Astron. Astrophys. 319, 225–234 (1997)

ASTRONOMY
AND
ASTROPHYSICS

Accretion disk boundary layers in classical T Tauri stars

M. v. Rekowski and H.-E. Fröhlich

Astrophysikalisches Institut Potsdam, An der Sternwarte 16, D-14482 Potsdam, Germany

Received 3 April 1996 / Accepted 29 July 1996

Abstract. The dynamical and thermal structure of an opaque boundary layer, connecting a classical T Tauri star (CTTS: $M_* = 1\,M_\odot$, $R_* = 4.3\,R_\odot$, $T_{\rm eff} = 5000\,{\rm K}$) with its accretion disk ($\dot{M} = 10^{-7}\,M_\odot$/yr), has been computed, including particularly the effect of a turbulence-driven global heat transport. The inner boundary has been shifted deeper and deeper into the star's envelope in order to determine how this affects the amount of angular momentum transfer. If the optical depth in the star's envelope, where the disk meets the star, exceeds in our model example $\gtrsim 10$, then there is an energy flux from the star into the boundary layer. For even larger ($\gtrsim 30$) optical depths this outward flux approaches the star's radiation flux and the boundary layer becomes a "hot" one.

The angular momentum parameter $\mathscr{C}$ alone does not determine the solution uniquely. For a given $\mathscr{C}$ there are, sometimes, two solutions.

An argument is given why $\mathscr{C} \approx 1$.

Key words: accretion, accretion disks – hydrodynamics – stars: pre-main sequence – stars: rotation

1. Motivation

A slowly rotating stellar object, gaining matter via a geometrical thin accretion disk, is – as a rule – accumulating angular momentum, too. Recent interest in this area stems from the realization that, if most stars form via protostellar disks, then rather short rotational periods near to break up are expected for newly born non-magnetic stars (Popham & Narayan 1991).

In order to write down the spin history of such an accreting star, one has to consider the exchange of angular momentum between the rotating disk and the rotating star. In that context the question arises, what determines ultimately the total flux of angular momentum as conveniently parametrized by the so called angular momentum parameter $\mathscr{C}$, i.e. the second integration constant in the theory of stationary accretion disks (cf. Pringle 1981).

Send offprint requests to: M. v. Rekowski

The amount of angular momentum transport is closely related with the width of the boundary layer (BL). If it is infinitely small in radial direction, as assumed in the standard theory of thin disk accretion onto a star (Lynden-Bell & Pringle 1974; Pringle 1981), and if the disk would rotate everywhere else with Keplerian speed, then $\mathscr{C}$ would be exactly unity.

In order to tackle — in general terms — the problem of a radial extent of the BL, Duschl & Tscharnuter (1991) have introduced a parameter ζ (our $\mathscr{C}$). The authors claim that this ζ measures the characteristic extent of the BL as well as the layer's characteristic temperature. In principle, ζ can be constrained analysing e.g. T Tauri spectra (Bertout et al. 1993).

A drawback of this method is that it neglects any energy flow from the disk into the star or vice versa.

The study of the *detailed* structure of BLs dates back to the pioneering work of Regev (1983). For recent reviews on this subject, with further references therein, see e.g. Narayan & Popham (1994) and Regev & Bertout (1995). In the case of a polytropic equation of state, i.e. if one avoids the thermal problem, even analytical solutions of the BL problem are possible for slowly rotating stars as well as for fast rotators (e.g. Paczyński 1991; Bisnovatyi-Kogan 1993, 1994).

In this paper the detailed accretion disk BL structure for a CTTS has been obtained numerically with the help of a relaxation method. We depart from related computations (e.g. Popham et al. 1993; Lioure & Le Contel 1994) by insisting on that it is the structure of the transition zone that must provide $\mathscr{C}$ self-consistently (cf. Paczyński 1991; Popham & Narayan 1991). *It is shown that additionally to the temperature at the inner boundary the energy flux condition can be fulfilled too, provided the optical depth in the star's envelope, where this (artificial) boundary is located, is large enough.*

Further improvements, as compared with related computations (e.g. Popham et al. 1993; Lioure & Le Contel 1994; Godon et al. 1995; Godon 1996), concern the usage of a more realistic vertical structure when deriving the vertically integrated equations and the admission of a radial flow of energy due to the turbulence. The main assumptions are:

1. The BL is geometrically thin and optically thick, i.e. the diffusion approximation holds.

2. The viscosity prescription is the usual α-prescription of Shakura & Sunyaev (1973), but scaled down by $\Omega/\Omega_{\rm K}$, i.e. the ratio of the actual angular velocity to its Keplerian value.
3. At the inner boundary temperature and pressure values have been assigned, characteristic for the unperturbed stellar envelope at a given optical depth.
4. The vertical structure, relying on the assumption of hydrostatical and radiative equilibrium, is obtained using a power-law representation for the opacity and assuming that the dissipative contribution to the vertical radiative flux at height z is proportional to the pressure there. It is shown, by altering the heat source distribution, that the outcome is merely dependent on the last assumption.

Some of these assumptions (e.g. geometrical and optical thickness) can be proved only a posteriori, others (e.g. the neglect of any z-dependence of the matter inflow) can be checked only by two-dimensional hydrodynamical computations (e.g. Kley & Hensler 1987; Kley 1989; Kley & Lin 1992; Hujeirat 1995; Kley & Lin 1996).

In a recent paper Narayan and Yi (1995), comparing their results relying on a height-integrated system of equations with those from a two-dimensional formulation, find an "excellent quantitative agreement between the two approaches, suggesting that the height-integration approximation may be much better than previously thought".

The reliability of our numerical model is limited by the necessity of appealing to the vertical integration procedure (cf. Appendix A). Whether it works well even in the stellar envelope can not be justified within the framework of our 1D-approach.

We do not consider the possibility of supersonic infall. (Arguments against the case of supersonic radial motions are given by Glatzel (1992) and Narayan & Popham (1994).) If centrifugal support is lost, the disk is balanced predominantly by a strong radial pressure gradient. For the present, magnetic effects are ignored. Their influence will be considered in a forthcoming paper.

The rest of the paper is organized as follows: We set out the basic equations in Sect. 2. The next sections (3 and 4) deal with the viscosity prescription and the boundary conditions. The numerical method and the normalizations used are described in Sect. 5. The final Sect. 6 presents the results and the discussion. That section also comments on the question, why the angular momentum flow parameter $\mathscr{C}$ is so often very near to unity. In the Appendix A the vertical structure equations are solved, in order to obtain some constants of order unity relating z-integrated quantities, like surface density and integrated pressure, with the mid-plane temperature.

2. Basic equations

2.1. Hydrodynamics

In cylindrical coordinates (R, z), assuming axisymmetry, the conservation law of mass reads

$$\frac{1}{R}\frac{\partial}{\partial R}(R\rho u_R) + \frac{\partial}{\partial z}(\rho u_z) = 0\,. \tag{1}$$

After an integration over z one obtains (with $\Sigma = \int_{-\infty}^{\infty} \rho\, dz$)

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}\,R}\left(R\Sigma\bar{u}_R\right) = 0 \quad \text{or} \quad 2\pi R\Sigma(-\bar{u}_R) = \dot{M}\,, \tag{2}$$

where $\dot{M} > 0$ denotes the constant inflow of matter, the accretion rate, and $\bar{u}_R < 0$ the z-averaged radial flow velocity. The radial part of the momentum equation provides

$$\frac{\partial p}{\partial R} + \rho\bar{u}_R\frac{\partial \bar{u}_R}{\partial R} - \rho R\Omega^2 = -GM_*\frac{\rho}{R^2}\,. \tag{3}$$

Here M_* is the central mass and p the gas pressure. (A viscous term in radial direction has been omitted.) If all terms can be neglected except centrifugal force and central gravitation, the basic Keplerian law, $\Omega_{\rm K} = \sqrt{GM_*/R^3}$, results. Integration over z yields

$$\frac{\mathrm{d}\,\Pi}{\mathrm{d}\,R} + \Sigma\bar{u}_R\frac{\mathrm{d}\,\bar{u}_R}{\mathrm{d}\,R} = R(\Omega^2 - {\Omega_{\rm K}}^2)\Sigma\,, \tag{4}$$

with the z-integrated pressure $\Pi = \int_{-\infty}^{\infty} p\, dz$.

The discrepancy between Ω and $\Omega_{\rm K}$ can be balanced either by a radial pressure gradient or by the inertial term. The conservation law of angular momentum is the ϕ-component of the Reynolds equation,

$$\mathrm{div}\,\boldsymbol{t} = 0\,, \tag{5}$$

with

$$\boldsymbol{t} = \rho R^2\Omega\bar{\boldsymbol{u}} - \nu_{\rm T}\,\rho R^2\frac{\mathrm{d}\,\Omega}{\mathrm{d}\,R}\,. \tag{6}$$

The net (inward) flow of angular momentum through the disk is given by the total torque

$$C\dot{M} = -2\pi R\int_{-\infty}^{\infty} t_R\, dz\,. \tag{7}$$

What determines C is addressed in this paper.

After the z-integration (and elimination of the radial inflow velocity through Eq. (2))

$$\frac{\mathrm{d}}{\mathrm{d}\,R}\left(R^3\left(\Omega\frac{\dot{M}}{2\pi R} + \langle\nu_{\rm T}\Sigma\rangle\frac{\mathrm{d}\,\Omega}{\mathrm{d}\,R}\right)\right) = 0 \tag{8}$$

results. The term $\langle\nu_{\rm T}\Sigma\rangle$ denotes the z-integrated dynamical eddy viscosity.

2.2. Thermodynamics

The general heat transport equation is

$$\mathrm{div}(\rho\, U\boldsymbol{u} + \boldsymbol{F}) + p\ \mathrm{div}\,\boldsymbol{u} = \epsilon\,, \tag{9}$$

with the internal energy U for an ideal gas ($U = \frac{\mathscr{R}}{\mu}\frac{T}{\gamma-1}$, γ denote the ratio of specific heats C_p/C_v, μ the mean molecular weight), the heat flux $\boldsymbol{F}$, and the source ϵ of the frictional heat. $\boldsymbol{F}$ consists of the radiative part $\boldsymbol{F}^{\rm D}$ and the eddy heat flux $\boldsymbol{F}^{\rm T}$ (S denote the entropy):

$$\boldsymbol{F}^{\rm D} = -\rho\, C_p \chi_{\rm D} \nabla T \tag{10}$$

$$\begin{aligned}\boldsymbol{F}^{\rm T} &= -\rho\, \chi_{\rm T}\, T\, \nabla\, S \\ &= -\rho\, C_p \chi_{\rm T} \left(\nabla\, T - \frac{\nabla\, p}{\rho\, C_p} \right) .\end{aligned} \tag{11}$$

Because thermal conductivity can be neglected, the coefficient for conduction is that for radiative transport: $\chi_{\rm D} = 16\sigma \cdot T^3/(3\kappa\rho^2 C_p)$. The symbols σ and κ denote the Stefan-Boltzmann constant and the opacity, respectively. The coefficient $\chi_{\rm T}$ is related to the turbulent viscosity $\nu_{\rm T}$ and can be parametrized by the Prandtl number: $\chi_{\rm T} = \nu_{\rm T}/{\rm Pr}$.
For the source term the usual formulation is adopted,

$$\epsilon = \rho\, \nu_{\rm T} \left(R \frac{{\rm d}\,\Omega}{{\rm d}\,R} \right)^2 , \tag{12}$$

i.e. the mean flow mainly consist on a global differential rotation. The kinematic viscosity $\nu_{\rm T}$ is of turbulent origin. (For further details and limitations of this approach cf. e.g. Rüdiger (1989).) Integrating Eq. (9) over z and observing Eq. (12) yields the basic energy equation

$$\begin{aligned}&\bar{u}_R \frac{{\rm d}\,\Pi}{{\rm d}\,R} + \gamma \frac{\Pi}{R} \frac{{\rm d}\,(R \bar{u}_R)}{{\rm d}\,R} \\ &\quad = (\gamma - 1) \left(\langle \nu_{\rm T} \Sigma \rangle \left(R \frac{{\rm d}\,\Omega}{{\rm d}\,R} \right)^2 - 2F_{\rm surf} - \mathscr{F}^{\rm T} - \mathscr{F}^{\rm D} \right),\end{aligned} \tag{13}$$

with the radial radiative transport

$$\mathscr{F}^{\rm D} = \frac{1}{R} \frac{\rm d}{{\rm d}\,R} \left(R \int_{-\infty}^{\infty} F_R^{\rm D} dz \right) \tag{14}$$

and the radial turbulent transport

$$\mathscr{F}^{\rm T} = \frac{1}{R} \frac{\rm d}{{\rm d}\,R} \left(R \int_{-\infty}^{\infty} F_R^{\rm T} dz \right). \tag{15}$$

It is as usual $F_{\rm surf} = \sigma T_{\rm eff}^4$. (16)

Here is

$$\begin{aligned}\mathscr{F}^{\rm T} = &\frac{C_p \alpha_{\rm SS}}{\rm Pr} \frac{1}{R} \frac{\rm d}{{\rm d}\,R} \left(R \frac{\Omega}{\Omega_{\rm K}^2} \left[\left(2 - \frac{1}{\gamma} \right) c_1 \frac{{\rm d}\,\Pi}{{\rm d}\,R} T_{\rm c} \right.\right. \\ &\left.\left. - c_2 \frac{\mathscr{R}}{\mu} \frac{\rm d}{{\rm d}\,R} (T_{\rm c}^2 \Sigma) \right] \right) .\end{aligned} \tag{17}$$

The temperature $T_{\rm c}$ is the mid-plane temperature. The constants c_1 and c_2 are of order unity (cf. Appendix A). We have used the viscosity law of Sect. 3 and the approximation

$$\frac{\partial p}{\partial R} = \rho\, A\, , \tag{18}$$

with A being the acceleration (cf. Eq. (3))

$$A = -\frac{GM_*}{R^2} + R\Omega^2 - \bar{u}_R \frac{\partial \bar{u}_R}{\partial R} . \tag{19}$$

Because any meridional circulation is neglected (i.e. $\partial \bar{u}_R / \partial z = 0$), A does not depend on z.

3. Viscosity prescription

Detailed modelling of a BL's structure requires an ad hoc viscosity prescription, which is an obvious drawback of this kind of approach. It is known from the pioneering work of Lynden-Bell & Pringle (1974) that for a BL to be thin either the viscosity must be small, i.e. strongly reduced as compared with the usual Shakura-Sunyaev prescription (1973), or the absolute value of the radial velocity must be very high.

We use an α-prescription for the turbulent viscosity:

$$\nu_{\rm T} = \alpha_{\rm SS}\, H^2 \Omega\, . \tag{20}$$

The disk's semi-thickness be defined by $H = C_{\rm s}/\Omega_{\rm K}$, where $C_{\rm s}$ is the isothermal sound velocity at mid-plane.

Eq. (20) is a straightforward extension of the Shakura-Sunyaev prescription in that the correlation time for the turbulence is assumed here to scale with the rotational period. The ansatz (20) leads by itself to a viscosity which drops considerably close to the stellar surface. Neither an interpolating formula nor a criterion which decides on the viscosity prescription is needed. As a matter of fact, the viscosity is prescribed by two parameters: $\alpha_{\rm SS}$ and the stellar spin Ω_*. (To put it simply, for a slow rotator, i.e. $\Omega_* \ll \Omega_{\rm K}(R_*)$, the actual stellar spin Ω_* is quite unimportant for the radial impulse balance (4), but — according to Eq. (20) — it determines the value of the kinematic viscosity near the star.)

In the outer regions the ansatz (20) converges to the standard prescription of Shakura & Sunyaev. For the calculations $\alpha_{\rm SS} = 0.01$ and $\Omega_* = 0.1 \cdot \Omega_{\rm K}$ are presumed. The latter value is compatible with the observed photometric variability (Bouvier et al. 1993), the former one follows from estimated viscous time-scales as well as disk masses.

By chance the choice $\Omega_* = 0.1 \cdot \Omega_{\rm K}$ makes our results comparable with those using a viscosity prescription relying on the radial pressure scale length and the sound velocity as the essential factors (for instance Papaloizou & Stanley 1986; Lioure & Le Contel 1994). Considering the radial pressure scale length as the characteristic length, one has to scale down the usual Shakura-Sunyaev viscosity by a factor H/R at the inner boundary, which is just of order 0.1 .

Nevertheless, to check whether the results depend strongly on that particular Ω_* a computation with Ω_* ten times lower has been run, i.e. $\Omega_* = 0.01 \cdot \Omega_{\rm K}$. As expected neither the dynamical structure nor the thermal one of the BL is changed noticeably, simply because our standard Ω_* is already low. But, because the advected angular momentum flow is strongly reduced, the torque at the star is now much higher.

Recently, using scaling arguments, even more sophisticated viscosity prescriptions for non-Keplerian disks have been developed by Godon (1995).

4. Boundary conditions

The above system of ordinary differential equations ((4), (8), (13)) contains five differential equations of first order. Of course, in the stationary case, Eq. (8) could be integrated at once, receiving an integration constant C, which represents the specific total flux of angular momentum (cf. Eq. (7)):

$$C = \frac{2\pi \langle \nu_\mathrm{T} \Sigma \rangle R^3}{\dot{M}} \frac{\mathrm{d}\,\Omega}{\mathrm{d}\,R} + R^2 \Omega \,. \tag{21}$$

However, for the numerical calculations we do not have integrated Eq. (8), but have converted it into two equations of first order. Note that our non-dimensional $\mathscr{C} \equiv C/(\Omega_{\mathrm{K}*} R_*^2)$ is an eigenvalue which is to be determined by boundary conditions. The parameter $\mathscr{C}$ will then be determined a posteriori from Eq. (21).

At issue is the appropriate inner boundary condition (cf. Godon et al. 1995). If one chooses, on the one hand, the temperature, as for instance Lioure & Le Contel (1994), one gets a "cool" BL. On the other hand, if one imposes the star's energy flux on that boundary, as for instance Narayan & Popham (1993), a "hot" BL results. Here we have tried to mediate between these two points of view, in that the inner boundary has been shifted to larger and larger optical depths until both requirements are (at least approximately) fulfilled: the temperature as well as its gradient. Hence, the inner boundary of the computational domain be within the stellar envelope, with the location being specified by the optical depth τ_* there. The outer boundary is taken to be 1000 stellar radii.

We specify the following five boundary conditions: At the inner edge, $R = R_*$, the angular velocity of the disk should match the angular velocity of the star, $\Omega(R_*) = \Omega_*$ (i.e. no slip!). Furthermore we prescribe the mid-plane temperature $T_\mathrm{c} = T_*(\tau_*)$, adopting Eddington's (1926) approximation for a (flat) grey atmosphere with radiation equilibrium:

$$T_* \approx T_{\mathrm{eff}*} [3(\tau_* + 17/24)/4]^{1/4} \,. \tag{22}$$

In thermodynamical equilibrium the gas will take up this radiation temperature approximately.

To fix a third boundary condition at the inner edge, the disk's mid-plane gas pressure is fitted to the star's gas pressure in that depth, with the latter following from integrating the equation of the hydrostatical equilibrium for a given gravity $g_* = GM_*/R_*^2$, observing the temperature law (22):

$$p_*^{n+1} = \frac{4(n+1) T_{\mathrm{eff}*}^{n-q} g_*}{\kappa_0 (n-q+4)} \left(\frac{\mathscr{R}}{\mu}\right)^n \left(\frac{4}{3}\right)^{(q-n)/4} f(\tau_*) \,, \tag{23}$$

with the abbreviation
$f(\tau_*) = \left[(\tau_* + 17/24)^{(n-q+4)/4} - (17/24)^{(n-q+4)/4}\right]$. Kramers' opacity law ($\kappa_0 = 6.6\,10^{22}$ cgs, $n = 1$ and $q = -3.5$) is taken. This envelope proves stable against convection.

One should be aware that taking Eddington's temperature stratification makes sense if the flux at the inner boundary is roughly the unperturbed stellar flux $\sigma T_{\mathrm{eff}*}^4$. How it will be shown later, for too small a τ_* the numerical computations provide, strictly speaking, a "false" temperature gradient there (e.g. an inward flux of energy). In that cases our approach lacks, of course, physically justification, but can be regarded nevertheless as a rule how to get boundary values. (Unfortunately, the relaxation code fails if one specifies at the inner boundary additionally to the temperature its gradient.)

Because the inner boundary of the computational domain is artificial (as well as the outer one), the resulting angular momentum flow should be independent of the exact location within the stellar envelope, provided it is well inside the star proper. We therefore have tried to go as deep as necessary into the star's envelope, but without violating the thin disk approximation. (In principle, the star and its disk should be considered in toto as for instance by Paczyński (1991).)

At the outer edge of the computational domain we fix the logarithmic derivations of the angular velocity, which is assumed to be Keplerian, and of the mid-plane temperature which is dependent on the used opacity law (Kramers' law).

The use of the outer boundary conditions is of technical nature. At least one has to be defined using a relaxation method (see Sect. 5). The second is, as already mentioned, necessary because otherwise one doesn't reach convergence.

5. Numerical method and normalizations

In order to solve the system of differential equations numerically, we use a relaxation method because this system has proven to be very stiff. Before doing this, it is necessary to write the equations in a non-dimensional form. This has been done in nearly the same manner as by Lioure & Le Contel (1994), i.e. radii are in units of the star's radius, $r = R/R_*$, the angular velocity in units of the Keplerian angular velocity at R_*, $\omega = \Omega(R)/\Omega_{\mathrm{K}*}$, and the z-integrated dynamical viscosity $\langle \nu_\mathrm{T} \Sigma \rangle$ in units of $\dot{M}/(2\pi)$. Departing from Lioure & Le Contel, the surface density unit and the velocity unit $\dot{M}\Omega_{\mathrm{K}*}/[2\pi\alpha_{\mathrm{SS}} c_1 (\mathscr{R}/\mu) T_\mathrm{unit}]$ and $\sqrt{c_1 (\mathscr{R}/\mu) T_\mathrm{unit}}$, respectively, have been chosen. Fixing the temperature unit by

$$T_\mathrm{unit} = \left[\frac{3\kappa_0 c_6^{n+1} \left(\mu/\mathscr{R}\right)^{\frac{3}{2}n+1}}{2^{2n+7} \pi^{n+2} c_1^{n+1} \sigma} \frac{\dot{M}^{n+2} \Omega_{\mathrm{K}*}^{2n+3}}{\alpha_{\mathrm{SS}}^{n+1}} \right]^{\frac{2}{10-2q+3n}} ,$$

one gets the following system of five first-order equations: (The second equation can be decomposed into two equations of first order.)

$$\begin{aligned}
\frac{\mathrm{d}\,\Sigma}{\mathrm{d}\,r} &= \left[\mathscr{M}^2 \Sigma (r\omega^2 - \tfrac{1}{r^2}) + \frac{\alpha_{\mathrm{SS}}^2}{\mathscr{M}^2 r^3 \Sigma} - \Sigma \frac{\mathrm{d}\,t}{\mathrm{d}\,r} \right] \\
&\quad \Big/ \left[t - \frac{\alpha_{\mathrm{SS}}^2}{\mathscr{M}^2 r^2 \Sigma^2} \right] , \\
\frac{\mathrm{d}^2 \omega}{\mathrm{d}\,r^2} &= -\left(\frac{\mathrm{d}\,\ln(\nu_\mathrm{T} \Sigma r^3)}{\mathrm{d}\,r} + \frac{1}{\nu_\mathrm{T} \Sigma r} \right) \frac{\mathrm{d}\,\omega}{\mathrm{d}\,r} - \frac{2\omega}{\nu_\mathrm{T} \Sigma r^2} \,, \\
\frac{\mathrm{d}\,t}{\mathrm{d}\,r} &= \left[\frac{g}{r} - \chi_\mathrm{T} \frac{2\gamma - 1 - \gamma c_2/c_1^2}{\gamma - 1} t \frac{\mathrm{d}\,\Sigma}{\mathrm{d}\,r} + \frac{t^4 z_0^2}{\tau} \frac{3}{2\,r} c_5 \right] \\
&\quad \Big/ \left[\chi_\mathrm{T} \frac{2\gamma - 1 - 2\gamma c_2/c_1^2}{\gamma - 1} \Sigma - \frac{t^3 z_0^2}{\tau} (c_4 + \tfrac{1}{2} c_5) \right] , \\
\frac{\mathrm{d}\,g}{\mathrm{d}\,r} &= r\mathscr{M}^2 \left[-\frac{2K c_1}{c_6} \frac{t^4}{\tau} + \nu_\mathrm{T} \Sigma \left(r \frac{\mathrm{d}\,\omega}{\mathrm{d}\,r} \right)^2 \right. \\
&\quad \left. + \frac{1}{r\mathscr{M}^2} \left(\frac{1}{\gamma - 1} \frac{\mathrm{d}\,t}{\mathrm{d}\,r} - \frac{t}{\Sigma} \frac{\mathrm{d}\,\Sigma}{\mathrm{d}\,r} \right) \right] ,
\end{aligned} \tag{24}$$

with the abbreviations

$$\nu_{\rm T} = t\,\omega/(c_1\omega_{\rm K}^2), \;\; \chi_{\rm T} = \nu_{\rm T}/{\rm Pr}\,, \;\; \tau = \Sigma^{n+1}\, t^{q-\frac{n}{2}}\, \omega_{\rm K}^n\,,$$

$$z_0 = \sqrt{t/c_1}/\omega_{\rm K}\,.$$

(Be not confused by the different meanings of Σ! Unless otherwise stated, it measures from now on the dimensionless surface density.) In the normalized equations the azimuthal Mach number appears:

$$\mathscr{M} = R_*\Omega_{{\rm K}*}/\sqrt{c_1(\mathscr{R}/\mu)T_{\rm unit}}\,. \quad (25)$$

The total energy flux (in ergs/s) through the disk is $g\cdot \dot{M}c_1(\mathscr{R}/\mu)T_{\rm unit}$.
Because Kramers' opacity law has been taken, $\kappa_0 = 6.6\,10^{22}$ cgs, $n = 1$ and $q = -3.5$ hold. The constants c_i, depending on the opacity law, have the following values: $c_1 = 0.884$, $c_2 = 0.801$, $c_3 = 1.104$, $c_4 \approx 2.9$, $c_5 \approx 1.0$, and $c_6 = 0.772$. Further, the values $K = 0.2687$, $\gamma = 5/3$, and $\mu = 0.615$ have been used.

The system (24) had been written explicitly, were it nor for convenience' sake. The resulting formulas would be simply too long.

6. Results and discussion

The numerical computations have been performed assuming a T Tauri star of one solar mass, with a radius $R_* = 4.3\,R_\odot$, and an accretion rate of $10^{-7}\,M_\odot$/yr, which is representative for a classical T Tauri star (Bertout et al. 1993). The star's angular velocity is fixed to one tenth of its Keplerian value at the surface, i.e. the slow-rotator case is considered. The latter is essential for the viscosity near the inner boundary.

6.1. Numerical results

Despite the smallness of the range for $\mathscr{C}$, found by others too (cf. for an explanation Sect. 6.3), the most interesting result is that the dependence of the angular momentum flow parameter $\mathscr{C}$ on the optical depth τ_* is not 'bijective' (see Fig. 1). (This holds true if the temperature T_* is fixed and the inner surface density Σ_* has been varied.) So it is by no means justified to prescribe the value of $\mathscr{C}$ a priori as a 'boundary condition' because in this formulation the resulting solution will not be determined uniquely.

It is not astonishing that there are sometimes two solutions for a given $\mathscr{C}$. The BL can be a hydrostatical one, i.e. it is the high bottom pressure, that supports the BL radially against the star's gravity. On the other hand, the inertial term, i.e. the second term on the left side of Eq. (4), can bear a part in the radial impulse balance. (Admittedly, we were unable with our code to follow this branch into the supersonic regime.)

The angular momentum flow parameter $\mathscr{C}$ has a lower limit, which is — with our viscosity prescription in the BL — larger than unity. This is contrary to Glatzel's (1992) "unconventional procedure" which excludes $\mathscr{C} > 1$. Hence, his so-called "generalized standard viscosity" (his Eq. 4.3.) is not unambiguous.

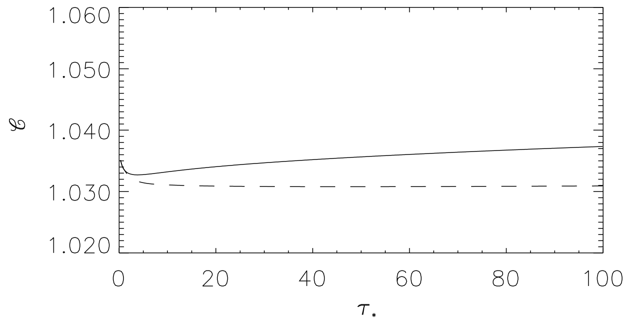

Fig. 1. Here it is shown, how the specific angular momentum flow $\mathscr{C}$, normalized to that radius which belongs to the optical depth τ_*, depends on that optical depth. The rectified $\mathscr{C}$-value (referring to that reference radius, where the optical depth is unity) shows the expected saturation for $\tau_* \gg 1$ (dashed line)

Other solutions for the dynamical viscosity $\nu_{\rm T}\Sigma(R)$ exist compatible with the Navier Stokes equation, too.

That $\mathscr{C}(\tau_*)$ does not show saturation for large optical depths, is simply due to the shrinking R_* for increasing optical depth. In other words, that the star's atmosphere is of finite size is no longer negligible. In order to have an τ_*-independent measure of the physical angular momentum flow, we have to refer $\mathscr{C}$ to a fixed radius. We have regarded the mentioned $4.3\,R_\odot$ as that radius where the optical depth is unity. The angular momentum parameter referring to that reference point is given by $\mathscr{C}_{\rm corr} = \mathscr{C}\cdot\sqrt{1-\Delta R/R_*}$. The depth difference ΔR between the actual τ_* and $\tau_* = 1$ has been found, integrating the equation for hydrostatical equilibrium, observing the Eddington temperature stratification (22) and Kramers' opacity rule. The depth-corrected angular momentum parameter $\mathscr{C}_{\rm corr}$ levels indeed for our model example at 1.031.

Lioure & Le Contel mentioned that the minimum of $\mathscr{C}$ occurs when the BL radiates away exactly all the energy dissipated in it. We cannot agree. As Fig. 2 demonstrates: at the minimum of $\mathscr{C}$ (i.e. at the turning point) the radiated power is noticeably lower than the dissipated one.

Radial runs of ω, T and Π, computed for $\tau_* = 30$, are given in the Figs. 4 – 6.

Obviously, in order to settle our question concerning $\mathscr{C}_{\rm corr}$, one has to decide on that minimal optical depth, where the inner boundary should be meaningfully located. As Fig. 3 reveals, for $\tau_* \gtrsim 30$ the radial radiation flux is directed outwards and is approaching even the stellar flux! The value of $\mathscr{C}_{\rm corr}$ is saturated even for smaller optical depths (see Fig. 1). That the inner boundary can be located at the star's outskirts indicates, as we see it, that the star proper may not be strongly influenced by an accretion rate of $\dot{M} = 10^{-7}\,M_\odot$/yr. At a ten times lower rate the flux constraint can be fulfilled even for τ_* of order unity.

It is noteworthy that for large τ_* the radial run of the midplane temperature shows one local minimum only (Fig. 5). *In our view, so-called cool BLs, which imply a temperature minimum below the inner boundary, result from somewhat artificial boundary conditions and have to be regarded suspicious.* Recent observations indicate indeed the existence of hot BL in CTTS

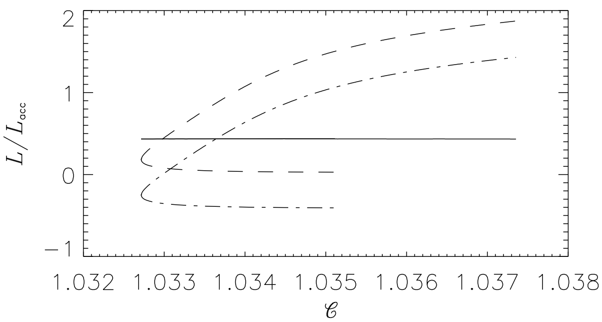

Fig. 2. The luminosity of the thermal BL (dashed line), radial luminosity (dot-dashed line) and dissipated energy (solid line) against $\mathscr{C}$. The optical depth varies between 0.3 (inward flow of energy) and 100 (outward flow of energy). For $\tau_* \geq 10$ the BL becomes a hot one, i.e. it is heated by the star. The unit is $L_{\rm acc} = GM_*\dot{M}/R_*$

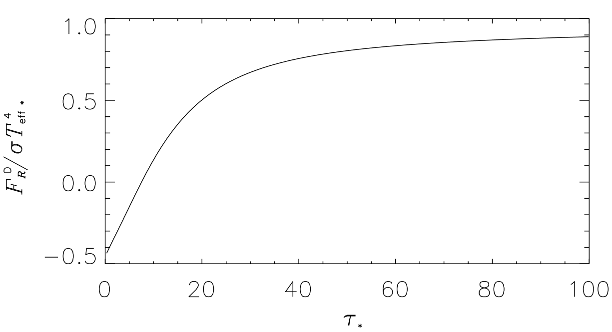

Fig. 3. Dependence of the radiative mid-plane flux (in units of $\sigma T_{\rm eff*}^4$) on the optical depth τ_*

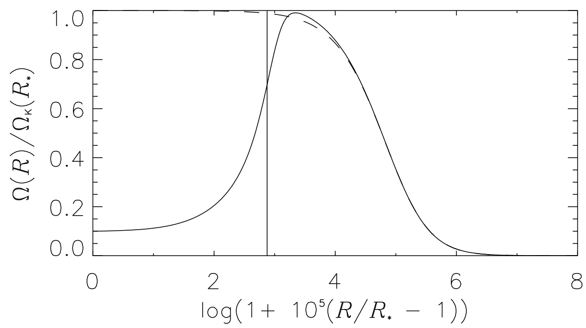

Fig. 4. The scaled angular velocity ω for $\omega_* = 0.1$ (solid line) in comparison to the Keplerian rotation rate (dashed line); note that there is a broad region where ω exceeds $\omega_{\rm K}$ slightly. The vertical marks the location of the reference radius, where $\tau_* = 1$

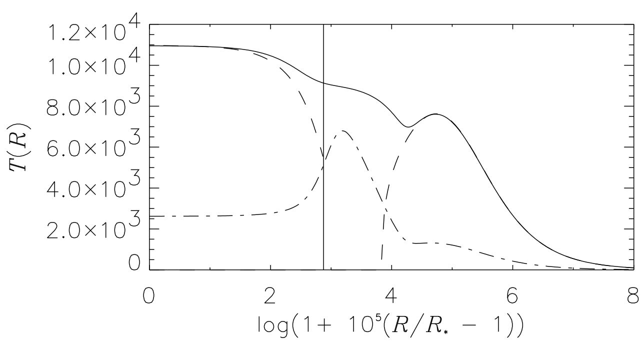

Fig. 5. The mid-plane temperature $T_{\rm c}$ (solid line) and the effective temperature $T_{\rm eff}$ (dot-dashed line); the dashed lines show the disk standard solution (without BL) and the temperature stratification in the unperturbed star's envelope, respectively

(cf. Regev & Bertout 1995), where "hot" indicates merely that the star radiates into the BL.

Inclusion of a turbulence-driven flow of energy does not alter the results significantly. To demonstrate this (Fig. 8), we have computed a characteristic effective temperature $T_{\rm eff}^{\rm BL}$ of the *thermal* BL by dividing the "vertical" luminosity of the BL by two times the area of it. (As by Lioure & Le Contel the thermal boundary layer extend to the local minimum of the mid-plane temperature.) In one calculation the Prandtl number has been set unity, in another one it has been given a very high value. No differences in the computed BL temperatures appear. Thus it seems to be impossible to constrain the value of the Prandtl number by spectral observations. (By the way, the BL spectra — resulting from the superposition of black-body spectra according to the actual radial distribution of $T_{\rm eff}$ — do not differ significantly from that computed spectrum resulting from $T_{\rm eff}^{\rm BL}$ alone. Hence, $T_{\rm eff}^{\rm BL}$ characterizes indeed the spectral energy distribution of the light coming from the BL.)

In any case, a $\mathscr{C}$ of almost unity, as found here for a slowly rotating star, would lead unavoidably to an increase of the star's angular momentum and, correspondingly, — if the star does not compensate for it by expanding — to a spin-up.

6.2. A scaling law for the width of the viscous BL

Integrating Eq. (8) at once and using dimensionless quantities one gets

$$\frac{{\rm d}\,\omega}{{\rm d}\,r} = \frac{\mathscr{C} - r^2\omega}{\langle\nu_{\rm T}\Sigma\rangle r^3}\,. \tag{26}$$

This can be formally integrated over the range $1 \leq r \leq r_{\rm m}$, where $r = 1$ denotes that radius where the optical depth τ_* is unity (in order to have a reference point as the extent of the star's atmosphere is no longer negligible) and $r_{\rm m}$ is that radius where the angular velocity ω reaches its maximum $\omega_{\rm m}$:

$$\omega_{\rm m} - \omega_* = \frac{\mathscr{M}}{\alpha_{\rm SS}} \int_1^{r_{\rm m}} \frac{\mathscr{C} - r^2\omega}{r^2} \cdot \left(\frac{-\bar{u}_r}{\nu_{\rm T}}\right) dr\,, \tag{27}$$

where the mass conservation law $-\bar{u}_r\Sigma r = \alpha_{\rm SS}/\mathscr{M}$ has been used. The first factor within the integral is a monotonic function. With the help of a well-known integration theorem and remembering that $\omega_{\rm m} = \mathscr{C}/r_{\rm m}^2$, one finds

$$\frac{\mathscr{C}}{r_{\rm m}^2} - \omega_* = (\mathscr{C} - \omega_*) \cdot \frac{\mathscr{M}}{\alpha_{\rm SS}} \int_1^{1+\delta_{\rm visc}/q_1} \frac{-\bar{u}_r}{\nu_{\rm T}}\, dr\,, \tag{28}$$

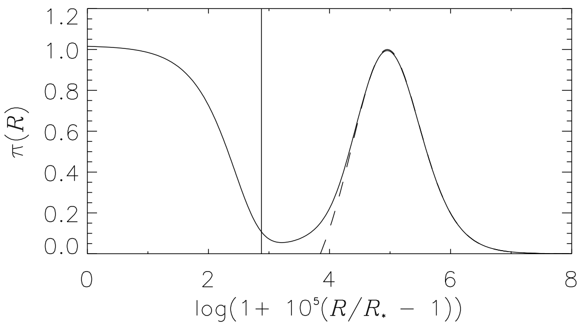

Fig. 6. The height-integrated pressure Π in units of its maximal value $9\dot{M}\Omega_{K*}/(256\pi\alpha_{SS}\mathscr{C}^3)$. The pressure minimum lies within the viscous BL. The dashed curve gives the integrated pressure for the standard solution

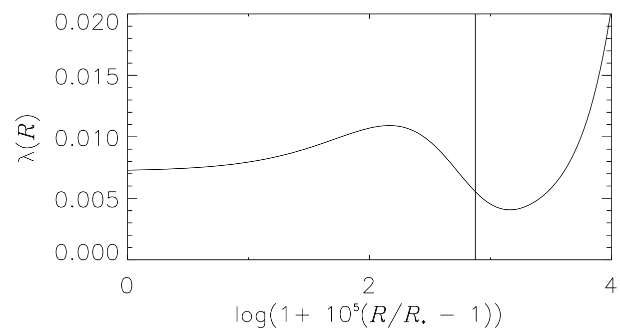

Fig. 7. The radial variation of the characteristic radial length-scale λ (in units of R_*); the local minimum corresponds to the pressure minimum seen in Fig. 6 and is just inside the point, where the angular velocity reaches its maximum. τ_* = 30 has been adopted

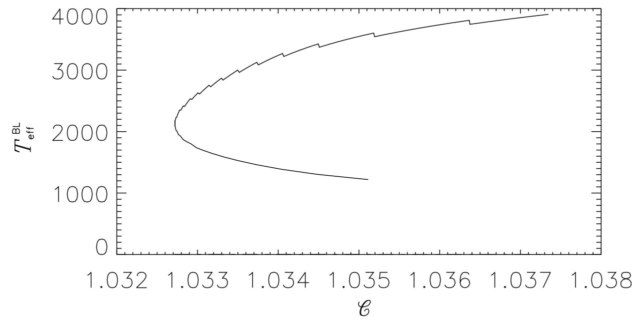

Fig. 8. The characteristic effective temperature of the thermal BL against the angular momentum parameter $\mathscr{C}$. For $\tau_* \geq 10$ the BL is additionally heated by the star. Two values of the Prandtl number Pr have been taken, 1 and ∞, but no difference is discernible

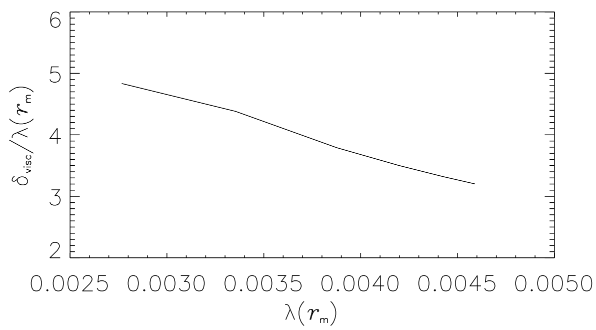

Fig. 9. The width of the viscous BL $\delta_{\rm visc}$ in units of the characteristic length scale $\lambda(r_{\rm m})$, measured at that radius where the angular velocity becomes maximal. The accretion rate has been varied from $10^{-8} \ldots 10^{-7}\, M_\odot$/yr. τ_* = 30 has been adopted

with $q_1 \geq 1$. The $\delta_{\rm visc}$ appearing in the upper limit of the integral denotes the width $r_{\rm m} - 1$ of the viscous BL.

The remaining integral can be expressed, using a further integration theorem, as $\delta_{\rm visc}/q_1 \cdot (-\bar{u}_r/\nu_{\rm T})_{r_2}$. The radius r_2, where the quotient $-\bar{u}_r/\nu_{\rm T}$ has to be taken, is now restricted through $r_2 = 1 + \delta_{\rm visc}/q_2$, whereby $q_2 \geq q_1$.

Let us assume for the sake of simplicity $\omega_* \approx 0$ and introduce the (dimensionless) length-scale $\lambda = \nu_{\rm T}\Sigma r = \alpha_{\rm SS}\nu_{\rm T}/(\mathscr{M}\bar{u}_r)$. Then the width of the viscous boundary obeys

$$\delta_{\rm visc} = \frac{q_1}{(1+\delta_{\rm visc})^2}\,\lambda(r_2)\,. \tag{29}$$

For a small $\delta_{\rm visc}$ ($\ll 1$) and because $q_1 \geq 1$ there is in any case a minimal width: the minimal value of λ within the BL. In our computations this minimal λ is reached at a slightly smaller radius than that one, where the angular velocity is maximal, i.e. $r_{\rm m}$.

The run of λ with r is shown in Fig. 7. Far outside the BL $\lambda \propto r$ holds. Provided the unknowns q_1 and q_2 turn out not to be crucial, a scaling relation like

$$\delta_{\rm visc} \propto \lambda(r_2) \tag{30}$$

is suggested. As an inspection of Fig. 9 reveals, where the dependence of the found $\delta_{\rm visc}$ on $\lambda(r_{\rm m})$ for different accretion rates is shown, a scaling like Eq. (30) is indeed indicated, despite a small downwards trend for higher accretion rates.

6.3. *Why is $\mathscr{C} \approx 1$?*

Let us assume that it is the radial pressure gradient that compensates for the deviation between gravitational attraction and centrifugal force. The radial impulse equation (4) takes the form

$$\frac{{\rm d}\,\Pi}{{\rm d}\,r} = -\mathscr{M}^2 \cdot r\Sigma\,(\omega_{\rm K}^2 - \omega^2)\,. \tag{31}$$

Integration yields the bottom pressure

$$\Pi_* = \Pi(r) + \mathscr{M}^2 \cdot \int_1^r r\Sigma\,(\omega_{\rm K}^2 - \omega^2)\,dr\,. \tag{32}$$

It is the overwhelming $\mathscr{M}^2$ in front of the integral that is essential in what follows. In brief, in order to keep the pressure excursions due to the imbalance of attractive and centrifugal force within bounds, either $\omega_{\rm K}^2 - \omega^2$ has to be small or, if this is impossible (e.g. near the surface of the star), at least the range of r must be small, where this difference is unavoidably important.

Now let us consider this in more detail: If the viscosity is prescribed by the Shakura-Sunyaev formula and the rotation law is Keplerian, there is a pressure maximum at $r(\Pi_{\max}) = \frac{16}{9}\,\mathscr{C}^2$. It is essential to note that this maximum is (for $\mathscr{C} \approx 1$) far outside the BL itself. Immediately below that point the pressure must drop if approaching the centre (cf. Fig. 6). The rotation is now super-Keplerian. The amount of pressure decrease is, of course, limited. Because onset of super-Keplerian rotation happens so far away from the real BL, the deviation from the Keplerian value must be very small to prevent the pressure to become at once negative.

Somewhere the angular velocity must fall below its Keplerian value. From now on the pressure raises again monotonically up to the fixed boundary value Π_*. Because now, due to the boundary condition for the angular velocity, ω deviates strongly from $\omega_{\rm K}$, the only way to bound the pressure is to keep the range for r small. Both demands together, a slightly super-Keplerian rotation over a large range as well as a small range for the innermost sub-Keplerian region, lead inevitably to a maximal angular velocity near its Keplerian value *and* near to the surface of the star, i.e. $\mathscr{C} \approx 1$.

Of course, extended BLs around slow rotators with $\mathscr{C} \neq 1$ are not refused categorically by this line of reasoning. For instance $\mathscr{M}$ could be small (thick disk!), or the bottom pressure could be high, or other forces (e.g. magnetic ones) could become important too.

Acknowledgements. We are grateful to acknowledge the support of one of the authors (M. v. R.) by the Deutsche Forschungsgemeinschaft.

The anonymous referee is thanked for well-meant suggestions for improvement and helpful comments.

Appendix A: vertical structure

The equations governing the steady state radial disk structure contain z-integrated quantities — e.g. Π, $T_{\rm eff}$ —, which to compute from $\Sigma, \Omega_{\rm K}$, and $T_{\rm c}$ needs the knowledge of the disk's vertical structure. The latter can be dealt with separately — at least in thin disks. Integrating out the z-coordinate is a standard approximation in accretion disk theory.

In what follows it is shown, how we have extracted six constants, c_i, all of order unity, relying on the assumption of hydrostatical equilibrium in a thin disk and on treating the radiative transfer in the diffusion approximation.

Because all material functions, especially that for the Rosseland mean of the opacity, are power laws, the vertical structure proves to be self-similar, i.e. scaling relations hold. (Of course, we must generally pay for it by neglecting the radiation pressure as compared with the thermal one, but this is justified in the CTTS case.) For instance, independent of the radial distance, R, the same normalized temperature profile holds, i.e. $T(z)/T_{\rm c} = f_T(z/z_0)$, with $T_{\rm c}$ and z_0 being functions of R. Given then central temperature and density and scale height, the vertical radiation flux follows immediately. In the steady state this loss of energy is to be balanced by viscous heat generation and/or by tapping the radial energy flow (cf. Eq. (13)).

Of course, in order to get the looked for temperature profile $f_T(z/z_0)$, the z-distribution of the source for the vertical flux is to be specified. For the sake of simplicity the dissipative contribution to the vertical radiative flux at height z be always proportional to the pressure there. (This assumption may be relaxed, as will be shown immediately afterwards.)

Hence, the following set of equations is to be solved:

$$\frac{\partial p}{\partial z} = -\rho\,\Omega_{\rm K}^2\,z\,, \quad z \ll R\,, \tag{A1}$$

$$\frac{\partial \Sigma_z}{\partial z} = 2\,\rho\,, \tag{A2}$$

$$\frac{\partial F_z^{\rm D}}{\partial z} = p\,\Omega_{\rm diss}\,, \tag{A3}$$

$$\frac{\partial T}{\partial z} = -\frac{3\,F_z^{\rm D}\kappa\rho}{16\sigma T^3}\,. \tag{A4}$$

The dissipation rate $\Omega_{\rm diss}$ — $(9/4)\alpha_{\rm SS}\Omega_{\rm K}$ in the case of Kepler rotation — depend on R only! Besides, there are the equation of state and an opacity law: $p = \mathscr{R}\rho T/\mu$ and $\kappa = \kappa_0\rho^n T^q$.

Introducing the length unit $z_0 = \sqrt{p_{\rm c}/\rho_{\rm c}}/\Omega_{\rm K}$, the surface density unit $2\rho_{\rm c}z_0$, the flux unit $p_{\rm c}z_0\,\Omega_{\rm diss}$, and expressing pressure, density and temperature in units of their corresponding mid-plane values (hence $z = \xi z_0$, $\Sigma_z = f_\Sigma\cdot(2\rho_{\rm c}z_0)$, $F_z^{\rm D} = f_{\rm F}\cdot p_{\rm c}z_0\,\Omega_{\rm diss}$, $p = f_p p_{\rm c}$, $\rho = f_\rho\rho_{\rm c}$, $T = f_T T_{\rm c}$), one receives instead of Eq. (A1) – (A4) the conveniently normalized equations:

$$\frac{\partial f_p}{\partial \xi} = -f_\rho\,\xi\,, \tag{A5}$$

$$\frac{\partial f_\Sigma}{\partial \xi} = f_\rho\,, \tag{A6}$$

$$\frac{\partial f_{\rm F}}{\partial \xi} = f_p\,, \tag{A7}$$

$$\frac{\partial f_T}{\partial \xi} = -K\cdot f_{\rm F} f_\rho^{n+1} f_T^{q-3}\,, \tag{A8}$$

with $f_\rho = f_p/f_T$ and $K = 3\Omega_{\rm diss}\,p_{\rm c}z_0^2\kappa_c\rho_{\rm c}/(16\sigma T_{\rm c}^4)$, where $\kappa_{\rm c}$ is the mid-plane opacity. Obvious boundary conditions are: $f_p(0) = f_T(0) = 1$ and $f_\Sigma(0) = f_{\rm F}(0) = 0$.

In the case of Kramers' opacity (and for Thomson scattering too) the temperature profile becomes nearly flat for $\xi \gtrsim 2$, hence the pressure drops Gaussian-like for $\xi \to \infty$. In order to fix K there are two possibilities. Firstly, there seems to be a limiting value for K, making $T(\infty)$ zero. Secondly, and this seems to be a physically sound approach, an additional relation between the effective temperature and the outer temperature limit $T(\infty)$ could be imposed. Again the Eddington approximation (Eq. (22)) for a grey atmosphere is applied, resulting in the constraint: $T(\infty)^4 = 17\,T_{\rm eff}^4/32$. Obeying this and the definition of K and using the constants c_1, c_3 and c_6, defined thereafter, one finds a relation connecting the disk's optical depth with K: $\tau = \frac{17}{3}\frac{K c_1 c_3}{c_6^2}\left(\frac{T_{\rm c}}{T(\infty)}\right)^4$. Remember that the constants c_i as well as the temperature ratio $T_\mathrm{c}/T(\infty)$ are functions of K. Integrating the set of Eqs. (A5) – (A8) one finds that K approaches 0.2687... for $\tau \to \infty$. For $K = 0.26$ the optical depth τ proves already larger than 10. Hence, in order to take advantage of the disk's self-similarity that limiting value for K, appropriate for optical thick disks, has been chosen.

Afterwards, we have calculated numerically

$$c_1 = \int_{-\infty}^{\infty} f_p \, d\xi / (2 f_\Sigma(\infty)) \, , \tag{A9}$$

$$c_2 = \int_{-\infty}^{\infty} f_p f_T \, d\xi / (2 f_\Sigma(\infty)) \, , \tag{A10}$$

$$c_3 = \int_{-\infty}^{\infty} f_\rho^{n+1} f_T^q \, d\xi \, / (2 f_\Sigma(\infty)) \, , \tag{A11}$$

$$c_4 = \int_{-2}^{2} \frac{f_T(\xi)^{4-q}}{f_\rho(\xi)^{n+1}} \, d\xi \, , \tag{A12}$$

$$c_5 = \int_{-2}^{2} \frac{f_T(\xi)^{3-q}}{f_\rho(\xi)^{n+1}} \left(- \frac{\partial f_T}{\partial \xi} \right) \xi \, d\xi \, , \tag{A13}$$

$$c_6 = 1 / f_\Sigma(\infty) \, . \tag{A14}$$

The following equations make clear the meaning of these constants:

$$\Pi = \int_{-\infty}^{\infty} p \, dz = c_1 \frac{\mathscr{R}}{\mu} \Sigma T_\mathrm{c} = \frac{2\, c_1}{c_6} \, z_0 \, p_\mathrm{c} \, , \tag{A15}$$

$$\int_{-\infty}^{\infty} \rho T^2 \, dz = c_2 \Sigma T_\mathrm{c}^2 \, , \tag{A16}$$

$$\tau = \int_{-\infty}^{\infty} \rho \kappa \, dz = c_3 \kappa_\mathrm{c} \Sigma \, , \tag{A17}$$

$$\int_{-\infty}^{\infty} F_R^\mathrm{D} \, dz = - \frac{16 \sigma T_\mathrm{c}^4 z_0}{3 \kappa_\mathrm{c} \rho_\mathrm{c}} \left[c_4 \frac{\partial \ln T_\mathrm{c}}{\partial R} + c_5 \frac{\partial \ln z_0}{\partial R} \right] . \tag{A18}$$

From the definition of K follows, how the optical depth τ connects the effective temperature T_eff with the mid-plane temperature T_c:

$$T_\mathrm{eff}^4 = \frac{32 K c_1 c_3}{3 c_6^2} \cdot \frac{T_\mathrm{c}^4}{\tau} \, . \tag{A19}$$

The two integrals (A12) and (A13) appear when integrating the radial radiative flux in the non-dimensional form. Unfortunately, we had to cut the integration at around two length-scales so as to prevent unlimited radial fluxes in the disk's atmosphere due to even minimal radial temperature variations in the optical thin region above the disk 'surface'.

It has been proven that the vertical structure is dynamically stable, i.e. the Schwarzschild criterion for stability is fulfilled.

There are for general reasons restrictions for some of these constants c_i, independent of the opacity law assumed. For instance, to insure a monotonic decline of the temperature with height z the relations $0 < c_2 < c_1 \leq 1$ and $c_5 \geq 0$ hold. Considering Eq. (A11) one can easily derive that provided $q \geq n$, $0 < c_3 < 1$ would follow. Perhaps most strongly restricted is c_6. In the case of a polytropic relation between pressure and density, i.e. $f_p = f_\rho^{1+1/m}$, the solution of Eq. (A5) reads $f_\rho = \left(1 - \xi^2 / (2\,(m+1))\right)^m$ and c_6 varies therefore between $1/\sqrt{2} \approx 0.707$ (box-shaped density profile) and $\sqrt{2/\pi} \approx 0.797$ (Gaussian density profile in case of an isothermal disk) to consider the most extreme cases only.

By the way, the vertical structure following from Kramers' opacity rule can be approximated very well — apart from the outermost layers — with the polytropic relation $f_p = f_\rho^{4/3}$. Remember that $m = 3$ is the special case where the radiation pressure is proportional to the gas pressure. Therefore, the effect of the radiation pressure could easily incorporated too, at least approximately.

We have further tested the sensitivity of the vertical structure to the distribution of the heat sources, assuming that the viscous heat production is proportional to the local density instead of being proportional to the local pressure (cf. Eq. (A7)). (This, perhaps, is even more in the spirit of our viscosity prescription (20).) Then the kinematical viscosity would be independent of z. The modifications are almost negligible. Now $K = 0.2607\ldots$, i.e. a reduction by 3 per cent. The changes for the c_i's are lower than 5 per cent.

An even more extreme model has been considered: the heat source be in the mid-plane only, i.e. a delta-function like distribution is assumed. Even then the c_i's are only moderately altered: $c_1 = 0.835, c_2 = 0.719, c_3 = 1.467$ and $c_6 = 0.755$. Only K is reduced distinctly to $0.1282\ldots$. But, of course, such a concentration of the heat input to the very mid-plane would make the layer above (and below) $z = 0$ convectively unstable.

References

Bertout C., Bouvier J., Duschl W.J., Tscharnuter W.M., 1993, A&A 275, 236

Bisnovatyi-Kogan G.S., 1993, A&A 274, 796

Bisnovatyi-Kogan G.S., 1994, MNRAS 269, 557

Bouvier J., Cabrit S., Fernández M., Martín E.L., Matthews J.M., 1993, A&A 272, 176

Duschl W.J., Tscharnuter W.M., 1991, A&A 241, 153

Eddington A.S., 1926, The Internal Constitution of the Stars. Cambridge Univ. Press, Cambridge

Glatzel W., 1992, MNRAS 257, 572

Godon P., 1995, MNRAS 277, 157

Godon P., 1996, MNRAS 279, 1071

Godon P., Regev O., Shaviv G., 1995, MNRAS 275, 1093

Hujeirat A., 1995, A&A 295, 268

Kley W., 1989, A&A 222, 141

Kley W., Hensler G., 1987, A&A 172, 124
Kley W., Lin D.N.C., 1992, ApJ 397, 600
Kley W., Lin D.N.C., 1996, ApJ 461, 933
Lioure A., Le Contel O., 1994, A&A 285, 185
Lynden-Bell D., Pringle J.E., 1974, MNRAS 168, 603
Narayan R., Yi I., 1995, ApJ 444, 231
Narayan R., Popham R., 1993, Nature 362, 820
Narayan R., Popham R., 1994, Accretion Disk Boundary Layers. In: Duschl W.J. et al. (eds.) Theory of Accretion Disks – 2. Kluwer, Dordrecht, p. 293
Paczyński B., 1991, ApJ 370, 597
Papaloizou J.C.B., Stanley G.Q.C., 1986 MNRAS 220, 593
Popham R., Narayan R., 1991, ApJ 370, 604
Popham R., Narayan R., Hartmann L., Kenyon S., 1993, ApJ 415, L127
Pringle J.E., 1981, ARA&A 19, 137
Regev O., 1983, A&A 126, 146
Regev O., Bertout C., 1995, MNRAS 272, 71
Rüdiger G., 1989, Differential Rotation and Stellar Convection: Sun and Solar-type Stars. Gordon & Breach, New York
Shakura N.I., Sunyaev R.A., 1973, A&A 24, 337

This article was processed by the author using Springer-Verlag LaTeX A&A style file *L-AA* version 3.