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Formation of MACHO-primordial black holes in inflationary cosmology

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Abstract. As a nonbaryonic explanation of massive compact halo objects, a phenomenological model is presented which predicts formation of primordial black holes at a desired mass scale. The required feature of initial density fluctuation is realized making use of primordially isocurvature fluctuations generated in an inflationary universe model with multiple scalar fields.

Key words: cosmology: theory – early universe – dark matter – Black hole physics

1. Introduction

If overdensity of order of unity exists in the hot early universe, a black hole can be formed when the perturbed region enters the cosmological horizon. The primordial black holes (hereafter PBHs) thus produced were a subject of active research decades ago (Zel'dovich & Novikov 1967; Hawking 1971) and various observational constraints have been obtained against their mass spectrum —with no observational evidence of their existence at that time (Novikov et al. 1979).

Recently, however, several independent projects reported observation of massive compact halo objects (MACHOs) through gravitational microlensing (Alcock et al. 1993, 1995; Aubourg et al. 1993, 1995). It is estimated that their mass is around $0.01 - 0.1 M_{\odot}$ and that they occupy $\sim 20\%$ of the galactic halo mass which makes up about $\mathscr{O}(10^{-3})$ of the critical density (Alcock et al. 1995). While the primary MACHO candidates are substellar baryonic objects such as brown dwarfs, it is difficult to reconcile such a large amount of these objects with the observed mass function of low mass stars (Richer & Fahlman 1992) and with the infrared observation of dwarf component (Boughn & Uson 1995), unless the mass function is extrapolated to the lower masses in an extremely peculiar manner or Population III stars are produced abundantly at the relevant mass scale (Carr et al. 1984). Therefore it is also an interesting and potentially important theoretical issue to consider nonbaryonic candidates.

In the present paper we consider the possibility that MACHOs consist of PBHs produced in the early universe and present a simple model which generates the desired spectrum of primordial density perturbations in the context of inflationary cosmology (Guth 1981; Sato 1981; for a review see, *e.g.* Olive 1990). In the simplest models of inflation with one *inflaton* scalar field, the predicted adiabatic density fluctuation has an almost scale-invariant spectrum (Hawking 1982; Starobinsky 1982; Guth & Pi 1982) unless the inflaton has a peculiar potential. Hence they do not predict PBH formation in general[1]. In models with multiple scalar fields, on the other hand, not only adiabatic but also isocurvature fluctuations are generated during inflation. The latter can be cosmologically important if the energy density of its carrier becomes significant in a later epoch (Linde 1985; Kofman & Linde 1987). Furthermore it is relatively easy to imprint a nontrivial feature on the spectral shape of the isocurvature fluctuations. Making use of this property, we here construct a model which possesses a peak in the spectrum of the total density fluctuation at horizon crossing. A significant amount of PBHs can then be produced on that scale. While our goal is to produce PBHs with mass $M_{\rm BH} \sim 0.1 M_{\odot}$ and density $\Omega_{\rm BH} \sim 10^{-3}$, our model can also be applied to the formation of PBHs with different mass and density by choosing different values of the model parameters.

The onset of the formation of PBHs has been discussed by Carr (1975), who considered evolution of a high density region, which is spherically symmetric and larger than the horizon scale initially, in a spatially-flat and radiation-dominant background. He concluded that a black hole is formed soon after the perturbed region starts contraction and enters the horizon with the density contrast δ larger than about $1/3$, and that the resulting black

[1] In a particular class of inflationary models predicting fluctuations with the spectral index $n > 1$, PBHs may be produced on small mass scales and one can impose an upperbound on n using various constraints on the abundance of PBHs (Carr & Lidsey 1993; Carr et al. 1994). We also note that, after we submitted the original version of this paper, Randall et al. (1995) proposed an inflation model which predicts adiabatic fluctuations with a large amplitude on small scales. But their spectral shape is not suitable for formation of the MACHO-PBHs as it is.

hole has of order the horizon mass, $M_H \simeq 10^5(t/1\,\mathrm{sec})M_\odot$ at the formation epoch t. If fluctuations are Gaussian distributed, the initial mass fraction of PBHs, $\beta(M)$, is given by

$$\beta(M) \simeq \bar{\delta}(M) \exp\left(-\frac{1}{18\bar{\delta}^2(M)}\right), \quad (1)$$

where $\bar{\delta}(M)$ is the root-mean-square amplitude of density fluctuations on a mass scale M at horizon crossing (Carr 1975).

In order to produce PBHs with mass $M_{\rm BH} \sim 0.1 M_\odot$ and present density $\Omega_{\rm BH} \sim 10^{-3}$, the initial fraction should be $\beta(M_{\rm BH}) \simeq 10^{-11}$ which implies $\bar{\delta}(M_{\rm BH}) \simeq 0.05$. Note that for $\bar{\delta} = 0.05$ the threshold of PBH formation, $\delta = 1/3$, corresponds to 6.4 standard deviation. It has been argued by Doroshkevich (1970) that such a high peak has very likely a spherically symmetric shape. Thus the assumption of spherical symmetry in the above discussion is justified and it is also expected that gravitational wave produced during PBH formation is negligibly small.

The rest of the paper is organized as follows. In Sect. 2 we consider the possibility of generating the necessary fluctuations stated above in the context of inflation and a model Lagrangian is proposed. Sects. 3-5 are devoted to a detailed description of the evolution of the universe in this model and in Sect. 6 constraints on the parameters of the model are obtained. Sect. 7 contains our conclusions.

2. Non-flat perturbation in inflationary cosmology

Since the amplitude of density perturbations on the large scales probed by the anisotropy of the background radiation (Bennet et al. 1996) is known to be $\bar{\delta} \simeq 10^{-5}$, the primordial fluctuations at horizon crossing must have a spectral shape such that they have an amplitude of 10^{-5} on large scales, sharply increase by a factor of 10^4 on the mass scale of the PBHs, and decrease again on smaller scales. It is difficult to produce such a spectrum of fluctuations in inflationary cosmology with a single component.

In generic inflationary models with a single scalar field ϕ, which drives inflation with a potential $V[\phi]$, the root-mean-square amplitude of the adiabatic fluctuations is given by

$$\frac{\delta\rho}{\rho}(r) \equiv \bar{\delta}\,(r(\phi)) \cong \frac{8\sqrt{6\pi}V[\phi]^{\frac{3}{2}}}{V'[\phi]M_{Pl}^3}, \quad (2)$$

on the comoving scale $r(\phi)$ when that scale reenters the Hubble radius (Hawking 1982; Starobinsky 1982; Guth & Pi 1982). The right-hand-side is evaluated when the same scale leaves the horizon during inflation. Because of the slow variation of ϕ and rapid cosmic expansion during inflation, (2) implies an almost scale-invariant spectrum in general. Nonetheless, one could in principle obtain various shapes for the fluctuation spectra by making use of the nontrivial dependence of $\bar{\delta}\,(r(\phi))$ on $V[\phi]$ (Hodges & Blumenthal 1990). In order to obtain the spectrum required for PBH formation, with a peak on a particular scale, we must employ a scalar potential with two breaks and a plateau in between (Ivanov et al. 1994). Such a solution is not aesthetically appealing.

Here we consider an inflation model with multiple scalar fields in which not only adiabatic but also primordially isocurvature fluctuations are produced. As mentioned in the Introduction it is much easier to imprint nontrivial structure on an isocurvature spectrum.

We introduce three scalar fields ϕ_1, ϕ_2, and ϕ_3 in order to generate the desired spectrum of density fluctuation. ϕ_1 is the inflaton field which induces the new inflation (Linde 1982; Albrecht & Steinhardt 1982) with a double-well potential, starting its evolution near the origin where its potential is approximated as

$$U[\phi_1] \cong V_0 - \frac{\lambda_1}{4}\phi_1^4. \quad (3)$$

Linde (1994) and Vilenkin (1994) give the natural realization of this initial condition. The Hubble parameter during inflation, H_I, is given by

$$H_I^2 = \frac{8\pi V_0}{3M_{Pl}^2}. \quad (4)$$

ϕ_2 is a long-lived scalar field which induces primordially isocurvature fluctuations that later lead to black hole formation later. ϕ_3 is an auxiliary field coupled to both ϕ_1 and ϕ_2, which changes the effective mass of the latter in order to imprint a specific feature on the spectrum of its initial fluctuations.

We adopt the following model Lagrangian:

$$\mathscr{L} = -\frac{1}{2}(\partial\phi_1)^2 - \frac{1}{2}(\partial\phi_2)^2 - \frac{1}{2}(\partial\phi_3)^2 - V[\phi_1, \phi_2, \phi_3] + \mathscr{L}_{\rm int}, \quad (5)$$

where $\mathscr{L}_{\rm int}$ represents interaction of the ϕ_j with the other fields. Here $V[\phi_1, \phi_2, \phi_3]$ is the effective scalar potential governing the dynamics of the fields:

$$V[\phi_1, \phi_2, \phi_3] = U[\phi_1] + \frac{\epsilon}{2}(\phi_1^2 - \phi_{1c}^2)\phi_3^2 + \frac{\lambda_3}{4}\phi_3^4 - \frac{\nu}{2}\phi_3^2\phi_2^2 + \frac{\lambda_2}{4}\phi_2^4 + \frac{1}{2}m_2^2\phi_2^2, \quad (6)$$

where $\lambda_j, \epsilon, \nu, \phi_{1c}$, and m_2 are positive constants. m_2 is assumed to be much smaller than the scale of inflation, H_I, and it does not affect the dynamics of ϕ_2 during inflation. Hence we ignore it for the moment.

Let us briefly outline how the system evolves before presenting a detailed description. In the early inflationary stage ϕ_1 is smaller than ϕ_{1c}, and ϕ_3 has its potential minimum off the origin. Then ϕ_2 also settles down to a nontrivial minimum, where it can have an effective mass larger than H_I so that its quantum fluctuations are suppressed. As ϕ_1 becomes larger than ϕ_{1c}, ϕ_3 rolls down to the origin. The potential of ϕ_2 then also becomes convex and its amplitude gradually decreases due to the quartic term. However, since its potential is now nearly flat, its motion is extremely slow, with its effective mass being smaller than H_I well until the end of inflation. During this stage quantum fluctuations are generated in ϕ_2 with a nearly scale-invariant spectrum. Thus the initial isocurvature fluctuations due to ϕ_2 have a scale-invariant spectrum with a cut-off on a large scale.

After inflation, the Hubble parameter starts to decrease in the reheating processes. As it becomes smaller than the effective mass of ϕ_2, the latter starts rapid coherent oscillations. At first, when its oscillations are governed by the quartic term, ϕ_2 dissipates its energy like radiation in the beginning. But later on, when $\lambda_2\phi_2^2$ becomes smaller than m_2^2, its energy density decreases more slowly like nonrelativistic matter. Thus the fraction of ϕ_2 contribution to the total energy density increases continuously, which implies that the total density fluctuations due to the isocurvature fluctuations in ϕ_2 grow. Since what is relevant for PBH formation is the magnitude of the fluctuations at horizon crossing, we thus obtain a spectrum with an amplitude which increases with scale until the cut-off scale in the initial spectrum is reached. Thus one has a single peak on the mass-scale of PBH formation.

In the above scenario, we have assumed that ϕ_2 survives until after PBH formation. On the other hand, were ϕ_2 stable, it would soon dominate the total energy density of the universe in conflict with the successful nucleosynthesis. As a natural way out we consider the possibility that ϕ_2 decays through gravitational interactions, so that it does not leave any unwanted relics, its only trace being the tiny fraction of PBHs produced.

In the subsequent sections, we describe the evolution of the above model in more detail and obtain constraints on the model in order to produce the right amount of PBHs on the right scale.

3. Background evolution

First we consider the evolution of the homogeneous part of the fields. During inflation, the behavior of the inflaton is governed by the $U[\phi_1]$ part of the potential. Solving the equation of motion with the slow-roll approximation,

$$3H_I\dot{\phi}_1 \cong -U'[\phi_1] \cong \lambda_1\phi_1^3, \tag{7}$$

we find

$$\lambda_1\phi_1^2(t) = \frac{\lambda_1\phi_{1i}^2}{1+\frac{2\lambda_1\phi_{1i}^2}{3H_I^2}H_I(t-t_i)}, \tag{8}$$

where ϕ_{1i} is the field amplitude at some initial epoch t_i. This approximate solution remains valid until $|U''[\phi_1]|$ becomes as large as $9H_I^2$ at $t \equiv t_f$, when inflation is terminated, and we find $\phi_1^2(t_f) = 3H_I^2/\lambda_1$. (8) can also be written as

$$\lambda_1\phi_1^2(t) = \frac{3H_I^2}{2H_I(t_f-t)+1} \equiv \frac{3H_I^2}{2\tau(t)+1} \simeq \frac{3H_I^2}{2\tau(t)}, \tag{9}$$

where $\tau(t)$ is the number of exponential expansion e-folds after $t(< t_f)$ and the last approximation is valid for $\tau \gg 1$. From now on, we often use the time variable $\tau(t)$ to refer to the comoving scale leaving the Hubble radius at t. Note that it is a decreasing function of t.

As stated in the last section, we take the view that ϕ_1 determines fate of ϕ_3 and that ϕ_3 controls the evolution of ϕ_2 but not vice versa. In order that ϕ_3 does not affect evolution of ϕ_1, the inequality

$$\lambda_1\lambda_3 \gg \epsilon^2 \tag{10}$$

must be satisfied, while we must have

$$\lambda_2\lambda_3 \gg \nu^2 \tag{11}$$

so that ϕ_2 does not affect the motion of ϕ_3. We assume these inequalities hold below.

When $\phi_1 < \phi_{1c}$, both ϕ_2 and ϕ_3 have nontrivial minima, which we denote by ϕ_{2m} and ϕ_{3m}, respectively. From

$$V_2 = -\nu\phi_3^2\phi_2 + \lambda_2\phi_2^3 = 0 \tag{12}$$

and

$$V_3 = \epsilon(\phi_1^2 - \phi_{1c}^2)\phi_3 + \lambda_3\phi_3^3 - \nu\phi_2^2\phi_3 = 0, \tag{13}$$

with $V_j \equiv \partial V/\partial\phi_j$, we find

$$\lambda_2\phi_{2m}^2(t) = \nu\phi_{3m}^2(t) \tag{14}$$

$$\begin{aligned}\lambda_3\phi_{3m}^2(t) &= \epsilon\left(\phi_{1c}^2 - \phi_1^2(t)\right) + \nu\phi_{2m}^2(t)\\ &\cong \epsilon\left(\phi_{1c}^2 - \phi_1^2(t)\right),\end{aligned} \tag{15}$$

where (11) was used in the last expression. In the early inflationary stage, when $\phi_1 \ll \phi_{1c}$, the effective mass-squared for ϕ_j, V_{jj}, at the potential minimum is given by

$$\begin{aligned}V_{22}[\phi_1,\phi_{2m},\phi_{3m}] &= 2\lambda_2\phi_{2m}^2 = \frac{2\nu\epsilon}{\lambda_3}(\phi_{1c}^2-\phi_1^2)\\ &\simeq \frac{2\nu\epsilon}{\lambda_3}\phi_{1c}^2 = \frac{3\nu\epsilon}{\lambda_1\lambda_3\tau_c}H_I^2,\end{aligned} \tag{16}$$

$$V_{33}[\phi_1,\phi_{2m},\phi_{3m}] = \frac{3\epsilon}{\lambda_1\tau_c}H_I^2, \tag{17}$$

where τ_c is the epoch when $\phi_1 = \phi_{1c}$. We choose parameters such that

$$\nu\epsilon > \lambda_1\lambda_3\tau_c, \qquad \epsilon > \lambda_1\tau_c. \tag{18}$$

Then V_{22} and V_{33} are initially larger than H_I^2 at the potential minimum, so both ϕ_2 and ϕ_3 settle down to $\phi_{2m}(t)$ and $\phi_{3m}(t)$, respectively.

$\phi_{3m}(t)$ decreases down to zero at $\tau = \tau_c$ when $V_{33}[\phi_{3m}]$ also vanishes. Then $V_{33}[\phi_{3m} = 0]$ starts to increase according to ϕ_1 and soon acquires a large positive value. This implies that ϕ_3 practically traces the evolution of $\phi_{3m}(t)$ down to zero without delay. On the other hand, ϕ_2 evolves somewhat differently because it does not acquire a positive effective mass from ϕ_3 at the origin. Although $\phi_2(t)$ traces $\phi_{2m}(t)$ initially, as $V_{22}[\phi_{2m}]$ becomes smaller it can no longer catch up with $\phi_{2m}(t)$. From a generic property of scalar fields with small mass in a de Sitter background, one can show that this happens when the inequality

$$\left|\frac{1}{\phi_{2m}(t)}\frac{d\phi_{2m}(t)}{dt}\right| > \frac{V_{22}[\phi_{2m}]}{3H} \tag{19}$$

is satisfied, or at

$$\tau = \left(1+\sqrt{\frac{\lambda_1\lambda_3}{2\nu\epsilon}}\right)\tau_c \equiv \tau_l \cong \tau_c \tag{20}$$

with

$$\phi_2^2(\tau_l) \equiv \phi_{2l}^2 = \sqrt{\frac{\nu\epsilon}{2\lambda_1\lambda_3}} \frac{3H_I^2}{2\lambda_2\tau_l}. \tag{21}$$

Thus ϕ_2 slows down its evolution and, when ϕ_3 vanishes, it is governed by the quartic potential. We can therefore summarize its evolution during inflation by

$$\phi_2^2(\tau) \cong \begin{cases} \phi_{2\mathrm{m}}^2(\tau) = \frac{3\nu\epsilon H_I^2}{2\lambda_1\lambda_2\lambda_3}\left(\frac{1}{\tau_c} - \frac{1}{\tau}\right), \ \tau \gtrsim \tau_l, \\ \\ \phi_{2l}^2\left[1 + \frac{2\lambda_2\phi_{2l}^2}{3H_I^2}(\tau_l - \tau)\right]^{-1}, \quad 0 < \tau \lesssim \tau_l. \end{cases} \tag{22}$$

Since $\lambda_2\phi_{2l}^2$ is adequately smaller than H_I^2, ϕ_2 remains practically constant in the second regime. For later use we also write down time dependence of the effective mass-squared:

$$V_{22}[\phi_2] \cong \begin{cases} \frac{\nu\epsilon H_I^2}{\lambda_1\lambda_3}\left(\frac{1}{\tau_c} - \frac{1}{\tau}\right), & \tau \gtrsim \tau_l, \\ \\ 3\lambda_2\phi_{2l}^2\left[1 + \frac{2\lambda_2\phi_{2l}^2}{3H_I^2}(\tau_l - \tau)\right]^{-1}, & 0 < \tau \lesssim \tau_l. \end{cases} \tag{23}$$

4. Generation of fluctuations

We now consider fluctuations in both the scalar fields and the metric variables in a consistent manner. We adopt Bardeen's (1980) gauge-invariant variables Φ_A and Φ_H, in terms of which the perturbed metric can be written as

$$ds^2 = -(1 + 2\Phi_A)dt^2 + a(t)^2(1 + 2\Phi_H)d\mathbf{x}^2, \tag{24}$$

in the longitudinal gauge (Kodama & Sasaki 1984; Mukhanov et al. 1992). In this gauge the scalar field fluctuation $\delta\phi_j$ coincides with the corresponding gauge-invariant variable.

Assuming an $\exp(i\mathbf{kx})$ spatial dependence and working in Fourier space, the perturbed Einstein and scalar field equations are given by

$$\Phi_A + \Phi_H = 0, \tag{25}$$

$$\dot{\Phi}_H + H\Phi_H = -4\pi G(\dot{\phi}_1\delta\phi_1 + \dot{\phi}_2\delta\phi_2 + \dot{\phi}_3\delta\phi_3), \tag{26}$$

$$\begin{aligned} &\delta\ddot{\phi}_j + 3H\delta\dot{\phi}_j + \left(\frac{k^2}{a^2(t)} + V_{jj}\right)\delta\phi_j \\ &= 2V_j\Phi_A + \dot{\Phi}_A\dot{\phi}_j - 3\dot{\Phi}_H\dot{\phi}_j - \sum_{i\neq j} V_{ji}\delta\phi_i, \end{aligned} \tag{27}$$

where all the fluctuation variables are functions of $\mathbf{k}$ and t.

This system of equations can be simplified using constraints (10), (11) and (18). First, since ϕ_3 has an effective mass larger than H_I^2 during inflation except in the vicinity of $\tau = \tau_c$, quantum fluctuations in $\delta\phi_3$ are suppressed and, moreover, its energy density practically vanishes by the end of inflation. We can therefore neglect fluctuations in ϕ_3. On the other hand, with the help of (11) and (18) we can show that

$$|\dot{\phi}_2| \sim \sqrt{\frac{\nu\epsilon}{\lambda_2\lambda_3}}|\dot{\phi}_1| \ll |\dot{\phi}_1|. \tag{28}$$

Hence (26) and (27) with $j = 1$ reduce to

$$\dot{\Phi}_H + H\Phi_H = -4\pi G\dot{\phi}_1\delta\phi_1, \tag{29}$$

$$\begin{aligned} &\delta\ddot{\phi}_1 + 3H\delta\dot{\phi}_1 + \left(\frac{k^2}{a^2(t)} + V_{11}\right)\delta\phi_1 \\ &= 2V_1\Phi_A - 4\dot{\Phi}_H\dot{\phi}_1. \end{aligned} \tag{30}$$

Thus only ϕ_1 contributes to adiabatic fluctuations and these can be calculated in the same manner as in the new inflation model with a single scalar field. This is as expected because ϕ_1 dominates the energy density during inflation. In fact, since we are only interested in the growing mode in the super-horizon regime, which turns out to be weakly time-dependent as can be seen from the final result, we can consistently neglect time derivatives of metric perturbations during inflation and terms with two time derivatives in (29) and (30). We thus find

$$\Phi_H = -\Phi_A \cong -\frac{4\pi G}{H_I}\dot{\phi}_1\delta\phi_1. \tag{31}$$

The resulting amplitude of scale-invariant adiabatic fluctuations depends on λ_1 and one can use COBE observations (Bennet et al. 1996) to normalize its value as

$$\lambda_1 = 1.3 \times 10^{-13}. \tag{32}$$

On the other hand, $\delta\phi_2$ satisfies (27) with $j = 2$. From quantum field theory in de Sitter spacetime, it has a root-mean-square amplitude $\delta\phi_2 \cong (H^2/2k^3)^{1/2}$ when the k-mode leaves the Hubble radius if V_{22} is not too large. Since V_2 vanishes when $\phi_2 = \phi_{2\mathrm{m}}$ and $V_2\Phi_A$ remains small even for $\tau \leq \tau_l$, we can neglect all the terms on the right-hand side to yield

$$\delta\ddot{\phi}_2 + 3H_I\delta\dot{\phi}_2 + V_{22}\delta\phi_2 \cong 0, \tag{33}$$

when $k \ll a(t)H_I$. We can find a WKB solution with the appropriate initial condition:

$$\begin{aligned} \delta\phi_2(k,t) \ &\cong \ \left(\frac{H_I^2}{2k^3}\right)^{\frac{1}{2}}\left(\frac{S(t_k)}{S(t)}\right)^{\frac{1}{2}} \\ &\times \ \exp\left\{\int_{t_k}^{t}\left[S(t')H_I - \frac{3}{2}H_I\right]dt'\right\}, \end{aligned} \tag{34}$$

$$S(t) \ \equiv \ \frac{3}{2}\left(1 - \frac{4V_{22}}{9H^2}\right)^{\frac{1}{2}},$$

where t_k is the time when the k-mode leaves the Hubble radius: $k = a(t_k)H_I$. The above expression is valid when $|\dot{S}| \ll S^2$. In terms of τ, (34) can be expressed as

$$\begin{aligned} \delta\phi_2(k,\tau) \ &\cong \ \left(\frac{H_I^2}{2k^3}\right)^{\frac{1}{2}}\left(\frac{S(\tau_k)}{S(\tau)}\right)^{\frac{1}{2}} \\ &\times \ \exp\left\{\int_{\tau}^{\tau_k} S(\tau')d\tau' - \frac{3}{2}(\tau_k - \tau)\right\}, \end{aligned} \tag{35}$$

Table 1. Evolution of the background fields ϕ_j and generation of fluctuations in ϕ_2 as a function of τ.

Time	$\tau > \tau_c$	$\tau_c > \tau \gtrsim \tau_l$	$\tau_l > \tau$
ϕ_1	$< \phi_{1c}$	$> \phi_{1c}$	$> \phi_{1c}$
ϕ_2	$= \phi_{2m}$	$= \phi_{2m}$	$> \phi_{2m}$
ϕ_3	$= \phi_{3m}$	$= 0$	$= 0$
$\delta\phi_2$	suppressed	marginal	scale-invariant

$$S(\tau) = \begin{cases} \frac{3}{2}\left(1 - \frac{\tau_*}{\tau_c} + \frac{\tau_*}{\tau}\right)^{\frac{1}{2}}, & \tau \gtrsim \tau_l, \\ \frac{3}{2}\left(1 - \frac{2\lambda_2\phi_{2l}^2}{3H_I^2}\left[1 + \frac{2\lambda_2\phi_{2l}^2}{3H_I^2}(\tau_l - \tau)\right]^{-1}\right)^{\frac{1}{2}}, & 0 < \tau \lesssim \tau_l, \end{cases}$$

where $\tau_k \equiv \tau(t_k)$ and $\tau_* \equiv \frac{4\nu\epsilon}{3\lambda_1\lambda_3}$. The above equality is valid until the end of inflation at $t = t_f$ or $\tau = 0$.

Table 1 summarizes the evolution of the homogeneous background scalar fields ϕ_j as a function of τ, which is a decreasing function of the time, and spectrum of fluctuations generated in ϕ_2 on the comoving horizon scale at each epoch.

5. Evolution of the universe after inflation

To avoid further complexity let us assume the universe is rapidly and efficiently reheated at $t = t_f$. (See Kofman et al. (1994), Yoshimura (1995), Shtanov et al. (1995), and Boyanovsky et al. (1995) for recent discussion on efficient reheating.) Then the reheat temperature is given by

$$T_R \simeq 0.1\sqrt{H_I M_{Pl}}. \tag{36}$$

If there is no further significant entropy production, one can calculate the epoch $\tau(L)$ when the comoving length scale corresponding to L pc today left the Hubble horizon during inflation as

$$\tau(L) = 37 + \ln\left(\frac{L}{1\,\mathrm{pc}}\right) + \frac{1}{2}\ln\left(\frac{H_I}{10^{10}\,\mathrm{GeV}}\right). \tag{37}$$

Thus the comoving horizon scale at the PBH formation $t = 10^{-6}$sec, $L = 0.03$ pc, corresponds to $\tau \cong 34 \equiv \tau_m$ and the present horizon scale, $L \simeq 3000$Mpc, corresponds to $\tau \cong 59$.

On the other hand, $\phi_2(t)$ and $\delta\phi_2(\mathbf{k}, t)$ evolve according to

$$\ddot{\phi}_2 + 3H\dot{\phi}_2 + \lambda_2\phi_2^3 + m_2^2\phi_2 = 0, \tag{38}$$

$$\delta\ddot{\phi}_2 + 3H\delta\dot{\phi}_2 + (3\lambda_2\phi_2^2 + m_2^2)\delta\phi_2 \cong 0, \tag{39}$$

where the Hubble parameter is now time-dependent, $H = 1/2t$, and the second equation is valid for $k \ll aH$.

When H^2 becomes smaller than $\lambda_2\phi_2^2$ ($\gg m_2^2$), both ϕ_2 and $\delta\phi_2$ start rapid oscillations around the origin. Using (35), one can express the amplitude of the gauge-invariant comoving fractional density perturbation of ϕ_2 as

$$\Delta_2 = \frac{1}{\rho_2}\left(\dot{\phi}_2\delta\dot{\phi}_2 - \ddot{\phi}_2\delta\phi_2 - \dot{\phi}_2{}^2\Phi_A\right) \simeq 4\left.\frac{\delta\phi_2}{\phi_2}\right|_{\tau=0} \equiv 4\frac{\delta\phi_{2f}}{\phi_{2f}}, \tag{40}$$

at the beginning of the oscillations, where

$$\rho_2 \equiv \frac{1}{2}\dot{\phi}_2{}^2 + \frac{\lambda_2}{4}\phi_2^4 + \frac{1}{2}m_2^2\phi_2^2 \tag{41}$$

is the energy density of ϕ_2.

Using the virial theorem, one can easily show that ρ_2 decreases in proportion to $a^{-4}(t)$ as long as $\lambda_2\phi_2^2 \gtrsim m_2^2$. Thus the amplitude of ϕ_2 decreases as $a^{-1}(t)$. On the other hand, $\delta\phi_2$ has a rapidly oscillating mass term when $\lambda_2\phi_2^2 \gtrsim m_2^2$, which causes parametric amplification. We have numerically solved equations (38) and (39) with various initial conditions with $|\phi_2| \gg |\delta\phi_2|$. We have found that in all cases the amplitude of $\delta\phi_2$ remains constant as long as m_2 is negligible. Thus Δ_2 increases as $t^{1/2}$ in this regime and it reaches a value

$$\Delta_2 \cong 4\frac{\delta\phi_{2f}}{\phi_{2f}}\left(\frac{\lambda_2\phi_{2f}^2}{m_2^2}\right)^{\frac{1}{2}}, \tag{42}$$

while the ratio of ρ_2 to the total energy density, ρ_{tot}, which is now dominated by radiation, remains constant:

$$\frac{\rho_2}{\rho_{tot}} = 2\pi\left(\frac{\phi_{2f}}{M_{Pl}}\right)^2. \tag{43}$$

As $\lambda_2\phi_2^2$ becomes smaller than m_2^2, ϕ_2 and $\delta\phi_2$ come to satisfy the same equations of motion, (38) and (39), and Δ_2 saturates to the constant value (42). At the same time ρ_2 starts to decrease like $a^{-3}(t)$ less rapidly than radiation. Since Δ_2 makes a contribution

$$\Delta \cong \frac{\rho_2}{\rho_{tot}}\Delta_2 \tag{44}$$

to the total comoving density fluctuation, it increases as $a(t) \propto t^{1/2}$. At the beginning of this stage, we find $H^2 \cong m_2^4/\lambda_2\phi_{2f}^2$ and so

$$\Delta \cong 8\pi\frac{\delta\phi_{2f}}{\phi_{2f}}\left(\frac{\lambda_2\phi_{2f}^2}{m_2^2}\right)^{\frac{1}{2}}\left(\frac{\phi_{2f}}{M_{Pl}}\right)^2\left(\frac{2m_2^2 t}{\sqrt{\lambda_2}\phi_{2f}}\right)^{\frac{1}{2}} \tag{45}$$

at a later time t.

In order to relate this to the initial conditions required for PBH formation, we must estimate it at the time the k-mode re-enters the Hubble radius, t_k^*, defined by

$$k = 2\pi a(t_k^*)H(t_k^*) = \frac{\pi a_f}{(t_k^* t_f)^{\frac{1}{2}}}. \tag{46}$$

Since k can also be expressed as $k = 2\pi a_f e^{-\tau_k}H_I$, the amplitude of the comoving density fluctuation at $t = t_k^*$ is given by

$$\Delta(k, t_k^*) \cong 8\pi\frac{\delta\phi_{2f}}{\phi_{2f}}\left(\frac{\sqrt{\lambda_2}\phi_{2f}}{H_I}\right)^{\frac{1}{2}}\left(\frac{\phi_{2f}}{M_{Pl}}\right)^2 e^{\tau_k}. \tag{47}$$

6. Constraints on model parameters

The mass fraction of PBHs has been estimated in (1) as a function of the root-mean-square amplitude of density fluctuations in the uniform Hubble constant gauge (Carr 1975). Hence we should calculate the predicted amplitude in this gauge and this is a linear combination of Δ and the gauge-invariant velocity perturbation. However, in the present case, in which Δ grows as $a(t)$ in the radiation-dominant universe, one finds that the latter quantity vanishes and that the density fluctuation in the uniform Hubble constant gauge coincides with Δ (Kodama & Sasaki 1984). Thus we finally obtain the quantity to be compared with $\bar{\delta}(M_{\rm BH})$ in (1), namely, the root-mean-square amplitude of density fluctuations on the scale $r = 2\pi/k$ at horizon crossing, $\bar{\delta}(r)$, as

$$\begin{aligned}\bar{\delta}(r) &= \left[\frac{4\pi k^3}{(2\pi)^3}|\Delta(k, t_k^*)|^2\right]^{\frac{1}{2}} \\ &\cong 4\left(\frac{\sqrt{\lambda_2}H_I\phi_{2\rm f}^3}{M_{Pl}^4}\right)^{\frac{1}{2}} e^{\tau_{\rm k}} C_f(\tau_{\rm k}, \tau_{\rm c}, \tau_*), \qquad (48)\end{aligned}$$

with

$$C_f(\tau_{\rm k}, \tau_{\rm c}, \tau_*) \equiv \left(\frac{S(\tau_{\rm k})}{S(0)}\right)^{\frac{1}{2}} \exp\left\{\int_0^{\tau_{\rm k}} S(\tau')d\tau' - \frac{3}{2}\tau_{\rm k}\right\}, \qquad (49)$$

where we have used (35). We also find

$$\phi_{2\rm f}^2 = \sqrt{\frac{3\tau_*}{8}}\left(1 + \sqrt{\frac{2}{3\tau_*}}\right)^{-1}\left(1 + \sqrt{\frac{3\tau_*}{8}}\right)^{-1}\frac{3H_I^2}{2\lambda_2\tau_{\rm c}}, \qquad (50)$$

from (21) and (22).

The remaining task is to choose values of parameters so that $\bar{\delta}(r)$ has a peak on the comoving horizon scale at $t = 10^{-6}$sec, which we denote by $r_{\rm m}$, corresponding to $\tau_{\rm k} = \tau_{\rm m} \cong 34$, and an amplitude $\bar{\delta}(r_{\rm m}) \cong 0.05$. We thus require

$$\begin{aligned}\frac{d\ln\bar{\delta}(r)}{d\tau_{\rm k}} &= \frac{S'(\tau_{\rm k})}{2S(\tau_{\rm k})} + S(\tau_{\rm k}) - \frac{1}{2} \\ &= -\frac{\tau_*\tau_{\rm c}}{4\tau_{\rm k}\left[\tau_{\rm c}\tau_{\rm k} + \tau_*(\tau_{\rm c} - \tau_{\rm k})\right]} + \frac{3}{2}\left(1 - \frac{\tau_*}{\tau_{\rm c}} + \frac{\tau_*}{\tau_{\rm k}}\right)^{\frac{1}{2}} - \frac{1}{2} \qquad (51)\end{aligned}$$

to vanish at $\tau_{\rm k} = \tau_{\rm m}$, which gives us a relation between $\tau_{\rm c}$ and τ_*. Since $\tau_{\rm c}$ roughly corresponds to the comoving scale where scale-invariance of the primordial fluctuations Δ_2 is broken, the peak at $\bar{\delta}(r_{\rm m})$ becomes the sharper, as $\tau_{\rm c}$ approaches $\tau_{\rm m}$.

For example, if we take $\tau_{\rm c} = 30$ we find

$$\tau_* = \frac{4\nu\epsilon}{3\lambda_1\lambda_2} = 200, \qquad (52)$$

so that

$$C_f = 0.13 \quad \text{and} \quad \phi_{2\rm f}^2 = 0.045\frac{H_I^2}{\lambda_2}. \qquad (53)$$

In order to have $\bar{\delta}(r_{\rm m}) = 0.05$, we find

$$\frac{1}{\sqrt{\lambda_2}}\left(\frac{H_I}{M_{Pl}}\right)^2 = 1.7 \times 10^{-15}, \qquad (54)$$

which can easily be satisfied with some reasonable choices of λ_2 and H_I. However, it is not the final constraint. Since we are assuming that the universe is dominated by radiation at this time, we require

$$\frac{\rho_2}{\rho_{\rm tot}} = 2\pi\left(\frac{\phi_{2\rm f}}{M_{Pl}}\right)^2\left(\frac{m_2^2}{\sqrt{\lambda_2}\phi_{2\rm f}H_I}\right)^{\frac{1}{2}} e^{\tau_{\rm m}} \ll 1. \qquad (55)$$

Furthermore ϕ_2 should decay some time after $t = 10^{-6}$sec so as not to dominate the energy density of the universe which would hamper the primordial nucleosynthesis. Assuming that it decays only through gravitational interaction, its lifetime is given by

$$\tau_{\phi_2} \cong \frac{M_{Pl}^2}{m_2^3} = 10^{-5.5}\left(\frac{m_2}{10^{6.5}\text{GeV}}\right)^{-3}\text{sec}. \qquad (56)$$

We have now given all the necessary equalities and inequalities which the model should satisfy. Since there is a wide range of allowed values in the multi-dimensional parameter space, we do not work out the details of the constraints but simply give one example which satisfies all the requirements:

$$\begin{aligned}H_I &= 1.7 \times 10^{10}\text{GeV}, \\ m_2 &= 3.2 \times 10^6\text{GeV}, \\ \lambda_1 &= 1.3 \times 10^{-13}, \\ \lambda_2 &= 1.4 \times 10^{-6}, \\ \lambda_3 &= \nu = 6.7 \times 10^{-8}, \\ \epsilon &= 2.0 \times 10^{-11}.\end{aligned} \qquad (57)$$

In this case $\rho_2/\rho_{\rm tot} = 0.1$ at $t = 10^{-6}$sec and inequalities (10) and (11) are maximally satisfied.

Fig. 1 gives the form of

$$\beta(M) = \bar{\delta}(r)\exp\left(-\frac{1}{18\bar{\delta}(r)^2}\right) \qquad (58)$$

for different values of $\tau_{\rm c}$ as a function of the horizon mass when the scale r reenters the horizon. This gives the qualitative form of the mass spectrum of PBHs produced.

7. Conclusion

In this paper we have considered the possibility of producing a significant number of PBHs on a specific mass scale by generating an appropriate spectrum of density fluctuations in inflationary cosmology. We have found a model with the desired feature by making use of a simple polynomial potential (6) without introducing any break in the potential of the scalar fields. We have chosen values of the parameters so that these PBHs can account for the claimed MACHO observations. In order to set the order of magnitude for the mass and abundance of the black holes correctly, we have had to tune some combinations of the parameters, such as (52) and (54) with two digit accuracy. However, a wide range of parameter values would also achieve this. The precise values quoted in (57) are not of much significance, because the formula (1) for the fraction of PBHs is only approximate. In this respect we have restricted ourselves to an analytic

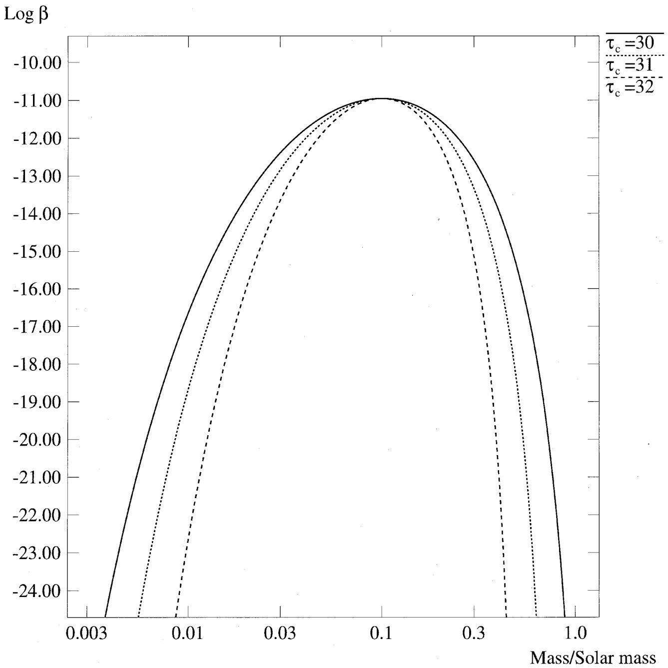

Fig. 1. Expected mass spectrum of primordial black holes with different values of τ_c.

treatment of the evolution of fluctuations, which is not exponentially accurate. When a more precise formula for PBH fraction is obtained, a full numerical analysis of fluctuations would be feasible. At present, however, an analytic treatment is more appropriate since this enables us to explain the dependence of the results on the physical parameters.

It is evident that our model can be applied to produce PBHs with a different mass and abundance by changing the parameters slightly. For example, we could produce black holes with mass $\sim 10^6 M_\odot$ which would act as a central engine for AGNs. These black holes are usually considered to have formed in the post-recombination universe (Loeb 1993), but they might have formed in the early universe at $t \sim 10$ sec corresponding to the onset of primordial nucleosynthesis. Note that this would not hamper successful nucleosynthesis because the root-mean-square amplitude of density fluctuation required for such black hole formation is still much smaller than unity.

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References

Albrecht, A. & Steinhardt, P.J., 1982, Phys. Rev. Lett. 48, 1220.
Alcock, C. et al., 1993, Nature 365, 621.
Alcock, C. et al., 1995, Phys. Rev. Lett. 74, 2867.
Aubourg, E. et al., 1993, Nature 365, 623.
Aubourg, E. et al., 1995, A& A 301, 1.
Bardeen, J.M., 1980, Phys. Rev. D22, 1882.
Bennet, C.L. et al., ApJ, 464, L1.
Bicknell, G.V. & Henriksen, R.N., 1979, ApJ 232, 670.
Boughn, S.P. & Uson, J.M., 1995, Phys. Rev. Lett. 74, 216.
Boyanovsky, D. et al., 1995, Preprint PITT-09-95.
Carr, B.J., 1975, ApJ 201, 1.
Carr, B.J., Bond, J.R. & Arnett, W.D., 1984, ApJ, 277.
Carr, B.J. & Lidsey, J.E., 1993, Phys. Rev. D48, 543.
Carr, B.J., Gilbert, J.H. & Lidsey, J.E., 1994, Phys. Rev. D50, 4853.
Doroshkevich, A.G., 1970, Afz 6, 581.
Guth, A.H. 1981, Phys. Rev. D23, 347.
Guth, A.H. & Pi, S-Y., 1982, Phys. Rev. Lett. 49, 1110.
Hawking, S.W., 1971, MNRAS 152, 75.
Hawking, S.W., 1982, Phys. Lett. B115, 295.
Hodges, H.M. & Blumenthal, G.R., 1990, Phys. Rev. D42, 3329.
Ivanov, P., Naselsky, P. & Novikov, I., 1994, Phys. Rev. D50, 7173.
Kodama, H. & Sasaki, M., 1984, Prog. Theor. Phys. Suppl. 78, 1.
Kofman, L.A. & Linde, A.D., 1987, Nucl. Phys. B282, 555.
Kofman, L., Linde, A.D. & Starobinsky, A.A., 1994, Phys. Rev. Lett. 73, 3195.
Linde, A.D., 1982, Phys. Lett. B108, 389.
Linde, A.D., 1985, Phys. Lett. B158, 375.
Linde, A.D., 1994, Phys. Lett. B327, 208.
Loeb, A., 1993, ApJ 403, 54.
Mukhanov, V.F., Feldman, H.A. & Brandenberger, R.H., 1992, Phys. Rep. 215, 203.
Novikov, I.D., Polnarev, A.G., Starobinsky, A.A., Zel'dovich, Ya.B., 1979, A&A 80, 104.
Olive, K.A., 1990, Phys. Rep. 190, 307.
Randall, L., Soljacic, M. & Guth, A.H., 1995 Preprint MIT-CTP-2501.
Richer, H.B. & Fahlman, G.G., 1992, Nature 358, 383.
Sato, K. 1981, MNRAS, 195, 467.
Shtanov, Y., Traschen, J., & Brandenberger, R., 1995, Phys. Rev. D51, 5438.
Starobinsky, A.A., 1982, Phys. Lett. B117, 175.
Vilenkin, A., 1994, Phys. Rev. Lett. 72, 3137.
Yoshimura, M., 1995, Prog. Theor. Phys. 94, 873.
Zel'dovich, Ya.B. & Novikov, I.D., 1967, SvA 10, 602.

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