Averaging inhomogeneous Newtonian cosmologies

Thomas Buchert[1] **and Jürgen Ehlers**[2]

[1] Theoretische Physik, Ludwig-Maximilians-Universität, Theresienstr. 37, D-80333 München, Germany
[2] Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, Schlaatzweg 1, D-14473 Potsdam, Germany

Abstract. Idealizing matter as a pressureless fluid and representing its motion by a peculiar-velocity field superimposed on a homogeneous and isotropic Hubble expansion, we apply (Lagrangian) spatial averaging on an arbitrary domain $\mathscr{D}$ to the (nonlinear) equations of Newtonian cosmology and derive an exact, general equation for the evolution of the (domain dependent) scale factor $a_{\mathscr{D}}(t)$. We consider the effect of inhomogeneities on the average expansion and discuss under which circumstances the standard description of the average motion in terms of Friedmann's equation holds. We find that this effect vanishes for spatially compact models if one averages over the whole space. For spatially infinite inhomogeneous models obeying the cosmological principle of large-scale isotropy and homogeneity, Friedmann models may provide an approximation to the average motion on the largest scales, whereas for hierarchical (Charlier-type) models the general expansion equation shows how inhomogeneities might appreciably affect the expansion at all scales. An averaged vorticity evolution law is also given. Since we employ spatial averaging, the problem of justifying ensemble averaging does not arise. A generalization of the expansion law to general relativity is straightforward for the case of irrotational flows and will be discussed. The effect may have important consequences for a variety of problems in large-scale structure modeling as well as for the interpretation of observations.

Key words: gravitation – instabilities – methods: analytical – cosmology: theory – large-scale structure of Universe

1. Outline of the problem

Traditionally, cosmological models have been based on the assumption that, on a sufficiently large scale, the Universe is isotropic and homogeneous. As long as inhomogeneities are small and described as linear perturbations off a Friedmann model which average to zero, this picture is consistent; by construction rather than derivation the averaged variables are given by a Friedmann model. However, in the observed part of the Universe the matter inhomogeneities are not small (e.g., the r.m.s.

Send offprint requests to: T. Buchert

density fluctuations are much larger than unity below the scale of clusters of galaxies), and contemporary theories of structure formation follow the evolution of inhomogeneities into the nonlinear regime, mostly in the framework of Newtonian cosmology, either with analytical approximations or numerical N-body simulations. Again, both methods to simulate nonlinear structure formation are constructed in such a way that the average flow obeys the homogeneous and isotropic Friedmann models.

Basic, nontrivial questions which are usually not even raised in the traditional approach are, whether and how a general anisotropic and inhomogeneous solution can be split into an isotropic and homogeneous average field and deviations thereof and whether, if so, the average variables obey Friedmann's equation. Because of the nonlinearity of Einstein's as well as Newton's laws for gravitationally interacting systems, the answers to the last questions are not obvious, as has been emphasized particularly by Ellis (1984).

In contemporary models of structure formation the time- and length-scales as well as the amplitude of the initial fluctuations are expressed in terms of quantities of their assumed Friedmann backgrounds such as $a(t)$, $H(t)$, $\Omega(t)$, etc. . Observational data for these parameters are interpreted accordingly. This procedure which rules a variety of problems in structure formation theories (like the question whether nonbaryonic dark matter is needed to explain present day structure) excludes by assumption all inhomogeneous models which do not obey Friedmann's equations if averaged on some large scale, even those which are kinematically isotropic and homogeneous.

In general relativity, the problem of averaging is very involved because (i) in a generic spacetime there are no preferred time-slices one could average over. (It should even be difficult to recognize a Friedmann model as such if one were given a complicated slice of it.) (ii) the metric is a dynamical variable entering the field equations nonlinearly, and it is difficult to average (or instead deform, see Carfora and Marzuoli 1988, Carfora et al. 1990) it, (iii) a gauge problem arises in relating the "true" and the averaged metric. Indications that the "backreaction" of inhomogeneities on the global expansion may have important consequences for the structure formation process have been put forward by Futamase (1989, 1996), Bildhauer (1990) and Bildhauer & Futamase (1991a,b) who have studied the "backreac-

tion" effect quantitatively based on perturbative calculations. In particular, they found that the expansion is accelerated significantly where inhomogeneities form. Recently, renormalization group techniques in relation to the averaging problem have been advanced by Carfora and Piotrkowska (1995) who investigated in detail the connection between (spatial) manifold deformations and spatial averaging at one instant, using the constraint equations of general relativity.

In this paper we address the problem of the effect of non-linear inhomogeneities on the average expansion in Newtonian cosmology where also the evolution equations can be averaged without uncontrolled approximations. This provides straightforward insight and some answers which are indicative also for the GR-case and uncovers possible limitations of current cosmological models. Moreover, we will show that for scalar quantities (like the expansion rate) the averaging procedure proposed in the present work carries over to the general relativistic case, if we restrict the equations to irrotational flows.

2. Averages in Newtonian cosmology

According to Newtonian physics, the motion of a self-gravitating, pressureless fluid ("dust") is governed by the *Euler-Poisson-system* of equations. Thus, with respect to a non-rotating Eulerian coordinate system[1] the fields of mass density $\varrho(x, t) > 0$, velocity $v(x, t)$ and gravitational acceleration $g(x, t)$ are required to satisfy

$$\partial_t v = -(v \cdot \nabla)v + g , \tag{1a}$$

$$\partial_t \varrho = -\nabla \cdot (\varrho v) , \tag{1b}$$

$$\nabla \times g = \mathbf{0} , \tag{1c}$$

$$\nabla \cdot g = \Lambda - 4\pi G \varrho , \tag{1d}$$

where Λ denotes the cosmological constant, here included for the sake of generality.

It is useful to introduce the rates of expansion $\theta = \nabla \cdot v$, shear $\underline{\sigma} = (\sigma_{ij})$ and rotation $\boldsymbol{\omega} = \frac{1}{2}\nabla \times v$ of the fluid flow via the decomposition*

$$v_{i,j} = \sigma_{ij} + \frac{1}{3}\delta_{ij}\theta + \omega_{ij} \; ; \; \omega_{ij} = -\eta_{ijk}\omega_k \; , \; \sigma_{[ij]} = 0 \; , \tag{2a}$$

of the velocity gradient, where δ_{ij} and η_{ijk} denote the (Euclidean) spatial metric and its volume form (Levi-Civita tensor), respectively. In contrast to v, the tensor fields $\underline{\sigma}$, θ and $\boldsymbol{\omega}$ are independent of the (non-rotating) coordinate system; v can be reconstructed from $\underline{\sigma}$, θ and $\boldsymbol{\omega}$ up to a spatially constant summand (compare Appendix A).

---

[1] notes are given at the end of the paper.

* a comma denotes partial differentiation with respect to Eulerian coordinates; we adopt the summation convention.

Using the Lagrangian time derivative operator $\frac{d}{dt} \equiv (...)^{\boldsymbol{\cdot}} =: \partial_t + v \cdot \nabla$, we may replace the system (1) by the *equivalent system* (2) consisting of Eq. (2a) and the transport equations[2]

$$\dot{\varrho} = -\varrho\theta , \tag{2b}$$

$$\dot{\boldsymbol{\omega}} = -\frac{2}{3}\boldsymbol{\omega}\theta + \underline{\sigma} \cdot \boldsymbol{\omega} , \tag{2c}$$

$$\dot{\theta} = \Lambda - 4\pi G\varrho - \frac{1}{3}\theta^2 + 2(\omega^2 - \sigma^2) , \tag{2d}$$

for ϱ, $\boldsymbol{\omega}$ and θ, in which σ and ω denote the magnitudes of shear and rotation, respectively:

$$\sigma = |\underline{\sigma}| := (\frac{1}{2}\sigma_{ij}\sigma_{ij})^{1/2} \; , \; \omega = |\boldsymbol{\omega}| =: (\frac{1}{2}\omega_{ij}\omega_{ij})^{1/2} \; .$$

((1) is equivalent to (2) as a system for ϱ and v. In fact, if (1a) is taken to define g in terms of v, then (1c) $\Leftrightarrow$ (2c) and (1d) $\Leftrightarrow$ (2d); (1b) $\Leftrightarrow$ (2b) is obvious.)

We wish to take spatial averages of the Eqs. (2b,c,d). For this purpose, let $f_t : X \mapsto x$ denote the mapping which takes initial positions X of fluid particles at time t_0 to their positions x at time t; in other words, let $x = f_t(X) \equiv f(X, t)$ be the field of trajectories on which the Lagrangian description of fluid motion is based. Then, the volume elements at t and t_0 are related by $d^3x = J d^3X$, where $J(X, t) := \frac{Df}{DX}$ is the Jacobian determinant of f_t. Therefore,

$$\dot{J} = J\theta , \tag{3}$$

and the "comoving" volume $V(t) =: a_{\mathscr{D}}^3(t)$ of a compact portion $\mathscr{D}(t)$ of the fluid changes according to

$$\dot{V} = \frac{d}{dt}\int_{\mathscr{D}(t)} d^3x = \int_{\mathscr{D}(t_0)} d^3X\, \dot{J} = \int_{\mathscr{D}(t)} d^3x\, \theta ,$$

which may be written

$$\langle\theta\rangle_{\mathscr{D}} = \frac{\dot{V}}{V} = 3\frac{\dot{a}_{\mathscr{D}}}{a_{\mathscr{D}}} . \tag{4}$$

Here and in the sequel, $\langle A\rangle_{\mathscr{D}} = \frac{1}{V}\int_{\mathscr{D}} d^3x\, A$ denotes the spatial average of a (spatial) tensor field $\mathscr{A}$ on the domain $\mathscr{D}(t)$ occupied by the amount of fluid considered, and $a_{\mathscr{D}}(t)$ is the scale factor of that domain. Suppressing temporarily the index $\mathscr{D}$ for notational simplicity, we note the useful "commutation rule"

$$\langle\mathscr{A}\rangle^{\boldsymbol{\cdot}} - \langle\dot{\mathscr{A}}\rangle = \langle\mathscr{A}\theta\rangle - \langle\mathscr{A}\rangle\langle\theta\rangle . \tag{5}$$

(Proof:
$\langle\mathscr{A}\rangle^{\boldsymbol{\cdot}} = \frac{d}{dt}(V^{-1}\int d^3x\, \mathscr{A}) = -\frac{\dot{V}}{V}\langle\mathscr{A}\rangle + V^{-1}\int d^3X\, (\dot{\mathscr{A}}J + \mathscr{A}\dot{J})$; using (3) and (4) gives (5).)

Applied in turn to ϱ, $\boldsymbol{\omega}$ and θ, Eq. (5) combined with (2b,c,d), respectively, leads to

$$\langle\varrho\rangle^{\boldsymbol{\cdot}} = -\langle\varrho\rangle\langle\theta\rangle , \tag{6}$$

$$\langle\boldsymbol{\omega}\rangle^{\boldsymbol{\cdot}} = \langle\boldsymbol{\omega} \cdot \nabla v\rangle - \langle\boldsymbol{\omega}\rangle\langle\theta\rangle , \tag{7}$$

$$\langle\theta\rangle^{\boldsymbol{\cdot}} = \Lambda - 4\pi G\langle\varrho\rangle + \frac{2}{3}(\langle\theta^2\rangle) - \langle\theta\rangle^2 + 2(\langle\omega^2\rangle - \langle\sigma^2\rangle) , \tag{8}$$

where we have also used Eq. (2a).

In terms of the scale factor a, Eq. (6) says that $a^3 \langle \varrho \rangle =: M$ is the constant total mass of the fluid portion considered. Eq. (8) may be rewritten as

$$3 \frac{\ddot{a}}{a} + 4\pi G \frac{M}{a^3} - \Lambda = \frac{2}{3} \left(\langle \theta^2 \rangle - \langle \theta \rangle^2 \right) + 2 \langle \omega^2 - \sigma^2 \rangle \ . \qquad (9a)$$

The averaged Raychaudhuri equation (9a) can also be written as a standard Friedmann equation for the "effective mass density" ϱ_{eff},

$$4\pi G \varrho_{\text{eff}} := 4\pi G \langle \varrho \rangle - \frac{2}{3} \langle (\theta - \langle \theta \rangle)^2 \rangle + 2 \langle \sigma^2 - \omega^2 \rangle \ . \qquad (9b)$$

Eq. (9) shows that inhomogeneities have an accelerating effect on the averaged volume expansion rate $\langle \theta \rangle$, if the shear term dominates the vorticity and contraction terms in that equation. Eqs. (9) can also be used to discuss anisotropies: If the domain $\mathscr{D}$ is chosen as a cone with the observer at the vertex, (9a) shows that the average expansion in two such cones will be different if the averages involving θ, ω and σ differ.

From Eq. (7) we conclude that

$$\langle \boldsymbol{\omega} \rangle = \mathbf{0} \ \Rightarrow \ \langle \boldsymbol{\omega} \cdot \nabla \boldsymbol{v} \rangle = \mathbf{0} \ . \qquad (10a)$$

The converse only holds asymptotically for positive average expansion:
if $\langle \boldsymbol{\omega} \cdot \nabla \boldsymbol{v} \rangle = \mathbf{0}$, then

$$\langle \boldsymbol{\omega} \rangle = \langle \boldsymbol{\omega}(t_0) \rangle \, e^{- \int_{\mathscr{D}(t)} dt \, \langle \theta \rangle \, (t - t_0)} \ ; \qquad (10b)$$

for a universe which collapses on average, $\langle \boldsymbol{\omega} \rangle$ blows up (compare Olson & Sachs 1973)[3].

The Eqs. (6)-(9) hold for any part of any pressureless fluid.

We want to apply the foregoing results to a large, typical part of a cosmological model. For this purpose we split the velocity field into a *Hubble flow* and a *peculiar-velocity u*. Recall that a Hubble flow can be characterized as having vanishing rates of shear and rotation and a spatially constant rate of expansion $3H(t)$ (cf. Eq. (2a)). Therefore,

$$v_{i,j} = H(t) \delta_{ij} + u_{i,j} \ , \qquad (11a)$$

hence

$$\theta = 3H + \nabla \cdot u \ . \qquad (11b)$$

We may choose[4]

$$H := \frac{\dot{a}_{\mathscr{D}}}{a_{\mathscr{D}}} \ , \qquad (12a)$$

or, equivalently,

$$\langle \nabla \cdot u \rangle_{\mathscr{D}} = 0 \ . \qquad (12b)$$

$H(t)$ and u then depend, of course, on the choice of the averaging region $\mathscr{D}$.

From eqs. (11a,b) and (2a) we get by a little computation an expression for the quadratic principal invariant of the velocity gradient $(v_{i,j})$:

$$2\mathbf{II}(v_{i,j}) \equiv \frac{2}{3} \theta^2 + 2(\omega^2 - \sigma^2)$$

$$= 6H^2 + 4H \, \nabla \cdot u + \nabla \cdot (u \, \nabla \cdot u - u \cdot \nabla u) \ . \qquad (13)$$

We use it and Eq. (12a) to rewrite the *averaged Raychaudhuri equation* (9a) in the form

$$3 \frac{\ddot{a}_{\mathscr{D}}}{a_{\mathscr{D}}} + 4\pi G \frac{M_{\mathscr{D}}}{a_{\mathscr{D}}^3} - \Lambda = a_{\mathscr{D}}^{-3} \int_{\partial \mathscr{D}(t)} dS \cdot (u \, \nabla \cdot u - u \cdot \nabla u), \qquad (14)$$

where we have reinserted the index $\mathscr{D}$ to exhibit the dependence on the comoving domain chosen, and we have applied Gauß's theorem to transform the volume integral in the average to a surface integral over the boundary $\partial \mathscr{D}$ of the domain.

We note that the *averaged Helmholtz vorticity equation* (7) can similarly be brought into a more transparent form. Since the Hubble flow is irrotational, we have $\boldsymbol{\omega} = \frac{1}{2} \nabla \times u$; then, using Green's formula,

$$\langle \boldsymbol{\omega} \rangle_{\mathscr{D}} = \frac{1}{2} a_{\mathscr{D}}^{-3} \int_{\partial \mathscr{D}(t)} dS \times u \ , \qquad (15)$$

$$(\langle \boldsymbol{\omega} \rangle_{\mathscr{D}})^{\cdot} + 2H \langle \boldsymbol{\omega} \rangle_{\mathscr{D}} = \langle \boldsymbol{\omega} \cdot \nabla u \rangle_{\mathscr{D}} - \langle \boldsymbol{\omega} \rangle_{\mathscr{D}} a_{\mathscr{D}}^{-3} \int_{\partial \mathscr{D}(t)} dS \cdot u \ . \qquad (16)$$

(Olson & Sachs (1973) have derived an evolution equation for the ensemble average of ω^2 assuming homogeneous-isotropic turbulence.)

We emphasize that, given a solution $(\varrho, \boldsymbol{v})$ of the basic equations (1), the terms in Eqs. (12)-(16) depend on the choice of an "initial" domain $\mathscr{D}(t_0)$ and a choice of a Hubble function $H(t)$.

So far, all equations refer to a single non-rotating, rectangular Eulerian coordinate system x^a. Such a description holds globally, if space is assumed to be the standard Euclidean $\mathbb{R}^3$. However, we wish to consider also spatially compact Newtonian models as defined and analyzed in (Brauer et al. 1994). As shown there, space can then be considered without loss of generality as a torus $\mathbb{T}^3$, which cannot be covered by a single coordinate system. It is important (especially for Sect. 3.1) that nevertheless the Eqs. (14)-(16) remain valid for such models as intrinsic, coordinate-free relations for any compact, orientable domain $\mathscr{D}(t_0)$ and its evolution $\mathscr{D}(t)$, provided the models admit a Hubble flow. The reason is that, although the inhomogeneous fluid motion (locally given by $\boldsymbol{v}$) as well as the Hubble flow cannot be described globally by smooth vector fields on the torus, the peculiar-velocity u, being the *relative* velocity of those two motions, *is* a smooth, global 3−vector field on $\mathbb{T}^3$ (see Appendix A for a detailed discussion of the kinematics on toroidal models).

3. Interpretation for different cosmologies

Let us now assume that the Hubble flow represents the mean motion on some domain $\mathscr{D}$ and, correspondingly, that the peculiar-velocity field u represents the inhomogeneous flow with $\langle u_{i,j} \rangle_{\mathscr{D}} = 0$ which, by Eq. (11a), is equivalent to Eq. (4) and $\langle \sigma_{i,j} \rangle_{\mathscr{D}} = 0$, $\langle \omega_{i,j} \rangle_{\mathscr{D}} = 0$. Eq. (14) then shows how this field u determines the deviation of the expansion, on the scale $a_{\mathscr{D}}$, from Friedmann's law. The "perturbing terms" are quadratic in u and are in general nonzero (even for isotropic expansion); a similar remark applies to Eq. (16).

We now consider three theoretically possible cosmologies.

3.1. Spatially compact cosmologies

These models with a *closed* (i.e., compact without boundary) 3-space admit either none or exactly one (global) Hubble flow, for the existence of such a flow means that the toroidal space expands self-similarly without rotation (see Appendix A). In this latter case we may choose the compact domain $\mathscr{D}(t)$ to be the whole space. Then, $\mathscr{D}(t)$ has no boundary, and we obtain from eqs. (11a) and (14)-(16):

$$3\frac{\ddot{a}}{a} + 4\pi G \frac{M_{tot}}{a^3} - \Lambda = 0 \ ,$$

$$\langle \sigma_{ij} \rangle = 0 \ ; \ \langle \boldsymbol{\omega} \rangle = \mathbf{0} \ , \ \langle \boldsymbol{\omega} \cdot \nabla u \rangle = \mathbf{0} \ , \tag{17a}$$

and, from (9a) and (14),

$$\frac{1}{3}\langle (\theta - \langle \theta \rangle)^2 \rangle + \langle \omega^2 \rangle \ = \ \langle \sigma^2 \rangle \ . \tag{17b}$$

(Here we have used that u is globally well-defined.)

For inhomogeneous models based on such Friedmann backgrounds Brauer et al. (loc.cit.) have established some exact qualitative results, and in (Ehlers & Buchert 1997) we develop a perturbation scheme the terms of which are well-defined to arbitrary order and are uniquely determined by initial data; that scheme is also useful for numerical work.

Toroidal models with a background Hubble flow are, in effect, used in all current structure formation simulations, since they employ periodic boundary conditions for the density and peculiar-velocity fields on some large scale. As we have just shown, such (Newtonian) models can always be regarded as finite perturbations of toroidal Friedmann models. The perturbations, whether small or large, cannot influence the overall expansion.

3.2. Cosmologies based on the principle of 'large-scale homogeneity and isotropy'

If we take space to be infinite Euclidean $\mathbb{R}^3$, we can assume the boundary term in (14) to be small compared to $\frac{\ddot{a}_{\mathscr{D}}}{a_{\mathscr{D}}}$. This may be the case if the inhomogeneities are substantially smaller than the size $a_{\mathscr{D}}$ of the averaging domain and if their peculiar-velocities are small on the averaging scale ($|u| << a_{\mathscr{D}} H$); it corresponds to the cosmological principle of 'statistical homogeneity and isotropy'. Then, the average motion may be *approximately* given by a Friedmann model on a scale which is considerably larger than the largest existing inhomogeneities. This is the generally held view.

3.3. Hierarchical (Charlier-type) cosmologies

Observational evidence for structures on scales of hundreds of Megaparsecs indicates that the "closure scale" - the circumference of the Universe if it is closed - has certainly not been reached by observed samples of the Universe. Power spectrum estimates of the density contrast show a negative slope close to the limit of the deepest available surveys. However, the COBE microwave background measurement suggests that the spectrum bends over to positive slopes on larger scales, supporting the assumption of large-scale homogeneity. Thus, if the COBE measurement is indeed a detection of *primordial* density fluctuations, then (up to the cosmic variance of this measurement) we may use Friedmann models as approximate models for the average flow on the largest scales.

The expansion law (14), however, indicates that Friedmann's law may be appreciably modified by peculiar-velocity terms; for large-scale homogeneity and isotropy the sum of these terms will approach a limiting value which may or may not be negligible, in contrast to the spatially compact case. Only in the case where this limiting value is negligible, global values of, e.g., the Hubble constant and the density parameter are related to the Friedmann parameters on that "asymptotic" scale. If such a scale doesn't exist, the average flow will evolve into an anisotropic and (after development of multi-streaming) rotational flow, in spite of isotropic and irrotational initial data. The extreme opposite, a (globally) *hierarchical cosmology*, where the spectrum of fluctuations continues to rise on large scales, could *only* be treated by including the source terms.

3.4. Supplementary remarks

Eqs. (9a) and (14) imply that the expression

$$\mathcal{Q} := \frac{2}{3}\langle (\theta - \langle \theta \rangle)^2 \rangle + 2\langle \omega^2 \rangle - 2\langle \sigma^2 \rangle \tag{18}$$

is a divergence. From this we conclude:

i) If the shear vanishes in a toroidal model which admits a background Hubble flow, the model is homogeneous and isotropic; in other words:

ii) In order to contain any inhomogeneities at all, compact models admitting a background Hubble flow must have nonvanishing shear. This statement applies to the shear scalar, the shear tensor as well as to the average $\langle \sigma^2 \rangle$.

iii) If, on *all* sufficiently large scales - e.g., because of the structure formation process - the expression $\mathcal{Q}$ in nonzero, then the model either is not compact or does not admit a background Hubble flow.

iv) Even if the expression Q is zero on some large scale A, then still $\langle\sigma^2\rangle_A \neq 0$ and, in general, $\theta \neq \langle\theta\rangle_A$, provided inhomogeneities are present.

These four statements are equivalent; they only emphasize different aspects of i). To prove i), note that if $\sigma = 0$ and the model is compact, the integral over the nonnegative function $\frac{2}{3}\langle(\theta - \langle\theta\rangle)^2\rangle + 2\langle\omega^2\rangle$ vanishes; hence $\theta = \langle\theta\rangle$ and $\langle\boldsymbol{\omega}\rangle = \boldsymbol{0}$, therefore, $\boldsymbol{v}$ itself represents a Hubble flow, q.e.d.

Note that Remarks ii) and iv) concerning the shear follow, because the average over σ^2 is performed over non-negative terms and can only vanish, if the shear scalar vanishes pointwise and, hence, if each component of the shear tensor itself is zero. Turning this argument around, nonvanishing components of the shear tensor result in a nonzero average of σ^2 on the scale A. Since this average does not vanish, the average fluctuation of the expansion does neither, unless the average vorticity exactly compensates the average shear (see also Yodzis 1974). In general we neither expect the summands in (18) to vanish individually, nor to decrease by going to larger volumes[5]. This indicates that the "conspiracy assumption" $Q = 0$ on some large scale (if the Universe is not genuinely compact) must be considered a strong restriction of generality. The existence of a Hubble flow (as a reference flow) does not imply that the *average flow* is a Hubble flow.

As we have indicated in the introduction, the averaging problem in general relativity is very involved. However, for scalar quantities like the expansion rate θ we may derive an expansion law in full analogy to the Newtonian case. For this purpose we may foliate the spacetime into a family of space-like hypersurfaces with spatial metric g_{ij} which are flow-orthogonal (this is only possible, if the flow is irrotational). Introducing Gaussian (normal) coordinates we can define spatial averages (e.g. the average of θ) in a spatial domain $\mathscr{D}$ with volume $V = \int_{\mathscr{D}} d^3X\sqrt{g}$, $g := \det(g_{ij})$, as follows:

$$\langle\theta\rangle_{\mathscr{D}} := \frac{1}{V}\int_{\mathscr{D}}\theta\sqrt{g}d^3X \quad . \tag{19}$$

This definition agrees with Eq. (4), if J is replaced by $\sqrt{g}$. (Indeed, if the metric is written as the quadratic form $g_{ij} = \eta_i^a\eta_j^a$ involving the one-forms $\eta^a = \eta_i^a dX^i$, then $\sqrt{g} = \det(\eta_i^a) =: J$. J is identical to the Jacobian determinant used in the present work, if $\eta_i^a \equiv \nabla_0 f$.) Since Raychaudhuri's equation is the same in GR we conclude that the expansion law (9) is also valid in general relativity.

Yodzis (1974) has shown (Theorem 2) that, for general-relativistic, irrotational dust models for which the hypersurfaces orthogonal to the dust world lines are closed and orientable, Eq. (9) holds if the averages are performed over the whole space. Our statement is more general in the sense that Eq. (9) holds in general relativity for the same assumptions, but for arbitrarily chosen spatial domains.

An important difference to the Newtonian treatment, besides spatial curvature, arises due to the fact that it may not be in general possible to represent the term (18) as a divergence in

GR. We stress that this would imply a strong challenge for the standard cosmologies, since we can no longer argue, except for non-generic situations, that there exist cases in which the average obeys Friedmann's law. Even more, we don't expect the previously discussed arguments (after Eq. (18)) to hold, since the valid theory on the large scales under consideration is general relativity.

4. Notes

1) For spatially unbounded mass distributions there are no inertial coordinate systems related by Galilean transformations. Instead, the preferred coordinate systems (t, x^a) are related by the more general transformations $x^{a'} = R_b^{a'}x^b + d^{a'}(t)$, where $R_b^{a'}$ denotes a constant rotation matrix and $d^{a'}(t)$ represents a translation depending arbitrarily on time t. The basic equations (1) are covariant with respect to these transformations, provided ϱ, $\boldsymbol{v}$ and $\boldsymbol{g}$ are transformed in the obvious way. These coordinate systems are called (dynamically) non-rotating, since with respect to them, no Coriolis forces occur (Heckmann and Schücking 1955,56).

2) We call Eqs. (2b,c,d) transport equations since they describe how ϱ, ω, θ change along a fluid trajectory. In Lagrangian coordinates (t, X) Eqs. (2) are ordinary differential equations with X as parameters.

3) The assumption of homogeneous-isotropic turbulence employed by Olson & Sachs (1973) imposes from the start a strong restriction which excludes the relevant terms discussed here. Also, they use ensemble averaging instead of spatial averaging. Of course, statistical statements for spatial averages can be investigated by averaging over statistical ensembles of spatial domains $\mathscr{D}$ centered at random points in space.

4) Given v_i and $H(t)$, (11a) defines u up to a spatially constant term. One can fix u uniquely by requiring either $\langle u\rangle = \boldsymbol{0}$ or $\langle\varrho u\rangle = \boldsymbol{0}$. A weaker assumption than Eq. (12b) is to require $\langle\nabla \cdot u\rangle = 0$ only on the largest scales. In this case $H(t)$ could be defined in terms of a Friedmann solution, whereas $H_{\mathscr{D}} := \frac{\dot{a}_{\mathscr{D}}}{a_{\mathscr{D}}}$ could be interpreted as the value of Hubble's constant as measured on smaller scales (after averaging over statistical ensembles of spatial domains). The inhomogeneous term on the r.h.s. of Eq. (14) would then contain additionally a surface integral of the flux of u through the boundary $\partial\mathscr{D}$. Since $a_{\mathscr{D}}$ is defined via the volume, it could be interpreted also as an "effective" (D-dependent) scale factor of a possibly anisotropically expanding domain (compare also Appendix B and, for further discussion, Buchert (1996)).

5) Imagine we divide a large portion of the Universe $\mathscr{D}$ with volume $V(t)$ into N subdomains $\mathscr{D}_i$ with volumes $V_i(t)$.

Then, e.g., $\langle\sigma^2\rangle_\mathscr{D} = \frac{1}{N}\sum_i\langle\sigma^2\rangle_{\mathscr{D}_i}$. If the subdomains are 'typical', then we conclude $\langle\sigma^2\rangle_\mathscr{D} = \langle\sigma^2\rangle_{\mathscr{D}_i}$, i.e., the average value of σ^2 attained in the subdomains is "frozen" and cannot become smaller by averaging over larger domains. This statement also applies to any positive semi-definite quantity such as ω^2 and $(\theta - \langle\theta\rangle)^2$.

Appendix A: kinematics on toroidal spacetimes

Consider any "Newtonian" toroidal spacetime $\mathbb{S} = \mathbb{T}^3 \times \mathbb{R}$. The "global motion" of the space $\mathbb{T}^3$ of $\mathbb{S}$ is given by a time-dependent triad $e_i(t)$ of spatial vectors generating those translations $n^i e_i(t)$ which map each point of the torus $\mathbb{T}^3$ onto itself at time t, n^i being arbitrary integers. The rates of change $\dot{e}_i$, measured with respect to a non-rotating orthonormal basis, uniquely determine a matrix $H_{ij}(t)$ via $\dot{e}_i = H_{ij}e_j$.

With respect to any particle as origin, any point of $\mathbb{T}^3$ has infinitely many position vectors $x + n^i e_i(t)$ at time t. Hence, any kinematically possible motion covering the universe $\mathbb{S}$ can be represented by a velocity field $v(x, t)$ on the covering spacetime $\mathbb{R}^3 \times \mathbb{R}$ of $\mathbb{S}$ which obeys the *torus condition*

$$v(x + n^i e_i(t), t) = v(x, t) + n^i \dot{e}_i(t) \; . \tag{T}$$

It relates the velocities associated with the position vectors of any one particle. The difference of two such velocity fields, *but not these fields themselves*, define global spatial vector fields on the torus, as (T) shows. This is illustrated in Fig. 1.

If v obeys (T), then $v_{i,j}(x + n^i e_i(t), t) = v_{i,j}(x, t)$, hence σ_{ij}, ω_{ij} and θ *are* tensor fields on $\mathbb{T}^3$.

A velocity field V obeys the cosmological homogeneity principle iff, for all y and x,

$$V(y, t) - V(x, t) = A(t)(y - x) \tag{C}$$

(Heckmann & Schücking 1955, 1956). This property is compatible with (T) iff $A = H$, as follows by putting $y = x + n^i e_i(t)$ in equation (C) and using (T). This shows: Except for a spatially constant additive contribution, there exists exactly one homogeneous motion on a toroidal spacetime; a representative of its velocity field is determined by the matrix H_{ij} defined above, as $V(x, t) = H(t)x$. Its shear Σ_{ij}, rotation Ω_{ij} and expansion Θ are spatially constant and given by the decomposition

$$H_{ij} = \Omega_{ij} + \Sigma_{ij} + \frac{1}{3}\Theta\delta_{ij} \; .$$

If v is any inhomogeneous flow on $\mathbb{S}$ and $V + A(t)$ the general homogeneous one, then a unique peculiar-velocity field u is defined by $u := v - V$, with $\langle u\rangle_{\mathbb{T}^3} = 0$. Since u is a global vector field on $\mathbb{T}^3$, $\langle u_{i,j}\rangle_{\mathbb{T}^3} = 0$, hence

$$\langle\omega_{ij}\rangle_{\mathbb{T}^3} = \Omega_{ij} \;\;, \;\; \langle\sigma_{ij}\rangle_{\mathbb{T}^3} = \Sigma_{ij} \;\;, \;\; \langle\theta\rangle_{\mathbb{T}^3} = \Theta \; .$$

The homogeneous motion of a toroidal model is a Hubble flow if and only if $\Sigma_{ij} = 0$ and $\Omega_{ij} = 0$ or, equivalently, $H_{ij} = \frac{1}{3}\Theta\delta_{ij}$;
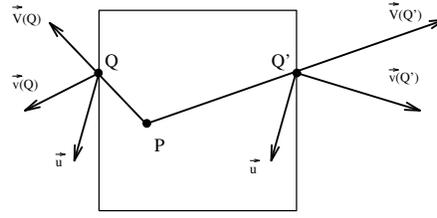

Fig. 1.

the last condition is in turn equivalent to $e_i(t) = \frac{a(t)}{a(t_0)}e_i(t_0)$, if we put $\Theta = 3\frac{\dot{a}}{a}$.

Thus, a Hubble flow exists and is then uniquely determined on a toroidal model if and only if the torus expands self-similarly without rotation.

Appendix B: general expansion law with global shear and vorticity

We start from Raychaudhuri's equation, averaged on a simply connected domain (Eq. (9a)),

$$3\frac{\ddot{a}_\mathscr{D}}{a_\mathscr{D}} + 4\pi G\frac{M}{a_\mathscr{D}^3} - \Lambda = \frac{2}{3}\left(\langle\theta^2\rangle_\mathscr{D} - \langle\theta\rangle_\mathscr{D}^2\right) + 2\langle\omega^2 - \sigma^2\rangle_\mathscr{D} \; .$$

By introducing on $\mathscr{D}$ arbitrary fields of expansion $\Theta = \Theta(t)$, shear $\Sigma_{ij} = \Sigma_{ij}(t)$ and vorticity $\Omega_{ij} = \Omega_{ij}(t)$ (which are not necessarily average values, but merely define a time-dependent standard of reference, e.g., a homogeneous solution of the basic system of equations), we may define a linear "background velocity field" V by $V_i := H_{ij}x_j$ with

$$V_{i,j} = \Sigma_{ij} + \frac{1}{3}\Theta\delta_{ij} + \Omega_{ij} =: H_{ij} \; . \tag{B.1}$$

Also, the inhomogeneous deviations from these reference values are introduced,

$$\theta = \Theta + \hat{\theta} \;\;; \;\; \sigma_{ij} = \Sigma_{ij} + \hat{\sigma}_{ij} \;\;; \;\; \omega_{ij} = \Omega_{ij} + \hat{\omega}_{ij} \; . \tag{B.2a,b,c}$$

Eq. (9a) may then be cast into the form

$$3\frac{\ddot{a}_\mathscr{D}}{a_\mathscr{D}} + 4\pi G\frac{M}{a_\mathscr{D}^3} - \Lambda =$$

$$2\left(\Omega^2 - \Sigma^2\right) + 2\left(\Omega_{ij}\langle\hat{\omega}_{ij}\rangle_\mathscr{D} - \Sigma_{ij}\langle\hat{\sigma}_{ij}\rangle_\mathscr{D}\right)$$

$$+\frac{2}{3}(\langle\hat{\theta}^2\rangle_\mathscr{D} - \langle\hat{\theta}\rangle_\mathscr{D}^2) + 2\langle\hat{\omega}^2 - \hat{\sigma}^2\rangle_\mathscr{D} \; .$$

$$\tag{B.3}$$

Using (13) to express deviations from homogeneity in terms of the peculiar-velocity gradient $(u_{i,j})$ with $H^2 = H_{ij}H_{ij} = 2(\Sigma^2 + \Omega^2) + \frac{1}{3}\Theta^2$, we arrive at the (most general) expansion law which holds for general inhomogeneities relative to a non-isotropic and rotational "background velocity field":

$$3\frac{\ddot{a}_\mathscr{D}}{a_\mathscr{D}} + 4\pi G\frac{M}{a_\mathscr{D}^3} - \Lambda =$$

$$2 \left(\Omega^2 - \Sigma^2 \right) + 2 \left(\Omega_{ij} \langle \hat{\omega}_{ij} \rangle_{\mathscr{D}} - \Sigma_{ij} \langle \hat{\sigma}_{ij} \rangle_{\mathscr{D}} \right)$$

$$+ \langle \nabla \cdot [u(\nabla \cdot u) - (u \cdot \nabla)u] \rangle_{\mathscr{D}} - \frac{2}{3} \langle \nabla \cdot u \rangle_{\mathscr{D}}^2 \quad . \qquad (B.4)$$

Instead of considering inhomogeneities relative to a (*D*-independent) arbitrary reference velocity field *V*, we may alternatively define the inhomogeneous fields in (B.2a,b,c) as deviations from the average flow within the domain $\mathscr{D}$ (which we identify with the global background velocity field). This corresponds to the point of view of an observer who maps a finite region of space and assumes that the average values of that region are 'typical' for the Universe. In this case all global variables $\Theta(t)$, $\Sigma_{ij}(t)$ and $\Omega_{ij}(t)$ are averages and depend on content, shape and position of the spatial domain $\mathscr{D}$:

$$\Theta = \langle \theta \rangle_{\mathscr{D}} \quad ; \quad \Sigma_{ij} = \langle \sigma_{ij} \rangle_{\mathscr{D}} \quad ; \quad \Omega_{ij} = \langle \omega_{ij} \rangle_{\mathscr{D}} \quad . \qquad (B.5a, b, c)$$

The averages $\langle \hat{\theta} \rangle_{\mathscr{D}} = \langle \nabla \cdot u \rangle_{\mathscr{D}}$, $\langle \hat{\sigma}_{ij} \rangle_{\mathscr{D}}$ and $\langle \hat{\omega}_{ij} \rangle_{\mathscr{D}}$ then vanish by definition, and the expansion law (B.4) simplifies to

$$3 \frac{\ddot{a}_{\mathscr{D}}}{a_{\mathscr{D}}} + 4\pi G \frac{M}{a_{\mathscr{D}}^3} - \Lambda =$$

$$2(\Omega^2 - \Sigma^2) + \langle \nabla \cdot [u(\nabla \cdot u) - (u \cdot \nabla)u] \rangle_{\mathscr{D}} \quad . \qquad (B.6)$$

The last term in (B.6) is, via Gauß's theorem, a surface integral over the boundary of $\mathscr{D}$. In case of a toroidal model we may choose $\mathscr{D}$ to be the whole torus. Then, the background flow is *unique* (see Appendix A) and, since there is no boundary, we obtain the global expansion law (in agreement with the result of Brauer et al. loc. cit., Eq. (2.15)):

$$3 \frac{\ddot{a}}{a} + 4\pi G \frac{M}{a^3} - \Lambda = 2 \left(\Omega^2 - \Sigma^2 \right) \quad . \qquad (B.6')$$

References

Bildhauer S., 1990, Prog. Theor. Phys. 84, 444

Bildhauer S., Futamase T., 1991a, MNRAS 249, 126

Bildhauer S., Futamase T., 1991b, GRG 23, 1251

Brauer U., Rendall A., Reula O., 1994, Class. Quant. Grav. 11, 2283

Buchert T., 1996, in "Mapping, Measuring and Modelling the Universe", València 1995, P. Coles et al. (eds), ASP conference series, pp. 349-356

Carfora M., Marzuoli A., 1988, Class. Quant. Grav. 5, 659

Carfora M., Isenberg J., Jackson M., 1990, J. Diff. Geom. 31, 249

Carfora M., Piotrkowska K., 1995, Phys. Rev. D 52, 4393

Ehlers J., Buchert T., 1997, GRG, in press

Ellis G.F.R., 1984, in Proc. 10th international conference on General Relativity and Gravitation, B. Bertotti et al. (eds), Reidel Dordrecht, 215–228

Futamase T., 1989, MNRAS 237, 187

Futamase T., 1996, Phys. Rev. D 53, 681

Heckmann O., Schücking E., 1955, Zeitschrift für Astrophysik 38, 95 (in German)

Heckmann O., Schücking E., 1956, Zeitschrift für Astrophysik 40, 81 (in German)

Olson D.W., Sachs R.K., 1973, ApJ 185, 91

Yodzis P., 1974, Proc. Roy. Irish Acad. 74A, 61