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ASTRONOMY AND ASTROPHYSICS

Current sheets in two-dimensional potential magnetic fields

III. Formation in complex topology configurations and application to coronal heating

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Abstract. We study the spontaneous formation of a *current sheet* (CS) in an x-invariant y-symmetric magnetic field $\mathbf{B}(y,z,t)$ occupying the half-space $\{z>0\}$, and embedded in a pressureless perfectly conducting plasma. At the initial time t = 0, $\mathbf{B}(y,z,0)$ is potential and *quadrupolar*, and therefore its lines in a poloidal plane have a complex topology: there is either one separatrix, which contains a neutral X-point or is tangent to the y-axis (X- and U-topology, respectively), or two separatrices extending to infinity (I-topology). For $t \geq 0$, the field is made to evolve quasi-statically by imposing its footpoints on the boundary $\{z = 0\}$ to move parallel to the y-axis at the slow velocity $v(y,t)$. It thus passes through a sequence of configurations which are either *potential equilibria* or *quasi-potential singular equilibria*, the latter containing a CS, assumed a priori to be vertical.

We compute analytically $\mathbf{B}(y,z,t)$ and its free-energy contents $\delta W(t)$ as functionals of $B_z(y,0,t)$ (this boundary value depending on $B_z(y,0,0)$ and $v(y,t)$), and also, when there is a CS, of the unknown heights $z_1(t)$ and $z_2(t)$ of its bottom and top, respectively. We derive equations satisfied by the latter quantities, and use them to show that: (i) When the initial field is of the U- or I-type, a CS – and a vertical one indeed – is actually present at time t if and only if the potential field $\mathbf{B}^p(y,z,t)$ associated to $B_z(y,0,t)$ has a X-topology. (ii) When the initial field is of the X-type, a CS exists in general at each time $t>0$, but it is vertical if and only if a quite specific condition is satisfied – which may not be the case for arbitrarily chosen data and puts a limit on the generality of our model. Finally, we derive for $z_1(t)$, $z_2(t)$, $\mathbf{B}(y,z,t)$ and $\delta W(t)$ useful approximate explicit expressions, which are valid just after the CS has started forming at some time $t_c \geq 0$.

As an application, we consider a plasma heating process in which a field evolving through a sequence of singular equilibria as described above, relaxes at each time $t_k = k\tau_D$ ($k = 1, 2, ..., N$) to a new potential equilibrium, the vertical CS being destroyed by some reconnection process. We present an estimate of the resulting heating rate, which is found to depend on the ratio τ_D/τ_{ev} (assumed to be $\ll 1$) of a given phenomenological dissipation time τ_D to the ideal evolution time τ_{ev} of the system. The relevance of this process for heating a stellar corona is briefly discussed.

Key words: MHD – plasmas – Sun: coronae – stars: coronae

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1. Introduction

It is well known that a magnetic field occupying some domain D filled up with a perfectly conducting plasma and having its lines connected to the boundary ∂D of D, may spontaneously develop electric *current sheets* (CS) when it is brought into a quasi-static evolution by motions imposed on ∂D. This process occurs when the field lines have a *complex topology* – i.e., when the magnetic mapping $\mathbf{f} : \partial D \to \partial D$ associating together the points at which an arbitrary line emerges and disappears, respectively, is discontinuous across some curves of ∂D, the lines originating from the latter forming singular surfaces called *separatrices* (Aly 1987a). The appearance of CS can be physically ascribed to an "incompatibility" between the frozen-in law and the force-balance equation, the latter turning out in general to have no regular solutions compatible with the topology of the field, which is preserved during the evolution as a consequence of the former (see, e.g., the reviews by Syrovatskii (1980), Aly (1987b), Low (1990) and Amari (1991), and the monograph of Parker (1995)).

The formation of CS has been often considered in the framework of a simple model in which the field is x-invariant, the thermal pressure of the ideal plasma is neglected and the imposed boundary velocity field has no x-component (v_x = 0). Initially, the field is in *potential equilibrium* (PE), and it stays current-free everywhere during its evolution, except on the CS which may appear: the configuration at each time can thus be called after Aly & Amari (1989; paper I hereafter) a *quasi-potential* *singular equilibrium* (QPSE). The literature provides numerous instances of the formation of QPSE when the initial configuration has a separatrix generated by a line of neutral hyperbolic points (X-point) (Priest & Raadu 1975; Tur & Priest 1976; Low & Hu 1983; Hu & Low 1982; Hu 1986). Later on, it was noted by Low (1987) that the presence of an X-point is not necessary for CS to develop: they may also form when the initial configuration has a magnetic surface tangent to the boundary – such a surface being a separatrix too. Low reported two explicit examples of CS formation in fields having this type of topology. Later on, his work was further elaborated by Sneyd (1993).

The mathematical properties of QPSE have been carefully established at a general level in Paper I. In particular, we have given therein the general equations satisfied by a QPSE (including the conditions of equilibrium at the extremities of the CS), and proven the covariance of these equations by the group of conformal mappings (a property which turned out to be valuable to build up non-symmetric models of solar quiescent prominences (Aly & Amari 1988)). In a second paper (Amari & Aly 1990, Paper II hereafter), we have computed analytically the QPSE which is approached *asymptotically* (and not instantaneously, as in the process previously described) by a sequence of x-invariant nonlinear force-free arcades occupying the half-space $D = \{z > 0\}$ and indefinitely sheared by a boundary velocity field parallel to the x-axis ($v_x \neq 0$, $v_y = 0$) (Aly 1985). The asymptotic QPSE – which has a CS extending up to infinity – was explicitly obtained as the solution of a so-called *Hilbert Problem*. It is fully determined by the values of the normal component of the magnetic field on the plane ∂D, and by a number depending on the shearing velocity and characterizing the magnetic surface on which the sheet is located.

In this paper, we reconsider the simple quasi-potential model described above, assuming in addition that: (i) The field occupies the half-space $D = \{z > 0\}$. (ii) The lines of the initial PE have a quadrupolar structure – which insures indeed that their topology is complex. (iii) The field is y-symmetric and the CS which forms is located in the plane $\{y = 0\}$. Our aim is thus to compute the sequence of evolving QPSE for quite arbitrary boundary conditions. For that, we shall adapt the method used in Paper II for treating QPSE with a semi-infinite CS.

Our plan is the following one. In Sect. 2, we study the properties of quadrupolar potential fields. We show in particular that the topology of their lines can be of three possible types, referred to as U, X and I, respectively. In Sect. 3, we consider the evolution of such fields resulting from the change in the boundary conditions induced by the velocity field. Thus we analyze in Sect. 4 the evolution of the field when the frozen-in constraint is enforced (in particular, we determine the conditions in which a CS is expected to form), and we write the equations it does satisfy. In Sect. 5, we compute analytically the structure of a QPSE obtained at some time $t > 0$ as a functional of the normal component of the field on the boundary, and of the heights z_1 and z_2 of the bottom and top of the CS. We also write the equations satisfied by the last two quantities. These equations are discussed in details in Sects. 6 (X-case) and 7 (U- and I-cases), and explicit approximate solutions are given, which are valid for a short time after the CS has started forming. In Sect. 8, we derive analytical expressions (containing the unknown z_1 and z_2) for the free magnetic energy $\delta W(t)$ stored in a QPSE and for its variation, and we establish an explicit formula for $\delta W(t)$ which is valid at small time in the X-case. In Sect. 9, we propose a plasma heating mechanism based upon the generation of a double sequence of PE and QPSE. We start with a quadrupolar PE having a X-topology and we apply to its footpoints a velocity field during a short period of time τ_D. This results in the production of a QPSE, which is assumed to relax instantaneously to a new PE, the CS being destroyed by reconnection. The process thus repeats. Here, τ_D is taken as a given phenomenological time, which is just imposed to be small compared to the time scale of evolution of the system, τ_{ev}. We give a formula for the resulting heating rate (which is found to depend on the ratio τ_D/τ_{ev}). The relevance of our model for explaining the heating of the solar and stellar coronas is briefly discussed in our final Sect. 10.

It should be noted that a preliminary account of this work was presented at the NATO Workshop on Cosmical Magnetism held at Cambridge (UK) in July 1993. Unfortunately, the corresponding paper was not included in the Proceedings because of a material error.

2. Quadrupolar potential magnetic fields

In this section, we study some general properties of x-invariant potential magnetic fields occupying the half-space $D = \{z > 0\}$ and having a vanishing magnitude at infinity (this last assumption is given up in Appendix E, where we consider fields approaching asymptotically a constant value $\mathscr{B}_0\hat{\mathbf{y}}$). After introducing some general formulas, we restrict our attention to y-symmetric *quadrupolar* fields, whose evolution will be the main concern of this paper. We shall denote as Ω the half-plane $\{x = 0\} \cap D$ (owing to the x-invariance, our mathematical problems will be set in this two-dimensional domain), and as ∂D and $\partial\Omega$ the boundaries of D and Ω, respectively.

2.1. General formulas for potential magnetic fields

$g(y)$ being a given regular function tending sufficiently fast to zero at infinity, let us consider the unique classical solution $A^p(y, z)$ to the boundary value problem (BVP)

$$-\triangle A^p(y,z) = 0 \qquad \text{in } \Omega\,, \tag{1}$$

$$A^p(y,0) = g(y) \qquad \text{on } \partial\Omega\,, \tag{2}$$

$$\lim_{r\to\infty} A^p(y,z) = 0\,, \tag{3}$$

where $r := (y^2 + z^2)^{1/2}$. A^p is given explicitly by (Paper II)

$$A^p(y,z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{z}{(y-y')^2 + z^2}\, g(y')\,\mathrm{d}y'\,. \tag{4}$$

Then

$$\mathbf{B}^p(y,z) = \nabla A^p(y,z) \times \hat{\mathbf{x}} \tag{5}$$

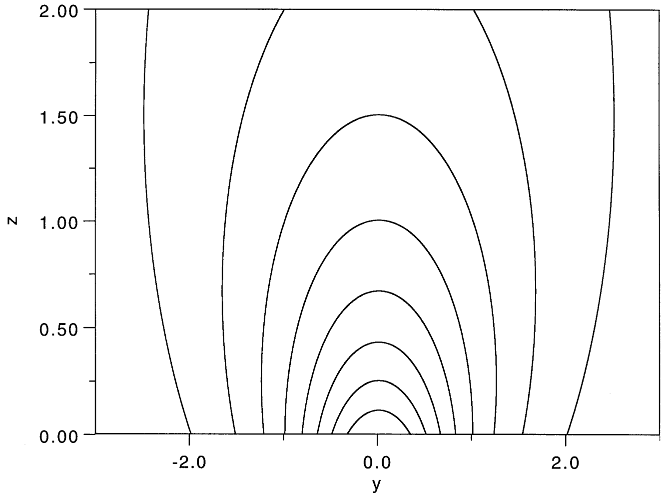

Fig. 1. Topology of the field lines of $\mathbf{B}^p$ when $yg'(y) \leq 0$. The lines are simple arcades.

is the unique x-invariant potential magnetic field satisfying $B_z^p(y,0) = -(\mathrm{d}g/\mathrm{d}y)(y) =: -g'(y)$ and tending to zero at infinity. This field can be given the complex representation (Paper II)

$$B^p(\eta) := B_y^p(y,z) - iB_z^p(y,z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{g'(y')}{y'-\eta}\,\mathrm{d}y' \,, \qquad (6)$$

B^p thus being an holomorphic function of $\eta := y + iz$ in Ω.

The fields $\mathbf{B}^p$ can be classified according to the topology of their lines in Ω, which are the level contours of A^p. Two regular fields $\mathbf{B}^p$ and $\mathbf{B}^{p\prime}$ are said to be topologically equivalent if there is an homeomorphism $\mathbf{h} : \Omega \to \Omega$ (bijective and bicontinuous application) which maps the lines of $\mathbf{B}^p$ into those of $\mathbf{B}^{p\prime}$. The topology of a poloidal field is determined by the structure of singular curves, the so-called *separatrices*, which divide the field lines into topological cells (Aly 1987a; Paper I). Generically, separatrices $\mathscr{S}$ of the potential fields considered here are of three possible types (remember that a potential field does not admit closed lines, with associated neutral O-point):

a. X-separatrix: $\mathscr{S}$ is X-shaped, being constituted of four branches meeting at a X-point (a point of Ω at which $\mathbf{B}^p = 0$).

b. U/V-separatrix: $\mathscr{S}$ is constituted of two branches meeting the boundary $\partial\Omega$, either tangentially at a U-point, or transversally at a V-point, where $\mathbf{B}^p = 0$.

c. I-separatrix: $\mathscr{S}$ connects the boundary to infinity, in which case $A(y,z) = 0$ along it.

Note that the U-V-I terminology is proposed here for the first time.

Then, for instance, all the fields considered in Paper II belong to the class of the so-called arcades, in which there are no separatrices. The corresponding topology is shown in Fig. 1. Fields of this type are easily produced: it is just necessary to impose the function g to satisfy either $yg' \leq 0$ or $yg' \geq 0$ (then g keeps the same sign over the y-axis, where it has a single maximum (minimum) value in the former (latter) case).

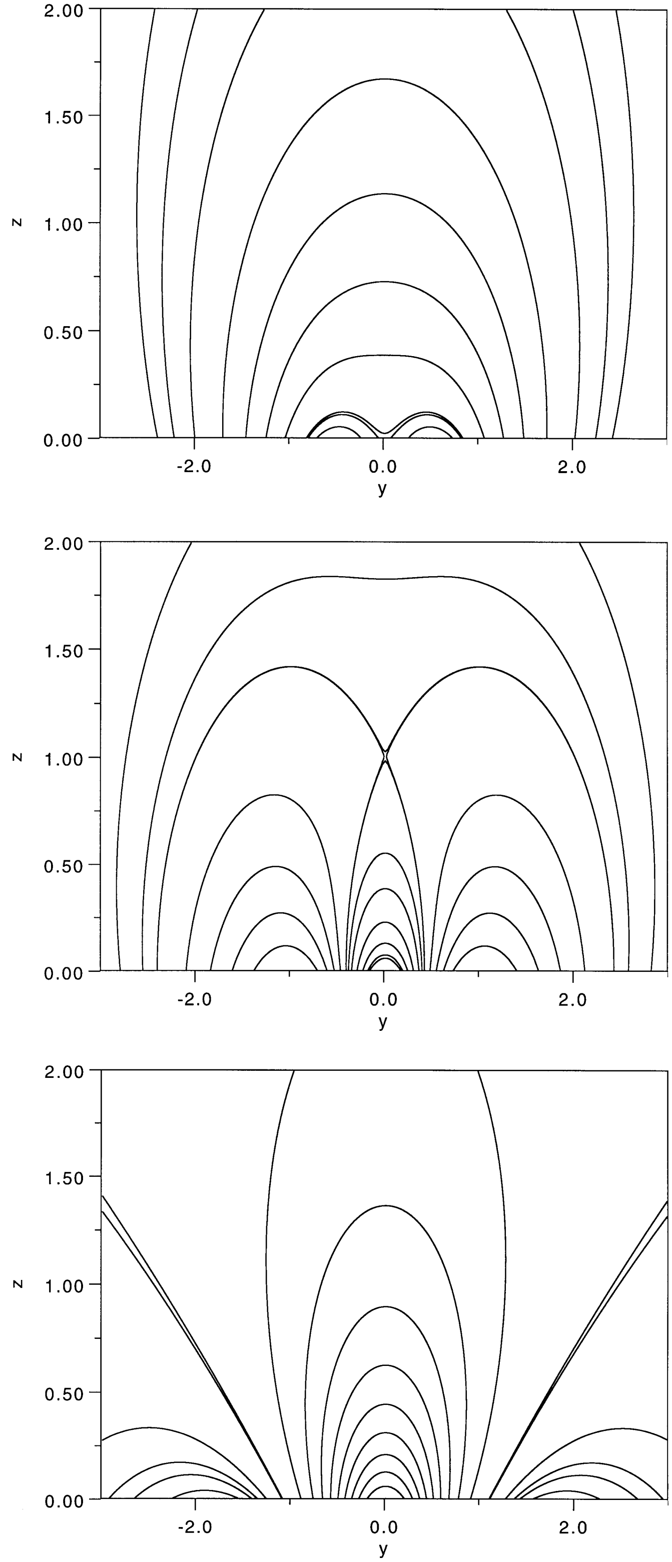

Fig. 2a–c. Topology of the field lines of the potential field $\mathbf{B}^p$ whose flux function is given by Eq. (A9), for three values of μ. **a** μ = 0.375: the field has a U-topology. **b** μ = 0: the field has a X-topology. **c** $\mu = -1.2$: the field has a I-topology.

2.2. y-symmetric quadrupolar potential fields

We now concentrate on the potential magnetic fields which are both y-symmetric and quadrupolar. Such fields are obtained by imposing the boundary function g' to satisfy

$$g'(-y) = -g'(y)\,, \quad \text{whence} \quad g(-y) = g(y)\,, \tag{7}$$

and to change its sign three times. Without any loss of generality, we shall assume here that, for some $d > 0$,

$$\begin{array}{llll} g' \geq 0 & \text{for} \ -\infty < y \leq -d & \text{and} & 0 \leq y \leq d\,, \\ g' \leq 0 & \text{for} \ -d \leq y \leq 0 & \text{and} & d \leq y < \infty\,, \end{array} \tag{8}$$

with $g' \not\equiv 0$ on both the intervals $[0, d]$ and $[d, \infty[$. This implies that, on $[0, \infty[$, the even function g increases from a minimum value $m := g(0)$ (positive or nonpositive) up to a maximum value $M := g(d) > 0$ and thus decreases to 0.

An important property of such quadrupolar fields is the existence or the non-existence of an X-point on the positive z-axis. As $B_z^p(0, z) = 0$ because of the imposed y-symmetry, such a point exists at the height Z if the y-component of the field,

$$B_y^p(0, z) = \frac{2}{\pi} \int_0^\infty \frac{yg'}{y^2 + z^2}\, \mathrm{d}y\,, \tag{9}$$

vanishes for $z = Z > 0$, i.e.,

$$\int_0^\infty \frac{yg'}{y^2 + Z^2}\, \mathrm{d}y = 0 \tag{10}$$

(Eq. (9) results immediately from Eqs (6)-(7)). As shown in Appendix A, we have

$$\exists Z \in]0, \infty[: \quad B_y^p(0, Z) = 0 \iff \begin{cases} P > 0\,, \\ Q > 0\,, \end{cases} \tag{11}$$

where

$$P := B_y^p(0, 0) = \frac{2}{\pi} \int_0^\infty \frac{g'}{y}\, \mathrm{d}y \tag{12}$$

(see Eq. (9)), and

$$Q := -\int_0^\infty y g'(y)\, \mathrm{d}y = \int_0^\infty g(y)\, \mathrm{d}y\,. \tag{13}$$

Using this result and some simple considerations on the possible ways of connecting by continuous curves the points of the boundary having the same value of g, it is easy to see that the topology of the lines of $\mathbf{B}^p$ can be of three possible types (Vekstein et al. 1991), whose occurence is determined by the signs of P and Q (the role of Q seems to have not been recognized thus far):

a. U-topology: If $P \leq 0$, the field does not vanish at any point of Ω. There is one separatrix $\mathscr{S}$, which is of the U(V)-type when $P < 0$ ($P = 0$), and meets the boundary at three points, the U(V)-branching point coinciding with the origin. $\mathscr{S}$ divides the lines into three topological cells, and

$$\gamma := A|_{\mathscr{S}} = m > 0\,. \tag{14}$$

Note that this last relation implies $g \geq 0$ on $\partial\Omega$, and then $Q > 0$.

b. X-topology: If $P > 0$ and $Q > 0$, the field vanishes only at $(0, Z)$. There is one X-separatrix $\mathscr{S}$ containing that point and dividing Ω into four topological cells. Moreover,

$$\gamma = \frac{2}{\pi} \int_0^\infty \frac{Z g(y)}{y^2 + Z^2} \mathrm{d}y \tag{15}$$

(by Eqs (4) and (7)), and

$$\max(0, m) < \gamma := A|_{\mathscr{S}} = A(0, Z) < M\,. \tag{16}$$

c. I-topology: If $P > 0$ and $Q \leq 0$, the field does not vanish at any point of Ω. There are two I-separatrices $\mathscr{S}_1$ and $\mathscr{S}_2$ located symmetrically on either sides of the z-axis. They divide Ω into three topological cells, and

$$\gamma := A|_{\mathscr{S}_1} = A|_{\mathscr{S}_2} = 0\,. \tag{17}$$

The three types of topology are represented in Fig. 2a,b and c, respectively.

3. Evolution of a quadrupolar potential field

3.1. Statement of the problem

Let us assume that:

a. We are given at some initial time $t = 0$ a quadrupolar potential field

$$\mathbf{B}_0(y, z) = \nabla A_0(y, z) \times \hat{\mathbf{x}} \tag{18}$$

of the type considered in Sect. 2. To $\mathbf{B}_0$ are attached the quantities $g_0(y)$, d_0, m_0, M_0, γ_0 and possibly Z_0.

b. The boundary ∂D is made of perfectly conducting plasma, which is imposed to move at the velocity

$$\mathbf{v}(y, t) = v(y, t)\hat{\mathbf{y}}\,, \tag{19}$$

with v being a bounded regular function satisfying

$$v(-y, t) = -v(y, t)\,. \tag{20}$$

Then, on the boundary, the normal component of the field (which is frozen in the plasma) and the associated flux function are imposed to change – actually in a way that is independant of the physical processes occuring in D.

c. The field in D stays potential for $t > 0$. Physically, we may think of D as a vacuum containing a field evolving sufficiently slowly for electromagnetic effects to be negligible.

Our problem is to compute the field at each time t and to study its properties.

3.2. Evolution of the boundary conditions

The evolution of the boundary flux function $g(y, t)$ is determined by the equation (Paper I)

$$\mathrm{d}_t g(y, t) := \partial_t g(y, t) + v(y, t) g'(y, t) = 0\,, \tag{21}$$

which expresses the frozen-in law (a "$'$" now denotes a partial derivative with respect to y), and by the initial condition

$$g(y,0) = g_0(y)\,. \tag{22}$$

Eq. (21) has an important property (Sneyd 1993): it preserves the y-symmetry of g (under condition (20)) and the qualitative aspect of its graph, as well as the values of its maximum and of its minimum at the origin, i.e.,

$$m(t) := g(0,t) = m_0\,, \tag{23}$$

$$M(t) := \max g(y,t) = M_0\,. \tag{24}$$

Eq. (21) can be solved by the method of characteristics. Let $y(y_0,t)$ denotes the solution of the ordinary differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = v(y,t)\,, \tag{25}$$

with the initial condition

$$y(0) = y_0\,. \tag{26}$$

Then

$$g(y(y_0,t),t) = g_0(y_0)\,. \tag{27}$$

By differentiating Eq. (21) with respect to y, we obtain

$$\partial_t g'(y,t) + [v(y,t)g'(y,t)]' = 0\,, \tag{28}$$

whose solution writes

$$g'(y(y_0,t),t) = g_0'(y_0)\left(\frac{\partial y}{\partial y_0}(y_0,t)\right)^{-1}\,. \tag{29}$$

3.3. Evolution of the magnetic field

Of course, the field in D is given by

$$\mathbf{B}^p(y,z,t) = \nabla A^p(y,z,t)\times\hat{\mathbf{x}}\,, \tag{30}$$

with $A^p(y,z,t)$ determined by Eqs (1)-(3), in which the boundary function g is taken to be the solution $g(y,t)$ to Eqs (21)-(22). Because $g(y,t)$ is an even function whose graph is qualitatively preserved, it is clear from the previous section that $\mathbf{B}^p(y,z,t)$ stays a y-symmetric quadrupolar field during its evolution. But there will be topological transitions:

a. Of the type $U \leftrightarrow X$, if $m_0 > 0$ and $P(t)$ changes its sign.

b. Of the type $X \leftrightarrow I$, if $m_0 < 0$ and $Q(t)$ changes its sign.

It is then important to know the variations of the control functions $P(t)$ and $Q(t)$, which are determined by their derivatives with respect to time (denoted hereafter by a dot). Using Eqs (21), (28) and (13), and effecting an integration by part, we obtain

$$\dot{P}(t) = -\frac{2}{\pi}\int_0^\infty \frac{(vg')(y,t)}{y^2}\mathrm{d}y\,, \tag{31}$$

$$\dot{Q}(t) = -\int_0^\infty (vg')(y,t)\mathrm{d}y\,. \tag{32}$$

On the other hand, if $\mathbf{B}^p$ has either a U- or a I-topology, we have

$$\dot{\gamma}(t) = 0\,, \tag{33}$$

while the value of γ and the height Z of the X-point evolve in time according to

$$\dot{\gamma}(t) = -\frac{2}{\pi}\int_0^\infty \frac{Z(t)(vg')(y,t)}{y^2+Z^2(t)}\mathrm{d}y\,, \tag{34}$$

$$\dot{Z}(t) = \int_0^\infty (vg')(y,t)\frac{Z^2(t)-y^2}{[y^2+Z^2(t)]^2}\mathrm{d}y \times\left(2Z(t)\int_0^\infty \frac{yg'(y,t)}{[y^2+Z^2(t)]^2}\mathrm{d}y\right)^{-1}\,, \tag{35}$$

when $\mathbf{B}^p$ has a X-topology (here, use has been made of Eqs. (12), (28), (15) and (9)).

3.4. Particular classes of velocity fields

We now introduce two classes of stationary boundary velocity field that will be useful hereafter to get sufficient conditions for a CS to form (other classes ensuring the appearance of a CS can be easily found):

(i) Class C: v is compressive (C), being given by

$$v(y) = -y\psi(y)\,, \tag{36}$$

with ψ an even positive decreasing function. This implies

$$\dot{d}(t) < 0\,, \tag{37}$$

i.e., the region of negative polarity on the right of the origin has a decreasing size.

(ii) Class CE: v is imposed to satisfy

$$\begin{array}{lll} v(y) > 0 & \text{for} & 0 < y < d_0\,,\\ v(y) \le 0 \;\text{ and }\; \not\equiv 0 & \text{for} & d_0 \le y\,. \end{array} \tag{38}$$

Then, for $y \ge 0$, the velocity is compressive (C) in the inner region of negative polarity, and expansive (E) (or noncompressive) in the outer region of positive polarity. Whence in particular

$$\forall\, y,\, t:\;\; v(y)g'(y,t) \le 0 \;\text{ and }\; \not\equiv 0\,, \tag{39}$$

$$\partial_t g(y,t) \ge 0 \;\text{ and }\; d(t) = d_0\;: \tag{40}$$

the boundary flux function is an increasing or constant function of time at any fixed point, while the positions of the regions of positive and negative polarities do not change in time. In some case, it will be convenient to add the further condition (defining the subclass CE')

$$-v(y) \ge \zeta y^2 \quad\text{for}\quad y\in[0,\delta]\,, \tag{41}$$

where ζ and $\delta < d_0$ are two positive constants.

Let us look at the effect of these particular velocities on the evolution of the potential magnetic field. We first note the following relations satisfied by $\dot{P}$ and $\dot{Q}$ (see Eqs (31) and (32)):

a. If **v** is of the CE-type:

$$\dot{P}(t) > 0 \quad \text{and} \quad \dot{Q}(t) > 0 \,. \tag{42}$$

If **v** is a CE'-field, the first relation can be easily strengthen into

$$\dot{P}(t) \geq c_1 > 0 \,, \tag{43}$$

where c_1 is a constant (this relation possibly holding only after some time has elapsed).

b. If **v** is of the C-type:

$$\dot{P}(t) > \psi(d_0) P(t) \,, \tag{44}$$

$$\dot{Q}(t) > -\psi(d_0) Q(t) \,, \tag{45}$$

where we have used Lemma A (Appendix A.1), and Eq. (37).

Then:

(i) If $\mathbf{B}_0$ is a U-field and **v** a CE'-field, $P(t)$ increases monotonically, reaching arbitrarily large values, and $\mathbf{B}^p$ becomes necessarily of the X-type at a time t_c, staying of this type afterwards.

(ii) If $\mathbf{B}_0$ is a X-field and **v** either a C- or a CE-field, $\mathbf{B}^p$ stays of the X-type forever: indeed, Eqs (42)-(45) and the initial conditions $P(0) > 0$ and $Q(0) > 0$ imply that $P(t) > 0$ and $Q(t) > 0$. On the other hand, γ increases monotonically, as

$$\dot{\gamma}(t) > 0 \,. \tag{46}$$

The last relation is obvious when **v** is of the CE-type. In the C-case, it results from Lemma A, which implies

$$\begin{aligned} \dot{\gamma}(t) &= \frac{2}{\pi} \int_0^\infty \frac{yZ}{y^2 + Z^2} \psi g' \,\mathrm{d}y \\ &> \frac{2Z\psi(d)}{\pi} \int_0^\infty \frac{yg'}{y^2 + Z^2} \,\mathrm{d}y = 0 \,. \end{aligned} \tag{47}$$

(iii) If $\mathbf{B}_0$ is a I-field and **v** either a CE- or a C-velocity, Q increases indefinitely by Eqs (42) and (45). In the former case, Q becomes positive at some critical time t_c and keeps its positive sign for $t > t_c$: indeed, the negative contribution to Q decreases to zero (it arises from a shrinking region in which $|g| < m_0$), while the positive contribution to Q increases (see Eq. (40)). Then the field becomes of the X-type at t_c and stays of this type afterwards. In the latter case, however, it seems that we cannot conclude without further conditions that Q is going to change its sign (it could stay negative, tending to 0 when $t \to \infty$).

4. Magnetic evolution under the frozen-in constraint

4.1. Statement of the problem

We now reconsider the previous evolution problem under the following new assumptions:

a. Rather than being a vacuum, D is filled up with a perfectly conducting plasma, whose thermal pressure p is completely negligible compared to the magnetic pressure (in other words, $\beta \ll 1$, where β is the standard plasma beta).

b. The evolution of the field is quasi-static – the typical value of the velocity imposed on the boundary being much smaller than the typical value of the Alfven speed in D. It is then determined by the following principles (Paper I):

- Because there is no pressure acting on the poloidal structure ($p = 0$ by assumption, while B_x, which vanishes initially, stays equal to zero), there is no continuous distribution of x-currents. Such currents can develop only on singular surfaces – CS – in mechanical equilibrium. Then, at each time, the field is either a PE or a QPSE (which are particular instances of more general *force-free* configurations).
- Because of the frozen-in law:
  - The topology of the field is preserved during the evolution, at least in some generalized sense. This last qualification is necessary here because the definition of two topologically equivalent configurations given in Sect. 2.1 applies only for smooth fields, while the previous principle indicates the possibility of development of singularities. Here, we can argue with Moffat (1985) that the frozen-in law just requires the field $\mathbf{B}(t)$ to be *accessible* from $\mathbf{B}_0$. This means that it can be considered as the limit (in some sense!) of a sequence of regular fields $\mathbf{B}_k(t)$ ($k = 1, 2, ...$) topologically equivalent to $\mathbf{B}_0$. A very tentative mathematical definition of this notion of general topology preservation has been given in Paper I.
  - The value of A on the separatrix keeps its initial value γ_0.
- The field stays a PE if $\mathbf{B}^p(t)$ is related to $\mathbf{B}_0$ by the two previous constraints. Otherwise – i.e., if one of the latter is violated –, a CS $\Sigma(t)$ has to develop along some part of the separatrix. We shall assume in this paper that $\Sigma(t)$ is vertical, being contained in the plane $\{y = 0\}$. Then its trace $\Gamma(t)$ on Ω (that will be also called the CS) appears to be a straight segment $]z_1(t), z_2(t)[$ ($0 \leq z_1(t) < z_2(t) \leq \infty$) of the z-axis.

4.2. Equations

Hereafter, we shall represent the field either by the usual expression in term of a flux function A,

$$\mathbf{B}(y, z, t) = \nabla A(y, z, t) \times \hat{\mathbf{x}} \,, \tag{48}$$

or by the complex function

$$B(\eta, t) := B_y(y, z, t) - iB_z(y, z, t) \,. \tag{49}$$

As indicated above, we have two possible situations:

a. If the evolving potential field $\mathbf{B}^p(t)$ satisfies

$$\text{topology}\ [\mathbf{B}^p(t)] \sim \text{topology}\ [\mathbf{B}_0] \tag{50}$$

and

$$\gamma(t) = \gamma_0 \,, \tag{51}$$

then

$$\mathbf{B}(t) = \mathbf{B}^p(t) \,. \tag{52}$$

b. If one of the two conditions (50)-(51) is not satisfied, and the CS which forms is vertical indeed, then (Paper I):

– A satisfies the equation

$$-\triangle A(y, z, t) = 0 \quad \text{in } \Omega\backslash\Gamma \tag{53}$$

(which expresses the equilibrium of the field outside the CS), and the boundary and asymptotic conditions

$$A(y, 0, t) = g(y, t)\,, \tag{54}$$

$$\lim_{r\to\infty} A(y, z, t) = 0\,. \tag{55}$$

This latter condition will insure in particular that the field remains closed (no open lines do form).

– On the CS, i.e., on $\Gamma(t) =]z_1(t), z_2(t)[$, we have:

$$A\,|_\Gamma = \gamma_0\,, \tag{56}$$

$$\left[|\nabla A|^2\right]_\Gamma = 0\,, \tag{57}$$

$$\nabla A(0, z_1(t), t) = \nabla A(0, z_2(t), t) = 0\,. \tag{58}$$

Eq. (56) expresses the fact that the normal component of the field vanishes on $\Gamma(t)$, which is the condition for the sheet to be in vertical force-balance, and the fact that the value of A on the separatrix (where the CS does appear) stays constant in time. Eq. (57), where $[.]_\Gamma$ denotes the jump of a quantity across $\Gamma(t)$, is the condition for the CS to be in horizontal equilibrium (pressure balance), and Eq. (58) is the condition for the equilibrium of the two end-points of $\Gamma(t)$.

4.3. When do we expect a priori a CS to form?

Let us consider in turn each of the three possible types of initial configurations:

a. If $\mathbf{B}_0$ has a U-topology, $\mathbf{B}^p(t)$ satisfies Eqs (50)-(51) – and then Eq. (52) holds – as long as $P(t) \leq 0$ (see Eq. (33)). However, if at some critical time t_c a V-point spontaneously appears at the origin ($P(t_c) = 0$) and the topology of $\mathbf{B}^p$ becomes of the X-type, then a CS develops, and the field becomes of the QPSE type. It seems physically obvious that $\Gamma(t)$ forms on the z-axis, where it extends between $z_1(t) = 0$ and $z_2(t) > 0$ (Fig. 3a).

b. If $\mathbf{B}_0$ has a X-topology, we have for most choices of v: $\dot\gamma(0) \neq 0$, and then the second topological constraint is violated at once. Then $\mathbf{B}(t) \neq \mathbf{B}^p(t)$, but rather $\mathbf{B}(t)$ is a QPSE, with a CS forming and developing near the X-point. A priori, however, we do not know if $\Gamma(t)$ is located on the z-axis (Fig. 3b). We could as well get a curved CS extending symmetrically on either side of that axis. Then we expect some conditions on the velocity to be necessary for our assumption of a vertical CS to be consistent. Such a condition will be derived indeed in Sect. 6.1 below.

c. If $\mathbf{B}_0$ has a I-topology, the situation is quite similar to the U-case. As long as $Q(t) \leq 0$, Eqs (50)-(51) are satisfied – and then the evolution of the field is determined by Eq. (52). A CS forms at some critical time t_c if $Q(t_c) = 0$ and the topology of $\mathbf{B}^p$ becomes of the X-type. It is physically expected to extend vertically between $z_1(t)$ and $z_2 = \infty$ (Fig. 3c).

4.4. Stability, uniqueness and validity of the solutions

We now conclude this general section by a few remarks:

a. At each time t, the solution $\mathbf{B}(t)$ to the evolution problem is linearly stable with respect to both x-invariant (Paper I) and x-dependent perturbations (Aly et al. 1994).

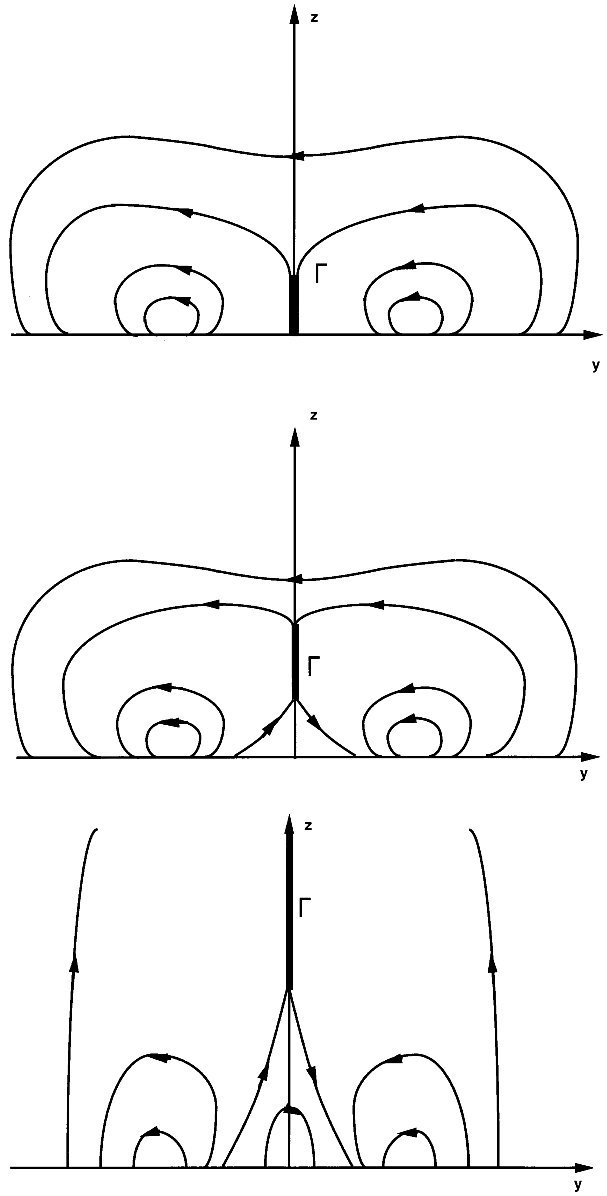

Fig. 3a–c. Structure of the field lines of the QPSE with a vertical CS obtained at $t > 0$ by applying a velocity field $v(y, t)\hat{\mathbf{y}}$ to the footpoints of the field $\mathbf{B}_0$ whose topology is given: **a** in Fig. 2a; **b** in Fig. 2b; **c** in Fig. 2c.

b. Moreover, by a general heuristic argument developed in Paper I, it is expected to be unique. Then, when we find a solution $\mathbf{B}(t)$ which contains a vertical CS at time t, we are guaranteed of the nonexistence of a topologically equivalent field $\mathbf{B}'(t)$ either with no CS, or with a curved or an asymmetric one. Our results below will be in accordance with this general result – without however furnishing a complete independant proof of it, as they are obtained under the restricted assumption of a vertical CS (see Appendix C).

c. To be valid in an actual physical situation (where inertia is always present and motions are not infinitely slow), the quasi-static approximation requires the Alfven time to be much shorter than the evolution time externally imposed by the velocity field.

- If $\mathbf{B}_0$ is of the X-type, this condition can be easily satisfied on the average, but not locally: a wave propagating from the boundary along the X-separatrix takes an infinite time to reach the X-point. However, the interesting region around the latter is reached by the waves in a finite time: the field can get deformed, and it is intuitively expected that the configuration that is obtained at some time t is well approximated by our static configuration. Actually, checking rigorously that this is the case requires solving the full system of ideal MHD equations, which appears as a formidable task (note that some progress in that direction have been made most recently by Priest et al. (1995), who used an intermediate approximation, the so-called strong field limit).
- If $\mathbf{B}_0$ is of the I-type, it would take an infinite time to a signal propagating from the boundary at a finite Alfven speed to reach the infinitely remote region where the CS is expected to form. Then the quasi-static approximation is certainly not valid in this case, and in a redhibitory way. In spite of this point – that the reader should keep in mind –, we shall present the formal calculation of the I-case as it is useful for a complete theoretical understanding of the problem (fortunately, our results for a I-topology will be a byproduct of the study of the other cases, and then will be obtained costless).

4.5. General method for computing the evolution of the field

It results from the previous analysis that the evolution of the field corresponding to the two boundary data g_0 and v, can be computed in several steps:

a. We first solve the equation for g, as explained in Sect. 3.2.

b. We thus compute the potential field $\mathbf{B}^p(t)$ for $t \geq 0$, which can be done by merely applying Eq. (5).

c. In the intervals of time where $\mathbf{B}^p(t)$ satisfies Eqs (50)-(51) (which can be easily checked by computing $P(t)$, $Q(t)$ and $\gamma(t)$), $\mathbf{B}(t)$ is given by Eq. (52).

d. In the intervals of time where one of the two constraints is not satisfied, $\mathbf{B}(t)$ must be a QPSE determined by Eqs (53)-(58). To solve these equations, we are now going to proceed in two steps:

- First we shall solve the equations at a given time, assuming that the positions $z_1(t)$ and $z_2(t)$ of the endpoints of $\Gamma(t)$ are known.
- Thus we shall write the equations which have to be satisfied by these two quantities. Unfortunately, it will not be possible to solve them in closed form, but we shall be able to discuss whether they do admit a solution or not. In the former case, we shall conclude that a vertical CS does develop indeed – our assumption being verified a posteriori. In the latter case, this assumption will appear to be incorrect: the calculation has to be stopped, our theory being not general enough to deal with the more complex situation which appears at this point.

5. Computation of a QPSE

In this section, we solve the BVP defined by Eqs (53)-(58), taking the quantities $g(y,t)$, $z_1(t)$ and $z_2(t)$ to be given. Thus we write the equations which have to be satisfied by $z_1(t)$ and $z_2(t)$. We first assume that $z_2(t) \neq \infty$, the case $z_2(t) = \infty$ being treated in the last subsection. To shorten a little bit the notations, dependence of the various quantities on t will be systematically understood in this section.

5.1. Solution to the BVP and a first relation between z_1 and z_2

To solve our BVP, we transform it into a problem for the complex field $B(\eta)$. Owing to the y-symmetry of the system, the latter can be formulated as follows: Determine in the quarter plane $\{\Re(\eta) > 0, \Im(\eta) > 0\}$ an holomorphic function $B(\eta)$ satisfying the boundary conditions:

$$\begin{aligned} \Im(B)(y) &= -B_z(y,0) = g'(y) \quad \text{for } 0 < y < \infty\,, & (59)\\ \Im(B)(iz) &= -B_z(0,z) = 0 \quad \text{for } 0 < z < z_1\,, & (60)\\ \Re(B)(iz) &= B_y(0,z) = 0 \quad \text{for } z_1 < z < z_2\,, & (61)\\ \Im(B)(iz) &= B_z(0,z) = 0 \quad \text{for } z_2 < z\,, & (62)\\ B(\eta_1) &= B(\eta_2) = 0\,, & (63)\\ \lim_{|\eta|\to\infty} B(\eta) &= 0\,, & (64) \end{aligned}$$

where $\eta_1 = iz_1$ and $\eta_2 = iz_2$. Actually, this problem for B is not strictly equivalent to the one for A set above: the value of the constant γ_0 is lost when we go from Eq. (56) to Eq. (61), and condition (64) is weaker than (55). We shall forget about these problems for a while, and return to them at the end of this subsection and in the next one.

We transform BVP (59)-(64) by effecting the conformal mapping

$$\xi = \eta^2\,, \tag{65}$$

which maps the quarter plane $\{\Re(\eta) > 0, \Im(\eta) > 0\}$ onto the half plane $\{\Im(\xi) > 0\}$. Then, if we set

$$H(\xi) = iB(\sqrt{\xi})\,, \tag{66}$$

BVP (59)-(64) is equivalent to find a holomorphic function H in $\{\Im(\xi) > 0\}$ which satisfies the following boundary conditions:

$$\begin{aligned} \Re(H)(y) &= g'(\sqrt{y}) \quad \text{for } 0 < y < \infty\,, & (67)\\ \Re(H)(y) &= 0 \quad \text{for } -z_1^2 < y < 0\,, & (68)\\ \Im(H)(y) &= 0 \quad \text{for } -z_2^2 < y < -z_1^2\,, & (69)\\ \Re(H)(y)) &= 0 \quad \text{for } -\infty < y < -z_2^2\,, & (70)\\ H(-z_1^2) &= H(-z_2^2) = 0\,, & (71)\\ \lim_{|\eta|\to\infty} H(\eta) &= 0\,. & (72) \end{aligned}$$

If instead of Eq. (71) we only impose H be bounded near $-z_1^2$, this problem admits a unique solution, which is given by Keldish-Sedov's formula (Laurentiev & Chabat 1972; Muskhelishvili 1953, p. 279)

$$H(\xi) = \frac{-1}{i\pi}\frac{(\xi+z_1^2)^{1/2}}{(\xi+z_2^2)^{1/2}}\int_0^\infty \frac{g'(\sqrt{s})}{s-\xi}\frac{(s+z_2^2)^{1/2}}{(s+z_1^2)^{1/2}}\,\mathrm{d}s\,. \tag{73}$$

Actually this solution turns out to vanish at $\xi = -z_1^2$ as required by Eq. (71), but, in general, it is not even bounded near $\xi = -z_2^2$. To get boundedness, we need to have a special relation between z_1 and z_2. Obviously a necessary condition to get this property is that the integral in the RHS of Eq. (73) vanishes for $\xi \to -z_2^2$, i.e.,

$$\int_0^\infty \frac{g'(\sqrt{s})}{(s+z_2^2)^{1/2}(s+z_1^2)^{1/2}}\,\mathrm{d}s = 0\,. \tag{74}$$

Fortunately, this condition turns out to be also sufficient and it even implies the required condition (71). One has indeed, assuming Eq. (74) to hold,

$$\begin{aligned} H(\xi) &= \frac{-1}{i\pi}\frac{(\xi+z_1^2)^{1/2}}{(\xi+z_2^2)^{1/2}}\int_0^\infty \left\{\frac{g'(\sqrt{s})}{s-\xi}\frac{(s+z_2^2)^{1/2}}{(s+z_1^2)^{1/2}} - \frac{g'(\sqrt{s})}{(s+z_2^2)^{1/2}(s+z_1^2)^{1/2}}\right\}\mathrm{d}s \\ &= \frac{-1}{i\pi}(\xi+z_1^2)^{1/2}(\xi+z_2^2)^{1/2}\times \int_0^\infty \frac{g'(\sqrt{s})}{s-\xi}\frac{\mathrm{d}s}{(s+z_2^2)^{1/2}(s+z_1^2)^{1/2}}\,. \end{aligned} \tag{75}$$

Using inverse relations, we obtain eventually

$$B(\eta) = \frac{2}{\pi}(\eta^2+z_1^2)^{1/2}(\eta^2+z_2^2)^{1/2}\times \int_0^\infty \frac{sg'(s)}{s^2-\eta^2}\frac{\mathrm{d}s}{(s^2+z_1^2)^{1/2}(s^2+z_2^2)^{1/2}}\,, \tag{76}$$

with z_1 and z_2 being related by

$$\int_0^\infty \frac{yg'(y)}{(y^2+z_2^2)^{1/2}(y^2+z_1^2)^{1/2}}\,\mathrm{d}y = 0\,. \tag{77}$$

Let us conclude this subsection by three remarks:

a. Formula (76) has been derived for $y > 0$, but it is still valid in the whole Ω provided the square root is defined so that Γ is a cut and $[(y^2+z_1^2)(y^2+z_2^2)]^{1/2}$ is a positive real number. $B(\eta, t)$ given by Eq. (76) is of the general form for a QPSE derived in Paper I: in particular, the integral in the RHS is holomorphic in $\{\Im(\eta) > 0\}$ and the CS Γ coincides with the cut used for defining the square root.

b. For large values of $|\eta|$, we have, taking Eq. (77) into account,

$$B(\eta) \sim_{|\eta|\to\infty} \frac{-2}{\pi\eta^2}\int_0^\infty \frac{(s^2+z_1^2)^{1/2}}{(s^2+z_2^2)^{1/2}}sg'(s)\,\mathrm{d}s\,, \tag{78}$$

which means that $B(\eta)$ behaves like a dipolar field at large distances. Therefore, the solution we have computed satisfies the asymptotic condition (55), as required.

c. Clearly, for g, z_1 and z_2 given and satisfying the condition (77), the solution (76) to BVP (59)-(64) is the unique one.

5.2. A second relation between z_1 and z_2

From Eqs (48), (23) and (56), we have

$$\int_0^{z_1} B_y(0,z)\,\mathrm{d}z = \gamma_0 - m_0\,. \tag{79}$$

Taking the limit $y \to 0$ in Eq. (76), we obtain for the value of B_y in the interval $]0, z_1[$ (see Appendix B)

$$B_y(0,z) = \frac{2}{\pi}\left[(z_2^2-z^2)(z_1^2-z^2)\right]^{1/2}\times \int_0^\infty \frac{sg'(s)}{s^2+z^2}\frac{\mathrm{d}s}{(s^2+z_1^2)^{1/2}(s^2+z_2^2)^{1/2}}\,. \tag{80}$$

Whence the further relation between z_1 and z_2

$$\gamma_0 - m_0 = \frac{2}{\pi}\int_0^\infty \frac{yg'(y)}{(y^2+z_1^2)^{1/2}(y^2+z_2^2)^{1/2}}\times \left(\int_0^{z_1}\frac{(z_1^2-z^2)^{1/2}(z_2^2-z^2)^{1/2}}{y^2+z^2}\,\mathrm{d}z\right)\mathrm{d}y\,. \tag{81}$$

The latter reintroduces the value of γ_0 that was lost in the previous subsection. Therefore, we can now assert that, as far as conditions (77) and (81) are satisfied, the flux function A associated to the field B given by Eq. (76), is the solution to the BVP set in Sect. 4.2.

5.3. The case $z_2 = \infty$

When $\mathbf{B}_0$ has a I-topology, we have seen that the vertical CS extends to infinity, i.e., $z_2 = \infty$ and Eq. (62) disappears. This case can be also treated by the techniques used above, and one gets for the field

$$B(\eta) = \frac{2}{\pi}(\eta^2+z_1^2)^{1/2}\int_0^\infty \frac{sg'(s)}{s^2-\eta^2}\frac{\mathrm{d}s}{(s^2+z_1^2)^{1/2}}\,, \tag{82}$$

which behaves at infinity as

$$B(\eta) \sim_{|\eta|\to\infty} -\frac{2}{\pi\eta}\int_0^\infty \frac{sg'(s)}{(s^2+z_1^2)^{1/2}}\,\mathrm{d}s - \frac{2}{\pi\eta^3}\int_0^\infty \frac{s^3g'(s)}{(s^2+z_1^2)^{1/2}}\,\mathrm{d}s + o(|\eta|^{-3})\,. \tag{83}$$

For the asymptotic condition (55) to be satisfied, the first term in the RHS needs to vanish, whence the relation

$$\int_0^\infty \frac{sg'(s)}{(s^2+z_1^2)^{1/2}}\,\mathrm{d}s = 0\,, \tag{84}$$

which replaces Eq. (77) and allows to determine z_1. Note that: (i) Eq. (82) can be formally deduced from Eq. (76) by merely taking the limit $z_2 \to \infty$. (ii) Eq. (82) is similar to Eq. (22) of Paper II. In the latter, however, we had not to impose Eq. (84), as we were interested in a partially open bipolar field, for which A does not vanish at infinity.

6. Formation of a CS when $\mathbf{B}_0$ is a X-field

In Sects. 4 and 5, we have explained the general principles which determine the formation of a CS, and computed the structure of the field when this process does occurs, assuming a priori the CS to be vertical. To check the consistency of this assumption and to determine the position of the CS, we now discuss the existence and the properties of a solution to the system of equations satisfied by the heights $z_1(t)$ and $z_2(t)$ of the bottom and top, respectively, of a vertical CS. The calculations are a little bit involved, and then, for the clarity of the presentation, we give here only the results, referring the interested reader to Appendix C for proofs. We first suppose that $\mathbf{B}_0$ is of the X-type, which is is the most important case for practical applications.

6.1. Existence of a vertical CS

The results of Sect. 4.3-4 and of Appendix C can be summarized in the following

◇ Proposition X: If $\mathbf{B}_0$ is of the X-type, then:

• $\mathbf{B}(t)$ contains a CS if and only if there is a violation of the topological constraints (50)-(51), which occurs most generally at any time t.

• This CS is vertical if and only if the potential field $\mathbf{B}^p(t)$ associated to $g(t)$ has a X-topology and a critical value $\gamma(t)$ satisfying

$$\gamma(t) > \gamma_0 \,. \tag{85}$$

• The heights of CS bottom and top are thus given by the unique solution $(z_1(t), z_2(t))$ (with $z_1(t) < z_2(t)$) to Eqs (77) and (81), and we have

$$z_1(0) = z_2(0) = Z_0 > 0 \tag{86}$$

in general,

$$0 < z_1(t) < Z(t) < z_2(t) < z_2^U(t) \tag{87}$$

if $m_0 > 0$, and

$$0 < z_1^I(t) < z_1(t) < Z(t) < z_2(t) < \infty \tag{88}$$

if $m_0 \leq 0$. Here, $z_2^U(t)$ and $z_1^I(t)$ are the solutions to Eq. (77) with $z_1 = 0$ and to Eq. (84), respectively (solutions for the U- and I-case; see next section). Then the CS develops around the position of the X-point of $\mathbf{B}^p(t)$. ◇

Using Proposition X and Eq. (46), we obtain the following

◇ Corolarry X: When $\mathbf{B}_0$ is a X-field and $\mathbf{v}$ is either a C- or a CE-velocity, the field $\mathbf{B}(t)$ contains a vertical CS at all time $t > 0$. ◇

6.2. Differential equations for z_1 and z_2

Let us set

$$T_{\mu\nu}[f, z_1, z_2] := \int_0^\infty \frac{y f(y)}{(y^2 + z_1^2)^{\mu/2}(y^2 + z_2^2)^{\nu/2}}\,\mathrm{d}y \,, \tag{89}$$

$$I_1(z_1, z_2) := \int_0^{z_1} \frac{(z_2^2 - z^2)^{1/2}}{(z_1^2 - z^2)^{1/2}}\,\mathrm{d}z \,, \tag{90}$$

$$I_2(z_1, z_2) := \int_0^{z_1} \frac{(z_1^2 - z^2)^{1/2}}{(z_2^2 - z^2)^{1/2}}\,\mathrm{d}z \,, \tag{91}$$

$$I(z_1, z_2, y) := \int_0^{z_1} \frac{(z_1^2 - z^2)^{1/2}(z_2^2 - z^2)^{1/2}}{y^2 + z^2}\,\mathrm{d}z \,. \tag{92}$$

(I_1, I_2 and I can be expressed in terms of Elliptic Integrals; see Appendix D). Then we can rewrite Eqs (77) and (81) as:

$$T_{11}[g', z_1, z_2] = 0 \,, \tag{93}$$

$$T_{11}[g' I[z_1, z_2, .], z_1, z_2] = \frac{\pi}{2}(\gamma_0 - m_0) \,. \tag{94}$$

Differentiating the latter with respect to t, and using Eqs (21) and (28), we obtain:

$$z_1 \dot{z}_1 T_{31}\left[g', z_1, z_2\right] + z_2 \dot{z}_2 T_{13}\left[g', z_1, z_2\right] = -T_{11}\left[(vg')', z_1, z_2\right] \,, \tag{95}$$

$$z_1 \dot{z}_1 I_1(z_1, z_2) T_{31}\left[g', z_1, z_2\right] + z_2 \dot{z}_2 I_2(z_1, z_2) \times T_{13}\left[g', z_1, z_2\right] = T_{11}\left[(vg')' I(z_1, z_2, .), z_1, z_2\right] \,. \tag{96}$$

Whence

$$\dot{z}_1 = \frac{1}{T_{31}\left[g', z_1, z_2\right]\mathbf{K}(z_1/z_2)}\left\{I_2(z_1, z_2) T_{11}\left[(vg')', z_1, z_2\right] + T_{11}\left[(vg')' I(z_1, z_2, .), z_1, z_2\right]\right\}\frac{1}{z_2^2 - z_1^2}\frac{z_2}{z_1} \,, \tag{97}$$

$$\dot{z}_2 = \frac{-1}{T_{13}\left[g', z_1, z_2\right]\mathbf{K}(z_1/z_2)}\left\{I_1(z_1, z_2) T_{11}\left[(vg')', z_1, z_2\right] + T_{11}\left[(vg')' I(z_1, z_2, .), z_1, z_2\right]\right\}\frac{1}{z_2^2 - z_1^2} \,, \tag{98}$$

where $\mathbf{K}$ is the complete elliptic integral of the first kind and use has been made of Eq. (D4).

6.3. Computation of the position of the CS at small time

Let us now look at the behaviour of $z_k(t)$ for small values of t. Since

$$\mathbf{K}\left(\frac{z_1}{z_2}\right) \sim -\frac{1}{2}\ln\left[\frac{2(z_2 - z_1)}{Z_0}\right] \quad \text{for} \;\; z_1 \sim z_2 \sim Z_0 \,, \tag{99}$$

we can first note that $\dot{z}_k \to (-)^k \infty$ when $t \to 0$ and $z_k \to Z_0$. Then a Taylor-Young expansion in the neighbourhood of $t = 0$ is not valid. To get a sensible expansion, we set

$$z_k(t) = Z_0(1 + \epsilon_k(t)) \,. \tag{100}$$

Thus, for small values of t, ϵ_1 and ϵ_2 are approximate solutions to the nonlinear differential system

$$\epsilon_1 = -\epsilon_2 \tag{101}$$

$$(\epsilon_2 - \epsilon_1)\dot{\epsilon}_2 \ln\frac{1}{\epsilon_2 - \epsilon_1} = \frac{K_1}{Z_0^2 K_2} \,, \tag{102}$$

where

$$K_1 := -\int_0^\infty \frac{v_0(y)g_0'(y)}{y^2+Z_0^2}\,\mathrm{d}y\,, \tag{103}$$

$$v_0(y) := v(y,0)\,, \tag{104}$$

$$K_2 := \int_0^\infty \frac{yg_0'(y)}{(y^2+Z_0^2)^2}\,\mathrm{d}y = -\frac{\pi}{4Z_0}\frac{\partial B_{0y}(0,Z_0)}{\partial z}(0,Z_0)\,. \tag{105}$$

Solving the equations to the lowest order, we obtain

$$\epsilon_k(t) \simeq (-1)^k\left[\frac{2t}{\tau_{ev}}\frac{1}{\ln(8\tau_{ev}/t)}\right]^{1/2}, \tag{106}$$

where the evolution time τ_{ev} is given by

$$\tau_{ev} := \frac{K_2 Z_0^2}{K_1}\,. \tag{107}$$

Of course, for this calculation to be consistent, we need to have $\tau_{ev} > 0$, i.e., $K_1K_2 > 0$. To get the condition of validity of this relation, we note that

$$K_1 = \pi\dot{\gamma}(0)/2Z_0 \tag{108}$$

(see Eq. (34)) and

$$K_2 > \frac{1}{(d^2+Z_0^2)}\frac{\pi}{2}B_{0y}(0,Z_0) = 0 \tag{109}$$

(where use has been made of Lemma A of Appendix A and Eq. (10); we could also note that $B_y^p(0,z,t)$ has to decrease above the X-point owing to the shape of the lines). Then the condition $K_1K_2 > 0$ is equivalent to $\dot{\gamma}(0) > 0$, which is satisfied in general as a consequence of (85) (actually, the latter implies only $\dot{\gamma}(0) \geq 0$, but we expect to have $\dot{\gamma}(0) = 0$ only for special boundary conditions. If this is the case, the expansion has to be pushed to a higher order).

7. Formation of a CS when $\mathbf{B}_0$ is either a U- or a I-field

7.1. Existence of a vertical CS

The results of Sect. 4.3-4 and of Appendix C imply the following

◇ Proposition UI: Assume that $\mathbf{B}_0$ has either a U- or a I-topology. Then:

• $\mathbf{B}(t)$ contains a CS at time t if and only if the potential field $\mathbf{B}^p(t)$ is of the X-type. This CS is vertical.

• In the U-case, the heights of CS bottom and top are given, respectively, by $z_1(t) = 0$ and by the unique solution $z_2(t)$ to the equation

$$\int_0^\infty \frac{g'}{\sqrt{y^2+z_2^2}}\,\mathrm{d}y = 0 \tag{110}$$

(Eq. (77) with $z_1 = 0$). $z_2(t)$ satisfies

$$z_1(t) = 0 < Z(t) < z_2(t)\,. \tag{111}$$

• In the I-case, the CS extends between $z_1(t)$ and $z_2(t) = \infty$, with $z_1(t)$ the unique solution to Eq. (84). $z_1(t)$ does satisfy

$$z_1(t) < Z(t) < z_2(t) = \infty\,. \tag{112}$$

Then in both cases the CS develops around the position of the X-point of $\mathbf{B}^p(t)$. ◇

By using the results of Sect. 3.2 on the evolution of the potential field, we obtain at once the following

◇ Corollary UI: If $\mathbf{B}_0$ is either of the U- or I-type, and $\mathbf{v}$ of the CE'- and CE-type, respectively, a vertical CS starts forming at some time t_c and stays present at all $t > t_c$. ◇

7.2. *Differential equations for z_1 and z_2 and approximate solutions*

Consider first the U-case. Differentiating Eq. (110) with respect to time, using Eqs (28) and (89), and effecting an integration by part, we obtain

$$\dot{z}_2 = -\frac{T_{03}\left[vg',0,z_2\right]}{z_2T_{03}\left[g'/y,0,z_2\right]}\,. \tag{113}$$

From this equation, we can easily obtain an approximate explicit form for $z_2(t)$ just after the CS has started developing at t_c. For simplicity, we shall assume here that $\zeta_g := g_0''(0,t_c) \neq 0$ (then $\zeta_g > 0$). For small values of $t-t_c$, $z_2(t)$ stays a small quantity. The numerator in Eq. (113) remains finite for $z_2 \to 0$ (we can evaluate the limit under the integral) while the denominator is unbounded (improper integral at 0), behaving like ζ_g/z_2. Whence

$$\dot{z}_2(t_c) \sim V_2\,, \tag{114}$$

where

$$V_2 := -\frac{1}{\zeta_g}\int_0^\infty \frac{(vg')(y,t_c)}{y^2}\,\mathrm{d}y = \frac{\dot{\gamma}(t_c)}{\zeta_g}\,, \tag{115}$$

and use has been made of Eq. (34). Integrating this equation, we get eventually

$$z_2(t) \sim V_2(t-t_c) \quad \text{for } t \geq t_c\,. \tag{116}$$

As required for the consistency of our calculation, we have $V_2 > 0$. Indeed, $\gamma(t) > m_0 = \gamma_0$ when $\mathbf{B}^p$ becomes of the X-type, and then $\dot{\gamma}(t_c) > 0$ in general (see the remark at the end of Sect. 6.3). We have thus shown that the top of the CS rises initially at a velocity of the order of the typical value of v on the boundary.

Similarly, we have in the I-case

$$\dot{z}_1 = z_1\frac{T_{30}[vg'/y,z_1,0]}{T_{30}[g',z_1,0]}\,. \tag{117}$$

Whence, for $t-t_c > 0$ not too large,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{z_1^2}\right) \simeq 4\frac{\int_0^\infty (vg')(y,t_c)\,\mathrm{d}y}{\int_0^\infty y^3g'(y,t_c)\,\mathrm{d}y} = 4\frac{\dot{Q}(t_c)}{\int_0^\infty y^3g'(y,t_c)\,\mathrm{d}y} > 0\,, \tag{118}$$

and

$$z_1(t) \simeq \frac{1}{2\sqrt{t-t_c}} \left(\frac{\int_0^\infty y^3 g'(y,t_c)\,\mathrm{d}y}{\dot{Q}(t_c)} \right)^{1/2}, \tag{119}$$

where we have used the fact that $Q(t_c) = 0$, while $\dot{Q}(t_c) > 0$ in general.

8. Free magnetic energy

8.1. Free magnetic energy stored in a QPSE

The free magnetic energy (per unit of x-length) available in the field $\mathbf{B}(t)$ is defined by

$$\begin{aligned} \delta W(t) &:= W(t) - W^p(t) \\ &= \frac{1}{8\pi} \int_\Omega [|\mathbf{B}(t)|^2 - |\mathbf{B}^p(t)|^2]\,\mathrm{d}y\mathrm{d}z\,, \end{aligned} \tag{120}$$

where $W(t)$ and $W^p(t)$ are the magnetic energies stored in the fields $\mathbf{B}(t)$ and $\mathbf{B}^p(t)$, respectively. Of course, when Eq. (52) holds at t, we have

$$\delta W(t) = 0\,. \tag{121}$$

When the field is a QPSE, we have (Paper I)

$$W(t) = \frac{1}{4\pi} \int_0^\infty [\gamma_0 - g(y,t)] B_y(y,0,t)\,\mathrm{d}y \tag{122}$$

and

$$W^p(t) = -\frac{1}{4\pi} \int_0^\infty g(y,t) B_y^p(y,0,t)\,\mathrm{d}y\,, \tag{123}$$

whence

$$\begin{aligned} \delta W(t) &= \frac{\gamma_0}{4\pi} \int_0^\infty B_y(y,0,t)\,\mathrm{d}y \\ &- \frac{1}{4\pi} \int_0^\infty g(y,t)[B_y - B_y^p](y,0,t)\,\mathrm{d}y\,. \end{aligned} \tag{124}$$

To evaluate this expression, we note that, when $z_2(t) \neq \infty$,

$$\begin{aligned} B_y(y,0,t) &= \frac{2}{\pi} \frac{(y^2+z_1^2)^{1/2}}{(y^2+z_2^2)^{1/2}} \times \\ & \oint_0^\infty \frac{sg'(s,t)}{s^2-y^2} \frac{(s^2+z_2^2)^{1/2}}{(y^2+z_1^2)^{1/2}}\mathrm{d}s \end{aligned} \tag{125}$$

by Eqs (B3) and (77), while

$$B_y^p(y,0,t) = \frac{2}{\pi} \oint_0^\infty \frac{sg'(s,t)}{s^2-y^2}\,\mathrm{d}s \tag{126}$$

by Eq. (6) ($\oint$ means that the integral has to be understood as a principal part). This last quantity satisfies (Ampère law)

$$\int_0^\infty B_y^p(y,0,t)\,\mathrm{d}y = 0\,. \tag{127}$$

Then, using Eqs (37), (53) and the previous relations, we get

$$\begin{aligned} \delta W(t) &= \frac{1}{2\pi^2} \int_0^\infty \oint_0^\infty [\gamma_0 - g(y,t)] \frac{sg'(s,t)}{s^2-y^2} \times \\ & \left(\sqrt{\frac{(y^2+z_1^2)(s^2+z_2^2)}{(y^2+z_2^2)(s^2+z_1^2)}} - 1 \right) \mathrm{d}y\,\mathrm{d}s\,. \end{aligned} \tag{128}$$

This relation applies when $\mathbf{B}_0$ is either of the X- or U-type, z_1 being set equal to zero in the latter case.

When $\mathbf{B}_0$ is of the I-type, we have $z_2 = \infty$, and we can show similarly that

$$\begin{aligned} \delta W(t) &= \frac{1}{2\pi^2} \int_0^\infty \oint_0^\infty [\gamma_0 - g(y,t)] \frac{sg'(s,t)}{s^2-y^2} \\ & \times \left(\sqrt{\frac{y^2+z_1^2}{s^2+z_1^2}} - 1 \right) \mathrm{d}y\,\mathrm{d}s\,. \end{aligned} \tag{129}$$

8.2. Variation of the energy

The time-derivative of $\delta W(t)$ can be directly computed by using the expressions of the previous subsection. It is simpler, however, to note that the variation of $W(t)$ and $W^p(t)$ are equal to the fluxes of the appropriate Pointing vectors through the perfectly conducting boundary $\partial\Omega$. Using Ohm's law, we have

$$\mathbf{E}_h = -(\mathbf{v} \times \mathbf{B})_h \quad \text{and} \quad \mathbf{E}_h^p = -(\mathbf{v} \times \mathbf{B}^p)_h \tag{130}$$

for the horizontal components on the boundary of the electric fields associated to $\mathbf{B}(t)$ and $\mathbf{B}^p(t)$, respectively. Whence

$$\begin{aligned} \delta \dot{W}(t) &= \int_{-\infty}^\infty \frac{\hat{\mathbf{z}}}{4\pi}.[\mathbf{E}_h \times \mathbf{B}_h - \mathbf{E}_h^p \times \mathbf{B}_h^p](y,0,t)\,\mathrm{d}y \\ &= \frac{1}{2\pi} \int_0^\infty (vg')(y,t)(B_y - B_y^p)(y,0,t)\,\mathrm{d}y\,. \end{aligned} \tag{131}$$

Then, using Eqs (125) and (126), we have

$$\begin{aligned} \delta \dot{W}(t) &= \frac{1}{\pi^2} \int_0^\infty \oint_0^\infty (vg')(y,t) \frac{sg'(s,t)}{s^2-y^2} \\ & \left(\sqrt{\frac{(y^2+z_1^2)(s^2+z_2^2)}{(y^2+z_2^2)(s^2+z_1^2)}} - 1 \right) \mathrm{d}y\,\mathrm{d}s \end{aligned} \tag{132}$$

when $z_2 \neq \infty$, and

$$\begin{aligned} \delta \dot{W}(t) &= \frac{1}{\pi^2} \int_0^\infty \oint_0^\infty (vg')(y,t) \frac{sg'(s,t)}{s^2-y^2} \\ & \left(\sqrt{\frac{y^2+z_1^2}{s^2+z_1^2}} - 1 \right) \mathrm{d}y\,\mathrm{d}s \end{aligned} \tag{133}$$

when $z_2 = \infty$.

8.3. Free energy at small time in the X-case

We now compute a small time approximation of the free magnetic energy stored in the QPSE. To avoid the paper becoming too long, we present only the X-case.

Let us set

$$z_k^2 = Z_0^2(1 + \eta_k)\,, \tag{134}$$

with $\eta_k = 2\epsilon_k + \epsilon_k^2$ small. Then

$$\sqrt{\frac{(y^2 + z_1^2)(s^2 + z_2^2)}{(y^2 + z_2^2)(s^2 + z_1^2)}} - 1 \simeq \frac{Z_0^2}{2}(\eta_2 - \eta_1)\frac{s^2 - y^2}{(y^2 + Z_0^2)(s^2 + Z_0^2)} \times \left[1 - \frac{Z_0^2}{4}\left(\frac{3\eta_1 + \eta_2}{y^2 + Z_0^2} + \frac{\eta_1 + 3\eta_2}{s^2 + Z_0^2}\right)\right]. \tag{135}$$

Injecting this expression into Eq. (132), developing v and g' in Taylor series, and using Eqs (10) and (106), we see that the dominant term in $\delta\dot{W}(t)$ is given by

$$\delta\dot{W}(t) \simeq \frac{Z_0^4}{8\pi^2}(\eta_2 - \eta_1)(3\eta_2 + \eta_1)K_1K_2 \simeq \frac{4Z_0^2K_1^2t}{\pi^2\ln(8\tau_{ev}/t)}\,, \tag{136}$$

where we have taken $\eta_2 \simeq -\eta_1 \simeq 2\epsilon_2$. Whence

$$\delta W(t) \simeq \frac{2Z_0^2}{\pi^2}\frac{K_1^2t^2}{\ln(8\tau_{ev}/t)}\,. \tag{137}$$

To make the scaling of τ_{ev} (and then of ϵ_k) and of δW more transparent, we introduce the typical magnitudes $\widetilde{B}_0$ and $\widetilde{v}_0$, and length scales l_B and l_v, of g_0' and v_0, respectively, and we define two dimensionless numbers k_1 and k_2 and a characteristic length l by

$$K_1 =: k_1\frac{\widetilde{v}_0\widetilde{B}_0}{Z_0}\frac{l}{l + Z_0}\,, \tag{138}$$

$$l := \frac{l_Bl_v}{l_B + l_v}\,, \tag{139}$$

$$K_2 =: k_2\frac{\widetilde{B}_0}{Z_0^2}\,. \tag{140}$$

Then

$$\tau_{ev} = \frac{k_2}{k_1}\frac{Z_0}{\widetilde{v}_0}\frac{l + Z_0}{l}\,. \tag{141}$$

and

$$\delta W(t) \simeq \frac{\widetilde{B}_0^2\widetilde{v}_0}{4\pi}\widetilde{v}_0t^2\left(\frac{l}{l + Z_0}\right)^2\frac{2k_1^2}{\pi\ln(8\tau_{ev}/t)}\,. \tag{142}$$

For reasonable boundary conditions, we can safely expect that k_1 and $k_2 \sim 1$.

8.4. Example

Let us assume that:

a. The initial field $\mathbf{B}_0$ is created by two horizontal dipoles of moment $\tilde{B}_0h^2\hat{\mathbf{y}}$ ($\tilde{B}_0$ and $h > 0$), located at $(\pm\mu h, -h)$, respectively, with $1 < \mu$ (then $\tilde{B}_0$ has a X-topology indeed, as shown in Appendix A.4).

b. The velocity field is of the C-type, being given by

$$v(y) = -\tilde{v}_0\frac{yZ_0}{y^2 + Z_0^2}\,, \tag{143}$$

where $\tilde{v}_0$ is a positive constant and Z_0 is the height of the X-point in the initial configuration (the calculations below can also be done in closed form if we substitute an arbitrary length l_v for Z_0, but the results are too cumbersome to be illuminating).

Making use of the relations of Appendix A.4, we obtain

$$K_1 = \frac{\pi\tilde{v}_0\tilde{B}_0}{4\mu^3h}\,, \tag{144}$$

$$K_2 = \frac{\pi}{4}\tilde{B}_0\frac{1}{\mu^3hZ_0} = \frac{\pi}{4}\tilde{B}_0\frac{1}{\mu^3(\mu - 1)h^2}\,. \tag{145}$$

Whence

$$\tau_{ev} = \frac{Z_0}{\tilde{v}_0} = \frac{(\mu - 1)h}{\tilde{v}_0}\,, \tag{146}$$

and

$$\delta W(t) = \frac{\tilde{v}_0\tilde{B}_0^2}{4\pi}\tilde{v}_0t^2\left(\frac{\mu - 1}{\mu^3}\right)^2\frac{\pi}{8\ln[8(\mu - 1)h/\tilde{v}_0t]}\,. \tag{147}$$

It is interesting to compute the variation of the height of the X-point of the associated field $\mathbf{B}^p(t)$. Using Eq. (35), we obtain for t not too large

$$Z(t) \simeq Z_0 + \dot{Z}(0)t = Z_0 + \frac{\tilde{v}_0}{2}\left(\frac{3}{2\mu} - 1\right)t\,. \tag{148}$$

Then, for t not too large, the X-point rises if $1 < \mu < 3/2$ and sinks if $3/2 < \mu$. This shows that there is no relation between the formation of a vertical CS and the sign of the motion of the X-point – a point which was not completely clear a priori.

9. Plasma heating by a relaxation mechanism

9.1. Description of the model

In the presence of resistivity, it is clear that an infinitely thin CS cannot form. But it is intuitively expected that there will be a concentration of electric currents near the location of the ideal CS, and that these currents will be rapidly dissipated once the typical width of their distribution will become too small. To try to represent this complex process, we propose here to use the following simple model. Assume that it takes some characteristic time τ_D to destroy a concentration of current by resistive effects (diffusion, tearing, reconnection). Then we replace the continuous process by the following discontinuous one:

- During the intervals of time $[k\tau_D, (k+1)\tau_D[$ ($k = 0, 1, 2, ..., N$), the field evolves quasi-statically according to the ideal picture presented in the previous sections.
- At each time $(k+1)\tau_D$, the field relaxes instantaneously to the potential configuration associated to the value $g[y, (k+1)\tau_D)]$ of g at that time, the stored free-energy being eventually converted into heat.

Let us assume for definiteness that we start at $t = 0$ with a X-potential field, and that we apply to its footpoints a C-velocity field during the period of time $N\tau_D$, with N not too big, for otherwise the compression of the boundary near the origin would become irrealistically large. In that case, we know that a vertical CS does form during each ideal phase, and the field is given by

$$B(t) = \mathscr{B}[B^p(k\tau_D), t - k\tau_D] \text{ for } k \leq t/\tau_D < (k+1)\,, \quad (149)$$

where $\mathscr{B}[B_0, t]$ is just a new notation (in which appears explicitly the dependence on the initial potential field) for the field given by Eq. (76). The energy which is released at each time $t = (k+1)\tau_D$ is thus given by

$$\begin{aligned} \delta W_{k+1} &:= \delta W((k+1)\tau_D) \\ &= W[\mathscr{B}[B^p(k\tau_D), \tau_D]] - W[B^p((k+1)\tau_D)]\,, \quad (150) \end{aligned}$$

with obvious notations.

In fact, it seems likely that the dissipation time τ_D should be in any actual case smaller than the evolution time τ_{ev}, and then we can use the approximate formula (142), which gives for the heating rate F_{k+1} averaged over one step – defined as the amount of heat released per unit of time and unit of boundary area –

$$\begin{aligned} F_{k+1} &:= \frac{\delta W_{k+1}}{\tau_D l_{Bk}} \\ &= \frac{\widetilde{B}_k^2 \widetilde{v}_k}{4\pi} \frac{\tau_D}{\tau_{trk}} \left(\frac{l_k}{l_k + Z_k}\right)^2 \frac{2k_{1k}^2}{\pi \ln(8\tau_{evk}/\tau_D)}\,. \quad (151) \end{aligned}$$

In this formula, the *transit time* τ_{trk} is defined by

$$\tau_{trk} := \frac{l_{Bk}}{\widetilde{v}_k}\,, \quad (152)$$

and the index k indicates that a quantity has to be computed by substituting $\mathbf{B}^p(k\tau_D)$ to $\mathbf{B}_0$ in the original definition. The mean heating rate over all the process is thus given by

$$F := \frac{\sum_0^{N-1} F_{k+1}}{N}\,. \quad (153)$$

For illustrating these results, let us suppose that the field of a strong solar active region evolves through configurations which do not differ too much from the particular $\mathbf{B}_0$ introduced in Sect. 8.4. Taking $\mu = 2$, $h = Z_0 = 10^9$ cm, $\tilde{v}_0 = 10^5$ cm s^{-1} and $\tilde{B}_0 = 500G$, we then get $\tau_{ev} \sim 10^4$ s and $F = 0.6\ 10^5$ ergs cm^{-2} s^{-1} for $\tau_D/\tau_{ev} = 0.1$, while $F = 10^6$ ergs cm^{-2} s^{-1} for $\tau_D/\tau_{ev} = 1$ (we assume here that our formula is still valid for this value of τ_D/τ_{ev}). Then, for our particular configuration, we can get heating rate comparable to the canonical $F_{obs} = 10^6$ ergs cm^{-2} s^{-1} only if τ_D turns out to be of the order of τ_{ev}, which may be difficult to admit. Other configurations, however, may lead to a more efficient heating.

To conclude this subsection, we note that if we start from a U-topology instead of a X-one, then we recover the previous case as soon as a CS has started forming and has been dissipated by reconnection – the field $\mathbf{B}^p$ to which the field relaxes and which is used as the new initial condition being necessarily of the X-type. The only difference between the U- and X-case is the delay in the start of the heating process which occurs for the former. It may be worth noticing here that some authors have challenged the view first expressed by Low (1987), according to which a CS forms in a U-field (Karpen et al. 1990). They do argue that this process is an artefact: it just results from neglecting the reaction of the phenomena occuring in D on those occuring below ∂D, and it disappears when the rigid infinitely thin boundary ∂D is replaced by a more realistic transition layer, whose detailed physics is taken into account. A rebuttal of these arguments was thus presented by Low (1991). Without entering this controversy, we just remark here that, as far as heating processes are concerned, it is a bit academic: owing to our remark above, we stay only for a very short while in the controversial regime, turning on very rapidly to fields with an X-point. But, of course, solving that point is important for a complete theoretical understanding of the magnetohydrodynamics of CS formation.

9.2. Comparison with the Heyvaerts-Priest model

Our model here is somewhat akin of a model devised by Heyvaerts & Priest (1984, HP hereafter), which is an adaptation to the solar corona context of a mechanism first proposed by Taylor (e.g., Taylor 1986) to account for the plasma behaviour in some laboratory machines. HP consider an x-invariant bipolar field in $\Omega_L := \{-L/2 < y < L/2, 0 < z\}$, and its evolution through a sequence of two-steps processes:

- During the intervals of time $[k\tau_D, (k+1)\tau_D[$ ($k = 0, 1, 2, ...$), the field $\mathbf{B}$ evolves quasi-statically through a sequence of nonlinear force-free arcade configurations as a result of a x-parallel boundary velocity field applied to its footpoints.
- Because of the development of an increasing amount of MHD turbulence, the field relaxes at each time $(k+1)\tau_D$ to the minimum energy state $\mathbf{B}^-[(k+1)\tau_D]$ having the same value of the relative helicity as $\mathbf{B}[(k+1)\tau_D]$ ($\mathbf{B}^-[(k+1)\tau_D]$ is a constant-alpha force-free field). The stored free-magnetic energy (now defined to be the difference between the energies of the nonlinear and the linear force-free fields, respectively) is thus converted into heat.

Keeping our notations above, the rate of heating computed in HP is given by

$$F(\tau_D) \sim \frac{\tilde{B}_0{}^2 \tilde{v}_0}{4\pi} \frac{l_B}{l_B + l_v} \frac{\tau_D}{\tau_{tr}}\,, \quad (154)$$

where it is assumed that $\tau_D/\tau_{tr} \ll 1$. Note that the quantity $\tau_{tr} = l_B/v_0$ appearing here is not (as in our model) a time characterizing the change in the boundary conditions – here, the normal component B_z of the field on the boundary is preserved by the x-motions.

In its spirit, our model appears to be quite similar to HP, in that sense that it is a global model, in which the details of the complicated dissipative physics are bypassed, being hidden in a phenomenological dissipation time τ_D. There is however a difference: our model is more specific, as it shows clearly the reason – the presence of the CS – why dissipation has necessarily to occur eventually. In HP, on the contrary, it is assumed that some MHD turbulence develops in the field for some reasons, but this point has not been proven thus far.

As far as the heating rate is concerned, the two formulas (151) and (154) looks quite similar, the main difference lying in the presence of a logarithmic term in the former.

10. Conclusions

Let us summarize the main results which have been obtained in this paper:

a. y-symmetric quadrupolar potential fields can be divided into three topological classes – denoted by U, X and I, respectively – that are characterized by the signs of two parameters P and Q. A U-field has a separatrix made of two branches meeting the boundary at the origin, a X-field has a separatrix made of four branches meeting at a neutral point of the X-type, and a I-field has two separatrices connecting the boundary to infinity.

b. When the footpoints of such a X-type $\mathbf{B}_0$ are submitted to y-symmetric motions parallel to the y-axis, a CS forms at once in general. This CS is vertical at time t if and only if the potential field $\mathbf{B}^p(t)$ associated to the value of $B_z(t)$ on the boundary is of the X-type and the value γ of the flux function at the X-point of the latter is larger than its initial value γ_0 (assuming that $B_z \geq 0$ for $y \geq d(t) > 0$). This certainly happens if the boundary velocity field is of the CE- or C-type (velocities of the former type are compressive in the inner region and expansive in the outer one; those of the latter type are compressive everywhere). If, on the contrary, $\gamma < \gamma_0$ (which would be the case for a v of the EC- or E-type), the CS is expected to develop transversally to the y-axis.

c. If $\mathbf{B}_0$ is of the U- or I-type, a CS – and always a vertical one – does form when $\mathbf{B}^p(t)$ becomes of the X-type (a X-point detaching from the boundary and from infinity, respectively). This certainly happens if the boundary velocity field is of the CE'- and CE type, respectively. The CS extends around the position of the X-point of $\mathbf{B}^p(t)$, between $z_1(t) = 0$ and $z_2(t) > 0$ in the U-case and between $z_1(t) > 0$ and $z_2(t) = \infty$ in the I-one.

d. When there is a vertical CS, the magnetic field $\mathbf{B}(t)$ (QPSE) can be computed analytically (by adapting a method introduced in Paper II) as a functional of the boundary value $B_z(t)$ and of the positions of the end-points $z_1(t)$ and $z_2(t)$ of the CS, the last two quantities being related to the first one by explicit equations. Unfortunately, it is not possible in general to solve them in closed form. This can be done in an approximate way, however, if a not too long time has elapsed after the formation of the CS at $t_c \geq 0$. In that case, $\mathbf{B}(t)$ can be expressed as a functional of only $B_z(t_c)$ and $v(t_c)$.

e. Similarly, the free-magnetic energy stored in $\mathbf{B}(t)$ can be explicitly computed as a functional of $B_z(t)$ on the boundary, $z_1(t)$ and $z_2(t)$ – at small time, of $B_z(t_c)$ and $v(t_c)$.

f. In the presence of inertia and resistivity, an infinitely thin CS cannot form, but it is expected that currents concentrate into a thin layer – the field being well approximated by our QPSE – before resistive dissipation starts on. This process may be tentatively represented by a discontinuous evolution, in which a quasi-ideal phase of some duration τ_D and describable by our quasi-static theory, is followed by an infinitely short phase of dissipation (by reconnection, ...) during which the field relaxes to a new potential configuration, these two steps repeating over and over. The energy released during such a process and the associated average heating rate F (heating per units of time and of boundary area) can then be estimated by using the formulas quoted in (e). This simple model furnishes an interesting result: it exhibits the explicit dependence of F on the time τ_D – basically, F is a linear function of τ_D. But it leaves open an important point, as it does not provide a way of estimating this characteristic timescale, which is just introduced by hand.

Our study may be of some use for discussing some aspects of the problem of the heating of the solar and stellar coronas, and more precisely of active regions. It is most generally admitted that coronal heating results from the dissipation of the electric currents which are continuously generated by the photospheric fluid motions applied to the footpoints of the field. However, the mechanisms actually at work are not yet clearly identified, and there is some ongoing debates on the relative efficiency of *wave heating* and *direct current heating* – the former resulting from the dissipation of short timescale AC disturbances, the latter from the dissipation of slowly evolving DC currents (timescales shorter and larger than the Alfven timescale, respectively). The problem with direct current heating is that Joule dissipation on coronal length scale (loops) leads to a very weak production of heat. The mechanism can be efficient only if the currents get concentrated into sufficiently thin layers, and theoreticians have long addressed the problem of the possibility of spontaneously generating small scales in a slowly evolving large scale field. The process we have studied (after many other authors) in this paper provides the simplest example of such a mechanism. Actually, more complicated situations have already been considered. For instance, the behaviour of a quadrupolar field submitted to shearing motions ($v_y = 0$, $v_x \neq 0$) and evolving quasi-statically through a sequence of nonlinear force-free configurations has been studied in details (e.g., Low & Wolfson 1988, Aly 1987a-b, Vekstein et al., 1990, Low 1991a), and it has been shown that CS form all along X- and U-separatrices in that case. Also, it has been suggested by Parker (1979, 1984) that CS form spontaneously in three-dimensional slowly evolving fields even in the absence of separatrices. Although this idea has been challenged by many authors (e.g., Van Ballegooijen 1985, Aly 1987a-b), it certainly deserves to be investigated more deeply (see the discussion in Amari 1991).

In the next paper of this series (Amari & Aly 1996, in preparation), we shall examine the case where the CS form transversally to the y-axis in an X-topology, as well as the case where the assumption of y-symmetry is given up.

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Appendix A: existence of a neutral point of $\mathbf{B}^p$ on the z-axis

A.1. Two elementary lemmas

We first give two simple lemmas which are repeatidly used in this paper.

⋄ Lemma A: Let g' satisfy the conditions (8), and Φ and Ψ be two positive functions (possibly vanishing for some isolated values of y), with Φ being strictly increasing. Then

$$\int_0^\infty (\Phi\Psi g')(y)\,\mathrm{d}y < \Phi(d)\int_0^\infty (\Psi g')(y)\,\mathrm{d}y\,. \tag{A1}$$

If Φ is strictly decreasing, the reversed inequality is satisfied. ⋄

To prove Eq. (A1), we just need to note that: (i) $\Psi g' \not\equiv 0$; (ii) $\Psi g' \geq 0$ and $\Phi(y) < \Phi(d)$ on $[0, d[$; (iii) $\Psi g' \leq 0$ and $\Phi(y) > \Phi(d)$ on $]d, \infty[$. The case where Φ is decreasing is handled in the same manner.

⋄ Lemma B: Let $F(\alpha)$ be a continuously differentiable function defined on $[a_1, a_2]$ ($0 \leq a_1 < a_2 < \infty$) and satisfying

$$\forall\alpha:\ F'(\alpha) < -cF(\alpha)\,, \tag{A2}$$

with $c > 0$ a constant. Then the equation $F(\alpha) = 0$ has a solution in $[a_1, a_2]$ if and only if

$$F(a_1) \geq 0 \ \text{ and } \ F(a_2) \leq 0\,, \tag{A3}$$

and this solution is unique. ⋄

Here, we first note that Eq. (A2) implies that $F'(\alpha) < 0$ if $F(\alpha) = 0$. Then the equation $F(\alpha) = 0$ has at most one solution. On the other hand, the same relation implies that Eq. (A3) is necessarily satisfied if such a solution does exist and belongs to $[a_1, a_2]$. Reciprocally, equation $F(\alpha) = 0$ has obviously a solution if Eq. (A3) holds true.

A.2. Existence of a neutral point of $\mathbf{B}^p$ on the z-axis

As noted in Sect. 3, $\mathbf{B}^p$ has a neutral point on the z-axis if the equation

$$f(\alpha) := \int_0^\infty \frac{yg'}{y^2+\alpha^2}\,\mathrm{d}y = 0 \tag{A4}$$

has a solution $\alpha = Z$. We have:

$$f'(\alpha) = -2\alpha\int_0^\infty \frac{yg'}{(y^2+\alpha^2)^2}\,\mathrm{d}y < -\frac{1}{d^2+\alpha^2}f(\alpha)\,, \tag{A5}$$

$$f(0) = \int_0^\infty \frac{g'}{y}\,\mathrm{d}y = \frac{\pi}{2}B_y^p(0,0) =: \frac{\pi}{2}P\,, \tag{A6}$$

$$\alpha^2 f(\alpha) \sim_{\alpha\to\infty} \int_0^\infty yg'\,\mathrm{d}y = -\int_0^\infty g(y)\,\mathrm{d}y =: -Q\,, \tag{A7}$$

where use has been made of Lemma A to get the first relation. Using Lemma B (with $a_1 = 0$ and a_2 some large number), we thus have:

- If $P < 0$, $f(\alpha) < 0$, and there is no neutral point.
- If $P = 0$, $f(0) = 0$, and we have a V-point at the origin.
- If $P > 0$ and $Q > 0$, the equation $f(\alpha) = 0$ has a unique positive solution $\alpha = Z$, and then there is one neutral point on the z-axis, located at height $Z > 0$. Note for later use that $f(\alpha) > 0$ for $0 < \alpha < Z$ and $f(\alpha) < 0$ for $Z < \alpha$.
- If $P > 0$ and $Q \leq 0$, $f(\alpha) > 0$, and there is no neutral point (for $Q = 0$, the neutral point of the previous case reaches infinity).

A.3. An example of quadrupolar potential configuration

Let us choose

$$g(y) = \widetilde{B}_0 h^3\frac{y^2+\mu h^2}{(y^2+h^2)^2}\,, \tag{A8}$$

where $\widetilde{B}_0$, $h > 0$ and μ are constants. Then we obtain after some algebra

$$A^p(y,z) = \widetilde{B}_0 h\left\{\frac{2h^2+(\mu+1)hz}{2[y^2+(h+z)^2]} + (\mu-1)h^2\frac{(h+z)^2}{[y^2+(h+z)^2]^2}\right\}, \tag{A9}$$

$$B_z^p(y,0) = 2\widetilde{B}_0 h^3 y\frac{y^2-h^2(1-2\mu)}{(y^2+h^2)^3}\,, \tag{A10}$$

$$B_y^p(0,z) = \widetilde{B}_0 h^2\frac{h(1-3\mu)-(1+\mu)z}{2(h+z)^3}\,, \tag{A11}$$

$$P = \widetilde{B}_0\frac{1-3\mu}{2}\,, \tag{A12}$$

$$Q = \pi\widetilde{B}_0 h^2\frac{\mu+1}{4}\,. \tag{A13}$$

Then the field is quadrupolar if

$$\mu < 1/2\,, \tag{A14}$$

in which case

$$d = h\sqrt{1-2\mu}\,, \tag{A15}$$

$$m = \widetilde{B}_0 h\mu\,, \tag{A16}$$

$$M = \widetilde{B}_0 h\frac{1}{4(1-\mu)}\,. \tag{A17}$$

On the other hand, its topology is of type

$$\begin{array}{lll} \bullet\, U & \text{if} & 1/3 \leq \mu < 1/2\,, \\ \bullet\, X & \text{if} & -1 < \mu < 1/3\,, \\ \bullet\, I & \text{if} & \mu \leq -1\,. \end{array} \tag{A18}$$

In the X-case,

$$Z = h\frac{1-3\mu}{1+\mu}\,, \tag{A19}$$

$$\gamma = \frac{\widetilde{B}_0 h}{8}\frac{(1+\mu)^2}{1-\mu}\,, \tag{A20}$$

A.4. A second example of quadrupolar potential configuration

A field which has been used quite often before (e.g., Priest & Raadu 1975, Low 1987, Sneyd 1993) is the one produced by two dipoles of moment $\tilde{B}_0 h^2 \hat{\mathbf{y}}$ located at $(\pm\mu h, -h)$, respectively (h, μ and $\tilde{B}_0 > 0$). For this field, we have

$$g(y) = \tilde{B}_0 h^2 \left(\frac{h}{(y+\mu h)^2 + h^2} + \frac{h}{(y-\mu h)^2 + h^2} \right) , \tag{A21}$$

$$\begin{aligned} A^p(y,z) &= \tilde{B}_0 h^2 (h+z) \left(\frac{1}{(y+\mu h)^2 + (h+z)^2} \right. \\ &\quad \left. + \frac{1}{(y-\mu h)^2 + (h+z)^2} \right) , \end{aligned} \tag{A22}$$

$$B(\eta) = \tilde{B}_0 h^2 \frac{(\eta + ih)^2 + (\mu h)^2}{\left[(\eta + ih)^2 - (\mu h)^2\right]^2} , \tag{A23}$$

$$\begin{aligned} B_z^p(y,0) &= 2\tilde{B}_0 y h^3 \times \\ &\quad \frac{[y^2 + (1+\mu^2)h^2]^2 - 4\mu^2(1+\mu^2)h^4}{[(y+\mu h)^2 + h^2]^2[(y-\mu h)^2 + h^2]^2} , \end{aligned} \tag{A24}$$

$$B_y^p(0,z) = 2\tilde{B}_0 h^2 \frac{(\mu h)^2 - (h+z)^2}{[(\mu h)^2 + (h+z)^2]^2} , \tag{A25}$$

$$P = 2\tilde{B}_0 \frac{\mu^2 - 1}{(\mu^2+1)^2} , \tag{A26}$$

$$Q = 2\pi \tilde{B}_0 h^2 > 0 . \tag{A27}$$

Then the field is quadrupolar if

$$1/\sqrt{3} < \mu , \tag{A28}$$

and dipolar otherwise. In the former case, we have

$$d = [2\mu\sqrt{1+\mu^2} - (1+\mu^2)]^{1/2} h , \tag{A29}$$

$$m = \tilde{B}_0 h \frac{2}{1+\mu^2} > 0 , \tag{A30}$$

$$M = \tilde{B}_0 h \frac{\sqrt{1+\mu^2} + \mu}{2\mu} , \tag{A31}$$

and the field has a topology of type

$$\begin{aligned} &\bullet\ U \quad \text{if} \quad \mu \leq 1 , \\ &\bullet\ X \quad \text{if} \quad 1 < \mu . \end{aligned} \tag{A32}$$

In the X-case

$$Z = (\mu - 1) h , \tag{A33}$$

$$\gamma = \frac{1}{\mu} \tilde{B}_0 h . \tag{A34}$$

Appendix B: computation of the field of a QPSE on the y- and z-axis

B.1. Field on the y-axis

The value of the field on the y-axis is obtained by computing the limit of $B(\eta)$ (Eq. (76)) when $z \to 0$. This can be done by using the relation

$$\frac{1}{s^2 - (y + i0^+)^2} = \mathscr{P} \frac{1}{s^2 - y^2} + \frac{i\pi}{2y} \{\delta(s-y) + \delta(s+y)\} , \tag{B1}$$

which is an immediate consequence of Plemelj's formula (Muskhelishvili 1953)

$$\frac{1}{s - (y + i0^\pm)} = \mathscr{P} \frac{1}{s-y} \pm i\pi\delta(s-y) . \tag{B2}$$

In these relations, $\mathscr{P}$ stands for the distribution *principal part* and δ for the Dirac measure.

Then, owing to the antisymmetry of g', we obtain after separating the real and imaginary parts of $B(y)$

$$\begin{aligned} B_y(y,0) &= \frac{2}{\pi} \left[(y^2 + z_1^2)(y^2 + z_2^2)\right]^{1/2} \times \\ &\quad \fint_0^\infty \frac{s g'(s)}{(s^2 + z_1^2)^{1/2}(s^2 + z_2^2)^{1/2}} \frac{\mathrm{d}s}{s^2 - y^2} , \end{aligned} \tag{B3}$$

$$B_z(y,0) = -g'(y) , \tag{B4}$$

where $\fint := \mathscr{P}\int$. No particular problem arises at $y = 0$ since $g'(y) \simeq_{y\to 0} \zeta_g y$, with $\zeta_g \geq 0$.

B.2. Field on the z-axis

To compute the value of the field on the z-axis, we take the limit of $B(\eta)$ when $y \to 0$. Here, the only point that needs to be treated properly is the computation of the square root $[(\eta^2 + z_1^2)(\eta^2 + z_2^2)]^{1/2}$ in Eq. (76), which requires the choice of a particular determination in the complex plane.

$s > 0$ being some arbitrary number, we take $\sqrt{\eta + is} = \sqrt{\rho} e^{i\theta/2}$, with $\theta \in]-\pi/2, 3\pi/2[$. Then:

• If $y = 0$ and $s < z$:

$$\theta = \pi/2 \ \text{ and } \ \sqrt{\eta + is} = \sqrt{\rho} e^{i\pi/4} . \tag{B5}$$

• If $y = 0^+$ and $z < s$:

$$\theta = -\pi/2 \ \text{ and } \ \sqrt{\eta + is} = \sqrt{\rho} e^{-i\pi/4} . \tag{B6}$$

• If $y = 0^-$ and $z < s$:

$$\theta = 3\pi/2 \ \text{ and } \ \sqrt{\eta + is} = -\sqrt{\rho} e^{-i\pi/4} . \tag{B7}$$

This implies:

• If $y = 0$ and $z_2 < z$:

$$\sqrt{(\eta^2 + z_1^2)(\eta^2 + z_2^2)} = -\sqrt{(z^2 - z_1^2)(z^2 - z_2^2)} . \tag{B8}$$

• If $y = 0^\pm$ and $z_1 < z < z_2$:

$$\sqrt{\eta^2 + z_1^2)(\eta^2 + z_2^2)} = \pm i \sqrt{(z_2^2 - z^2)(z^2 - z_1^2)} . \tag{B9}$$

• If $y = 0$ and $0 < z < z_1$:

$$\sqrt{(\eta^2 + z_1^2)(\eta^2 + z_2^2)} = +\sqrt{(z_2^2 - z^2)(z_1^2 - z^2)}\,. \tag{B10}$$

Therefore:
• If $z_2 \le z$:

$$B(iz) = B_y(0,z) = -\frac{2}{\pi}\sqrt{(z^2 - z_1^2)(z^2 - z_2^2)} \times \int_0^\infty \frac{sg'(s)}{s^2 + z^2} \frac{\mathrm{d}s}{\sqrt{(s^2 + z_1^2)(s^2 + z_2^2)}}\,. \tag{B11}$$

• If $z_1 \le z \le z_2$:

$$B(0^\pm + iz) = -iB_z(0^\pm, z) = \pm\frac{2i}{\pi}\sqrt{(z_2^2 - z^2)(z^2 - z_1^2)} \times \int_0^\infty \frac{sg'(s)}{s^2 + z^2} \frac{\mathrm{d}s}{\sqrt{(s^2 + z_1^2)(s^2 + z_2^2)}}\,. \tag{B12}$$

• If $0 \le z \le z_1$:

$$B(iz) = B_y(0,z) = \frac{2}{\pi}\sqrt{(z_2^2 - z^2)(z_1^2 - z^2)} \times \int_0^\infty \frac{sg'(s)}{s^2 + z^2} \frac{\mathrm{d}s}{\sqrt{(s^2 + z_1^2)(s^2 + z_2^2)}}\,. \tag{B13}$$

Appendix C: conditions of appearance of a vertical CS

We discuss here the existence of a solution to the equations determining the position at time t of a vertical CS in a QPSE: Eqs (77) and (81) when the initial field $\mathbf{B}_0$ is of the X-type, Eqs (110) and (84), respectively, when $\mathbf{B}_0$ is of the U- or I-type. We first discuss Eqs (77) and (81) from a general point of view.

C.1. Analysis of Eq. (77)

In the domain $\omega := \{0 \le \alpha \le \beta < \infty\}$ of the plane (α, β), we set

$$F(\alpha, \beta) := \int_0^\infty \frac{yg'}{\sqrt{y^2 + \alpha^2}\sqrt{y^2 + \beta^2}}\,\mathrm{d}y\,. \tag{C1}$$

Our aim is to determine the set of couples (α, β) for which

$$F(\alpha, \beta) = 0\,. \tag{C2}$$

Fixing the value of α in $[0, \infty[$, we obtain an equation for the unknown β, to which we shall apply Lemma B:

a. Differentiating F with respect to α and β, and applying Lemma A, we have

$$F'_\alpha(\alpha, \beta) = -\alpha \int_0^\infty \frac{yg'}{(y^2 + \alpha^2)^{3/2}(y^2 + \beta^2)^{1/2}}\,\mathrm{d}y < -\frac{\alpha}{d^2 + \alpha^2} F(\alpha, \beta)\,, \tag{C3}$$

$$F'_\beta(\alpha, \beta) = -\beta \int_0^\infty \frac{yg'}{(y^2 + \alpha^2)^{1/2}(y^2 + \beta^2)^{3/2}}\,\mathrm{d}y < -\frac{\beta}{d^2 + \beta^2} F(\alpha, \beta)\,, \tag{C4}$$

where $F'_\alpha := \partial F/\partial\alpha$, ... Then we can use Lemma B.

b. To apply the latter, we first determine the sign of F at the lower bound of the interval $[\alpha, \infty[$. Making $\beta = \alpha$, we obtain

$$F(\alpha, \alpha) = \int_0^\infty \frac{yg'}{y^2 + \alpha^2}\,\mathrm{d}y =: f(\alpha)\,, \tag{C5}$$

where f is the function already defined by Eq. (A4). As shown in Appendix A2:

- If $\mathbf{B}^p$ has a U-topology, $f(\alpha) < 0$ when $P < 0$, while $f(\alpha) \le 0$ and $f(0) = 0$ when $P = 0$.
- If $\mathbf{B}^p$ has a X-topology, $f(\alpha) > 0$ for $0 \le \alpha < Z$, $f(Z) = 0$ and $f(\alpha) < 0$ for $Z < \alpha$.
- If $\mathbf{B}^p$ has a I-topology, $f(\alpha) > 0$.

c. Next we determine the sign of f for large values of β. When $\beta \to \infty$, we have

$$\beta F(\alpha, \beta) \sim_{\beta\to\infty} \int_0^\infty \frac{yg'}{(y^2 + \alpha^2)^{1/2}}\,\mathrm{d}y =: h(\alpha)\,. \tag{C6}$$

h satisfies

$$h'(\alpha) < -\frac{\alpha}{d^2 + \alpha^2} h(\alpha)\,, \tag{C7}$$

$$\frac{-Q}{(d^2 + \alpha^2)^{1/2}} < h(\alpha) < (d^2 + \alpha^2)^{1/2} f(\alpha)\,, \tag{C8}$$

$$h(0) = \int_0^\infty g'\,\mathrm{d}y = -m_0\,, \tag{C9}$$

where Lemma A has been applied to get the two first relations. Then

- If $\mathbf{B}^p$ has a X-topology and $m_0 > 0$, $h(0) < 0$ by Eq. (C9), whence $h(\alpha) < 0$ by Lemma B.
- If $\mathbf{B}^p$ has a X-topology and $m_0 < 0$, then $h(0) < 0$ by Eq. (C9) while $h(Z) < 0$ by Eq. (C8) (in which we take $\alpha = Z$). Hence (Lemma B) there does exist a number $z_1^I \in]0, Z[$ such that $h(\alpha) < 0$ for $0 \le \alpha < z_1^I$, $h(z_1^I) = 0$, and $h(\alpha) > 0$ for $z_1^I < \alpha \le Z$. This result also holds true when $m_0 = 0$, in which case $z_1^I = 0$.
- If $\mathbf{B}^p$ has a I-topology, $Q \le 0$, whence $h(\alpha) > 0$ by Eq. (C8).

Putting together the results of (b) and (c) and applying Lemma B (which implies that Eq. (C2) has a solution $\beta(\alpha) \in [\alpha, \infty[$ if and only if $f(\alpha) \ge 0$ and $h(\alpha) < 0$), we can conclude that, for a given value of α:

- If $\mathbf{B}^p$ has a U-topology, Eq. (C2) has no solution if $P < 0$. If $P = 0$, it has the solution $\beta = 0$ if $\alpha = 0$ (O is a V-point).
- If $\mathbf{B}^p$ has a X-topology and $m_0 > 0$, Eq. (C2) has a solution (and a unique one) if and only if $0 \le \alpha \le Z$. For $\alpha = 0$, we denote the solution by z_2^U ($z_2^U := \beta(0) < \infty$). Of course, we have $\beta(Z) = Z$.
- If $\mathbf{B}^p$ has a X-topology and $m_0 \le 0$, Eq. (C2) has a solution (and a unique one) if and only if $z_1^I < \alpha \le Z$. For $\alpha \to z_1^I$, $\beta(\alpha) \to \infty$, while we still have $\beta(Z) = Z$.

– If $\mathbf{B}^p$ has a I-topology, Eq. (C2) has no solution.

Then, when $\mathbf{B}^p$ has a X-topology, a solution $\beta(\alpha)$ exists on an interval. An important point is that $\beta(\alpha)$ is a decreasing function. We have indeed

$$F'_\alpha(\alpha,\beta) < 0 \quad \text{and} \quad F'_\beta(\alpha,\beta) < 0 \tag{C10}$$

because of Eqs (C3)-(C4), and then

$$\frac{\mathrm{d}\beta}{\mathrm{d}\alpha}(\alpha) = -\frac{F'_\alpha[\alpha,\beta(\alpha)]}{F'_\beta[\alpha,\beta(\alpha)]} < 0\,. \tag{C11}$$

This implies in particular that (remember that $Z = \beta(Z)$)

$$\begin{aligned} &Z < \beta(\alpha) < z_2^U \quad \text{for } 0 < \alpha < Z \text{ and } m_0 > 0\,, \\ &Z < \beta(\alpha) < \infty \quad \text{for } z_1^I < \alpha < Z \text{ and } m_0 \le 0\,. \end{aligned} \tag{C12}$$

C.2. Analysis of Eq. (81)

Let us set

$$\begin{aligned} G(\alpha,\beta) \;:=\; &\int_0^\infty \frac{yg'}{(y^2+\alpha^2)^{1/2}(y^2+\beta^2)^{1/2}} \times \\ &\left(\int_0^\alpha \frac{(\beta^2-z^2)^{1/2}(\alpha^2-z^2)^{1/2}}{y^2+z^2}\,\mathrm{d}z\right)\mathrm{d}y \\ &-\frac{\pi}{2}(\gamma_0 - m_0)\,, \end{aligned} \tag{C13}$$

where $0 \le \alpha \le \beta < \infty$ and $\gamma_0 \ge \max(m_0, 0)$. Then

$$G'_\alpha(\alpha,\beta) = -F'_\alpha(\alpha,\beta) I_1(\alpha,\beta)\,, \tag{C14}$$

$$G'_\beta(\alpha,\beta) = -F'_\beta(\alpha,\beta) I_2(\alpha,\beta)\,, \tag{C15}$$

where I_1 and I_2 are defined by Eqs (90) and (91), respectively.

We now discuss the system of equations

$$F(\alpha,\beta) = G(\alpha,\beta) = 0\,. \tag{C16}$$

We thus have to consider the equation $G = 0$ on the curve $\mathscr{L}$ of ω on which $F(\alpha,\beta) = 0$. From Sect. C.1, $\mathscr{L}$ exists if and only if $\mathbf{B}^p$ has a X-topology, and it can be parametrized by α, with the latter variable belonging to some interval.

a. On $\mathscr{L}$, we have

$$\frac{\mathrm{d}G}{\mathrm{d}\alpha} = G'_\alpha + G'_\beta\frac{\mathrm{d}\beta}{\mathrm{d}\alpha} = -F'_\alpha\frac{\beta^2-\alpha^2}{\beta}\mathbf{K}\left(\frac{\alpha}{\beta}\right) > 0\,, \tag{C17}$$

where use has been made of Eqs (C14), (C15), (C11) and (D4). Then G vanishes at most once on $\mathscr{L}$.

b. At the right endpoint of $\mathscr{L}$, we have

$$\begin{aligned} G(Z,Z) \;:=\; &\int_0^\infty \frac{yg'}{y^2+Z^2}\left(\int_0^Z \frac{Z^2-z^2}{y^2+z^2}\,\mathrm{d}z\right)\mathrm{d}y \\ &-\frac{\pi}{2}(\gamma_0 - m_0) = \frac{\pi}{2}(\gamma-\gamma_0)\,. \end{aligned} \tag{C18}$$

c. At the left endpoint of $\mathscr{L}$, we have

$$G(0, z_2^U) = -\frac{\pi}{2}(\gamma_0 - m_0) \le 0 \tag{C19}$$

if $m_0 > 0$, and

$$\lim_{\alpha\to z_1^I,\beta\to\infty} G(\alpha,\beta) = -\frac{\pi}{2}\gamma_0 \le 0 \tag{C20}$$

if $m_0 \le 0$. Then, G vanishes once (and only once) at a point $(z_1, z_2) \ne (Z, Z)$ of $\mathscr{L}$ if and only if

$$\gamma_0 < \gamma \tag{C21}$$

(we convene here that (z_1^I, ∞) is an admissible solution when $\gamma_0 = 0$). Of course, it vanishes at (Z, Z) if $\gamma = \gamma_0$.

C.3. Conditions for the existence of a vertical CS

The previous results give at once the conditions which are necessary and sufficient for a vertical CS to be possibly present in the evolving field $\mathbf{B}(t)$.

a. If $\mathbf{B}_0$ is of the X-type, a vertical CS can exist when the system (C16) has a solution (z_1, z_2), with $z_1 < z_2$. This is the case if and only if $\mathbf{B}^p$ is of the X-type and condition (C21) is satisfied. The solution is unique, and because of Eqs (C12) and (16) ($\gamma_0 > \max(m_0, 0)$), it satisfies

$$\begin{aligned} &0 < z_1 < Z < z_2 < z_2^U \quad &&\text{if } m_0 > 0\,, \\ &0 \le z_1^I < z_1 < Z < z_2 < \infty \quad &&\text{if } m_0 \le 0\,. \end{aligned} \tag{C22}$$

b. If $\mathbf{B}_0$ is of the U-type, a vertical CS can exist if Eq. (C2) has a solution (z_1, z_2) with $z_1 = 0 < z_2$. This is the case if and only if $\mathbf{B}^p$ is of the X-type. This solution (which coincides with z_2^U) is unique, and it satisfies

$$0 = z_1 < Z < z_2 \tag{C23}$$

because of Eq. (C12). In fact, we could have also looked for a solution to Eq. (C16), without setting a priori $z_1 = 0$. As Eqs (14) and (16) imposes $0 < \gamma_0 = m_0 = m < \gamma$, this system has a solution if and only if $\mathbf{B}^p$ is of the X-type, and it is unique and of the form $(0, z_2^U)$ indeed. This can be considered as a formal proof of the fact that the CS starts from the origin.

c. If $\mathbf{B}_0$ is of the I-type, a vertical CS can exist if the equation $\lim_{\beta\to\infty} F(\alpha,\beta) =: h(\alpha) = 0$ has a solution z_1 – the CS extending between z_1 and $z_2 = \infty$. As $m_0 < 0$, the results of Sect. C.1.c implies that this is the case indeed if and only if $\mathbf{B}^p$ is of the X-type. This solution (coinciding with z_1^I) is unique, and it satisfies

$$0 < z_1 < Z < z_2 = \infty \tag{C24}$$

because of Eq. (C12). Here too, we could have looked for a solution of the system (C16) (F and G being defined for $\beta = \infty$ by taking appropriate limits) without imposing a priori $z_2 = \infty$. As we have $m_0 < 0 = \gamma_0 < \gamma$ by Eqs (17) and (16), this system has a solution if and only if $\mathbf{B}^p$ is of the X-type, and it is unique and of the form (z_1^I, ∞) – which provides a formal proof of the fact that the CS has to extend to infinity.

Two remarks are worth here. First, the equations for z_1 and z_2 have a solution only when the topological constraints are violated. Second, they have only one solution, and therefore, owing to remark (c) at the end of Sect. 5.1, there is at most one QPSE containing a vertical CS and satisfying the constraints imposed at some time t. These statements are in accordance with the uniqueness result of Paper I (quoted at the end of Sect. 4.2). But they do not provide a complete independant proof of uniqueness, as they have been obtained by assuming a priori a y-symmetric field and a vertical CS.

Appendix D: reduction formulas for I_1, I_2 and I

We express here the three quantities I_1, I_2 and I defined by Eqs (90)-(92), respectively, in terms of the complete elliptic integrals of first, second and third kind **K**, **E** and Π, respectively. For that, we set

$$k := \frac{z_1}{z_2} \quad \text{and} \quad n := \frac{z_1^2}{y^2}\,, \tag{D1}$$

and we use formulas 8.111.2-4 and 8.112.1-2 of Gradshteyn and Ryzhik (1965).

As for the two first quantities, we have:

$$\begin{aligned} I_1 &:= \int_0^{z_1} \left(\frac{z_2^2 - z^2}{z_1^2 - z^2}\right)^{1/2} \mathrm{d}z \\ &= z_2 \int_0^1 \left(\frac{1-k^2s^2}{1-s^2}\right)^{1/2} \mathrm{d}s = z_2 \mathbf{E}\left(\frac{z_1}{z_2}\right)\,, \end{aligned} \tag{D2}$$

$$\begin{aligned} I_2 &:= \int_0^{z_1} \left(\frac{z_1^2 - z^2}{z_2^2 - z^2}\right)^{1/2} \mathrm{d}z \\ &= \frac{z_1^2}{z_2} \int_0^1 \left(\frac{1-s^2}{1-k^2s^2}\right)^{1/2} \mathrm{d}s \\ &= \frac{z_1^2}{z_2} \int_0^1 \frac{1-s^2}{\sqrt{1-s^2}\sqrt{1-k^2s^2}} \mathrm{d}s \\ &= z_2 \left[\mathbf{E}\left(\frac{z_1}{z_2}\right) - \frac{z_2^2 - z_1^2}{z_2^2} \mathbf{K}\left(\frac{z_1}{z_2}\right)\right]\,. \end{aligned} \tag{D3}$$

Whence in particular

$$I_1 - I_2 = \frac{z_2^2 - z_1^2}{z_2} \mathbf{K}\left(\frac{z_1}{z_2}\right)\,. \tag{D4}$$

As for the third quantity, we have

$$\begin{aligned} I(z_1, z_2, y) &:= \int_0^{z_1} \frac{(z_1^2 - z^2)^{1/2}(z_2^2 - z^2)^{1/2}}{y^2 + z^2} \mathrm{d}z \\ &= \frac{z_1^2 z_2}{y^2} \int_0^1 \frac{(1-s^2)^{1/2}(1-k^2s^2)^{1/2}}{1+\nu^2 s^2} \mathrm{d}s\,. \end{aligned} \tag{D5}$$

Then, using the decomposition

$$\begin{aligned} \frac{(1-s^2)^{1/2}(1-k^2s^2)^{1/2}}{1+\nu^2 s^2} &= \\ &+ \frac{(n+1)(n+k^2)}{n^2} \frac{1}{(1+ns^2)\sqrt{(1-s^2)(1-k^2s^2)}} \\ &- k^2 \frac{n+1}{n^2} \frac{1}{\sqrt{(1-s^2)(1-k^2s^2)}} \\ &- \frac{1}{n} \left(\frac{1-k^2s^2}{1-s^2}\right)^{1/2}\,, \end{aligned} \tag{D6}$$

we obtain eventually

$$\begin{aligned} I(z_1, z_2, y) = &-\frac{y^2 + z_1^2}{z_2} \mathbf{K}\left(\frac{z_1}{z_2}\right) - z_2 \mathbf{E}\left(\frac{z_1}{z_2}\right) + \\ &\frac{(y^2 + z_1^2)(y^2 + z_2^2)}{y^2 z_2} \Pi\left(\frac{z_1^2}{y^2}, \frac{z_1}{z_2}\right)\,. \end{aligned} \tag{D7}$$

This expression allows us to reduce the double integral in the LHS of Eq. (81) to a single integral with respect to y.

Appendix E: computation of a QPSE when $B(\infty) \neq 0$

Some authors (e.g., Tur & Priest 1976, Low 1987, Sneyd 1993) have considered the problem of the formation of a CS in the case where the field approaches an horizontal constant value at infinity. Then the asymptotic condition (64) is replaced by

$$\lim_{r\to\infty} B(\eta) = \mathscr{B}_0\,, \tag{E1}$$

where $\mathscr{B}_0$ is a given nonzero constant.

In that case, the initial potential field has a complex topology even if the boundary condition on $\partial\Omega$ is of the dipolar type, which will be assumed hereafter. Rather than presenting a detailed study of the topology of $\mathbf{B}_0$, let us consider a simple example. Consider the complex field

$$B_0(\eta) = \mathscr{B}_0 \left(1 + \frac{\mu}{(\eta + ih)^2}\right)\,. \tag{E2}$$

where μ and h are real positive constants. Clearly, $B_0(\eta)$ has all the required virtues. The topology of its lines in Ω depends on the parameter μ and changes at the critical value $\mu_c = h^2$:

- If $\mu > \mu_c$, $\mathbf{B}_0$ has a neutral point in Ω, located at $(0, \sqrt{\mu} - h)$.
- If $\mu < \mu_c$, $\mathbf{B}_0$ has no neutral point in Ω, but its lines admit a separatrix of the U-type tangent to the boundary at the origin. The topology is as shown in Fig. 4a.

Then, when a velocity is applied to the footpoints on the boundary along the y-axis, a CS will appear either at once (first topology) or after an X-point has formed in the potential field $\mathbf{B}^p$ (second topology). The second situation has been considered by Low (1987), whose results are represented in Fig. 4b.

To compute the QPSE which may be produced in situations of this type (the CS being still assumed to be vertical), we thus need to solve BVP (59)-(64) whith the asymptotic condition (64) being replaced by Eq. (E1). It is readily shown that, if we give up for a while with the latter condition and merely assume that the field is bounded in Ω, then the solution to Eqs (59)-(63)

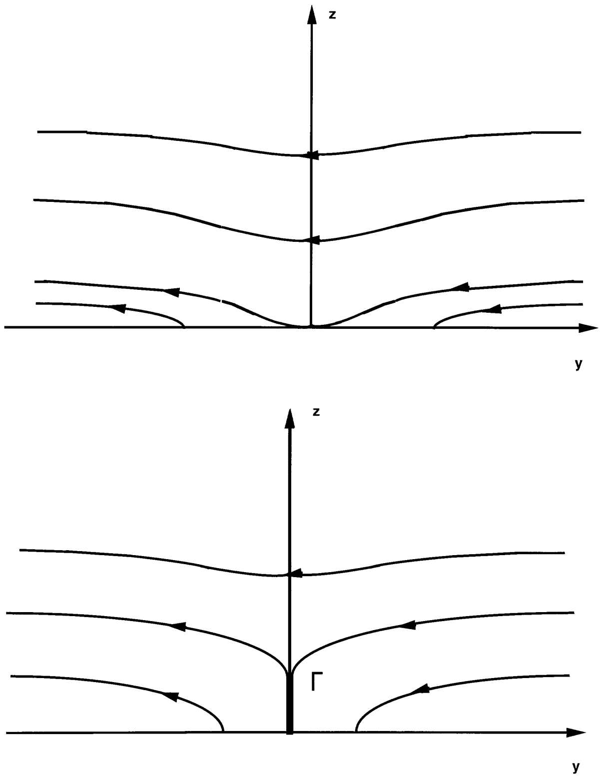

Fig. 4. a Topology of the field lines of $\mathbf{B}_0$ given by Eq. (D2) when $\mu < \mu_c$. $\mathbf{B}_0$ has no X-point above the photosphere, but has a U-separatrix tangent to the boundary at the origin. $\mathbf{B}_0$ tends towards a uniform field at infinity. **b** Structure of the field lines of the QPSE resulting from a compressive velocity field being applied to the footpoints of the potential configuration shown in (a). The resulting CS starts from the origin.

is given by

$$B(\eta) = \frac{2}{\pi}(\eta^2 + z_1^2)^{1/2}(\eta^2 + z_2^2)^{1/2} \times \int_0^\infty \frac{sg'(s)}{(s^2 + z_1^2)^{1/2}(s^2 + z_2^2)^{1/2}} \frac{\mathrm{d}s}{s^2 - \eta^2} . \quad \text{(E3)}$$

Taking the limit of $B(\eta)$ when $|\eta| \to \infty$ and applying condition (E1), we thus obtain

$$\lim_{|\eta|\to\infty} B(\eta) = \frac{2}{\pi}\int_0^\infty \frac{y'g(y)}{(y^2 + z_1^2)^{1/2}(y^2 + z_2^2)^{1/2}} \mathrm{d}y = \mathscr{B}_0 . \quad \text{(E4)}$$

This relation replaces Eq. (77) as the condition of existence of a solution to the BVP. Clearly, it reduces to Eq. (77) when $\mathscr{B}_0 = 0$. Hence Eq. (77) may be interpreted as implying either an asymptotic condition or a boundedness condition.

When $\mathbf{B}_0$ has a U-separatrix and a CS has formed, $z_1 = 0$ and z_2 can be determined from Eq. (E4). If the initial separatrix is of the X-type, we need one more equation to compute both z_1 and z_2. It is easy to see that Eq. (81) provides the seeked relation. Eqs (E4) and (81) can be analyzed by merely adapting the arguments of Appendix C. For avoiding this paper becoming too long, we leave to the interested reader the care of completing the discussion.

References

Aly, J.J., 1985, A&A, 143, 19
Aly, J.J., 1987a, Internal Report, Saclay
Aly, J.J., 1987b, in Interstellar Magnetic Fields, eds R. Beck & R. Gräve, Springer Verlag, Berlin, p. 240
Aly, J.J., Amari, T., 1988, A&A 207, 154
Aly, J.J., Amari, T., 1989, A&A 221, 287 (Paper I)
Aly, J.J., Colombi, S., Lepeltier, T., 1994, ApJ 432, 793
Amari, T., 1991, in Advances in Solar System Magnetohydrodynamics, eds E.R. Priest & H.W. Hood, Cambridge University Press, Cambridge, p. 173
Amari, T., Aly, J.J., 1990, A&A 208, 361 (Paper II)
Gradshteyn, I.S., Ryzhik, I.M., 1965, Table of Integrals, Series and Products, Academic Press, New-York
Hu, Y.Q., 1986, Scientia Sinica 29, 199
Hu, Y.Q., Low, B.C., 1982, Solar Phys. 81, 107
Karpen, J.T., Antiochos, S.K., De Vore, R.C., 1990, ApJ 356, L67
Laurentiev, M., Chabat, B., 1972, Méthode de la Théorie des Fonctions d'une Variable Complexe, Editions Mir, Moscou
Low, B.C., 1987, ApJ 323, 358
Low, B.C., 1990, ARA&A 28, 491
Low, B.C., 1991, ApJ 381, 295
Low, B.C., Hu, Y.Q., 1983, Solar Phys. 84, 83
Low, B.C., Wolfson, R., 1988, ApJ 324, 574
Moffat, H.K., 1985, J. Fluid Mech. 159, 359
Muskhelishvili, N.I., 1953, Singular Integral Equations, P. Nordhoff N.V., Groningen, Holland
Parker, E.N., 1972, ApJ 174, 449
Parker, E.N., 1979, Cosmical Magnetic Fields, Oxford University Press, New-York
Parker, E.N., 1994, Spontaneous Current Sheets in Magnetic Fields, with Applications to Stellar X-Rays, Oxford University Press, Oxford
Priest, E.R., Raadu, M.A., 1975, Solar Phys. 43, 177
Priest, E.R., Titov, V.S., Rickard, G., 1995, Phil. Trans. R. Soc. Lond. A 351, 1
Sneyd, A.D., 1993, Geophys. Astrophys. Fluid Dyn. 70, 195
Syrovatskii, S.I., 1981, ARA&A. 19, 163
Taylor, J.B., 1986, Rev. Mod. Phys. 58, 741
Tur, T.J, Priest, E.R., 1976, Solar Phys. 48, 89
Van Ballegooijen, A.A., 1985, ApJ 312, 423
Vekstein, G., Priest, E.R., Amari, T., 1991, A&A 243, 492

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