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ASTRONOMY
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Theory of the rotation of the rigid Earth

P. Bretagnon, P. Rocher, and J.L. Simon

Bureau des longitudes URA 707 du CNRS 77, Avenue Denfert-Rochereau, F-75014 Paris, France

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Abstract. We present the computation of the precession and nutation expressions built with the analytical theories of the motion of the Moon, the Sun and the planets of the Bureau des longitudes. We take into account the influence of the Moon, the Sun and all the planets on the potential of the Earth limited to $C_{j,0}$ for j from 2 to 5, $C_{2,2}$, $S_{2,2}$, $C_{3,k}$, $S_{3,k}$ for k from 1 to 3 and $C_{4,1}$, $S_{4,1}$. We have determined the analytical variations of the angles ψ and ω fixing the equator with respect to the ecliptic J2000. We find, in ω, a secular term of -265.036 μas per year. We use for the constant of the general precession the Williams' (1994) value $50''.2877/\text{year}$. The corresponding value of the dynamical flattening of the Earth $H_d = (2C - A - B)/(2C)$ is 0.003 273 7671. In this work we used a truncated solution for the motion of the Moon. We have compared our analytical solution with a numerical integration. The differences are smaller than 16 μas for ψ and 8 μas for ω.

Key words: astrometry – celestial mechanics – reference systems – Earth

1. Introduction

For the last twenty years, nutation series for a rigid Earth have been computed by different methods. Kinoshita (1975, 1977), Kinoshita and Souchay (1990), and Souchay and Kinoshita (1996) used Hamiltonian methods. Williams (1994, 1995), and Roosbeek and Dehant (1996) computed torques on the oblate Earth, using analytical theories for the motion of the Moon, the Sun and the planets. Note that in all these papers the perturbations by the disturbing bodies are taken into account step by step: for each body the authors first consider a Keplerian orbit which they later correct by applying the perturbations differentially. Hartmann and Soffel (1994) proceed similarly but used a numerically integrated ephemeris, so that the perturbations need not be applied separately.

In this paper we present a theory of the rotation of the rigid Earth by using analytical theories for the motion of the solar system bodies including the perturbations at initio. The equations given in Sect. 4 are integrated by an iterative method. Perturbations are thus computed as nutation on nutation terms.

This theory gives analytical solutions for each of the Earth's three Euler angles ψ, ω, φ. These angles are reckoned positively in positive rotation, in contrast to lunisolar (and general) precession. The solutions combine precession and nutation in longitude and in obliquity on the one and the diurnal rotation on the other hand.

2. Notations

2.1. Coordinate systems

Let $Oxyz$ be the coordinate system defined by the ecliptic and the equinox J2000. The axes Ox and Oy are in the ecliptic plane with the axis Ox pointing toward the equinox. In addition, one use the following coordinate systems:
$OXy'z$, originating from $Oxyz$ by a rotation of $R_3(\psi)$;
$OXYZ$, originating from $OXy'z$ by a rotation of $R_1(\omega)$;
$O\xi\eta\zeta$, originating from $OXYZ$ by a rotation of $R_3(\varphi)$. The various coordinate systems are illustrated in Fig. 1. Here, $O\xi\eta\zeta$ is a terrestrial coordinate system with $O\xi$ and $O\eta$ in the plane of the equator and with $O\xi$ pointing toward the intersection of the plane of the equator with the plane containing the Greenwich Meridian. Let α be the longitude of the major axis of the equatorial ellipse corresponding to the principal moment of inertia A. The coordinate system $O\tilde{\xi}\tilde{\eta}\tilde{\zeta}$, originating from $O\xi\eta\zeta$ by a rotation of α about the ζ axis is also used. For a rigid Earth α is a constant, $O\tilde{\xi}\tilde{\eta}\tilde{\zeta}$ is therefore also a terrestrial coordinate system.

2.2. Basic notations

In $O\tilde{\xi}\tilde{\eta}\tilde{\zeta}$, the Earth's angular velocity vector **R** has the components:

$$\mathbf{R} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}, \tag{1}$$

and the angular-momentum vector $\boldsymbol{\sigma}$ is given by:

$$\boldsymbol{\sigma} = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} \cdot \mathbf{R}, \tag{2}$$

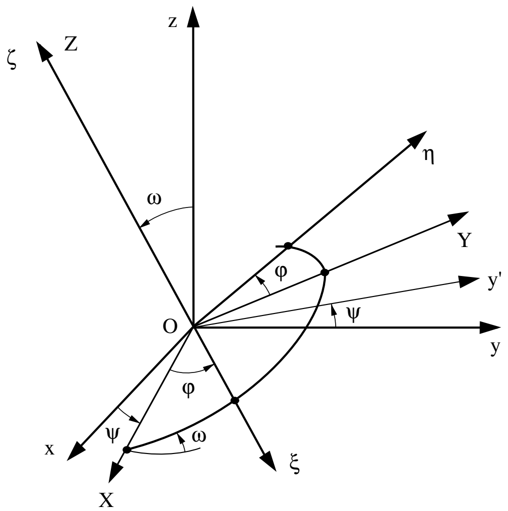

Fig. 1. Coordinate systems used for the rotation of the rigid Earth.

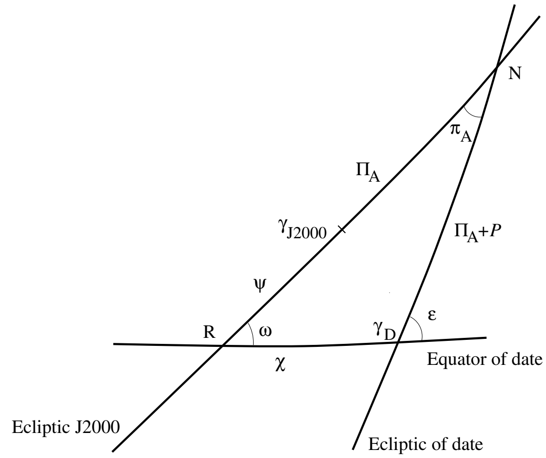

Fig. 2. Ecliptic and equator J2000 and of date.

where A, B, C are the principal moments of inertia of the Earth, given by the well-known formulas

$$A = \iiint_{\text{(Earth)}} (\tilde\eta^2 + \tilde\zeta^2)\, dm\,; \quad B = \iiint_{\text{(Earth)}} (\tilde\zeta^2 + \tilde\xi^2)\, dm\,;$$
$$C = \iiint_{\text{(Earth)}} (\tilde\xi^2 + \tilde\eta^2)\, dm.$$

Let $\mathscr{M}$ be the torque due to the external forces. Its components in different coordinate systems are

$$\begin{aligned} L, M, N &\text{ referred to } OXYZ, \\ \lambda, \mu, \nu &\text{ referred to } O\xi\eta\zeta, \\ \tilde\lambda, \tilde\mu, \tilde\nu &\text{ referred to } O\tilde\xi\tilde\eta\tilde\zeta. \end{aligned} \quad (3)$$

The rectangular coordinates of an attracting body are denoted by ξ, η, ζ in the coordinate system $O\xi\eta\zeta$, $\tilde\xi, \tilde\eta, \tilde\zeta$ in the coordinate system $O\tilde\xi\tilde\eta\tilde\zeta$ and X, Y, Z in the coordinate system $OXYZ$. In terms of spherical polar coordinates (r, Λ and β) we have

$$\begin{cases} \xi = r\cos\beta\cos\Lambda \\ \eta = r\cos\beta\sin\Lambda \\ \zeta = r\sin\beta \end{cases} \text{and} \quad \begin{cases} r &= (\xi^2+\eta^2+\zeta^2)^{\frac12} \\ \tan\beta &= \zeta(\xi^2+\eta^2)^{-\frac12} \\ \tan\Lambda &= \eta/\xi \end{cases} \quad (4)$$

$$\begin{cases} \tilde\xi = r\cos\beta\cos(\Lambda-\alpha) \\ \tilde\eta = r\cos\beta\sin(\Lambda-\alpha) \\ \tilde\zeta = r\sin\beta \end{cases} \quad (5)$$

$$\begin{cases} X = r\cos\beta\cos(\Lambda+\varphi) \\ Y = r\cos\beta\sin(\Lambda+\varphi) \\ Z = r\sin\beta. \end{cases} \quad (6)$$

Finally, we denote the coordinates of a point of the Earth by ξ', η', ζ'.

The potential energy U of the external body at (r, Λ, β) in the gravity field of the oblate Earth is:

$$U = \frac{Gm_E}{r}\Big[1 + \sum_{n=2}^{\infty}\frac{a_E^n}{r^n}C_{n,0}P_{n,0}(\sin\beta)$$
$$+ \sum_{n=2}^{\infty}\sum_{k=1}^{\infty}\frac{a_E^n}{r^n}(C_{n,k}\cos k\Lambda + S_{n,k}\sin k\Lambda)P_{n,k}(\sin\beta)\Big], \quad (7)$$

where G is the gravitational constant, m_E and a_E are Earth's mass and equatorial radius, respectively, $P_{n,k}$ are the Legendre polynomials, $C_{n,0}$ the zonal harmonics and $C_{n,k}$, $S_{n,k}$ the tesseral harmonics. The Legendre polynomials are given by the well-known relationship

$$P_{n,k}(x) = \frac{(1-x^2)^{\frac{k}{2}}}{2^n n!}\,\frac{d^{n+k}(x^2-1)^n}{dx^{n+k}}. \quad (8)$$

The moments of inertia A, B, C and the zonal harmonic $C_{2,0}$ are related to the dynamical ellipticity H_d by

$$H_d = \frac{2C-A-B}{2C} = -m_E a_E^2\frac{C_{2,0}}{C}. \quad (9)$$

2.3. Rotation of the Earth

In Fig. 2, γ_{J2000} is the equinox J2000 (direction of the axis Ox), γ_D the equinox of date, N the ascending node of the ecliptic of date on the ecliptic J2000 and R the intersection of the equator of date and the ecliptic J2000 (direction of the axis OX). The rotation of the Earth is measured by the angle φ from R or by the sidereal time from γ_D. φ is related to the sidereal time by

$$\varphi = \text{sidereal time} + \chi, \quad (10)$$

where χ is the arc $R\gamma_D$. In Sect. 6, we use φ in the split:

$$\varphi = \varphi_0 + \varphi_1\, t + \Delta\varphi. \quad (11)$$

Finally, the rotational displacement of the axis $O\tilde\xi$ is measured by the angle $\tilde\varphi$, which is given by

$$\tilde\varphi = \varphi + \alpha. \quad (12)$$

3. Constants used, and constants of integration

3.1. Units

We reckon mass in solar mass, length in astronomical units, (1 au = 149 597 870.61 km). The unit of time is one thousand Julian years, thus tjy = 365 250 days.

3.2. Basic constants

The following constants are taken from the IERS Standards 1992.
Equatorial Radius of the Earth: a_E = 6 378.1363 km = 42.635 207 801 10^{-6} au.
Ratio of the Sun's mass to that of the Earth:

$$\frac{m_S}{m_E} = 332\,946.045.$$

Ratio of the Sun's mass to the mass of the Moon:

$$\frac{m_S}{m_E}/\frac{m_M}{m_E} = 332\,946.045/0.012\,300\,034 = 27\,068\,709.322.$$

For the masses of Mercury, Venus, Mars, Jupiter, Saturn, Uranus and Neptune we use the IAU 1976 masses. The IERS Standards 1992 masses would give periodic differences smaller than 0.002 μas and secular differences smaller than 1 μas/tjy.

In these units the value of the gravitational constant is

$$\begin{aligned} G &= (k \times 365\,250)^2 = (0.017\,202\,098\,95 \times 365\,250)^2 \\ &= 39\,476\,926.421 \text{ au}^3 m_S^{-1} \text{tjy}^{-2}, \end{aligned}$$

so that

$$G m_E = 118.568\,539\,901 \text{ au}^3 \text{tjy}^{-2}$$

3.3. Precession constants

We use, for the constant of general precession, Williams' (1994) value of the constant

$$\mathscr{P} = 50\,287.7''/\text{tjy}, \tag{13}$$

and the value of Simon et al. (1994) for the rate $\dot\psi$ of lunisolar precession (at J2000)

$$\dot\psi = \left(\frac{d\psi}{dt}\right)_{t=\text{J2000}} = -50\,384.564\,895''/\text{tjy}, \tag{14}$$

and finally, the value of Brumberg et al. (1991) for the geodesic precession

$$\mathscr{P}_g = 19''\!.198\,830/\text{tjy}. \tag{15}$$

The equations of Sect. 4 must therefore be solved to yield:

$$\begin{aligned} \left(\frac{d\psi}{dt}\right)_{t=\text{J2000}} &= -50\,384''\!.564\,895 - 19''\!.198\,827 \\ &= -50\,403''\!.763\,722/\text{tjy} \\ &= -0.244\,364\,342\,318 \text{rad}/\text{tjy}. \end{aligned} \tag{16}$$

3.4. Principal moments of inertia

In order to satisfy (16) when we solve the equations of Sect. 4, we fix C at

$$C = 0.180\,548\,385\,370 \times 10^{-14}\ m_S \text{au}^2. \tag{17}$$

$B - A$ is given by:

$$\frac{B-A}{m_E a_E^2} = 4\sqrt{C_{2,2}^2 + S_{2,2}^2} = 7.261\,454 \times 10^{-6},$$

and we computed the values of A and B from (9), and thus obtained

$$A = 0.179\,955\,329\,763 \times 10^{-14}\ m_S \text{au}^2, \tag{18}$$

$$B = 0.179\,959\,294\,245 \times 10^{-14}\ m_S \text{au}^2, \tag{19}$$

and finally

$$\frac{C-A}{m_E a_E^2} = 1086.256\,801 \times 10^{-6}, \tag{20}$$

$$\frac{C-B}{m_E a_E^2} = 1078.995\,348 \times 10^{-6}, \tag{21}$$

$$\frac{B-A}{m_E a_E^2} = 7.261\,453 \times 10^{-6}. \tag{22}$$

These values agree well with those of Bursa (1992)

$$\begin{aligned} \frac{C-A}{m_E a_E^2} &= (1086.258 \pm 0.002) \times 10^{-6}, \\ \frac{C-B}{m_E a_E^2} &= (1078.996 \pm 0.002) \times 10^{-6}, \\ \frac{B-A}{m_E a_E^2} &= (7.262 \pm 0.006) \times 10^{-6}. \end{aligned}$$

Furthermore, we compute from (17)–(19)

$$\frac{A}{m_E a_E^2} = 0.329\,611\,083, \tag{23}$$

$$\frac{B}{m_E a_E^2} = 0.329\,618\,344, \tag{24}$$

$$\frac{C}{m_E a_E^2} = 0.330\,697\,340. \tag{25}$$

These values agree not too well with Bursa's which are

$$\frac{A}{m_E a_E^2} = 0.329\,591 \pm 10^{-6},$$

$$\frac{B}{m_E a_E^2} = 0.329\,599 \pm 10^{-6},$$

$$\frac{C}{m_E a_E^2} = 0.330\,678 \pm 10^{-6}.$$

Finally, we adopt Bursa's (1992) value for α

$$\alpha = 14^\circ\!.95 \quad \text{West}.$$

3.5. Geopotential

The coefficients $C_{n,k}$ and $S_{n,k}$ are computed from the GEM–T3 normalized coefficients given by the IERS Standards 1992:

$$
\begin{aligned}
C_{2,0} &= -1082.626\,074\,59 \times 10^{-6}\\
C_{3,0} &= 2.532\,516\,07 \times 10^{-6}\\
C_{4,0} &= 1.618\,563\,60 \times 10^{-6}\\
C_{5,0} &= 0.226\,669\,08 \times 10^{-6}
\end{aligned}
$$

$$
\begin{aligned}
C_{2,2} &= 1.574\,410\,20 \times 10^{-6} & S_{2,2} &= -0.903\,757\,18 \times 10^{-6}\\
C_{3,1} &= 2.190\,181\,66 \times 10^{-6} & S_{3,1} &= 0.269\,185\,23 \times 10^{-6}\\
C_{3,2} &= 0.308\,935\,56 \times 10^{-6} & S_{3,2} &= -0.211\,581\,67 \times 10^{-6}\\
C_{3,3} &= 0.100\,446\,96 \times 10^{-6} & S_{3,3} &= 0.197\,156\,77 \times 10^{-6}\\
C_{4,1} &= -0.508\,637\,59 \times 10^{-6} & S_{4,1} &= -0.449\,140\,83 \times 10^{-6}
\end{aligned}
$$

3.6. Fundamental arguments of the series

The values of the arguments of the nutation series are taken from the set of polynomial expressions by Simon et al. (1994) for the planetary mean longitudes with respect to the fixed equinox J2000 and for the Delaunay arguments D, F, l of the Moon:

$$
\begin{aligned}
\lambda_1 &= 4.402\,608\,842\,46 + 26\,087.903\,141\,5742\,t\\
\lambda_2 &= 3.176\,146\,696\,96 + 10\,213.285\,546\,2110\,t\\
\lambda_3 &= 1.753\,470\,459\,50 + 6\,283.075\,849\,9914\,t\\
\lambda_4 &= 6.203\,476\,112\,91 + 3\,340.612\,426\,6998\,t\\
\lambda_5 &= 0.599\,547\,105\,07 + 529.690\,962\,6406\,t\\
\lambda_6 &= 0.874\,016\,284\,02 + 213.299\,104\,9603\,t\\
\lambda_7 &= 5.481\,293\,871\,54 + 74.781\,598\,5673\,t\\
\lambda_8 &= 5.311\,886\,286\,68 + 38.133\,035\,6378\,t\\
D &= 5.198\,466\,741\,03 + 77\,713.771\,468\,1205\,t\\
F &= 1.627\,905\,233\,38 + 84\,334.661\,581\,3083\,t\\
l &= 2.355\,555\,898\,30 + 83\,286.914\,269\,5536\,t
\end{aligned}
\tag{26}
$$

where t is the time measured in tjy from J2000.

3.7. Integration constants

Six integration constants are necessary for solving the system (50) of differential equations.

The two constants φ_0 and φ_1 corresponding to the angle φ ($\varphi = \varphi_0 + \varphi_1 t + \ldots$) are taken from the expressions of the sidereal time and of the variable χ. We used the expression for sidereal time at $0^{\rm h}$ given by Aoki et al. (1982)

$$
\begin{aligned}
\text{sidereal time } (0^{\rm h}) &= 6^{\rm h}41^{\rm m}50^{\rm s}\!.54841\\
&+ 86\,401\,847^{\rm s}\!.928\,6012\,t + \cdots,
\end{aligned}
$$

where t is measured in tjy from J2000. Therefore,

$$
\begin{aligned}
\text{sidereal time } (t) &= \text{sidereal time } (0^{\rm h}) + 12^{\rm h}\\
&+ 86\,400^{\rm s} \times 365\,250\,t.
\end{aligned}
$$

Or, in units of radians

$$
\begin{aligned}
\text{sidereal time } (t) &= 4.\,894\,961\,212\,82\\
&+ 2\,301\,216.753\,139\,684\,t + \cdots
\end{aligned}
$$

We take the secular part of χ from Simon et al. (1994) including planetary mass and corrections to the precessional parameters:

$$
\begin{aligned}
\chi &= 105''\!.5769\,t + \cdots\\
&= 0.000\,511\,851\,t + \cdots
\end{aligned}
$$

The two integration constants φ_0 and φ_1 are computed from (10):

$$
\begin{aligned}
\varphi_0 &= 4.894\,961\,212\,82\\
\varphi_1 &= 2\,301\,216.753\,651\,535
\end{aligned}
$$

The two other integration constants K_S and K_C defined by (52) are obtained from observation. They are the amplitudes of the solution of the reduced system resulting from (50). The period of this solution is the Euler period (305 days) but no periodic term with this period is observed for a nonrigid Earth. Thus we get

$$K_S = K_C = 0.$$

The last two integration constants are the values ψ_0 and ω_0 of the secular parts of the angle ψ and ω for t = J2000. By definition

$$\psi_0 = \psi(t = \text{J2000}) = 0;$$

ω_0 is taken from Simon et al. (1994):

$$
\begin{aligned}
\omega_0 = \omega(t = \text{J2000}) &= -23^\circ 26' 21''\!.412\\
&= -0.409\,092\,629\,69.
\end{aligned}
$$

4. Equations of motion

4.1. Basic equations

From (1) and (2), the angular-momentum vector $\boldsymbol{\sigma}$ is

$$\boldsymbol{\sigma} = \begin{pmatrix} Ap \\ Bq \\ Cr \end{pmatrix}, \tag{27}$$

and we get from the theorem of the angular-momentum and from (3)

$$\frac{d\boldsymbol{\sigma}}{dt} = \begin{pmatrix} \tilde{\lambda} \\ \tilde{\mu} \\ \tilde{\nu} \end{pmatrix}. \tag{28}$$

Since the derivative of $\boldsymbol{\sigma}$ is

$$\frac{d\boldsymbol{\sigma}}{dt} = \begin{pmatrix} A\dot{p} \\ B\dot{q} \\ C\dot{r} \end{pmatrix} + \mathbf{R} \times \boldsymbol{\sigma}, \tag{29}$$

we obtain the Euler equations

$$
\begin{aligned}
A\dot{p} + (C - B)qr &= \tilde{\lambda}\\
B\dot{q} + (A - C)rp &= \tilde{\mu}\\
C\dot{r} + (B - A)pq &= \tilde{\nu}.
\end{aligned}
\tag{30}
$$

The derivatives $\dot\psi$ and $\dot\omega$ referred to $O\tilde\xi\tilde\eta\tilde\zeta$ are given by

$$R_3(\tilde\varphi)R_1(\omega)\begin{pmatrix}0\\0\\\dot\psi\end{pmatrix}_{OXy'z} = \begin{pmatrix}\dot\psi\sin\tilde\varphi\sin\omega\\\dot\psi\cos\tilde\varphi\sin\omega\\\dot\psi\cos\omega\end{pmatrix}_{O\tilde\xi\tilde\eta\tilde\zeta}$$

and

$$R_3(\tilde\varphi)\begin{pmatrix}\dot\omega\\0\\0\end{pmatrix}_{OXYZ} = \begin{pmatrix}\dot\omega\cos\tilde\varphi\\-\dot\omega\sin\tilde\varphi\\0\end{pmatrix}_{O\tilde\xi\tilde\eta\tilde\zeta},$$

where $\tilde\varphi$ is given by (12).

The components of **R** with respect to $O\tilde\xi\tilde\eta\tilde\zeta$ are therefore

$$\begin{aligned}p &= \dot\psi\sin\tilde\varphi\sin\omega + \dot\omega\cos\tilde\varphi\\q &= \dot\psi\cos\tilde\varphi\sin\omega - \dot\omega\sin\tilde\varphi\\r &= \dot\psi\cos\omega + \dot\varphi.\end{aligned}\tag{31}$$

The components L, M, N of the torque $\mathscr{M}$ referred to $OXYZ$ are related to $\tilde\lambda$, $\tilde\mu$, $\tilde\nu$ by

$$\begin{aligned}L &= \tilde\lambda\cos\tilde\varphi - \tilde\mu\sin\tilde\varphi\\M &= \tilde\lambda\sin\tilde\varphi + \tilde\mu\cos\tilde\varphi\\N &= \tilde\nu.\end{aligned}\tag{32}$$

From (30) and (32) we get

$$\begin{aligned}L &= \cos\tilde\varphi[A\dot p + (C-B)qr] - \sin\tilde\varphi[B\dot q + (A-C)rp]\\M &= \sin\tilde\varphi[A\dot p + (C-B)qr] + \cos\tilde\varphi[B\dot q + (A-C)rp]\\N &= C\dot r + (B-A)pq\end{aligned}\tag{33}$$

or,

$$\begin{aligned}L &= C(qr\cos\tilde\varphi + rp\sin\tilde\varphi) + A(\dot p\cos\tilde\varphi - rp\sin\tilde\varphi)\\&- B(qr\cos\tilde\varphi + \dot q\sin\tilde\varphi)\\M &= C(qr\sin\tilde\varphi - rp\cos\tilde\varphi) + A(\dot p\sin\tilde\varphi + rp\cos\tilde\varphi)\\&- B(qr\sin\tilde\varphi - \dot q\cos\tilde\varphi)\\N &= C\dot r + (B-A)pq\end{aligned}\tag{34}$$

and, by using (31),

$$\begin{aligned}L &= C\dot\psi\dot\varphi\sin\omega + A\ddot\omega + \frac{1}{2}(C-A)\dot\psi^2\sin 2\omega\\&- (B-A)F_1\\M &= -C\dot\omega\dot\varphi + A\ddot\psi\sin\omega + (A+B-C)\dot\psi\dot\omega\cos\omega\\&- (B-A)G_1\\N &= C\ddot\varphi + C\ddot\psi\cos\omega - C\dot\psi\dot\omega\sin\omega - (B-A)H_1,\end{aligned}\tag{35}$$

with:

$$\begin{aligned}F_1 &= \frac{1}{2}\ddot\psi\sin 2\tilde\varphi\sin\omega + \dot\psi\dot\varphi\cos 2\tilde\varphi\sin\omega - \dot\omega\dot\varphi\sin 2\tilde\varphi\\&- \frac{1}{2}\ddot\omega(1-\cos 2\tilde\varphi) + \frac{1}{4}\dot\psi^2(1+\cos 2\tilde\varphi)\sin 2\omega\\G_1 &= -\frac{1}{2}\ddot\psi(1+\cos 2\tilde\varphi)\sin\omega + \dot\psi\dot\varphi\sin 2\tilde\varphi\sin\omega\\&+ \dot\omega\dot\varphi\cos 2\tilde\varphi + \frac{1}{2}\ddot\omega\sin 2\tilde\varphi + \frac{1}{4}\dot\psi^2\sin 2\tilde\varphi\sin 2\omega\\H_1 &= \frac{1}{2}\dot\omega^2\sin 2\tilde\varphi - \dot\psi\dot\omega\cos 2\tilde\varphi\sin\omega\\&- \frac{1}{4}\dot\psi^2\sin 2\tilde\varphi(1-\cos 2\omega).\end{aligned}\tag{36}$$

4.2. *Torques due to the Moon, the Sun and the planets*

We used analytical solutions for the motion of the Moon, the Sun and the planets. They are expressed in terms of rectangular coordinates referred to $Oxyz$ (ecliptic and equinox J2000).

We first computed the components λ, μ, ν of the torque $\mathscr{M}$ referred to $O\xi\eta\zeta$ originating from a body P of mass m. Let Q (ξ', η', ζ') be a point of the Earth of mass dm_E. The disturbing force on Q, dF, due to P is then

$$dF = G\frac{m\,dm_E}{\Delta^3}\mathbf{QP},$$

where

$$\Delta = |\mathbf{QP}| = [(\xi-\xi')^2 + (\eta-\eta')^2 + (\zeta-\zeta')^2]^{\frac{1}{2}}.$$

The torque due to P is

$$\begin{aligned}d\mathscr{M} &= G\frac{m\,dm_E}{\Delta^3}\mathbf{OQ}\times\mathbf{QP}\\&= G\frac{m\,dm_E}{\Delta^3}\mathbf{OQ}\times\mathbf{OP},\end{aligned}$$

and the torque $\mathscr{M}$ on the Earth is

$$\mathscr{M} = Gm\iiint_{(\mathrm{Earth})}\frac{dm_E}{\Delta^3}\mathbf{OQ}\times\mathbf{OP}.$$

The component λ of $\mathscr{M}$ is given by

$$\begin{aligned}\lambda &= Gm\iiint_{(\mathrm{Earth})}\frac{1}{\Delta^3}(\eta'\zeta - \zeta'\eta)dm_E\\&= Gm\zeta\iiint_{(\mathrm{Earth})}\frac{\eta'}{\Delta^3}dm_E - Gm\eta\iiint_{(\mathrm{Earth})}\frac{\zeta'}{\Delta^3}dm_E.\end{aligned}\tag{37}$$

The components μ and ν are given by analogous expressions.

The geopotential U in P (ξ, η, ζ) is

$$U = \iiint_{(\mathrm{Earth})}\frac{G}{\Delta}dm_E,$$

and the partial derivative

$$\begin{aligned}\frac{\partial U}{\partial\xi} &= \iiint_{(\mathrm{Earth})} -G\frac{\xi-\xi'}{\Delta^3}dm_E\\&= -G\xi\iiint_{(\mathrm{Earth})}\frac{dm_E}{\Delta^3} + G\iiint_{(\mathrm{Earth})}\frac{\xi'}{\Delta^3}dm_E.\end{aligned}\tag{38}$$

$\frac{\partial U}{\partial\eta}$ and $\frac{\partial U}{\partial\zeta}$ are given by analogous expressions.

From (37) and (38) we get

$$\begin{aligned}\lambda &= m(\zeta\frac{\partial U}{\partial\eta} - \eta\frac{\partial U}{\partial\zeta})\\\mu &= m(\xi\frac{\partial U}{\partial\zeta} - \zeta\frac{\partial U}{\partial\xi})\\\nu &= m(\eta\frac{\partial U}{\partial\xi} - \xi\frac{\partial U}{\partial\eta}).\end{aligned}\tag{39}$$

We then compute λ, μ, ν in terms of the variables r, β, Λ defined in (4). Partial derivatives of r, β, Λ are found to be

$$\begin{aligned}
\frac{\partial r}{\partial \xi} &= \frac{\xi}{r}; \quad \frac{\partial r}{\partial \eta} = \frac{\eta}{r}; \quad \frac{\partial r}{\partial \zeta} = \frac{\zeta}{r};\\
\frac{\partial \beta}{\partial \xi} &= -\frac{\xi\eta}{r^2(\xi^2+\eta^2)^{\frac{1}{2}}}; \quad \frac{\partial \beta}{\partial \eta} = -\frac{\eta\zeta}{r^2(\xi^2+\eta^2)^{\frac{1}{2}}};\\
\frac{\partial \beta}{\partial \zeta} &= \frac{(\xi^2+\eta^2)^{\frac{1}{2}}}{r^2};\\
\frac{\partial \Lambda}{\partial \xi} &= -\frac{\eta}{\xi^2+\eta^2}; \quad \frac{\partial \Lambda}{\partial \eta} = \frac{\xi}{\xi^2+\eta^2}; \quad \frac{\partial \Lambda}{\partial \zeta} = 0.
\end{aligned} \tag{40}$$

$\frac{\partial U}{\partial \xi}$ can be written in the form

$$\frac{\partial U}{\partial \xi} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial \xi} + \frac{\partial U}{\partial \beta}\frac{\partial \beta}{\partial \xi} + \frac{\partial U}{\partial \Lambda}\frac{\partial \Lambda}{\partial \xi},$$

with analogous expressions for $\frac{\partial U}{\partial \eta}$ and $\frac{\partial U}{\partial \zeta}$. With the use of (39) and (40) we finally obtain

$$\begin{aligned}
\lambda &= -m\frac{\eta}{(\xi^2+\eta^2)^{\frac{1}{2}}}\frac{\partial U}{\partial \beta} + m\frac{\xi\zeta}{(\xi^2+\eta^2)}\frac{\partial U}{\partial \Lambda}\\
\mu &= m\frac{\xi}{(\xi^2+\eta^2)^{\frac{1}{2}}}\frac{\partial U}{\partial \beta} + m\frac{\eta\zeta}{(\xi^2+\eta^2)}\frac{\partial U}{\partial \Lambda}\\
\nu &= -m\frac{\partial U}{\partial \Lambda}
\end{aligned} \tag{41}$$

or

$$\begin{aligned}
\lambda &= -m\sin\Lambda\frac{\partial U}{\partial \beta} + m\tan\beta\cos\Lambda\frac{\partial U}{\partial \Lambda}\\
\mu &= m\cos\Lambda\frac{\partial U}{\partial \beta} + m\tan\beta\sin\Lambda\frac{\partial U}{\partial \Lambda}\\
\nu &= -m\frac{\partial U}{\partial \Lambda}
\end{aligned} \tag{42}$$

with

$$\begin{aligned}
\sin\Lambda &= \eta(\xi^2+\eta^2)^{-\frac{1}{2}}; \quad \sin\beta = \zeta(\xi^2+\eta^2+\zeta^2)^{-\frac{1}{2}};\\
\cos\Lambda &= \xi(\xi^2+\eta^2)^{-\frac{1}{2}}; \quad \cos\beta = (\xi^2+\eta^2)^{\frac{1}{2}}(\xi^2+\eta^2+\zeta^2)^{-\frac{1}{2}};\\
& \tan\beta = \zeta(\xi^2+\eta^2)^{-\frac{1}{2}}.
\end{aligned}$$

Lastly, we compute the components L, M, N of $\mathscr{M}$, referred to $OXYZ$

$$\begin{aligned}
L &= \lambda\cos\varphi - \mu\sin\varphi\\
M &= \lambda\sin\varphi + \mu\cos\varphi\\
N &= \nu,
\end{aligned}$$

and thus get

$$\begin{aligned}
L &= -m\sin(\Lambda+\varphi)\frac{\partial U}{\partial \beta} + m\tan\beta\cos(\Lambda+\varphi)\frac{\partial U}{\partial \Lambda}\\
M &= m\cos(\Lambda+\varphi)\frac{\partial U}{\partial \beta} + m\tan\beta\sin(\Lambda+\varphi)\frac{\partial U}{\partial \Lambda}\\
N &= -m\frac{\partial U}{\partial \Lambda}.
\end{aligned} \tag{43}$$

The derivatives of the geopotential are taken from (7):

$$\begin{aligned}
\frac{\partial U}{\partial \beta} &= \frac{Gm_E}{r}\sum_{n=2}^{\infty}\frac{a_E^n}{r^n}C_{n,0}\frac{\partial}{\partial \beta}[P_{n,0}(\sin\beta)]\\
&+ \frac{Gm_E}{r}\sum_{n=2}^{\infty}\sum_{k=1}^{\infty}\frac{a_E^n}{r^n}(C_{n,k}\cos k\Lambda + S_{n,k}\sin k\Lambda)\\
&\times\frac{\partial}{\partial \beta}[P_{n,k}(\sin\beta)]\\
\frac{\partial U}{\partial \Lambda} &= \frac{Gm_E}{r}\sum_{n=2}^{\infty}\sum_{k=1}^{\infty}\frac{a_E^n}{r^n}k(S_{n,k}\cos k\Lambda - C_{n,k}\sin k\Lambda)\\
&\times P_{n,k}(\sin\beta).
\end{aligned} \tag{44}$$

(35), (36), (43) and (44) are the equations to be solved.

5. Models used

5.1. Potential

The torques on the oblate rigid Earth due to the gravitational attraction of the Moon, the Sun and the planets from Mercury to Neptune are considered. The effects of the zonal harmonics $C_{n,0}$ with $2 \le n \le 5$ and of the tesseral harmonics $C_{2,2}$, $S_{2,2}$, $C_{3,k}$, $S_{3,k}$ with $1 \le k \le 3$, $C_{4,1}$, $S_{4,1}$ are computed.

5.2. Motion of the Sun and the planets

For the motion of the Sun and the planets we use the solution VSOP87A (Bretagnon, Francou, 1988). This solution gives heliocentric rectangular coordinates of the Earth (not the Earth-Moon barycenter) and the planets referred to the ecliptic and equinox J2000. Solutions are given in terms of Poisson series for the angles (26).

5.3. Motion of the Moon

We compute geocentric rectangular coordinates of the Moon referred to the ecliptic and equinox J2000 from the solution ELP 2000 (Chapront-Touzé, Chapront, 1983) truncated to $0''.001$. In ELP 2000, the solutions are Poisson series for the angles (26), for the mean anomaly l' of the Sun and for the mean longitude $\mathscr{L}$ of the Moon referred to the equinox of date. $\mathscr{L}$ is related to λ_3 and D by:

$$\mathscr{L} = \lambda_3 + D + \mathscr{P}t - 180°, \tag{45}$$

where $\mathscr{P}$ is the constant of precession. Value of $\mathscr{P}$ is given by (13). Here $\lambda_3 + D$ is substituted for $\mathscr{L}$ and $\mathscr{P}$ is developed in Poisson series (period of $\mathscr{P}$ is about 26 000 years). l' is related to λ_3 and to the mean longitude ϖ_3 of the perihelion of the Earth referred to the ecliptic and equinox J2000 by:

$$\lambda_3 = l' + \varpi_3. \tag{46}$$

According to Simon et al (1994)

$$\varpi_3 = 1.796\,595\,647\,27 + 0.056\,298\,2756\,t + \cdots .$$

Here λ_3 is substituted for l' and ϖ_3 (with a period of about 111 600 years) is developed as a Poisson series. The angles (26) are referred to the ecliptic and equinox J2000. So, in the solutions of the nutation presented in this paper, the argument of the most important term is not the longitude Ω_D of the node of the Moon referred to the equinox of date, but the longitude Ω_{J2000} of the node of the Moon referred to the equinox J2000, given in the form:

$$\begin{aligned} \lambda_3 + D - F &= \Omega_{J2000} + 180° \\ &= 5.324\,031\,967\,15 - 337.814\,263\,1964\,t. \end{aligned} \quad (47)$$

The period of this argument is 18.599 526 years = 6793.48 days.

With this representation, we find no arguments of similar periods in the series while, for instance, the following arguments are found in the classical nutation tables:

$$\begin{aligned} &\Omega_D && \text{(period 6798.38 days,)} \\ &-2l' + 2F - 2D + \Omega_D && \text{(period 6786.32 days.)} \end{aligned} \quad (48)$$

These arguments are in phase every 10 468 years. Also we find the arguments

$$\begin{aligned} &l - l' - \ D && \text{(period 3232.86 days,)} \\ \text{and } &l - F - \ \Omega_D && \text{(period 3231.50 days.)} \end{aligned} \quad (49)$$

These arguments are in phase every 20 937 years.

Many such couples of arguments can be found which need not be kept in expressions of nutation valid over some thousand years. Here, the argument $\lambda_3 + D - F$ (period 6793.48 days) is substituted for the arguments (48) and the argument $\lambda_3 + D - l$ (period 3232.61 days) is substituted for the arguments (49).

6. Solution of the equations

6.1. The method

We obtain the following differential equations of the second order from (35)

$$\begin{aligned} \ddot{\omega} + \frac{C}{A} \sin\omega_0\, \varphi_1\, \dot{\psi} &= \frac{L}{A} + F_2 + \frac{B - A}{A} F_1 \\ \sin\omega_0\, \ddot{\psi} - \frac{C}{A} \varphi_1\, \dot{\omega} &= \frac{M}{A} + G_2 + \frac{B - A}{A} G_1 \\ \ddot{\varphi} &= \frac{N}{C} + H_2 + \frac{B - A}{C} H_1 \end{aligned} \quad (50)$$

with

$$\begin{aligned} F_2 &= -\frac{C}{A} \varphi_1\, \dot{\psi}\, (\sin\omega - \sin\omega_0) - \frac{C}{A} \Delta\dot{\varphi}\, \dot{\psi}\, \sin\omega \\ &\quad -\frac{1}{2}\, \frac{C - A}{A}\, \dot{\psi}^2\, \sin 2\omega \\ G_2 &= \frac{C}{A} \Delta\dot{\varphi}\, \dot{\omega} - \ddot{\psi}\, (\sin\omega - \sin\omega_0) \\ &\quad -\frac{A + B - C}{A}\, \dot{\psi}\, \dot{\omega}\, \cos\omega \\ H_2 &= -\ddot{\psi}\, \cos\omega + \dot{\psi}\, \dot{\omega}\, \sin\omega, \end{aligned} \quad (51)$$

where we defined $\Delta\varphi$ in (11). F_1, G_1, H_1 are given by (36).

The solution of the reduced system derived from the first two equations of (50) is

$$\begin{aligned} \dot{\omega} &= K_S\, \sin\sigma t + K_C\, \cos\sigma t \\ \sin\omega_0\, \dot{\psi} &= K_C\, \sin\sigma t - K_S\, \cos\sigma t, \end{aligned} \quad (52)$$

where

$$\sigma = \frac{C}{A}\, \varphi_1 = 2\,308\,800.579\,286\,298 \text{ rd tjy}^{-1}.$$

σ is the Euler frequency in the coordinate system J2000. The period of this term is 0.997 day in the coordinate system J2000 and 305 days in the terrestrial coordinate system.

K_S et K_C are the two integration constants of the reduced system.

We find a particular solution of (50) by the method of the variation of the constants. One gets

$$\begin{aligned} \dot{K}_S &= \frac{L}{A} \sin\sigma t - \frac{M}{A} \cos\sigma t + \left(F_2 + \frac{B - A}{A} F_1\right) \sin\sigma t \\ &\quad - \left(G_2 + \frac{B - A}{A} G_1\right) \cos\sigma t \\ \dot{K}_C &= \frac{L}{A} \cos\sigma t + \frac{M}{A} \sin\sigma t + \left(F_2 + \frac{B - A}{A} F_1\right) \cos\sigma t \\ &\quad + \left(G_2 + \frac{B - A}{A} G_1\right) \sin\sigma t. \end{aligned} \quad (53)$$

The first two equations of (50) are solved by integrating (53), then substituting in (52). The third equation is solved by a double integration. In the Eqs. (50), the quantities F, G, H are related to ψ, ω and φ. For computing L, M, N the disturbing body must be referred in the coordinate system $O\xi\eta\zeta$ by a rotation also depending on ψ, ω and φ. It is thus necessary to recompute the right-hand sides of the Eqs. (50) and to proceed by iterations. The process is stopped when the internal convergence of the iterative method is 0.01 μas.

6.2. Results

Table 1 gives the most important periodic perturbation for each term in the nutation. The amplitude of the periodic terms with a period near 1 day due to the effect of the Moon on the harmonics $C_{n,1} - S_{n,1}$ must be noted. For the planets only the effect on $C_{2,0}$ is significant. The complements correspond to the quantities F and G of the Eqs. (50).

Table 2 gives secular terms of ψ and ω due to the various effects. As indicated in Sect. 1, ψ and ω are reckoned in the positive direction from the ecliptic and equinox J2000.

6.3. Form of the solution

Here, ψ, ω and φ are series of angles referred to the ecliptic and equinox J2000. The precession, which is an argument of the actual theories, has been developed as polynomials in time. The development corresponding to the periodic term of period

Table 1. Most important perturbations of the nutation. Amplitudes are in μas, periods in days.

Origin		Argument	Amplitude of ψ	Amplitude of ω	Period
Moon	$C_{2,0}$	$\lambda_3 + D - F$	17 292 173.10	9 227 878.02	6 793.48
	$C_{3,0}$	$\lambda_3 + D - l$	104.05	88.95	3 232.61
	$C_{4,0}$	$\lambda_3 + D - F$	0.73	6.84	6 793.48
	$C_{5,0}$	$\lambda_3 + D - l$	0.01	0.00	3 232.61
	$C_{2,2} - S_{2,2}$	$2\lambda_3 + 2D - 2\varphi$	29.44	11.71	0.52
	$C_{3,1} - S_{3,1}$	$\lambda_3 + D + \varphi$	38.44	15.25	0.96
	$C_{3,2} - S_{3,2}$	$\lambda_3 + D - 2\varphi$	0.39	0.14	0.51
	$C_{3,3} - S_{3,3}$	$3\lambda_3 + 3D - 3\varphi$	0.14	0.05	0.35
	$C_{4,1} - S_{4,1}$	φ	1.68	0.67	1.00
Sun	$C_{2,0}$	$2\lambda_3$	1 276 710.99	552 389.67	182.63
	$C_{3,0}$	λ_3	0.26	0.22	365.26
	$C_{2,2} - S_{2,2}$	$2\lambda_3 - 2\varphi$	12.32	4.90	0.50
	$C_{3,1} - S_{3,1}$	$\lambda_3 + \varphi$	2.79	1.11	1.00
Mercury	$C_{2,0}$	$\lambda_1 - 4\lambda_3$	1.03	0.43	2 432.11
Venus	$C_{2,0}$	$3\lambda_2 - 5\lambda_3$	216.71	90.76	2 959.21
Mars	$C_{2,0}$	$\lambda_3 - 2\lambda_4$	11.55	0.95	5 764.01
Jupiter	$C_{2,0}$	$2\lambda_5$	104.41	45.69	2 166.29
Saturn	$C_{2,0}$	$2\lambda_6$	12.15	5.16	5 379.61
Uranus	$C_{2,0}$	$2\lambda_7$	0.65	0.29	15 344.24
Neptune	$C_{2,0}$	$2\lambda_8$	0.40	0.16	30 091.15
Complements		$\lambda_3 + D - F$	15 361.12	0.04	6 793.48

Table 2. Secular term of ψ and of ω in arcseconds per thousand years.

Origin		$-\psi$	$-\omega$
Moon	$C_{2,0}$	34 454.955 449	−0.254 414
	$C_{3,0}$	−0.000 057	−0.000 011
	$C_{4,0}$	0.025 192	
Sun	$C_{2,0}$	15 948.701 343	0.002 923
	$C_{3,0}$	−0.000 026	−0.000 005
Mercury	$C_{2,0}$	0.003 698	−0.000 088
Venus	$C_{2,0}$	0.181 581	−0.016 814
Mars	$C_{2,0}$	0.005 999	0.000 357
Jupiter	$C_{2,0}$	0.117 059	0.002 804
Saturn	$C_{2,0}$	0.005 208	0.000 220
Uranus	$C_{2,0}$	0.000 100	0.000 001
Neptune	$C_{2,0}$	0.000 029	0.000 001
Complements		−0.231 853	
	p_g	−19.198 827	−0.000 010
		50 384.564 895	−0.265 036

18.6 years, truncated to 1 μas over 1900–2100 has therefore the following form:

$$\begin{aligned}\psi = &- 17''.280\,764\,99 \sin\beta + 0''.000\,439\,93 \cos\beta \\ &- 0''.084\,215\,93\, t \sin\beta - 4''.226\,495\,99\, t \cos\beta \\ &+ 0''.518\,035\,06\, t^2 \sin\beta - 0''.056\,363\,12\, t^2 \cos\beta \\ &+ 0''.010\,143\,60\, t^3 \sin\beta + 0''.042\,273\,00\, t^3 \cos\beta,\end{aligned}$$

where $\beta = \lambda_3 + D - F$.

7. Precision of the solution

7.1. Comparison to numerical integration

We used numerical integration to test the analytical development described in Sect. 6. We transform Eqs. (50) in the form

$$\begin{aligned}\ddot{\omega} = &\frac{L}{A} + \frac{B-A}{A}\sin\tilde{\varphi}\Big(\frac{M}{A}\cos\tilde{\varphi} - \frac{L}{A}\sin\tilde{\varphi}\Big) - \dot{\psi}\dot{\varphi}\sin\omega \\ &- \frac{C-B}{A}\dot{\psi}\sin\omega\,(\dot{\varphi} + \dot{\psi}\cos\omega) \\ &+ \frac{B-A}{A}\,\frac{C-A-B}{B} \\ &\quad\times(\dot{\psi}\sin\omega\sin^2\tilde{\varphi} + \frac{1}{2}\dot{\omega}\sin 2\tilde{\varphi})\,(\dot{\varphi} + \dot{\psi}\cos\omega) \\ \sin\omega\,\ddot{\psi} = &\frac{M}{B} + \frac{B-A}{A}\sin\tilde{\varphi}\Big(\frac{M}{B}\sin\tilde{\varphi} + \frac{L}{B}\cos\tilde{\varphi}\Big) \\ &+ \frac{C+B-A}{B}\dot{\varphi}\dot{\omega} + \frac{C-A-B}{B}\dot{\psi}\dot{\omega}\cos\omega \qquad (54) \\ &+ \frac{B-A}{A}\,\frac{C-A-B}{B}\end{aligned}$$

$$\times(\dot\omega \sin^2\tilde\varphi - \frac{1}{2}\dot\psi \sin\omega \sin 2\tilde\varphi)\,(\dot\varphi + \dot\psi\cos\omega)$$
$$\ddot\varphi = \frac{N}{C} + H_2 + \frac{B-A}{C} H_1$$

We used the Bulirsch-Stoer method (Stoer and Bulirsch, 1980). This integrator solves first order differential equations. The second order Eqs. (54) are therefore written as six first order equations:

$$\begin{aligned} \frac{d\omega}{dt} &= \dot\omega \;; & \frac{d\dot\omega}{dt} &= f_1(\omega,\tilde\varphi,\dot\omega,\dot\psi,\dot\varphi); \\ \frac{d\psi}{dt} &= \dot\psi \;; & \frac{d\dot\psi}{dt} &= f_2(\omega,\tilde\varphi,\dot\omega,\dot\psi,\dot\varphi); \\ \frac{d\varphi}{dt} &= \dot\varphi \;; & \frac{d\dot\varphi}{dt} &= f_3(\omega,\tilde\varphi,\dot\omega,\dot\psi,\dot\varphi). \end{aligned} \quad (55)$$

The right-hand sides of the equations and the constants are the same as in the analytical method. The only difference is that the coordinates of the Moon are computed with the full solution ELP 2000 (Chapront-Touzé, Chapront, 1983).

To test the accuracy of the integrator we integrated the problem over intervals of 100 yr, in a backward direction in time (2000–1900) and its reverse. We used a maximum stepsize of 0.5 day. We compared the results every 50 days. We only found a drift of 0.14 μas over 100 yr for the variable φ.

We have compared the analytical solution with a numerical integration run over 50 days, in order to test the accuracy of the diurnal terms and with a numerical integration run over 150 yr (1900–2050). The comparison with the integration over 50 days is illustrated by the Fig. 3. For the three variables the differences are smaller than 0.4 μas, compared to the amplitudes of the diurnal terms given in table 1 (40 μas). The comparison with the integration over 150 yr is illustrated by the Fig. 4. The differences are smaller than 16 μas for ψ, 8 μas for ω and 15 μas for φ.

7.2. Accuracy of the analytical developments

From the comparison with the numerical integration it can be seen that the accuracy achieved of our analytical theory is 16 μas for ψ and 8 μas for ω. This precision is essentially limited by the accuracy of the solution used for the motion of the Moon and by the accuracy of the computations.

The solution used for the motion of the Moon is truncated to 0″.001 and thus ensures an accuracy of 0″.09 for the motion of the Moon. In the future the full solution ELP 2000 will be used and we must achieve an accuracy of about 1 μas.

For the effect on $C_{2,0}$ due to the Moon, the right-hand side of the Eqs. (53) has been computed accurate to 6.15 10^{-4}tjy^{-2}. After integration, the terms of frequency γ are accurate to

$$6.66\,10^{-10}/\gamma = 0''.000\,137/\gamma \text{ for } \psi$$
$$2.66\,10^{-10}/\gamma = 0''.000\,055/\gamma \text{ for } \omega.$$

For ψ, all terms with a period $\leq$ 46 years are computed with an accuracy better than 1 μas. The term with period 18.6 years is computed accurate to 0.4 μas.

Table 3. Precision of the terms of period > 46 years in $\Delta\psi$ due to $C_{2,0}$ (Moon).

Argument	Precision (μas)	Frequency (rdi/tjy)	Period (years)
$2\lambda_5 - 5\lambda_6$	18	– 7.114	883
$3\lambda_3 - 2\lambda_5 + 3D - F - 2l$	6	22.670	277
$18\lambda_2 - 16\lambda_3 - l$	6	23.012	273
$8\lambda_2 - 13\lambda_3$	5	26.298	239
$3\lambda_3 + 3D - 2F - l$	4	34.305	183
$2\lambda_4 + D - F$	2	60.335	104
$3\lambda_2 - 4\lambda_3 + D - l$	2	– 65.590	96
λ_7	2	74.782	84

Table 3 gives the terms of period > 46 years which are computed with an accuracy worse than 1 μas. It must be noted that, for the effects on the other harmonics due to the Moon and for the effects due to the other disturbing bodies, the accuracy is much better.

7.3. Effect on the constant of precession

Some authors claim that an error on the amplitude of the long-period terms induces a modification of the determination of the constant of precession. This is not true. Let $\phi = \phi_0 + \gamma t$ be a long-period argument in the solution. In Sect. 6.4 we stated that the accuracy of this term is

$$\delta\psi = \frac{0''.000\,137}{\gamma}\;\sin(\phi_0 + \gamma t)$$

Developing $\sin(\phi_0 + \gamma t)$ in term of time t, we find

$$\begin{aligned}\delta\psi &= 0''.000\,137/\gamma\;\sin\phi_0 + 0''.000\,137\,t\cos\phi_0 \\ &- 0''.000\,069\,\gamma\,t^2\sin\phi_0 + \cdots\end{aligned}$$

The constant term is absorbed by the determination of the integration constant; thus we see that the modification of the constant of precession due to the t–term is about some 0″.0001 compared to the error of some 0″.1 in $\mathscr{P}$.

7.4. Nutation on nutation terms

The ELP82 solution contains the Earth figure perturbations. The Earth nutation terms produce in the longitude of the Moon perturbations the most important of which is (Chapront-Touzé, 1982):

$$-0''.002\,23\sin(F-\zeta),$$

where ζ is the mean longitude of the Moon reckoned from the equinox of date.

This perturbation is 40 times smaller than the error due to the truncation of the ELP solution (0″.090) and is taken into account. It gives a second order effect in the nutation of the terrestrial equator negligible with respect to the differences between the

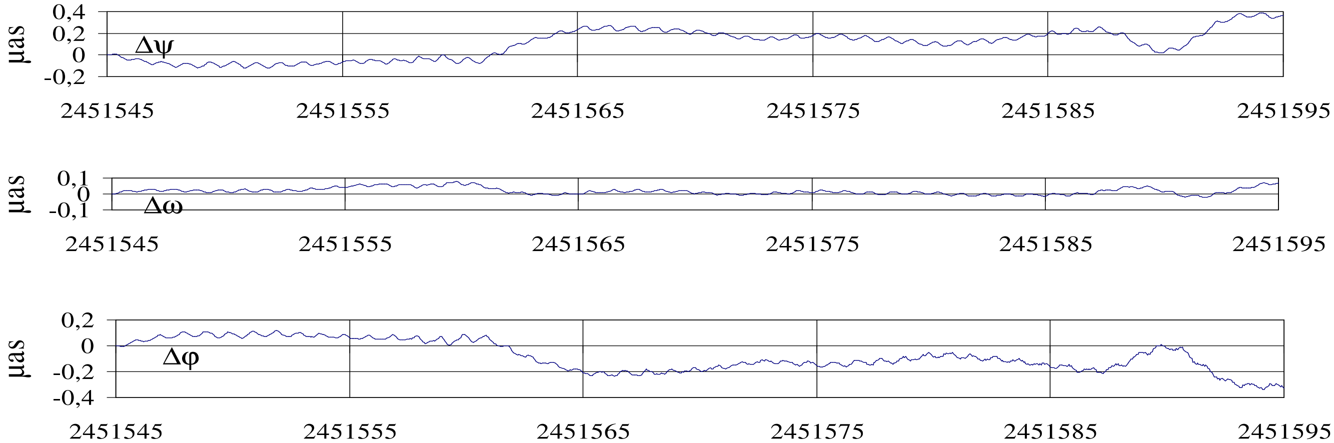

Fig. 3. Theory – numerical integration over 50 days

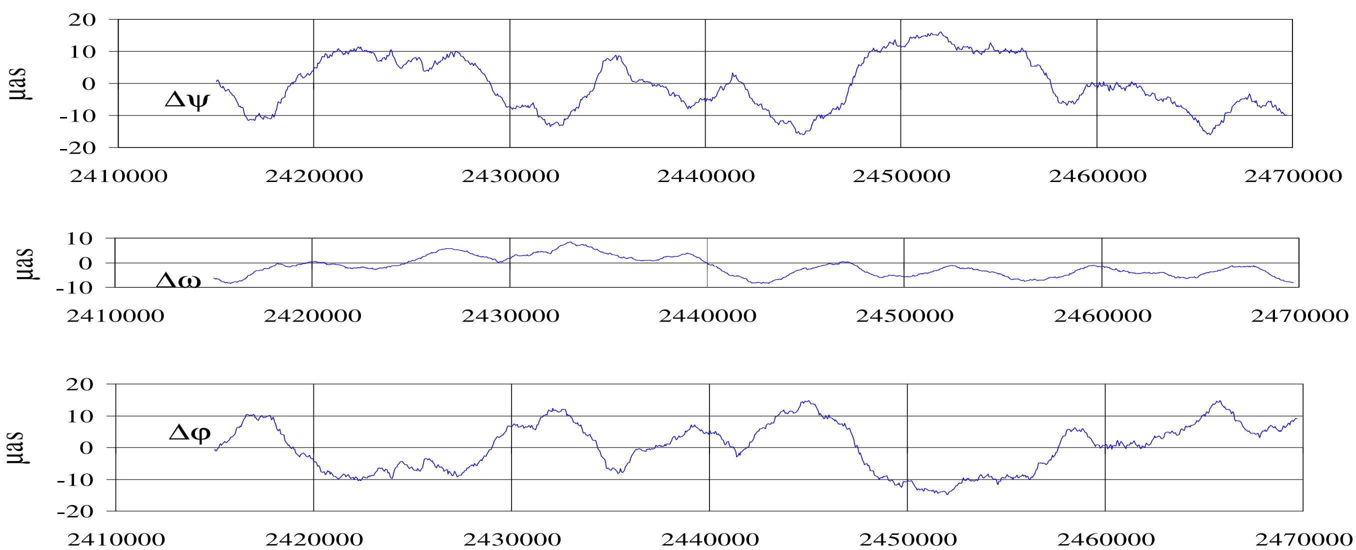

Fig. 4. Theory – numerical integration over 1900-2050

analytical solution and the numerical integration (16 μas). The third order effects are still smaller. They have not been computed neither in the analytical solution or in the numerical integration.

8. Computation of kinematical quantities $\psi_K, \omega_K, \varphi_K$

From the dynamical quantities ψ, ω, φ, we computed the kinematical quantities ψ_K, ω_K, φ_K, defined by the Brumberg's (1996) formulas:

$$\psi_K - \psi = c^{-2}[F^3 + \frac{\cos\omega}{\sin\omega}(-F^1 \sin\psi + F^2 \cos\psi)]$$
$$\omega_K - \omega = c^{-2}(F^1 \cos\psi + F^2 \sin\psi)$$
$$\varphi_K - \varphi = \frac{c^{-2}}{\sin\omega}(-F^1 \sin\psi + F^2 \cos\psi),$$

where c is the velocity of light. We use the Brumberg et al's (1992) expressions of $c^{-2}F^1$, $c^{-2}F^2$, $c^{-2}F^3$.

Finally we have, in μas:

$$\begin{aligned}
\psi_K - \psi = {} & 19\,198\,827.44t - 50\,386.33t^2 - 754.08t^3 \\
& -34.28\sin(\lambda_3) - 149.22\cos(\lambda_3) \\
& +3.01\sin(\lambda_3 + D - F) \\
& -1.73\sin(2\lambda_3) + 0.84\cos(2\lambda_3) + \ldots \\
& +t[-7.34\sin(\lambda_3) + 6.45\cos(\lambda_3) + \ldots] \\
& +t^2[-0.47\sin(\lambda_3 + D - F) \\
& \quad - 6.16\cos(\lambda_3 + D - F) + \ldots] \\
& +t^3[2.42\sin(\lambda_3 + D - F) \\
& \quad - 0.20\cos(\lambda_3 + D - F) + \ldots]
\end{aligned}$$

$$\begin{aligned}
\omega_K - \omega = {} & 9.55t + 1\,954.11t^2 - 4\,721.80t^3 \\
& -1.30\cos(\lambda_3 + D - F) \\
& +0.17\sin(2\lambda_5 - 5\lambda_6) - 0.09\cos(2\lambda_5 - 5\lambda_6)
\end{aligned}$$

$$
\begin{aligned}
&+\ldots\\
&+t[0.32\sin(\lambda_3+D-F)+\ldots]\\
&+t^2[-1.83\sin(\lambda_3+D-F)\\
&\quad+0.04\cos(\lambda_3+D-F)+\ldots]
\end{aligned}
$$

$$
\begin{aligned}
\varphi_K-\varphi \;=\;& -3.01t-54\,771.04t^2-802.04t^3\\
&+3.28\sin(\lambda_3+D-F)\\
&+t[0.21\sin(\lambda_3)\\
&\quad+0.83\cos(\lambda_3)+\ldots]\\
&+t^2[-0.51\sin(\lambda_3+D-F)\\
&\quad-5.65\cos(\lambda_3+D-F)+\ldots]\\
&+t^3[2.37\sin(\lambda_3+D-F)\\
&\quad-0.20\cos(\lambda_3+D-F)+\ldots]
\end{aligned}
$$

9. Computation of the variables $\mathscr{P}$, ε, χ and of the sidereal time

Now we note ψ, ω, φ the kinematical quantities computed in Sect. 8. The precessional quantities Π_A and π_A describe the motion of the mean ecliptic of date on the ecliptic J2000. The variables ψ and ω describe the motion of the mean equator of the date on the ecliptic J2000. From $\Pi_A, \pi_A, \psi, \omega$, we can compute the precession quantities $\mathscr{P}, \varepsilon$, and χ. Here, $\mathscr{P}, \varepsilon$, and χ together contain the secular terms of the precession and the periodic terms of the nutation. For the precession quantities θ, ζ, and z, we can only compute the secular parts.

From the spherical triangle NR$\gamma_{\rm D}$ of the Fig. 2, we obtain the relations:

$$\cos\varepsilon=\sin\omega\sin\pi_A\cos(\Pi_A-\psi)+\cos\omega\cos\pi_A, \tag{56}$$

$$\sin(\Pi_A+\mathscr{P})=-\sin(\Pi_A-\psi)\frac{\sin\omega}{\sin\varepsilon}. \tag{57}$$

Note that, on the Fig. 2, ψ is the arc γ_{J2000}R and ω is reckoned positive from the ecliptic to the equator. On the other hand, in (56) and (57), $\mathscr{P}, \varepsilon$ and χ = R$\gamma_{\rm D}$ are measured as in Lieske et al. (1977). The variables π_A and Π_A are according to Simon et al. (1994):

$$
\begin{aligned}
\pi_A \;=\;& 469''.9756\,t-3''.3505\,t^2-0''.1237\,t^3+0''.0003\,t^4\\
=\;& 0.002\,278\,5060\,t-16.2437\times10^{-6}\,t^2\\
&-\;0.5997\times10^{-6}\,t^3+0.0015\times10^{-6}\,t^4\\
\Pi_A \;=\;& 629\,543''.988-8\,679''.218\,t+15''.342\,t^2\\
&+0''.005\,t^3-0''.037\,t^4-0''.001\,t^5\\
=\;& 3.052\,115\,382-0.042\,078\,036\,t+74.380\times10^{-6}\,t^2\\
&+0.024\times10^{-6}\,t^3-0.018\times10^{-6}\,t^4\\
&-0.005\times10^{-6}\,t^5.
\end{aligned}
\tag{58}
$$

We then solve Eqs. (56) and (57) by successive approximations.

We find that the periodic terms are the same for ψ and $\mathscr{P}$ and for ω and ε. Only the Poisson terms are different. We give hereafter the differences between $\mathscr{P}$ and $-\psi$ up to $10^{-5}{''}$ per century and differences between ε and $-\omega$ up to 0.4 $10^{-5}{''}$ per century:

$$
\begin{aligned}
\mathscr{P}-(-\psi) \;=\;& -96''.864\,878\,t+218''.538\,688\,t^2\\
&+1''.222\,063\,t^3-1''.565\,199\,t^4\\
&+\;t\,[\,0''.090\,455\sin(\lambda_3+D-F)\\
&\quad-0''.011\,877\cos(\lambda_3+D-F)\\
&\quad-0''.006\,694\sin\,2\lambda_3+0''.000\,716\cos\,2\lambda_3\\
&\quad-0''.001\,159\sin(2\lambda_3+2D)\\
&\quad+0''.000\,122\cos(2\lambda_3+2D)\\
&\quad+0''.001\,100\sin(2\lambda_3+2D-2F)\\
&\quad-0''.000\,117\cos(2\lambda_3+2D-2F)\\
&\quad-0''.000\,161\sin\,\lambda_3-0''.000\,529\cos\,\lambda_3\\
&\quad+0''.000\,355\sin\,l\\
&\quad+0''.000\,086\sin\,3\lambda_3+0''.000\,249\sin\,3\lambda_3\\
&\quad+0''.000\,198\sin(\lambda_3+D+F)\\
&\quad-0''.000\,025\cos(\lambda_3+D+F)\\
&\quad-0''.000\,155\sin(2\lambda_3+2D+l)\\
&\quad+0''.000\,016\cos(2\lambda_3+2D+l)\,]\\
&+\;t^2\,[\,0''.004\,852\sin(\lambda_3+D-F)\\
&\quad+0''.048\,845\cos(\lambda_3+D-F)\\
&\quad-0''.000\,430\sin\,2\lambda_3-0''.004\,872\cos\,2\lambda_3\,]\\
&+\;t^3\,[\,-0''.011\,062\sin(\lambda_3+D-F)\\
&\quad+0''.001\,322\cos(\lambda_3+D-F)],
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon-(-\omega) \;=\;& -468''.095\,499\,t-5''.148\,267\,t^2+9''.735\,102\,t^3\\
&+\;t\,[\,-0''.003\,518\sin(\lambda_3+D-F)\\
&\quad+0''.000\,260\sin\,2\lambda_3\\
&\quad+0''.000\,045\sin(2\lambda_3+2D)\\
&\quad-0''.000\,043\sin(2\lambda_3+2D-2F)\,]\\
&+\;t^2\,[\,0''.007\,923\sin(\lambda_3+D-F)\\
&\quad-0''.000\,860\cos(\lambda_3+D-F)\\
&\quad-0''.000\,589\sin\,2\lambda_3+0''.000\,127\cos\,2\lambda_3\,],
\end{aligned}
$$

where t is counted in thousand of Julian years from J2000.

χ is given by

$$\sin\chi=-\sin\pi_A\frac{\sin(\Pi_A+\mathscr{P})}{\sin\omega}$$

The development of χ contains polynomials in time and Poisson terms but no periodic terms.

The sidereal time is given in (10). The periodic terms of the sidereal time are therefore the same as the periodic terms of φ.

Note a precessional quantity in the split

$$X=X_S+X_P,$$

where X_S is the secular part of X and where X_P is the periodic and Poisson part. The sidereal time TS can be then computed

Table 4. Secular parts of the precession quantities. Unit is arcsec. Time t is in thousand of Julian years (tjy) from J2000.

	t^0	t	t^2	t^3	t^4	t^5
P	0.00000	41.99605	19.39714	−0.22351	−0.01036	0.00019
Q	0.00000	−468.09553	5.10429	0.52233	−0.00567	−0.00013
π	0.00000	469.97563	−3.35058	−0.12374	0.00027	−0.00001
Π	629 543.988	−8679.218	15.342	0.005	−0.037	−0.001
$\mathscr{P}$	0.00000	50287.70000	111.24285	0.07709	−0.23465	0.00182
θ	0.00000	20041.82085	−42.66962	−41.82308	−0.07291	−0.01127
ζ	2.72767	23060.70257	30.23242	18.01728	−0.05708	−0.03040
z	−2.72767	23060.65855	109.56706	18.26652	−0.28276	−0.02486
ε	84381.41200	−468.36057	−0.01664	1.99906	−0.00512	−0.00259
$-\omega$	84381.41200	−0.26504	5.12764	−7.72726	−0.00484	0.03318
$-\psi$	0.00000	50384.56488	−107.19537	−1.14353	1.32847	−0.00578
χ	0.00000	105.57686	−238.13644	−1.21255	1.70239	−0.00769

as a function of $\mathscr{P}_P$ and ε_S. We give hereafter the terms greater than 10 μas:

$$
\begin{aligned}
TS = TS_S &+ \mathscr{P}_P \times \cos\varepsilon_S \\
&- 0''.002\,642\,57 \sin(\lambda_3 + D - F) \\
&- 0''.000\,000\,06 \cos(\lambda_3 + D - F) \\
&- 0''.000\,031\,49 \sin(\lambda_3) - 0''.000\,136\,93 \cos(\lambda_3) \\
&- 0''.000\,063\,95 \sin(2\lambda_3 + 2D - 2F) \\
&- 0''.000\,023\,39 \sin(2\lambda_3 + 2D - 2\varphi) \\
&- 0''.000\,013\,43 \cos(2\lambda_3 + 2D - 2\varphi) \\
&- 0''.000\,010\,16 \sin(2\lambda_3 - 2\varphi) \\
&- 0''.000\,005\,83 \cos(2\lambda_3 - 2\varphi) \\
&- 0''.000\,011\,44 \sin(3\lambda_3 + D - F) \\
&- 0''.000\,010\,91 \sin(\lambda_3 - D + F) \\
&+ t[0''.000\,008\,58 \sin(\lambda_3 + D - F) \\
&\quad - 0''.000\,644\,52 \cos(\lambda_3 + D - F) \\
&\quad + 0''.000\,000\,57 \sin(2\lambda_3 + 2D - 2F) \\
&\quad + 0''.000\,031\,19 \cos(2\lambda_3 + 2D - 2F) \\
&\quad + 0''.000\,006\,54 \sin(2\lambda_3 + 2D - 2\varphi) \\
&\quad - 0''.000\,011\,45 \cos(2\lambda_3 + 2D - 2\varphi)] \\
&+ t^2[0''.000\,077\,84 \sin(\lambda_3 + D - F) \\
&\quad - 0''.000\,018\,81 \cos(\lambda_3 + D - F)]
\end{aligned}
$$

In this development, the term of period 1 yr (λ_3) is due to the geodesic precession and the semidiurnal terms are due to the perturbations of the variable φ by the tesseral harmonics.

10. Polynomials

Table 4 gives the secular parts of the variables $P = \sin\pi\sin\Pi$, $Q = \sin\pi\cos\Pi$, π, Π, $\mathscr{P}$, θ, ζ, z, ε, $-\omega$, $-\psi$, χ.

The polynomial expressions in this table agree well with those of Williams (1994). For the ecliptic motion, the polynomial expressions are the same because they are taken from Simon et al. (1994). The slight difference on the constant part of Π arises from the fact that Williams computed this constant ($\arctan(P/Q)$) from the truncated digits of P and Q published in Simon et al. (1994) while we used the exact values of P and Q.

Table 5 gives the differences Table 4 (this paper) minus Table 5 (Williams, 1994). The most important part of these differences arises from the fact that Williams considers the tidal effects (which gives a contribution of 24 μas/yr in the obliquity rate) and the Earth's J_2 rate. The contribution of the J_2 rate gives −0.651 804 μas/yr^2 and Poisson terms in the variable $-\psi$ according to Bretagnon (1996). This correction has not been introduced here, the work being limited to the rigid Earth.

The values of the moments of inertia A, B, C and of the dynamical ellipticity H_d given in (9) are related to the secular part of ψ. We find

$$H_d = 0.003\,273\,7671.$$

This value agrees well with the Williams' value 0.003 273 7634.

The expression for Greenwich Mean Sidereal Time (GMST) at 0^h is

$$
\begin{aligned}
\mathrm{GMST}(0^h) &= 24\,110^s.548\,41 + 86\,401\,847^s.928\,601\,t \\
&+ 9^s.313\,216\,t^2 - 0^s.000\,288 t^3 \\
&- 0^s.019\,514\,t^4 + 0^s.000\,123\,t^5 + 0^s.000\,017\,t^6.
\end{aligned}
$$

11. Comparison with the Kinoshita-Souchay solution

Our solution have been compared with the Kinoshita-Souchay (1990) solution for the precession and nutation in longitude in Bretagnon (1996). Some errors of the KS90 solution have been

Table 5. Difference Table 4 – Williams (1994). Unit is arcsec. Time t is in thousand Julian years (tjy) from J2000.

	t^0	t	t^2	t^3	t^4
$\mathscr{P}$	0.00000	0.00000	0.7022	0.001	0.01
θ	0.00000	0.00062	0.2770	−0.001	0.00
ζ	0.21649	−0.00803	0.3297	0.000	−0.01
z	−0.21649	0.00776	0.3155	0.002	0.01
ε	0.00300	−0.02097	0.0008	−0.001	0.00
$-\omega$	0.00300	−0.02104	0.0008	0.000	0.00
$-\psi$	0.00000	−0.00013	0.7023	−0.003	0.00
χ	0.00000	−0.00014	0.0002	−0.005	0.00

corrected in the Souchay-Kinoshita (1996) solution. The most important difference is for the term of period 18.6 yr. The SK96 solution gives

$$\mathscr{P}_{SK96} = -\ 17''.280\,585\ \sin(\Omega_D) + 0''.000\,135\ \cos(\Omega_D) - 0''.000\,128\ \sin(-2l' + 2F - 2D + \Omega_D). \quad (59)$$

The periodic part of (59) can be expressed in terms of our arguments (see Sect. 5.3):

$$\mathscr{P}_{SK96} = +\ 17''.280\,700\ \sin(\lambda_3 + D - F) - 0''.000\,190\ \cos(\lambda_3 + D - F),$$

compared with our development given in Sect. 6.3. The difference is

$$\Delta\mathscr{P} = -65\,\mu\text{as}\ \sin(\lambda_3 + D - F) + 250\,\mu\text{as}\ \cos(\lambda_3 + D - F).$$

For the precession-nutation in obliquity the difference is

$$\Delta\varepsilon = -5\,\mu\text{as}\ \sin(\lambda_3 + D - F) + 38\,\mu\text{as}\ \cos(\lambda_3 + D - F).$$

12. Rotations

The sequence of rotations for the reduction from the mean equator and equinox J2000 to the true equator and equinox of date is

$$R_3(\chi) \cdot R_1(\omega) \cdot R_3(-\psi) \cdot R_1(\varepsilon_0),$$

where the solutions for the precession-nutation in the variables ψ, ω and χ include secular parts and periodic and Poisson perturbations.

13. Conclusion

This paper is a first step in the computation of the precession for a rigid Earth. We computed secular perturbations (precession) and periodic and Poisson pertubations (nutation) for the five quantities ψ, ω, χ, $\mathscr{P}$, ε. The comparison of our analytical solution with a numerical integration over 1900-2050 give differences smaller than 16 μas for ψ, 8 μas for ω.

Our solution agrees well with the secular developments of Williams (1994) and the periodic series of Souchay and Kinoshita (1996). However, there is in $\mathscr{P}$ an unexplained difference of about 250 μas in the term of period 18.6 yr. We intend to improve our solution by computing the right-hand sides of the equations with a better accuracy and by using the complete solution ELP for the motion of the Moon.

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