & G_0 = L_0\sqrt{1 - e_0^2}; \quad g_0 = \omega_0 \end{aligned} \quad (19)$$

Applying now the Deprit method inversely, we can calculate the value of the constants of motion: L^*, G^*, g^* and A_0 according to:

$$\begin{aligned} & L^* = L_0 + \left(\tfrac{\partial W}{\partial \ell}\right)_0 \quad ; \quad \ell_0{}^* = \ell_0 - \left(\tfrac{\partial W}{\partial L}\right)_0 \\ & G^* = G_0 + \left(\tfrac{\partial W}{\partial g}\right)_0 \quad ; \quad g^* = g_0 - \left(\tfrac{\partial W}{\partial G}\right)_0 \\ & A^*_0 = \ell_0{}^* - A^*_1 t_0 - A^*_2 t_0^2 - A^*_3 t_0^3 \end{aligned} \quad (20)$$

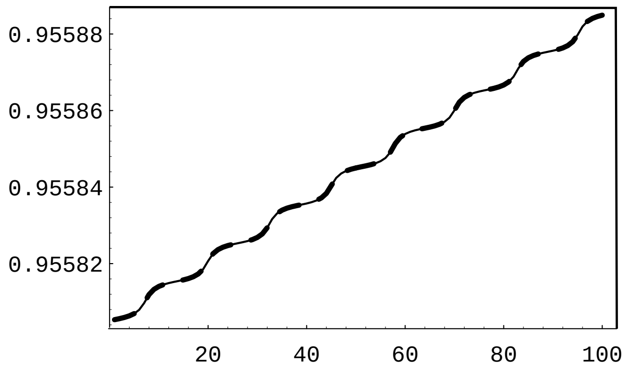

Fig. 1. a' against t in the last 100 instants.

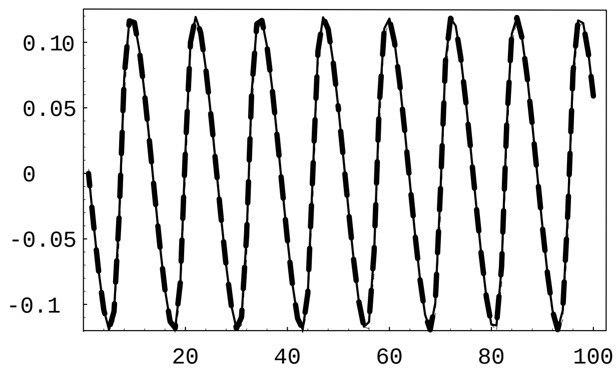

Fig. 2. e' against t in the last 100 instants.

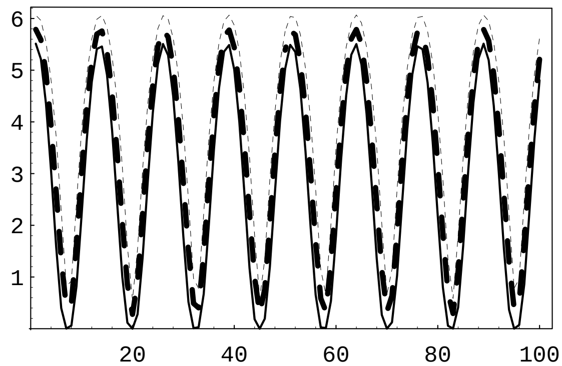

Fig. 3. ω' against t in the last 100 instants.

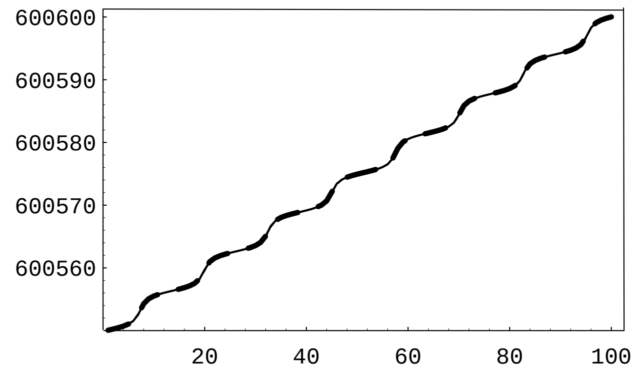

Fig. 4. f against t in the last 100 instants.

where $A^*{}_1, A^*{}_2$ and $A^*{}_3$ are calculated according to (18). So we are now able to obtain the values of the orbital elements in any instant t (ephemeris calculus).

In fact, in a t instant in which the value of the mass is m, we will determine the value of ℓ^* according to the corresponding expression of (17) and, since L^*, G^* and g^* are constant, we can calculate in t the values of the Delaunay variables using (16), and from this point the corresponding orbital elements in accordance with the equations:

$$\begin{aligned} a &= \frac{L^2}{m} \\ e &= \sqrt{1 - \frac{G^2}{L^2}} \\ T &= t - \frac{l}{2\pi}\sqrt{\frac{m}{a^3}} \\ \omega &= g \end{aligned} \tag{21}$$

Although the value of the time of periastron passage is usually substituted either by the mean anomaly or the true anomaly, the latter is the one we will use in practical cases.

5. Practical application.

As a test of our analytic solution we have considered the case of a double star in which the total mass loss is of the order of that of the Sun's, that is we will use $\alpha = 0.35 \times 10^{-14}$ from the Eddington-Jeans law as E. L. Schatzman y F. Praderie (Schatzman y Praderie 1993) suggest. Likewise, we will use $n = 2$, having chosen this value to enable us to avail of the Mestschersky exact solution, because for this value of n the Eq. (9) is equivalent to (5). The initial values taken for the mass and the orbital elements were the following: $m(t_0) = 1$, $a(t_0) = 1$, $e(t_0) = 0.5$, $\omega(t_0) = 0$ and $f(t_0) = 0$. From a certain time $t_0 = 0$, in an interval of 600.000 units of time (u.t.), we calculated the values of the orbital elements in 1.200.000 instances each two separated 0.5 u.t. An interval which would correspond to the Sun-Earth system approximately 100.000 years away.

With the aim of making the development of (13) convergent, the algorithm developed in parts 3 and 4 was reinitiated at each stage, that is, each 0.5 units of time.

The partial results of the last 100 instants have been sketched in (1), (2), (3) and (4) for each orbital element. Since the variations of a, e and ω are very small, we believe that it is advisable to place the quantities $a' = 10^8(a-1)$, $e' = 10^{12}(e-0.5)$ and $\omega' = 10^{14}\omega$ in the respective y-axis so as to permit better visualization. Although the distinct solutions practically fuse together in the graphs, the continuous line represents the analytic solution, the broken line the exact and the dots a numerical obtained using the Runge-Kutta method of eighth-order for systems of first-order ordinary differential equations.

We also include in Tables (1),(2), (3), (4) the values of the above quantities in the last ten instants, with the aim of facil-

Table 1. Values for a' obtained with the three solutions in the last ten instants.

t'	Deprit	Mestsersky	Numerical
-9	0.955875955	0.955875949	0.955876062
-8	0.955876369	0.955876365	0.955876456
-7	0.955876974	0.955876974	0.955877022
-6	0.955877966	0.955877972	0.955877931
-5	0.955879754	0.955879771	0.955879564
-4	0.955881963	0.955881974	0.955881827
-3	0.955883303	0.955883305	0.955883303
-2	0.955884071	0.955884069	0.955884135
-1	0.955884568	0.955884564	0.955884664
0	0.955884926	0.955884921	0.955885038

Table 2. Values for e' obtained with the three solutions in the last ten instants.

t'	Deprit	Mestsersky	Numerical
-9	-0.07629292	-0.07667316	-0.073098594
-8	-0.10490948	-0.10517251	-0.103246092
-7	-0.11917229	-0.11920844	-0.120465668
-6	-0.10449793	-0.10405719	-0.111955167
-5	-0.03005305	-0.02876442	-0.049162147
-4	0.075927670	0.076783236	0.0608531103
-3	0.116732596	0.116889793	0.1118441911
-2	0.114685454	0.114517295	0.1145881365
-1	0.092279355	0.091948671	0.0945563371
0	0.059439078	0.059024203	0.0629483051

itating a clearer analysis of the final results, where t' denotes $t - 1.200.000$.

6. Conclusions

Commenting on the results obtained we can state that by comparing the analytic solution and the numerical solution with the exact, ours gives smaller differences in the final instant, as can be seen in Table (5).

All these calculations were carried out on a FUJITSU vectorial computer model VP 2400/10 in the Supercomputation Center of Galicia (CESGA), the calculation times being 59^s12 for the exact solution, 1^m05^s33 for our analytic solution and $2^m04^s12$for the numerical method.

Acknowledgements. The authors wish to express their gratitude to Professors J. H. Hadjidemetriou, A. Deprit, A. Elipe y A. Abad for their invaluable observations on this study which was carried out as part of the research projects XUGA 24301B92 and PB92-1074 financed by the XUNTA DE GALICIA (Spain), and the Spanish D.G.I.C.Y.T. respectively.

Table 3. Values for ω' obtained with the three solutions in the last ten instants.

t'	Deprit	Mestsersky	Numerical
-9	4.313668624	4.921140988	5.528613352
-8	3.006971604	3.892191434	4.777411264
-7	1.539885786	2.608204073	3.676522359
-6	0.367943592	1.299380344	2.230817096
-5	0.001430274	0.389689972	0.777949670
-4	0.071892979	0.521136429	0.970379878
-3	0.862602414	1.678297370	2.493992327
-2	2.248691067	3.067964739	3.887238412
-1	3.681464626	4.304630172	4.927795717
0	4.805446980	5.213939541	5.622432101

Table 4. Values for f' obtained with the three solutions in the last ten instants.

t'	Deprit	Mestsersky	Numerical
-9	600594.449	600594.451	600594.351
-8	600594.699	600594.702	600594.593
-7	600595.036	600595.041	600594.911
-6	600595.586	600595.595	600595.411
-5	600596.743	600596.761	600596.434
-4	600598.292	600598.305	600598.035
-3	600599.081	600599.087	600598.929
-2	600599.501	600599.505	600599.384
-1	600599.787	600599.79	600599.685
0	600600.015	600600.017	600599.919

Table 5. Differences in the values obtained for each orbital element.

	a	e	ω	f
D-M	10^{-9}	10^{-4}	0.6	10^{-3}
N-M	10^{-7}	10^{-3}	1.0	10^{-2}

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