

# A general and practical method for calculating cosmological distances<sup>★</sup>

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**Abstract.** The calculation of distances is of fundamental importance in extragalactic astronomy and cosmology. However, no practical implementation for the general case has previously been available. We derive a second-order differential equation for the angular size distance valid not only in all *homogeneous* Friedmann-Lemaître cosmological models, parametrised by  $\lambda_0$  and  $\Omega_0$ , but also in *inhomogeneous* ‘on-average’ Friedmann-Lemaître models, where the inhomogeneity is given by the (in the general case redshift-dependent) parameter  $\eta$ . Since most other cosmological distances can be obtained trivially from the angular size distance, and since the differential equation can be efficiently solved numerically, this offers for the first time a practical method for calculating distances in a large class of cosmological models. We also briefly discuss our numerical implementation, which is publicly available.

**Key words:** cosmology: theory – methods: numerical – cosmology: distance scale – gravitational lensing

the density parameter  $\Omega_0$  and the inhomogeneity parameter  $\eta$ .<sup>1</sup> Usually, smaller distances are determined by the traditional ‘distance ladder’ technique and larger distances are calculated from the redshift, assuming some cosmological model. Since the redshift is for most purposes exactly measurable, knowledge of or assumptions about two of the factors (a) Hubble constant, (b) other cosmological parameters and (c) ‘astronomical distance’ (i.e. ultimately tied in to the local distance scale) determines the third. In this paper we discuss distances given the Hubble constant  $H_0$ , the redshifts  $z_x$  and  $z_y$  and the cosmological parameters  $\lambda_0$ ,  $\Omega_0$  and  $\eta$ . Traditionally, a simple cosmological model is often assumed for ease of calculation, although the distances thus obtained, and results which depend on them, might be false if the assumed cosmological model does not appropriately describe our universe. A general method allows one to look at cosmological models whether or not they are easy-to-calculate special cases and offers the possibility of determining cosmological distances which are important for other astrophysical topics once the correct cosmological model is known.

We stress the fact that the inhomogeneity can be as important as the other cosmological parameters, both in the field of more traditional cosmology and in the case of gravitational lensing, where, e.g. in the case of the time delay between the different images of a multiply imaged source, the inhomogeneity cannot be neglected in a thorough analysis (Kayser & Refsdal 1983). For an example involving a more traditional cosmological test, Perlmutter et al. (1995) (see also Goobar & Perlmutter (1995)) discuss using supernovae with  $z \approx 0.25$ – $0.5$  to determine  $q_0$ ; for  $z$  near the top of this range or larger, the uncertainty due to

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## 1. Introduction

The determination of distances is one of the most important problems in extragalactic astronomy and cosmology. Distances between two objects X and Y depend on their redshifts  $z_x$  and  $z_y$ , the Hubble constant  $H_0$ , the cosmological constant  $\lambda_0$ ,

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<sup>1</sup> When discussing the distance between *two* objects, one can always make a coordinate transformation such that the contribution from the  $\theta$  and  $\phi$  terms in Eq. (1) vanish. Then one simply needs the redshifts and cosmological parameters in order to determine the distance between them. When discussing the distances between several objects, for example QSOs with  $\alpha$ ,  $\delta$  and  $z$  as coordinates, this is no longer possible. In many cases, however, suitable geometrical approximations can be made so that the most complicated part of the problem is essentially a determination of a distance between two objects. This point is further discussed in Sect. 5.

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<sup>★</sup> All three appendices are *only* available in electronic form at the CDS via anonymous ftp to

ftp://cdsarc.u-strasbg.fr/pub/A+A/ (130.79.128.5) or via http://cdsweb.u-strasbg.fr/Abstract.html

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our ignorance of  $\eta$  is comparable with the other uncertainties of the method.

The plan of this paper is as follows. In Sect. 2 the basics of Friedmann-Lemaître cosmology are briefly discussed; this also serves to define our terms, which is important since various conflicting notational schemes are in use. (For a more thorough discussion using a similar notation see, e.g., Feige (1992).) Sect. 3 defines the various distances used in cosmology. In Sect. 4 our new differential equation is derived. Similar efforts in the literature are briefly discussed. Sect. 5 briefly describes our numerical implementation and gives the details on how to obtain the source code for use as a ‘black box’ (which however can be opened) for use in cosmology and extragalactic astronomy. The symmetry properties of the angular size distance, analytic solutions and methods of calculating the volume element are addressed in three appendices.

## 2. Basic theory

Considering for the moment *homogeneous* Friedmann-Lemaître cosmological models, we can write the familiar Robertson-Walker line element:

$$ds^2 = c^2 dt^2 - R^2(t) \times \left( \frac{d\sigma^2}{(1 - k\sigma^2)} + \sigma^2 d\theta^2 + \sigma^2 \sin^2 \theta d\phi^2 \right), \quad (1)$$

where the symbols are defined as follows (with the corresponding units):

|          |                        |                 |
|----------|------------------------|-----------------|
| $s$      | 4-dimensional interval | [length]        |
| $c$      | speed of light         | [velocity]      |
| $t$      | time                   | [time]          |
| $R$      | scale factor           | [length]        |
| $\sigma$ | radial coordinate      | [dimensionless] |
| $k$      | curvature constant     | [dimensionless] |
| $\theta$ | angular coordinate     | [dimensionless] |
| $\phi$   | angular coordinate     | [dimensionless] |

The dynamics of the universe is given by the Friedmann equations

$$\dot{R}^2(t) = \frac{8\pi G\rho(t)R^2(t)}{3} + \frac{\Lambda R^2(t)}{3} - kc^2 \quad (2)$$

and

$$\frac{\ddot{R}(t)}{R(t)} = -\frac{4\pi G\rho(t)}{3} + \frac{\Lambda}{3}, \quad (3)$$

where dots denote derivatives with respect to  $t$ ,  $G$  is the gravitational constant,  $\rho(t)$  the matter density (this paper assumes negligible pressure),  $\Lambda$  the cosmological constant and the sign of  $k$  determines the curvature of the 3-dimensional space.

Introducing the usual parameters

$$\begin{aligned} H &= \frac{\dot{R}}{R} && \text{(Hubble parameter)} \\ \Omega &= \frac{8\pi G\rho}{3H^2} && \text{(density parameter)} \\ \lambda &= \frac{\Lambda}{3H^2} && \text{(normalised cosmological constant)} \end{aligned} \quad (4)$$

( $\Omega$  and  $\lambda$  are dimensionless and  $H$  has the dimension  $t^{-1}$ ) we can use Eq. (2) to calculate

$$kc^2 = R^2 H^2 (\Omega + \lambda - 1), \quad (5)$$

so that

$$k = \text{sign}(\Omega + \lambda - 1). \quad (6)$$

Since  $R > 0$  we can write

$$R = \frac{c}{H} \frac{1}{\sqrt{|\Omega + \lambda - 1|}}; \quad (7)$$

this is the radius of curvature of the 3-dimensional space at time  $t$ . For  $k = 0$  it is convenient to *define* the scale factor  $R$  to be  $c/H$ . In the following the index 0 will be used to denote the present value of a given quantity, fixed, as usual, at the time  $t_0$  of observation.<sup>2</sup> The explicit dependence on  $t$  will be dropped for brevity. Taking matter conservation into account and using the present-day values, we have

$$\rho R^3 = \rho_0 R_0^3 \quad (8)$$

and so from Eqs. (2), (4), (5) and (8) follows

$$\dot{R}^2 = H_0^2 R_0^2 \left( \frac{\Omega_0 R_0}{R} + \frac{\lambda_0 R^2}{R_0^2} - (\Omega_0 + \lambda_0 - 1) \right). \quad (9)$$

Since below we want to discuss distances as functions of the cosmological redshift  $z$ , by making use of the facts that

$$z = \frac{R_0}{R} - 1 \quad (10)$$

and that  $R_0$  is fixed, we can use Eq. (9) to get

$$dz = \frac{dz}{dR} \dot{R} dt = -H_0(1+z) \sqrt{Q(z)} dt, \quad (11)$$

where

$$Q(z) = \Omega_0(1+z)^3 - (\Omega_0 + \lambda_0 - 1)(1+z)^2 + \lambda_0. \quad (12)$$

**Note:** Throughout this paper, the  $\sqrt{\quad}$  sign should be taken to signify the positive solution, except that  $\text{sign} \sqrt{Q(z)} = \text{sign}(\dot{R})$  always.

## 3. Distance measures

### 3.1. Distances defined by measurement

In a static Euclidean space, one can define a variety of distances according to the method of measurement, which are all equivalent.

<sup>2</sup> Note that this paper is concerned with the calculation of distances from redshift. We are not concerned with a change in redshift with  $t_0$ .

### 3.1.1. Angular size distance

Let us consider at position  $y$  two light rays intersecting at  $x$  with angle  $\theta$ . If  $l$  is the distance between these light rays, it is meaningful to define the angular size distance  $D_{xy}$  as

$$D_{xy} = \frac{l}{\theta}, \quad (13)$$

since an object of projected length  $l$  at position  $y$  will subtend an angle  $\theta = l/D_{xy}$  (for small  $\theta$ ) at distance  $D_{xy}$ .

### 3.1.2. Proper motion distance

The proper motion distance is similar to the angular size distance, except that  $l$  is given by  $vt$ , where  $v$  is the tangential velocity of an object and  $t$  the time during which the proper motion is measured.

### 3.1.3. Parallax distance

Parallax distance is similar to the proper motion distance, except that the angle  $\pi$  is at  $y$  instead of  $x$ , so that we have

$$D_{xy}^{\pi} = \frac{l}{\pi}. \quad (14)$$

In the canonical case,  $l = 1$  AU.

### 3.1.4. Luminosity distance

Since the apparent luminosity  $L$  of an object at distance  $D$  is proportional to  $1/D^2$ , one can define the luminosity distance as

$$D^L = D_0^L \sqrt{\frac{L_0}{L}}, \quad (15)$$

where  $L_0$  is the luminosity at some fiducial distance  $D_0^L$ .

### 3.1.5. Proper distance

By proper distance  $D^P$  we mean the distance measured with a rigid ruler.

### 3.1.6. Distance by light travel time

Finally, from the time required for light to traverse a certain distance, one can define a distance  $D^c$  by

$$D^c = ct \quad (16)$$

where  $t$  is the so-called look-back time.

## 3.2. Cosmological distances

### 3.2.1. General considerations

In a static Euclidean space, which was used above when defining the distances through a measurement description, these distance measures are of course equivalent. In the general case in cosmology, where the 3-dimensional space need not be flat ( $k = 0$ ) but can be either positively ( $k = +1$ ) or negatively ( $k = -1$ ) curved, and where the 3-dimensional space is scaled by  $R(t)$ , not only do the distances defined above differ, but also (in the general case)  $D_{xy} \neq D_{yx}$ . The definitions are still applicable, but different definitions will result in different distances.

In reality, of course, the universe is neither perfectly homogeneous nor perfectly isotropic, as one assumes when deriving Eq. (1). However, as far as the usefulness of the Friedmann equations in determining the global dynamics is concerned, this appears to be a good approximation. (See, for example, Longair (1993) and references therein for an interesting discussion.) The approximation is certainly too crude when using the cosmological model to determine distances as a function of redshift, since the angles involved in such cases can have a scale comparable to that of the inhomogeneities. In this paper, we assume that these inhomogeneities can be sufficiently accurately described by the parameter  $\eta$ , which gives the fraction of homogeneously distributed matter. The rest ( $1 - \eta$ ) of the matter is distributed clumpily, where the scale of the clumpiness is by definition of the same order of magnitude as the angles involved.

For example, a halo of compact MACHO type objects around a galaxy in a distant cluster would be counted among the homogeneously distributed matter if one were concerned with the angular size distance to background galaxies further away, but would be considered clumped on scales such as those important when considering microlensing by the compact objects themselves. Since we don't know exactly how dark matter is distributed, different  $\eta$  values can be examined to get an idea as to how this uncertainty affects whatever it is one is interested in. If one has no selection effects, then, due to flux conservation, the 'average' distance cannot change (Weinberg 1976);  $\eta$  introduces an additional uncertainty when interpreting observations. It is generally not possible to estimate this scatter by comparing the cases  $\eta = 0$  and  $\eta = 1$ , since, depending on the cosmological parameters and the cosmological mass distribution, not all combinations are self-consistent. For instance, if one looks at scales where galaxies are compact objects, and the fraction of  $\Omega_0$  due to the galaxies is  $x$ , then  $\eta$  must be  $\leq (1 - x)$ .

We further assume that light rays from the object whose distance is to be determined propagate sufficiently far from all clumps. (See Schneider et al. (1992) – hereafter SEF – for a more thorough discussion of this point.) Compared to the perfectly homogeneous and isotropic case, the introduction of the  $\eta$  parameter will influence the angular size and luminosity distances (as well as the proper motion and parallax distances) since these depend on angles between light rays which are influenced by the amount of matter in the beam, but not the proper distance and only negligibly the light travel time. The last two distances are discussed briefly in Sect. 3.2.2 and in Appendix B3 and B6.

Since there is a simple relation between the angular size distance and the luminosity distance (Sect. 3.2.2) which also holds for the inhomogeneous case (see Appendix A), for the general case it suffices to discuss the angular size distance, which we do in Sect. 4.

### 3.2.2. Relationships between different distances

Without derivation<sup>3</sup> we now discuss some important distance measures, denoting the redshifts of the objects with the indices  $x$  and  $y$ . Due to symmetry considerations (see Appendix A)

$$D_{yx} = D_{xy} \left( \frac{1+z_y}{1+z_x} \right), \quad (17)$$

where the term in parentheses takes account of, by way of Eq. (10), the expansion of the universe. It is convenient, in keeping with the meaning of angular size distance, to think of the expansion of the universe changing the angle  $\theta$  in Eq. (13) and not  $l$ , if one identifies  $l$  as the (projected) size of an object. The angle is defined at the time when the light rays intersect the plane of the observer. Thus  $D_{xy}$  with the observer at  $x = 0$  defines what one normally thinks of as an angular size distance. On the other hand,  $D_{xy}$  and  $D_{yx}$  with  $x$  in general  $\neq 0$  can be important in, for example, gravitational lensing.<sup>4</sup>

Although the angle between the rays (at the source) at the time of reception of the light is important for the luminosity distance, this distance is *not* simply  $D_{yx}$ , since in the cosmological case the observed flux is obtained by multiplying the ‘non-redshifted flux’ by the factor  $(1+z_x)^2/(1+z_y)^2$ . One factor of  $(1+z_x)/(1+z_y)$  occurs because a given wavelength is increased by  $(1+z_y)/(1+z_x)$ , which reduces the flux correspondingly; an additional factor of  $(1+z_x)/(1+z_y)$  occurs because the arrival rate of photons is also decreased. Therefore, since  $D^L$  is inversely proportional to the square root of the (observed, ‘redshifted’) flux the luminosity distance is

$$D_{xy}^L = D_{yx} \left( \frac{1+z_y}{1+z_x} \right). \quad (18)$$

From this and Eq. (17) follows the relation

$$D_{xy}^L = D_{xy} \left( \frac{1+z_y}{1+z_x} \right)^2. \quad (19)$$

This means that the surface brightness of a ‘standard candle’ is  $\sim (1+z)^{-4}$ , a result independent of the cosmological model

<sup>3</sup> See, e.g., Feige (1992) Berry (1986) or Bondi (1961) for a more general discussion. What we present in the rest of this section is not new, but is important in order to clarify the notation. The results are obvious from the definitions introduced above.

<sup>4</sup> Although not useful in cosmology or extragalactic astronomy, for completeness we mention the fact that the proper motion distance is equivalent to  $D_{yx}$  and the parallax distance is equivalent to  $D_{yx}/\sqrt{1-k\sigma^2}$ .

parameters, including  $\eta$ .<sup>5</sup> (This result also holds for the inhomogeneous case, since Eq. (17) still holds (see Appendix A) and the additional factor due to the expansion of the universe (given by the term in parentheses in Eq. (18)) is of course present in the inhomogeneous case as well.)

Of course, this applies only to the *bolometric* luminosity. Observing in a finite band introduces two corrections. The so-called *K*-correction as it is usually defined today (see, e.g., Coleman et al. (1980) or, for an interesting and thorough discussion, Sandage (1995)) takes account of these, both of which come from the fact that the observed wavelength interval is redshifted compared to the corresponding interval on emission. This means that, first, for a flat spectrum, less radiation is observed, because the bandwidth at the observer is  $(1+z)$  times larger than at the source. Second, the spectrum need not be flat, in which case additional corrections based on the shape of the spectrum have to be included.<sup>6</sup> Thus,

$$m = M + 5 \log \left( \frac{D^L[\text{pc}]}{10 \text{ pc}} \right) + K \quad (20)$$

where  $m$  is the apparent magnitude,  $M$  the absolute magnitude,  $D^L$  is the luminosity distance and  $K$  is the *K*-correction as defined in Coleman et al. (1980). Perhaps more convenient is

$$m = M + 5 \log D^L + K + N \quad (21)$$

where  $N$  is a normalisation term:  $N = -5$  for  $D^L$  in units of 1 pc,  $N = 25$  for  $D^L$  in units of 1 Mpc and  $N = x - 5 \log h$  for  $D^L$  in units of the Hubble length<sup>7</sup>  $c/H_0$ , where

$$x = 5 \log \left( \frac{\text{Hubble length}}{1 \text{ pc}} \right) - 5 \approx 42.384$$

and  $h$  is the Hubble constant in units of 100 km/s/Mpc. In practice one has to add terms to correct for various sources of extinction and consider the fact that  $M$  is the absolute magnitude of the object when the light was emitted, which of course could be different from the present  $M$  of similar objects at negligible redshift.

The light travel time (or lookback time)  $t_{xy} = t_x - t_y$  between  $z_x$  and  $z_y$  (where  $t_x = t(z_x) > t_y = t(z_y)$ ) is given by the integration of the reciprocal of Eq. (11):

$$t_{xy} = \int_{z_y}^{z_x} \left( \frac{dz}{dt} \right)^{-1} dz = \frac{1}{H_0} \int_{z_x}^{z_y} \frac{dz}{(1+z)\sqrt{Q(z)}}, \quad (22)$$

<sup>5</sup> Thus, a ‘surface brightness test’ can in principle show that cosmological redshifts are due to the expansion of the universe and not to some other cause. See, e.g., Sect. 6 in Sandage (1995).

<sup>6</sup> Since the observed objects generally evolve with time, and redshifted objects are necessarily observed as they were when the radiation was emitted, some authors include an evolutionary term in the *K*-correction. Still other authors prefer to absorb one or more of these terms into the definition of the luminosity distance. Our luminosity distance is a bolometric distance based on the geometry and includes the unavoidable dimming due to the redshift. Our *K*-correction takes account of both effects of a finite bandwidth. Evolutionary effects are considered separately from distances.

<sup>7</sup> For example, as given by our numerical implementation; see Sect. 5

where the minus sign from Eq. (11) is equivalent to the swapped limits of integration on the right-hand side so that the integral gives  $t_x - t_y$  instead of  $t_y - t_x$ , making the light travel time increase (for  $\dot{R} > 0$ ) with  $z$ ; thus  $D_{xy}^c = ct_{xy}$ .

Since the proper distance would be the same as  $D^c$  were there no expansion, the former can be calculated by multiplying the integrand in Eq. (22) by  $c(1+z)$ . Thus

$$D_{xy}^P = \frac{c}{H_0} \int_{z_x}^{z_y} \frac{dz}{\sqrt{Q(z)}}. \quad (23)$$

This gives the proper distance at the present time. Since  $D^P$  scales linearly with the expansion of the universe, the proper distance at some other time can be obtained by dividing Eq. (23) with  $(1+z_i)$ , where  $z_i$  is the redshift at the corresponding time. For homogeneous ( $\eta = 1$ ) cosmological models,<sup>8</sup> the propagation of light rays is determined by the global geometry, so that there is a simple relation between  $D^P$  and  $D$  and, thus,  $D^L$ . This is discussed in Sect. B3. Although not ‘directly’ observable, the proper distance is nevertheless important in cosmological theory, since it is the basic distance of general relativity. Although not useful as a distance, the light travel time is of course important when considering evolutionary effects.

For inhomogeneous models, where this relation between global geometry and local light propagation does not exist, another approach must be used, which takes account of both the expansion of the universe as well as the local propagation of light, when calculating angle-defined distances such as the angular size distance.

#### 4. The general differential equation for the angular size distance

In a series of papers Zeldovich (1964), Dashevskii and Zel-dovich (1965) and Dashevskii and Slysh (1966) developed a general differential equation for the distance between two light rays on the boundary of a small light cone propagating far away from all clumps of matter in an inhomogeneous universe:

$$\ddot{l} = -4\pi G\eta\rho l + \frac{\dot{R}}{R} \dot{l} \quad (24)$$

where  $\eta$  and  $\rho$  are functions of the time  $t$  (not the lookback time of Eq. 22). The first term can be interpreted as Ricci focusing due to the matter inside the light cone, and the second term is due to the expansion of space during the light propagation. We now have to transform this *time* dependent differential equation

<sup>8</sup> This includes *empty* models ( $\Omega_0 = 0$ ); although  $\eta$  has no meaning here, the same arguments apply.

into a *redshift* dependent differential equation. From Eq. (11) we obtain<sup>9</sup>

$$dt = - \left( H_0(1+z)\sqrt{Q} \right)^{-1} dz, \quad (25)$$

and thus

$$\frac{dl}{dt} = -H_0(1+z)\sqrt{Q} \frac{dl}{dz} \quad (26)$$

and

$$\frac{d^2l}{dt^2} = H_0^2(1+z)\sqrt{Q} \frac{d}{dz} \left( (1+z)\sqrt{Q} \frac{dl}{dz} \right) \quad (27)$$

$$= H_0^2 \left( \left( (1+z)Q + (1+z)^2 \frac{1}{2} \frac{dQ}{dz} \right) \frac{dl}{dz} + (1+z)^2 Q \frac{d^2l}{dz^2} \right). \quad (28)$$

Furthermore, since  $R = R_0/(1+z)$  (Eq. (10)), we obtain, using Eq. (25),

$$\frac{dR}{dt} = -H_0(1+z)\sqrt{Q} \frac{dR}{dz}. \quad (29)$$

From the definition of  $\Omega$  (Eq. (4)) and matter conservation (Eq. (8)) we obtain

$$4\pi G\rho = \frac{3}{2} H_0^2 \Omega_0 (1+z)^3. \quad (30)$$

If we now insert Eqs. (26), (28), (29) and (30) into Eq. (24), sort the terms appropriately and cancel  $H_0^2$ , which appears in all terms, we obtain

$$Q l'' + \left( \frac{2Q}{1+z} + \frac{1}{2} Q' \right) l' + \frac{3}{2} \eta \Omega_0 (1+z) l = 0, \quad (31)$$

where a prime denotes a derivative with respect to redshift and from Eq. (12) follows

$$Q'(z) = 3\Omega_0(1+z)^2 - 2(\Omega_0 + \lambda_0 - 1)(1+z). \quad (32)$$

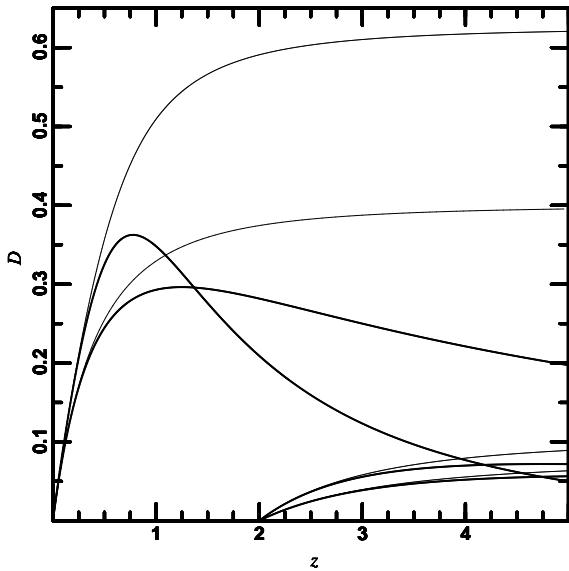
From the definition of the angular size distance (Eq. (13)) it is obvious that it follows the same differential equation as  $l$ :

$$Q D'' + \left( \frac{2Q}{1+z} + \frac{1}{2} Q' \right) D' + \frac{3}{2} \eta \Omega_0 (1+z) D = 0 \quad (33)$$

with special boundary conditions at the redshift  $z_x$  where the two considered light rays intersect. The first boundary condition is trivially

$$D = 0 \quad \text{for} \quad z = z_x, \quad (34)$$

<sup>9</sup> This transformation causes problems if the integration interval contains a point where  $\dot{R} = 0$  and thus  $\sqrt{Q}$  changes sign. In this case the integration interval  $(t_x, t_y)$  has to be transformed into *two* integration intervals, namely  $(z_x, z_{\max})$  and  $(z_{\max}, z_y)$ , where  $z_{\max}$  is the redshift at  $\dot{R} = 0$ , with the boundary conditions for the second integration interval chosen appropriately.



**Fig. 1.** The angular size distance from the observer ( $z_1 = 0$ ) and from  $z_1 = 2$  (lower right) as a function of the redshift  $z_2$  for different cosmological models. Thin curves are for  $\eta = 0$ , thick for  $\eta = 1$ . The upper curves near  $z = 0$  ( $z = 2$  at lower right) are for  $\lambda_0 = 2$ , the lower for  $\lambda_0 = 0$ .  $\Omega_0 = 1$  for all curves. The angular size distance  $D$  is given in units of  $c/H_0$

and the second boundary condition follows from the Euclidean approximation for small distances, i.e.

$$\left. \frac{dD}{dt} \right|_{z=z_x} = c \operatorname{sign}(t_x - t_y), \quad (35)$$

hence

$$D' = \frac{c}{H_0} \frac{1}{(1+z_x)\sqrt{Q(z_x)}} \operatorname{sign}(t_y - t_x) \quad \text{for } z = z_x, \quad (36)$$

where the sign has been chosen such that  $D$  is always  $> 0$  locally. We denote these special solutions of Eq. (33) with  $D_x(z)$ , and, following the definition (Eq. (13)), the angular size distance of an object at redshift  $z_y$  is then given as

$$D_{xy} = D_x(z_y) \quad . \quad (37)$$

Fig. 1 shows the influence of  $z$ ,  $\eta$  and  $\lambda$  on the angular size distance, calculated using Eq. (33) with our numerical implementation.

For completeness we note that after the original derivation by Kayser (1985) an equivalent equation was derived by Linder (1988) which, however, is difficult to implement due to the cumbersome notation.

Special mention must be made of the so-called bounce models, which expand from a finite  $R$  after having contracted from  $R = \infty$ . (See, e.g., Feige (1992).) A glance at Eq. (10) shows that in these cosmological models there must be *four* distances for an (ordered) pair of redshifts. If we denote the distances by  $D_{12}$ ,  $D_{14}$ ,  $D_{34}$  and  $D_{32}$ , where 1(2) und 3(4) refer to  $z_1(z_2)$

during the expanding (contracting) phase, then symmetry considerations dictate that  $D_{12} = D_{34}$  and  $D_{14} = D_{32}$  as long as the dependence of  $\eta$  on  $z$  is the same during both phases. In this case, there are *two* independent distances per (ordered) pair of redshifts. If this is not the case, the degeneracy is no longer present and there are *four* independent distances per (ordered) pair of redshifts.

## 5. Numerics and practical considerations

For the actual numerical integration of the differential equation, we have found the Bulirsch-Stoer method to be both faster and more exact than other methods such as Runge-Kutta. However, the conventional method of rational function extrapolation is rather unstable in this particular case; fortunately, using polynomial extrapolation solves the problem. Although programming the integration is rather straightforward in theory, in numerical practice considerable effort is needed to determine combinations of free parameters which work for all cases. We have tested the finished programme intensively and extensively, for example by comparing the results of calculations for  $\eta = 1$  (the value of  $\eta$  plays no special role in the integration of the differential equation) with those in Refsdal et al. (1967) or given by the method of elliptical integrals as outlined in Feige (1992) and have used it in Kayser (1995), Helbig (1996) and Helbig & Kayser (1996). For a general discussion of various methods of integrating second-order differential equations, see Press et al. (1992). Those interested in technical details can read the comments in our source code and the accompanying user's guide.

Since  $H_0$ , in contrast to the other cosmological parameters, merely inversely scales the angular size distance, our routine actually calculates the angular size distance in units of  $c/H_0$ . This *dimensionless* quantity must be multiplied by  $c/H_0$  (in whatever units are convenient) in order to obtain the actual distance. Other than reducing numerical overhead, this allows all distances to be calculated modulo  $c/H_0$ , which is convenient for expressing quantities in an  $H_0$ -independent manner. In practice,  $H_0$  cancels out of many calculations anyway.

Apart from auxiliary routines which the user does not have to be concerned with, our implementation consists of four FORTRAN77 subroutines. The first, INICOS, calculates  $z$ -independent quantities used by the other routines, some of which are returned to the calling programme. ANGSIK calculates the angular size distance. Normally,  $\eta$  is used as a  $z$ -independent cosmological parameter, on an equal footing with  $\lambda_0$  and  $\Omega_0$ . If desired, however, the user can let INICOS know that a variable (that is,  $z$ -dependent)  $\eta$  is to be used; this is given by the function VARETA. We supply an example; the user can modify this to suit her needs. In particular, many different dependencies of  $\eta$  on  $z$  can be included, and a decision made in the calling programme about which one to use. This feature is also included in our example. ANGSIK returns only the distance  $D_{12}$ ; if one is interested in the other distances in the bounce models, our subroutine BNGSIK returns all of these (though internally calculating only the independent distances, of course, depending on the dependence of  $\eta$  on  $z$ ).

Due to the fact that not everyone has a Fortran90 compiler at his disposal, we have coded the routines in FORTRAN77. Only standard FORTRAN77 features are used, and thus the routines should be able to be used on all platforms which support FORTRAN77. Since standard FORTRAN77 is a subset of Fortran90, the routines can be used without change in Fortran90 as well.

With the exception of  $D^c$ , all distance measures can be easily transformed into one another. Thus, it suffices to calculate the angular size distance for a given case.<sup>10</sup>

When discussing the distance between two objects other than the observer, rather than between the observer and one object, in many cases one of two simplifying assumptions can be made:

$D(\Delta z) \ll D(\beta)$  In this case, the *proper distance*  $D^P$  at the time of emission between the two objects is  $\beta D_{0x} \approx \beta D_{0y}$ , where  $\beta \ll 1$  is the angle in radians between the two objects on the sky.

$D(\beta) \ll D(\Delta z)$  In this case, the *angular size distance* between the two objects is  $D_{xy}$ .

$D(\Delta z)$  ( $D(\beta)$ ) refers to the distance due to  $\Delta z$  ( $\beta$ ) when setting  $\beta$  ( $\Delta z$ ) equal to zero. In the first case, where the two objects are practically at the same redshift, one uses the angular size distance to this redshift to transform the observed difference in angular position on the sky into the *proper distance* between the two objects at the time of emission. This follows directly from the definition of the angular size distance. Since the distance between the objects is much less than the distance from the observer to the objects, the differently defined distances between the objects are for practical purposes degenerate. A practical example of this case would be the distance between individual galaxies in a galaxy cluster at large redshift. Naturally, one should use *one* redshift, say, of the cluster centre; the individual redshifts will in most cases be overlaid with the doppler redshift due to the velocity dispersion of the cluster, so the difference in *cosmological* redshifts is negligible. (Of course, the *present* distance would be a factor of  $(1+z)$  larger, due to the expansion of the universe, were the objects comoving and not, as in a galaxy cluster, bound.) In the second case, which is typical of gravitational lensing, the angles on the sky between, for example, source and lens, are small enough to be neglected, so that the angular size distance between the objects is determined by the difference in redshift. If neither of these assumptions can be made, any sort of distance between the two objects is probably of no practical interest. (Of course, there is the trivial case where the redshifts are all  $\ll 1$  in which case one can simply use  $\alpha$ ,  $\delta$  and  $cz/H_0$  as normal spherical coordinates.)

<sup>10</sup> The proper distance, which is  $\eta$ -independent, can be calculated from the angular size distance assuming  $\eta = 1$ , by making use of the simple relation between proper distance and angular size distance in this case. The result holds of course for all values of  $\eta$ .

## 6. Summary

After discussing cosmological distances with an emphasis on practical distance measures for general use in cosmology and extragalactic astronomy, we have obtained a new differential equation, which gives the angular size distance for a class of ‘on average’ Friedmann-Lemaître cosmological models, that is, models described not only by  $\lambda_0$  and  $\Omega_0$  but also by  $\eta(z)$ , which describes the clumpiness of the distribution of matter. We have also developed a practical numerical method of solving this equation, which we have made publicly available. Since the equation is valid for *all* cases, this offers for the first time an efficient means of calculating distances in a large class of cosmological models.

The numerical implementation (in FORTRAN77), user’s guide and a copy of the latest version of this paper (including appendices) can be obtained from either of the following URLs:

<http://www.hs.uni-hamburg.de/english/persons/helbig/Research/Publications/Info/angsiz.html>

<ftp://ftp.uni-hamburg.de/pub/unihh/astro/angsiz.tar.gz>

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