Astron. Astrophys. 320, 920–928 (1997)

ASTRONOMY
AND
ASTROPHYSICS

Radiation hydrodynamics with many spectral lines

Analytical expressions for a differentially moving slab

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Received 21 May 1996 / Accepted 5 September 1996

Abstract. An analytical solution of the comoving frame radiative transfer equation for a differentially moving slab is obtained and written in terms of the wavelength-integrated extinction coefficient ψ, the spectral thickness. It is shown that ψ is much smoother than the extinction coefficient $\chi(\lambda)$ itself and that it can be well approximated e.g. by a simple piecewise linear function. Compared to conventional algorithms it allows speed-ups of factors $\geq 10^5$ in the calculation of wavelength-integrated quantities such as e.g. the total flux, the radiative force, or the energy balance with an accuracy of $\leq 1\%$. The total number of lines which can be taken into account is essentially unlimited. A simple expression for the total flux in the diffusion limit is derived and the dependencies on the velocity gradient are discussed.

Key words: radiative transfer – hydrodynamics – methods: analytical

1. Introduction

It is well known that spectral lines affect strongly the spectra and wavelength-integrated quantities of radiating media. We only mention winds from hot stars (e.g. Castor et al. 1975), where the radiation pressure in the lines causes the outward motion of the matter, and novae where the line opacity may be crucial for the ignition (Starrfield et al. 1988). In general, several ten millions of spectral lines have to be taken into account. As e.g. has been pointed out by Kurucz (1994) even $17 \cdot 10^6$ lines are still not enough for modelling the solar spectrum.

In the modelling of *static* configurations mean opacities, opacity distribution functions or opacity sampling methods (cf. Mihalas 1978) provide convenient ways to incorporate many lines. However, if the media are *moving*, the strong and complicated wavelength and angle couplings seem to prevent direct generalizations of these approaches. On the other hand, full numerical solutions of the transfer equation can be obtained rather easily for stationary configurations and a moderate number of lines and/or a limited part of the spectrum (e.g. Hauschildt & Wehrse 1991), whereas it is by far too time-consuming for *non-stationary* situations and/or the calculation of the whole spectrum even on most powerful machines.

Therefore in actual calculations usually Rosseland mean opacities or the diffusion approximation are applied, approaches which are of low accuracy for moving media. First attempts to find (under certain conditions) a simple solution by introducing an expansion opacity are due to Karp et al. (1977).

In this paper we present a new approach based on an exact analytical solution of the transfer equation in the two-stream approximation for a plane-parallel, differentially moving slab. If we introduce the *spectral thickness* ψ, i.e. the wavelength integral of the extinction coefficient, surprisingly simple explicit expressions for the total flux, the energy balance, and the radiative force result which provide insight into the effects of the spectral lines in differentially moving media.

After deriving the analytical solution for the moving slab (Sect. 2), we describe the properties of the spectral thickness (Sect. 3). In Sect. 4 the wavelength-integrated radiative quantities entering hydrodynamic modelling are summarized in a form suitable for our purpose. In Sect. 5 the radiative quantities are evaluated by the realistic approximation of the spectral thickness by a piecewise linear function, and in Sect. 6 the simple "one-piece approximation" of ψ is shown to suffice to demonstrate the basic effects of motions on the radiative quantities. Finally (Sect. 7), we discuss the range of applicability and future aspects of our approach.

2. Radiative transport

We consider – at any instant of time – a plane-parallel layer with lower boundary $z=0$ and upper boundary z_0, illuminated from below and moving stationarily with $\beta(z) = v(z)/c$ in the z direction. The plane-parallel geometry is chosen mainly since

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here a simple analytical solution can be obtained and since it is sufficient to demonstrate the essentials of our approach. We expect that in an analogous way a solution can be worked out also for spherical geometry.

We formulate the radiative transfer equation for the specific monochromatic intensity $I(z,\mu,\lambda)$ in the *comoving* frame (Mihalas & Weibel Mihalas 1984) keeping only terms of first order in β, i.e. $\beta \leq 0.1$ in practice. The wavelength λ, the cosine of the angle of the ray relative to the z-axis μ, the extinction coefficient χ, and the source function S refer to the comoving frame whereas, as usual, z is kept in the observer's frame. We use the comoving frame since thermodynamical quantities relevant for radiation hydrodynamics such as the temperature are only defined in this frame.

We simplify the problem by considering only the rays with $\mu = \pm 1$, i.e. adopt the two-stream-approximation, which is known to give reasonably accurate results (of the order of 10%). In this case the aberration and advection terms play no role since then the $\partial/\partial\xi$-term in the comoving transfer equation vanishes due to the factor $(1-\mu^2)$ (Mihalas & Weibel Mihalas 1984). We stress that this approximation is not essential; it has been introduced to keep the formulae as simple as possible. We give explicit formulae for the *outward* direction $I^+ = I(\mu = 1)$ only, since the solution for $I^- = I(\mu = -1)$ follows analogously.

Instead of the λ-scale we introduce $\xi = \ln\lambda$ to describe the wavelength dependencies since then the Doppler shifts are independent of the wavelength.

With the abbreviation $w = \partial\beta/\partial z$, the radiative transfer equation for the outward intensity $I \equiv I^+$ then reads

$$\frac{\partial I(z,\xi)}{\partial z} + w \frac{\partial I(z,\xi)}{\partial \xi} = -\big(\chi(\xi) + 5w\big) I(z,\xi) + \chi(\xi) S(z,\xi) \tag{1}$$

where we have assumed χ as well as w to be *independent* of z, but allowing S to depend on z. As it will turn out below, the assumption of depth-independent χ and w is essential for obtaining simple expressions for the radiative quantities. In practice, however, this poses no serious restriction, since in actual applications algorithms are available to add up many layers of such a type (e.g. Peraiah 1984).

As initial values we specify the (outward) intensity at the bottom layer $I(z{=}0,\xi)$ for all ξ, and, in principle, the intensity at the lowest wavelength point $I(z,\xi = -\infty)$ for all z (and $w \geq 0$). In practice, the latter condition is replaced by a restricted integration range depending on the Doppler shifts of the lines.

Eq. (1), though simplified, suffices to demonstrate the characteristic behaviour of a differentially moving medium with a large number of absorption lines, and also is a realistic approximation in many cases (Kalkofen & Wehrse 1982).

For the solution of Eq. (1), we now shift the argument, i.e. replace ξ by $\eta = \xi + wz$, and utilize the Lagrangean derivative

$$\frac{dI(z,\eta)}{dz} = \frac{\partial I(z,\eta)}{\partial z} + w \cdot \frac{\partial I(z,\eta)}{\partial \eta} \tag{2}$$

which physically reflects the connection of the depth scale z with the wavelength scale by the Doppler shift $d\lambda/\lambda = d\xi = dv/c = d\beta = w dz$. With Eq. (2) the transfer equation can be solved *analytically* (Kamke 1965) yielding

$$\begin{aligned} I(z,\xi + wz) = {} & I(0,\xi) \exp\left(-\int_0^z \Big(\chi(\xi + wz') + 5w\Big) dz'\right) \\ & + \int_0^z \exp\left(-\int_{z'}^z \Big(\chi(\xi + wz'') + 5w\Big) dz''\right) \\ & \times \chi(\xi + wz') S(z', \xi + wz') dz' . \end{aligned} \tag{3}$$

Of course, this expression reduces to the familiar solution in the static case.

Eq. (3) suggests that the role of the opacities in moving media is best described by means of the monotonic function

$$\psi(\xi) = \int_0^\xi \chi(\eta) d\eta . \tag{4}$$

We call this wavelength-integrated extinctinction coefficient *spectral thickness* in analogy to the depth-integrated optical thickness. It has the dimension of a reciprocal length. The choice of the lower integration limit 0 is not critical since only differences of $\psi(\xi)$ occur in the subsequent formulae. Introducing $\psi(\xi)$, Eq. (3) reads

$$\begin{aligned} I(z,\xi) = {} & I(0,\xi - wz) \exp\Big(-\tfrac{1}{w}(\psi(\xi) - \psi(\xi - wz))\Big) \mathrm{e}^{-5wz} \\ & + \int_{\xi - wz}^{\xi} \exp\Big(-\tfrac{1}{w}(\psi(\xi) - \psi(\eta))\Big) \mathrm{e}^{-5(\xi-\eta)} \\ & \times \chi(\eta) S(z - \tfrac{\xi-\eta}{w}, \eta) \frac{d\eta}{w} . \end{aligned} \tag{5}$$

(In this paper we assume $w \neq 0$.)

After partial integration, the specific intensity can fully be expressed in terms of $\psi(\xi)$, i. e. the strongly varying function $\chi(\xi)$ (cf. below) is completely eliminated:

$$\begin{aligned} I(z, \xi) = {} & I(0,\xi - wz) \exp\Big(-\tfrac{1}{w}(\psi(\xi) - \psi(\xi - wz))\Big) \mathrm{e}^{-5wz} \\ & - \int_{\xi - wz}^{\xi} \exp\Big(-\tfrac{1}{w}(\psi(\xi) - \psi(\eta))\Big) \\ & \times \frac{d}{d\eta}\Big(\mathrm{e}^{-5(\xi-\eta)} S(z - \tfrac{\xi-\eta}{w}, \eta)\Big) d\eta + S(z,\xi) \\ & - S(0,\xi - wz) \exp\Big(-\tfrac{1}{w}(\psi(\xi) - \psi(\xi - wz))\Big) \mathrm{e}^{-5wz} . \end{aligned} \tag{6}$$

We point out that the Doppler shifts of all the numerous lines influence the result effectively only over the *restricted* ξ-interval wz. Note that we require $wz \leq wz_0 \leq 0.1$ since $\beta = \int w dz \leq 0.1$.

The form of our solution (5) is similar to the expression on which the *Sobolev method* is based, a frequently used method to calculate an isolated spectral line in moving spherical configurations (cf. Mihalas 1978, Bastian et al. 1980), if we consider it for the plane-parallel case. Our approach, however, aims at a quite different direction, namely to efficiently deal (in the comoving frame) with many spectral lines in a differentially moving medium.

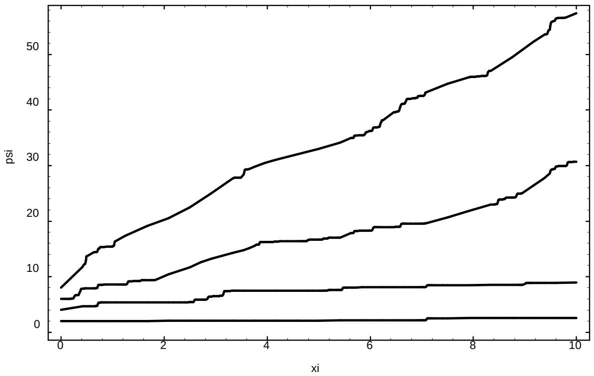

Fig. 1. Spectral thickness $\psi(\xi)$ for randomly chosen lines as function of the logarithmic wavelength $\xi = \ln\lambda$ for L = 1, 10, 50, 100 (bottom to top) spectral lines in the interval shown.

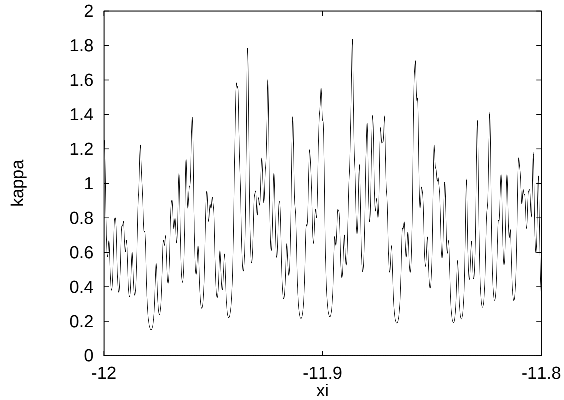

Fig. 2. Example of a typical run of the extinction coefficient χ with $\xi = \ln\lambda$ (χ and λ in arbitrary units)

3. Spectral thickness ψ

First we point out that $\psi(\xi)$ is *independent* of w, i.e. that it can be calculated independently of the dynamical state of the matter.

A further attractive property of the spectral thickness is, that – by definition – it is additive with respect to its contributing components as is the extinction coefficient χ:

$$\chi(\xi) = \chi_{\rm cont}(\xi) + \sum_{\ell=1}^{L} \chi_\ell(\xi)$$
$$\Longrightarrow \quad \psi(\xi) = \psi_{\rm cont}(\xi) + \sum_{\ell=1}^{L} \psi_\ell(\xi) \tag{7}$$

Here the index "cont" denotes the continuum contribution (absorption plus scattering) and the sum over ℓ the contribution of the lines.

For example, a wavelength-independent continuum would contribute $\psi_{\rm cont}(\xi) = a\cdot\xi$ with constant a, or a power-law dependence $\chi = \chi_0\lambda^\alpha$ would result in $\psi_{\rm cont}(\xi) = \chi_0(\exp(\alpha\xi) - 1)/\alpha$. Due to the additivity of ψ, this component could be evaluated independently of the line part.

Each term of the sum over the spectral line contributions is proportional to the occupation number of the lower level of the line ℓ, to its gf value, and to its integrated profile function. In many applications the line width is very small compared to the relevant Doppler shifts by the differential motion. Then ψ_ℓ can be well approximated by a Heaviside step function. For Lorentzian and Gaussian line shapes ψ_ℓ is given by arctangent and error functions, respectively.

For illustration, we show in Fig 1. how ψ is modified by an increasing number of randomly chosen narrow lines (without continuum). It is seen that already for a moderate number of lines ψ becomes essentially a simple smooth function. This well known *smoothing effect* of integration (up to now not yet exploited in radiative transfer modelling) becomes particularly strong when many lines contribute to the opacity: a distribution of extinction coefficients caused by about 4000 lines (Fig. 2) is transformed essentially into a *straight* line (Fig. 3).

Evidently, the details depend on the particular distribution of lines. Therefore, we have calculated the line contribution to ψ for an LTE plasma of solar composition, a temperature of 5000 K, and a gas pressure of $4.3\cdot10^4$ dyn cm^{-2} based on the line list "GFALL.DAT" of Kurucz (1995) with $1.3\cdot10^6$ entries. Since the broadening caused by the macroscopic velocities is much larger than thermal and pressure broadening we have assumed that the profile functions are given by Dirac delta functions. As is shown in Fig. 4, there is a small number of strong lines causing jumps in $\psi(\xi)$. The remaining fainter lines show up only collectively by non-zero gradients. It is therefore suggestive to approximate $\psi(\xi)$ by four straight lines in this example.

As will be discussed below, such a simple piecewise linear approximation makes sense indeed, since it speeds up the computation of wavelength integrated quantities by more than 5 orders of magnitude whereas the accuracy is better than 1%! We are convinced that such linear approximations also give good results for other combinations of the input parameters.

4. Wavelength-integrated radiative quantities

Hydrodynamic modelling requires the following (comoving frame) *wavelength-integrated* radiative quantities (cf. Mihalas & Weibel Mihalas 1984) which we denote by caligraphic letters: (i) the total flux

$$\mathscr{F}(z) = \int_0^\infty F(z,\lambda)d\lambda = \int_{-\infty}^{\infty} F(z,\xi){\rm e}^\xi d\xi\,, \tag{8}$$

(ii) the radiative force

$$\mathscr{K}(z) = \frac{1}{c}\int_0^\infty \chi(\lambda)F(z,\lambda)d\lambda\,, \tag{9}$$

and
(iii) the difference between absorbed and emitted radiative power

$$\Delta\mathscr{E}(z) = \int_0^\infty \chi(\lambda)\big(J(z,\lambda) - S(z,\lambda)\big)d\lambda$$
$$= \int_0^\infty \kappa(\lambda)\big(J(z,\lambda) - B(z,\lambda)\big)d\lambda \tag{10}$$

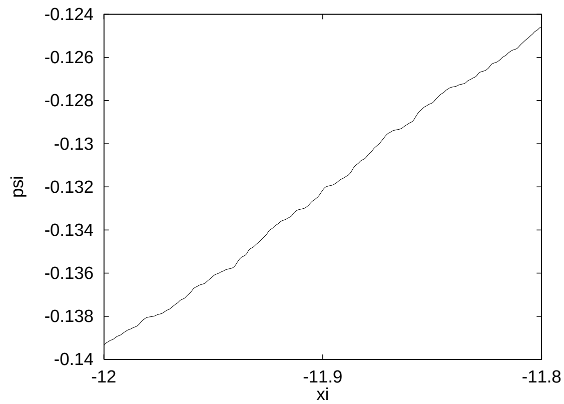

Fig. 3. Spectral thickness $\psi(\xi)$ corresponding to $\chi(\xi)$ of Fig. 2. Note the extreme smoothing effect of the wavelength integration.

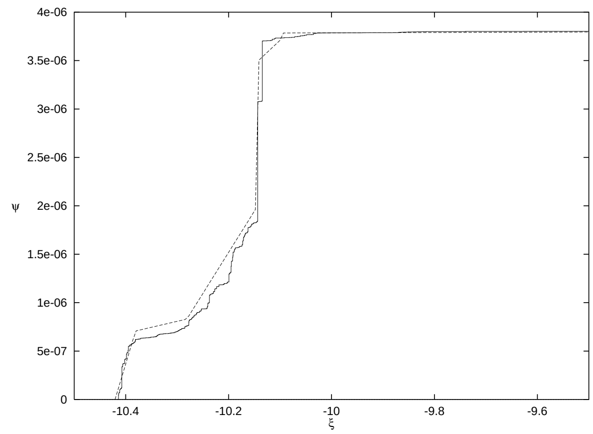

Fig. 4. $\psi(\xi)$ as function of $\xi = \ln \lambda$ for Kurucz's (1995) line list comprising $1.3 \cdot 10^6$ lines (broken curve) and its piecewise linear approximation (full curve)

where $J(z, \lambda)$ is the mean intensity, $\kappa(\lambda)$ the absorption coefficient, and $B(z, \lambda) = B(T(z), \lambda)$ the Planck function for the temperature T. The scattering terms cancel in the energy balance .

In the two-stream approximation the (monochromatic) mean intensity is $J = (I^+ + I^-)/2$ and the radiation flux $F = F^+ + F^- = I^+ - I^-$ so that all the radiative quantities can be calculated from $I^\pm$.

Since in this paper – for the purpose of illustration – we give only the expression for the outward intensity $I = I^+$, we have to restrict ourselves to present the total radiative flux as well as the radiative force only for the *outward* direction

$$\mathscr{F}^+(z) = \int_0^\infty I(z, \lambda) d\lambda \,, \tag{11}$$

$$\mathscr{K}^+(z) = \frac{1}{c} \int_0^\infty \chi(\lambda) I(z, \lambda) d\lambda \,, \tag{12}$$

and to give the radiative power difference for the *surface* layer z_0 which receives no irradiation from above,

$$\Delta\mathscr{E}(z_0) = \int_0^\infty \kappa(\lambda) \big(\tfrac{1}{2} I(z_0, \lambda) - B(z_0, \lambda) \big) d\lambda \,. \tag{13}$$

5. Piecewise linear approximation of ψ

In general the radiative quantities have to be calculated from Eqs. (5) or (6) by numerical integration . However, as we have shown in Sect. 3, the approximation of the spectral thickness by a piecewise linear function is realistic in many cases, and for this case of a piecewise linear approximation *analytical* expressions for the radiative quantities integrated over a corresponding ξ-interval can be given.

We write the spectral thickness in the interval $[\xi_k, \xi_{k+1}]$ as

$$\psi(\xi_k \le \xi \le \xi_{k+1}) = a_k \cdot \xi + b_k \,, \quad \frac{d\psi}{d\xi} = \chi(\xi) = a_k \tag{14}$$

with known a_k and b_k. We choose the ξ_k as the break points of ψ and furthermore require $|\xi_{k+1} - \xi_k| > wz$. Let N be the number of pieces, so that k runs from 0 to N with the special notation $\xi_0 = -\infty$ and $\xi_N = +\infty$ for the endpoints of the total spectral interval. At any ξ_k, a jump in the spectral thickness $\psi(\xi)$ is allowed, e.g. due to the almost infinitely steep increase caused by a very strong spectral line (cf. Fig. 1); the jump, however, is not counted as a piece, i.e. is not included in N.

We now have to realize that according to Eq. (5) the intensity $I(z, \xi)$ is given by an integral over the range wz of the logarithmic wavelength scale. If the interval $[\xi - wz, \xi]$ includes a point ξ_k, we have to split the interval: If the integration variable η is smaller than ξ_k, the preceding piece with the slope a_{k-1} has to be taken. If, however, $\xi_k \le \eta \le \xi$, then the integration involves only the slope a_k.

Introducing the abbreviations

$$U_k = a_k + 4w \,, \quad V_j = a_j + 5w \qquad (j = k-1, k) \,, \tag{15}$$

the solution (5) in the piecewise linear approximation reads

$$\begin{aligned} I(z, \xi) &= I(0, \xi - wz) \mathrm{e}^{-V_k z} + a_k \, \mathrm{e}^{-V_k \xi / w} \\ &\times \int_{\xi - wz}^{\xi} \mathrm{e}^{+V_k \eta / w} S(z + \tfrac{\xi - \eta}{w}, \eta) \frac{d\eta}{w} \quad (\xi \ge \xi_k) \,, \end{aligned} \tag{16}$$

$$\begin{aligned} I(z, \xi) &= I(0, \xi - wz) \\ &\times \left(\mathrm{e}^{[(a_k - a_{k-1})\xi + (b_k - b_{k-1})]/w} \mathrm{e}^{-V_{k-1} z} + \mathrm{e}^{-V_k z} \right) \\ &+ a_{k-1} \mathrm{e}^{-(b_k - b_{k-1})/w} \mathrm{e}^{-V_k \xi / w} \\ &\times \int_{\xi - wz}^{\xi_k} \mathrm{e}^{+V_{k-1} \eta / w} S(z + \tfrac{\xi - \eta}{w}, \eta) \frac{d\eta}{w} \\ &+ a_k \mathrm{e}^{-V_k \xi / w} \int_{\xi_k}^{\xi} \mathrm{e}^{+V_k \eta / w} S(z + \tfrac{\xi - \eta}{w}, \eta) \frac{d\eta}{w} \quad (\xi \le \xi_k), \end{aligned} \tag{17}$$

The *total outward* flux is given by

$$\mathscr{F}^+(z) = \sum_{k=0}^{N-1} \mathscr{F}^+_{k,k+1}(z) \tag{18}$$

with

$$\mathscr{F}^+_{k,k+1}(z) = \int_{\xi_k}^{\xi_{k+1}} \mathscr{F}^+(z, \xi) \mathrm{e}^{\xi} d\xi = \int_{\xi_k}^{\xi_{k+1}} I(z, \xi) \mathrm{e}^{\xi} d\xi \tag{19}$$

being the flux integrated over the interval $[\xi_k, \xi_{k+1}]$.

According to Eqs. (16, 17) we have to split the flux integration into two parts, denoted by the superscripts I and II, respectively:

$$\begin{aligned}\mathscr{F}^{+}_{k,k+1}(z) &= \mathscr{F}^{\rm I}_{k,k+1}(z) + \mathscr{F}^{\rm II}_{k,k+1}(z)\\ &= \int_{\xi_k+wz}^{\xi_{k+1}} I(z,\xi)\mathrm{e}^{\xi}d\xi + \int_{\xi_k}^{\xi_k+wz} I(z,\xi)\mathrm{e}^{\xi}d\xi\,. \qquad (20)\end{aligned}$$

The interval I $[\xi_k + wz, \xi_{k+1}]$ involves only the slope a_k. For the case II $[\xi_k, \xi_k + wz]$, however, the slopes a_{k-1} as well as a_k enter resulting in a more complicated evaluation. For the piece at the low-wavelength end of the spectrum, $\mathscr{F}^{+}_{0,1}(z)$, the calculation of part II is not necessary because the source function goes to zero for $\xi \to -\infty$.

In many applications, the Doppler shift wz is very small compared to the length of interval $[\xi_k, \xi_{k+1}]$ so that the contribution II can be neglected provided that no large jump in ψ occurs at ξ_k.

In the following we omit the irradiation term throughout this Section since its evaluation is trivial. We furthermore drop the depth variable in the source function as we will concentrate on the *wavelength dependence* in a sufficiently thin layer in which S may be assumed to be depth-independent. The resulting flux parts are

$$\begin{aligned}\mathscr{F}^{\rm I}_{k,k+1}(z) &= \int_{\xi_k+wz}^{\xi_{k+1}} a_k\,\mathrm{e}^{-U_k\xi/w}\\ &\times \int_{\xi-wz}^{\xi} \mathrm{e}^{V_k\eta/w} S(\eta)\,\frac{d\eta}{w}\,d\xi\,, \qquad (21)\end{aligned}$$

$$\begin{aligned}\mathscr{F}^{\rm II}_{k,k+1}(z) &= a_{k-1}\,\mathrm{e}^{-(b_k-b_{k-1})/w}\int_{\xi_k}^{\xi_k+wz} \mathrm{e}^{-U_k\xi/w}\\ &\times \int_{\xi-wz}^{\xi_k} \mathrm{e}^{V_{k-1}\eta/w} S(\eta)\frac{d\eta}{w}\cdot d\xi\\ &+ a_k\int_{\xi_k}^{\xi_k+wz} \mathrm{e}^{-U_k\xi/w}\int_{\xi_k}^{\xi} \mathrm{e}^{V_k\eta/w} S(\eta)\frac{d\eta}{w}\,d\xi. \qquad (22)\end{aligned}$$

In addition to a fully numerical integration, there are essentially two *practical procedures* to derive good approximations of the wavelength-integrated flux pieces $\mathscr{F}^{\rm I}_{k,k+1}$ and $\mathscr{F}^{\rm II}_{k,k+1}$ from Eqs. (21, 22):

In the first way we interchange the sequence of integrations in Eq. (21) which allows us to perform the first integration analytically. All subsequent integrations extend over the range wz only so that they can accurately be evaluated after linearization of the integrand.

The second way to evaluate the flux is based upon a *rational approximation* of the source function S so that the integrals occurring in Eqs. (21, 22) can be expressed analytically.

For the subsequent evaluation of the outward flux we assume *local thermodynamic equilibrium* (LTE), but our procedure can be generalized to an arbitrary source function S.

In LTE the source function is equal to the Planck function $B(T(z), \xi)$. Written as function of ξ, it is

$$\begin{aligned}B(\xi) &= 2hc^2\,\frac{\mathrm{e}^{-5\xi}}{\mathrm{e}^{\exp(\alpha-\xi)}-1} = \frac{15}{\pi^4}\,\frac{\mathrm{e}^{-\alpha}\mathrm{e}^{5(\alpha-\xi)}}{\mathrm{e}^{\exp(\alpha-\xi)}-1}\,\mathscr{B}(T)\\ &= 2hc^2\,\mathrm{e}^{-5\alpha}\cdot\Omega(\alpha-\xi) \qquad (23)\end{aligned}$$

with $\alpha = \ln(hc/kT)$, $\mathscr{B}(T) = \sigma_{\rm SB}/\pi T^4$, and $\sigma_{\rm SB}$ being the Stefan-Boltzmann constant. The function

$$\Omega(\eta) = \frac{\mathrm{e}^{5\eta}}{\mathrm{e}^{\exp(\eta)}-1} \qquad (24)$$

does not contain α, i.e. it does not depend on the temperature T. The constant $2hc^2$ is equivalent to $(15/\pi^4)\exp(4\alpha)\mathscr{B}(T)$.

5.1. Direct integration

For the evaluation of $\mathscr{F}^{\rm I}_{k,k+1}(z)$ we use an alternative form of Eq. (21) which is derived from Eqs. (16, 17) by substitution of the variables and interchanging the order of integrations:

$$\begin{aligned}\mathscr{F}^{\rm I}_{k,k+1}(z) &= a_k\cdot\mathrm{e}^{-U_kz}\int_0^{wz}\mathrm{e}^{U_k\eta/w}\int_{\xi_k+\eta}^{\xi_{k+1}+\eta-wz}\mathrm{e}^{\zeta}B(\zeta)d\zeta\,\frac{d\eta}{w}\\ &= a_k\cdot\mathrm{e}^{-U_kz}\int_0^{wz}\mathrm{e}^{U_k\eta/w}\\ &\times[\tilde{B}(\eta+\xi_{k+1}-wz)-\tilde{B}(\eta+\xi_k)]\frac{d\eta}{w}, \qquad (25)\end{aligned}$$

where the difference of the indefinite integral

$$\tilde{B}(\xi) = \int B(\xi)\mathrm{e}^{\xi}d\xi \qquad (26)$$

desribes the wavelength-integrated Planck function.

Using the algebraic code *Mathematica* (Wolfram Research 1992) we obtain an *analytical* expression for $\tilde{B}(x)$ in terms of the complex polylogarithmic functions $\mathrm{Li}_s(x)$ (Appendix A):

$$\begin{aligned}\frac{\tilde{B}(x)}{\mathscr{B}(T)} &= \frac{15}{\pi^4}\int\frac{x^3}{\mathrm{e}^x-1}dx\\ &= \frac{15}{\pi^4}\left[-\frac{x^4}{4} + x^3\ln(1-\mathrm{e}^x)\right.\\ &\left.+ 3x^2\,\mathrm{Li}_2(\mathrm{e}^x) - 6x\,\mathrm{Li}_3(\mathrm{e}^x) + 6\,\mathrm{Li}_4(\mathrm{e}^x)\right] \qquad (27)\end{aligned}$$

with

$$x = \mathrm{e}^{\alpha-\xi} = \frac{hc}{kT}\mathrm{e}^{-\xi} = \frac{hc}{kT\lambda}\,. \qquad (28)$$

For the evaluation of the remaining integral in Eq. (25) we now expand $\tilde{B}$ around the integration variable $\eta = 0$,

$$\tilde{B}(\eta) = \tilde{B}(0) + \eta\left.\frac{d\tilde{B}}{d\eta}\right|_0 = \tilde{B}(0) + \eta B(0) \qquad (29)$$

and obtain the flux

$$\begin{aligned}\mathscr{F}^{\rm I}_{k,k+1}(z) &= a_k\,\tilde{B}_0\,\frac{1}{U_k}\left(1-\mathrm{e}^{-U_kz}\right)\\ &+ a_k B_0\,\frac{w}{U_k^2}\left(U_kz - 1 + \mathrm{e}^{-U_kz}\right) \qquad (30)\end{aligned}$$

with $\tilde{B}_0 = \tilde{B}(\xi_{k+1} - wz) - \tilde{B}(\xi_k)$ and $B_0 = B(\xi_{k+1} - wz) - B(\xi_k)$. For the numerical evaluation of the integral in Eq. (27) we present a convenient approximation by a rational function in Appendix B.

$\mathscr{F}^{\rm II}_{k,k+1}(z)$ (Eq. 22) can be evaluated after linearization of the Planck function, $B(\eta) = B(\xi_k) + (\eta - \xi_k) \cdot B'(\xi_k)$, resulting in

$$\begin{aligned}\mathscr{F}^{\rm II}_{k,k+1}(z) &= -a_{k-1} e^{-(b_k - b_{k-1})/w} \\ &\times\Big(e^{-U_k z}\tfrac{1}{w}[\mathscr{R}_{k-1}(\xi_k) - \mathscr{R}_{k-1}(\xi_k - wz)] \\ &\quad -e^{-U_k\xi_k/w}\tfrac{1}{U_k}(1 - e^{-U_k z})\mathscr{P}_{k-1}(\xi_k)\Big) \\ &+a_k\Big(\tfrac{1}{w}[\mathscr{R}_k(\xi_k + wz) - \mathscr{R}_k(\xi_k)] \\ &\quad -e^{-U_k\xi_k/w}\tfrac{1}{U_k}(1 - e^{-U_k z})\mathscr{P}_k(\xi_k)]\Big)\,. \end{aligned} \tag{31}$$

with $B = B(\xi_k)$ and $B' = B'(\xi_k)$,

$$\begin{aligned}\mathscr{P}_j(\eta) &= \int e^{+V_j\eta/w} B(\eta) d\eta \\ &\simeq \frac{w}{V_j} e^{+V_j\eta/w}\left[B + \left(\eta - \xi_k - \frac{w}{V_j}\right) B'\right]\,, \end{aligned} \tag{32}$$

and

$$\begin{aligned}\mathscr{R}_j(\eta) &= \int e^{-U_k\eta/w}\mathscr{P}_j(\eta) d\eta \\ &\simeq \frac{w^2}{V_j(V_j - U_k)} e^{+(V_j - U_k)\eta/w} \\ &\times\left[B + \left(\eta - \xi_k - \frac{w}{V_j} - \frac{w}{V_j - U_k}\right) B'\right]\,. \end{aligned} \tag{33}$$

Note that in particular

$$\mathscr{R}_k(\eta) = \frac{w}{V_j} e^{+\eta}\left[B + \left(\eta - \xi_k - \frac{w}{V_j} - 1\right) B'\right]\,. \tag{34}$$

With Eqs. (32) to (34) the evaluation of Eq. (31) is straightforward.

We emphasize that the linearization is meaningful only for the contribution II where the integration interval wz is small. For part I, however, the interval will mostly be too large to give accurate results with full linearization.

5.2. Rational approximation

A second way for the numerical evaluation of the flux from Eqs. (25, 31) is to approximate the factor Ω (24) of the Planck function (23) by a rational function in the range $-2 \le x \le +4$:

$$\Omega(x) = \frac{\exp(5x)}{\exp\big(\exp(x)\big) - 1} \simeq \frac{\sum_{j=0}^{4} p_j x^j}{1 + \sum_{j=1}^{4} q_j x^j}\,. \tag{35}$$

The coefficients are given in Appendix B. We furthermore replace $\mathscr{R}_j$ and $\mathscr{P}_j$ by the (indefinite) integrals

$$\Xi_j(\xi) = \int e^{-V_j\xi/w}\Omega(\xi)d\xi = -\frac{e^{-a_j\alpha/w}}{2hc^2}\mathscr{P}_j(\alpha - \xi)\,, \tag{36}$$

$$\begin{aligned}\Theta_j(\xi) &= \int e^{+U_k\xi/w}\Xi_j(\xi)d\xi \\ &= +\frac{e^{-(a_j - a_k)\alpha/w}}{2hc^2} e^{4\alpha}\mathscr{R}_j(\alpha - \xi)\,, \end{aligned} \tag{37}$$

which are independent of α or T. Approximate expressions of Ξ and Θ, based on the rational approximation of Ω, are also given in Appendix B.

6. Basic effects of the motion upon radiative quantities

Once we have calculated the outward flux, the other radiative quantities required in hydrodynamic modelling, the energy balance and the outward radiative force, can be expressed by the outward flux.

Particularly simple expressions are obtained if we approximate ψ over the *total* spectral range by a *single* straight line ($N = 1$) so that $a_k = a$ and $b_k = 0$ in Eq. (14):

$$\psi(\xi) = \int_0^{\xi}\chi(\eta)d\eta = a\cdot\xi\,, \quad \frac{d\psi}{d\xi} = \chi(\xi) = a\,. \tag{38}$$

This admittedly not too realistic case nevertheless is well suited to discuss the *basic effects* of the differential motion on the wavelength-integrated radiative quantities.

In this case Eq. (16) simplifies, after a change of the integration variable, to

$$\begin{aligned}I(z,\xi) &= I(0, \xi - wz)e^{-Vz} + a\, e^{-V\xi/w} \\ &\times\int_0^{wz} e^{+V\eta/w} S(z + \tfrac{\eta - wz}{w}, \eta + \xi - wz)\frac{d\eta}{w} \end{aligned} \tag{39}$$

($V = a + 5w$) and the outward radiative flux – after interchanging the order of integrations – becomes

$$\begin{aligned}\mathscr{F}^+(z) &= \mathscr{F}^+(0)\, e^{-Uz} + a\cdot e^{-Uz} \\ &\times\int_0^{wz} e^{U\eta/w}\int_{-\infty}^{\infty} e^{\zeta} S(z + \tfrac{\eta - wz}{w}, \zeta)d\zeta\,\frac{d\eta}{w} \end{aligned} \tag{40}$$

with $U = a + 4w$ and $\mathscr{F}^+(0) = \int_{-\infty}^{\infty} I(0,\xi)\exp(\xi)d\xi$. For the following discussion, we introduce the parameters

$$u = 4\,w/a = 4\,\partial\beta/\partial\tau\,, \qquad \tau = \tau(z) = az\,, \tag{41}$$

i.e. essentially the derivative of the velocity $\partial\beta/\partial\tau$ and the usual (static) optical depth τ (which includes the spectral lines) which allow a more compact formulation of the results.

If the source function is regarded as *independent* of the depth, $S(z,\zeta) = S(\zeta)$, Eq. (40) can be written as

$$\mathscr{F}^+(z) = \mathscr{F}^+(0)\, e^{-(1+u)\tau} + \frac{1}{1+u}\left(1 - e^{-(1+u)\tau}\right)\mathscr{S} \tag{42}$$

where $\mathscr{S} = \int_0^{\infty} S\exp(\xi)d\xi$ is the wavelength-integrated source function. Note that $u\tau = 4wz \le 0.4$, but u itself may adopt large positive or negative values.

Finally, the total outward radiative force

$$\mathscr{K}^+(z) = \frac{a}{c}\mathscr{F}^+(z) \tag{43}$$

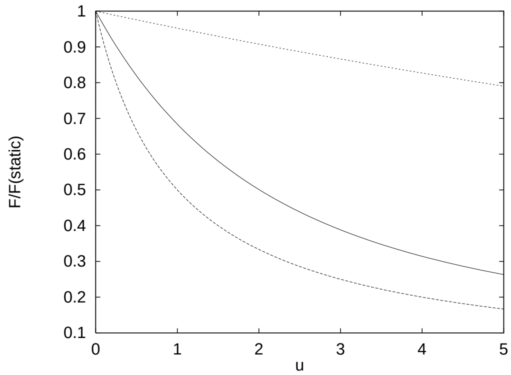

Fig. 5. Ratio of the total flux from a non-illuminated, differentially moving slab with depth-independent source function to that of a static slab of identical sorce function and extinction as a function of $u = 4\partial\beta/\partial\tau$ for optical depths τ=0.1 (···), 1 (—), 10 (- - -)

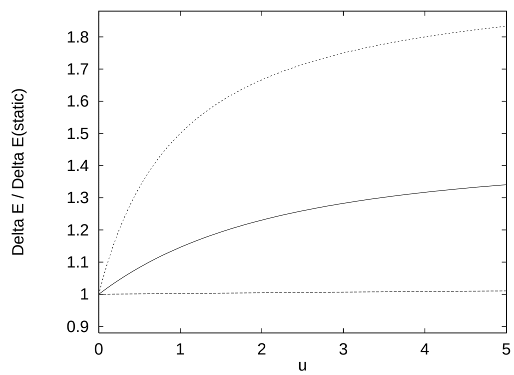

Fig. 6. Normalized energy balance $\Delta\mathscr{E}(z_0)$ in the surface layer of a non-illuminated, differentially moving slab with depth-independent source function as a function of $u = 4\partial\beta/\partial\tau$ for optical depths $\tau(z_0)$=0.1 (- - -), 1 (—), 10 (···)

as well as the energy balance

$$\Delta\mathscr{E}(z_0) = a\left(\tfrac{1}{2}\mathscr{F}^+(z_0) - \mathscr{B}(z_0)\right) \quad (44)$$

can be expressed in terms of the total outward flux $\mathscr{F}^+(z)$. Here $\mathscr{B}(z) = \mathscr{B}(T(z)) = \int_0^\infty B(T,\lambda)d\lambda = (\sigma_{\rm SB}/\pi)T^4(z)$ denotes again the wavelength-integrated Planck function.

To illustrate the results, we plot for the special case of a non-illuminated ($\mathscr{F}^+(0) = 0$) slab with *depth-independent* source function S in Fig. 5 the total outward flux $\mathscr{F}^+(z)$ (Eq. 42), and in Fig. 6 the radiative energy balance at the surface $\Delta\mathscr{E}(z_0)$ (Eq. 44), both normalized to the corresponding static value. As expected, the effect of the differential motion is small for small optical depths, whereas for large optical depths the factor $(1 + u)$ in the denominator considerably reduces the flux. Since we assume an outwardly accelerated motion, w and u are positive so that, seen from the surface, the bottom layers recede and the flux is decreased due to the redshift.

As a last illustrative example we discuss the *diffusion approximation* in a moving slab. To this purpose, we have to assume $S(z,\xi) = B(z,\xi)$ and linearize the Planck function with respect to *depth*,

$$B(z + \tfrac{\eta - wz}{w}, \zeta) = B(z,\zeta) + \frac{\partial B(z,\zeta)}{\partial T}\frac{dT}{dz}\cdot\frac{\eta - wz}{w} \quad (45)$$

and obtain

$$\begin{aligned}\mathscr{F}^+(z) &= \mathscr{F}^+(0)\,\mathrm{e}^{-(1+u)\tau} + \frac{1}{1+u}\left(1 - \mathrm{e}^{-(1+u)\tau}\right)\mathscr{B}(z)\\ &+ Q(u,\tau)\frac{1}{a}\frac{\partial\mathscr{B}(z)}{\partial T}\frac{dT}{dz}\end{aligned} \quad (46)$$

with

$$Q(u,\tau) = \frac{(1+u)\tau - \left(1 + (1+u)\tau\right)\left(1 - \mathrm{e}^{-(1+u)\tau}\right)}{(1+u)^2}. \quad (47)$$

For *large* optical depths, $\tau \to \infty$, we obtain from Eq. (46) the *net* flux in the diffusion approximation

$$\mathscr{F}(z) = -\frac{1}{(1+u)^2}\cdot\frac{1}{a}\frac{\partial\mathscr{B}(z)}{\partial T}\frac{dT}{dz} \quad (48)$$

since $Q \to -1/(1+u)^2$. Compared to the static case here the additional factor $1/(1+u)^2$ occurs.

Eq. (48) shows that for moving media the Rosseland mean opacity has to be replaced by a of Eq. (38). There is no direct relationship between these two quantities. Note that sharp lines hardly contribute to the Rosseland mean (Karp et al. 1977) whereas in our approach their full strength is accounted for in a. For the special case of approximation of the spectral thickness by a single straight line over the whole spectrum, the dependence of the *total* radiative quantities on the line extinction is entirely described by the *mean comoving-frame* extinction coefficient a of Eq. (38). The more (roughly equally distributed) lines are considered the better is this approximation. This reflects the fact that the more lines are taken into account, the more their contribution effectly forms a part of the continuum.

7. Discussion and outlook

In order to deal with the problem of many spectral lines in differentially moving media, we first have achieved an analytical solution of the radiative transport equation for a moving slab in the two-stream approximation, utilizing the comoving frame, i.e. the frame relevant for radiation hydrodynamics. The solution is valid to first order in β and comprises the optically thick, as well as the thin case.

The main advantage of our approach is that it allows the replacement of the highly varying function $\chi(\xi)$ by the much smoother and monotonic *spectral thickness* $\psi(\xi)$ which – for a large number of spectral lines – can be approximated once for all by a piecewise linear function or by simple polynomials.

The most important advantage of the *algorithm* presented here is the huge speed-up in the CPU time needed for the calculation of radiative quantities that are relevant for radiation hydrodynamics: No longer a partial differential equation with a highly wavelength-varying extinction coefficient has to be solved. Instead – if we use the algorithm of Sect. 5.2 – essentially only a few exponential-integral functions (each taking only about 10 μs on a workstation) have to be evaluated. In the case referred to in Fig. 4 we find a speed-up of more than 10^5 with some loss of accuracy. Nevertheless the error is less than 1% only which is much less than we have to expect from the errors in the extinction coefficient.

In general, this method is particularly suited to calculate *functions* of the intensities as demonstrated above, but not so well for the numerical determination of a spectrum. For this purpose a direct integration of the transport equation e.g. by means of finite differences and up-wind discretisations is much faster.

We are aware that in the present paper a number of simplifying assumptions have been made to stress the essence of our approach. We are confident, however, that all these restrictions can be removed without major difficulty so that more realistic modelling can be performed.

First we assume plane-parallel geometry, but a generalization to *spherical symmetry* appears possible if the integration is performed along characteristics. Furthermore we have only dealt with the direction $\mu = +1$. The solution for the case $\mu = -1$ is obvious. Also the consideration of *more angles* does not appear to cause additional principal difficulties.

Although we have assumed $w = \partial\beta/\partial z$ and the extinction coefficient $\chi(\xi)$ to be independent of depth, this poses no serious restriction because the build-up of a large configuration in which the absorption coefficient *varies* with depth can (at least numerically) be performed by adding up shells (e.g. Peraiah 1984).

In addition we have assumed pure absorption. However, for the two-stream approximation the inclusion of (coherent) *scattering* is possible by the generalization according to Shaviv and Wehrse (1991). In the general case, for not too large a scattering fraction, the specific intensities can be determined from our formulae by means of additional fixed point ("Lambda") iterations. Evidently, our expressions are also important in non-LTE calculations since they facilitate the computation of the level occupation numbers from the rate equations. It seems that scattering media can also be treated if – by Eq. (2) – the transfer equation is cast into a form very similar to that of Eq. (23) of Efimov et al. (1995) so that the corresponding resolvent can be evaluated along the lines described there. This means that in particular scattering in lines and scattering by electrons can be dealt with by straightforward generalization.

Acknowledgements. We thank G. Shaviv and S. Starrfield for persistently stressing the need for a proper inclusion of spectral lines in radiation hydrodynamical calculations, and L. Campbell for critically reading the manuscript. This work was supported in part by the Deutsche Forschungsgemeinschaft (SFB 359/C2).

Appendix A: polylogarithmic function

The complex *polylogarithmic function* $\mathrm{Li}_s(z)$ (Wolfram Research 1992), also denoted by $F(z, s)$ (Erdélyi et al. 1953), is defined by

$$\mathrm{Li}_s(z) = F(z, s) = \sum_{k=1}^{\infty} \frac{z^k}{k^s} \qquad (|z| < 1)\,. \tag{A1}$$

For the analytical continuation beyond its circle of convergence $|z| = 1$ see Eqs. (1.11.17,18) of Erdélyi et al. (1953). In particular, we have

$$\mathrm{Li}_1(z) = -\ln(1 - z)\,, \quad \mathrm{Li}_2(z) = \int_z^0 \frac{\ln(1 - t)}{t} dt \tag{A2}$$

(Li_2: Euler's dilogarithm). For $s \geq 2$ the recursive relations

$$\frac{d\mathrm{Li}_s(z)}{dz} = \frac{\mathrm{Li}_{s-1}(z)}{z}\,, \quad \frac{d\mathrm{Li}_s(\mathrm{e}^z)}{dz} = \mathrm{Li}_{s-1}(\mathrm{e}^z) \tag{A3}$$

hold. The polylogarithmic function is related to the Lerch transcendental function

$$\Phi(z, s, a) = \sum_{k=0}^{\infty} \frac{z^k}{(a + k)^s} \tag{A4}$$

by

$$\mathrm{Li}_s(z) = z \cdot \Phi(z, s, 1)\,. \tag{A5}$$

For $z = 1$ $(s > 1)$ the polylogarithmic function reduces to the Riemann zeta function $\zeta(s)$, and the Lerch function to the generalized (Hurwitz) Riemann zeta function $\zeta(s, a)$ (cf. Erdélyi et al. 1953, Wolfram Research 1992).

In our application to the *wavelength-integrated Planck function* (Eq. 26), we need the polylogarithmic functions of order $s = 1 \ldots 4$ for real values $z \geq 0$. The particular combination of terms in Eq. (27) involves a rather intricate cancelling of the imaginary parts as the imaginary part of $\ln(1 - \exp(z))$ cancels against that of Li_4, and the imaginary part of Li_2 against that of Li_3.

Appendix B: rational approximation of relevant integrals

All rational approximations given here have been calculated by means of Mathematica (Wolfram Research 1992).

For the calculation of the *wavelength-integrated Planck function* we present a rational approximation for the integral in Eq. (27) with an accuracy better than 1 per cent, valid in the range $0 \leq x \leq 10$:

$$\tilde{B}(x) = \int \frac{x^3}{\mathrm{e}^x - 1} dx \;\simeq\; \frac{\sum_{j=0}^{4} \tilde{p}_j x^j}{1 + \sum_{j=1}^{4} \tilde{q}_j x^j} \tag{B1}$$

with the coefficients

$$\begin{aligned}
\tilde{p}_0 &= +6.500856126180727,\\
\tilde{p}_1 &= -2.8152204951105375,\\
\tilde{p}_2 &= +1.0810674894800913,\\
\tilde{p}_3 &= -0.18330264847775468,\\
\tilde{p}_4 &= +0.012434782492776181,\\
\tilde{q}_1 &= -0.4159028704354005,\\
\tilde{q}_2 &= +0.11463706018762805,\\
\tilde{q}_3 &= -0.016352597999118857,\\
\tilde{q}_4 &= +0.0010175489193538605.
\end{aligned} \tag{B2}$$

For the numerical evaluation of the *flux* according to Eqs. (25, 31) we write the *Planck function* in the form given in Eq. (23) and approximate the factor Ω in the range $-2 \leq x \leq +4$ by the rational function

$$\Omega(x) = \frac{\exp(5x)}{\exp\big(\exp(x)\big) - 1} \simeq \frac{\sum_{j=0}^{4} p_j x^j}{1 + \sum_{j=1}^{4} q_j x^j} \tag{B3}$$

with the coefficients

$$\begin{aligned}
p_0 &= +0.5935880013471108, \\
p_1 &= +0.27067008305245199, \\
p_2 &= -0.12158564798187317, \\
p_3 &= -0.03718804962457788, \\
p_4 &= +0.010374838682226144, \\
q_1 &= -3.746609397313664, \\
q_2 &= +6.038006013083701, \\
q_3 &= -4.236921176411602, \\
q_4 &= +1.0137965337376775.
\end{aligned} \tag{B4}$$

Then the *integrals* (36) and (37), needed for the calculation of the fluxes, are given by

$$\begin{aligned}
\Xi(x;r) &= \int \mathrm{e}^{rx}\,\Omega(x)dx \\
&\simeq +c_0 \exp(c_1 r)\,\mathrm{Ei}\big(r(x-c_1)\big) \\
&\quad -c_2 \exp(c_3 r)\,\mathrm{Ei}\big(r(x-c_3)\big) \\
&\quad +c_4^* \exp(c_5 r)\,\mathrm{Ei}\big(r(x-c_5)\big) \\
&\quad +c_4 \exp(c_5^* r)\,\mathrm{Ei}\big(r(x-c_5^*)\big) + c_6\,\frac{\exp(rx)}{r}\,,
\end{aligned} \tag{B5}$$

$$\begin{aligned}
\Theta(x;r,s) &= \int \mathrm{e}^{sx}\,\Xi(x,r)dx \\
&\simeq \frac{1}{s}\,[\,c_0 \exp(c_1 r + sx)\,\mathrm{Ei}\big(r(x-c_1)\big) \\
&\quad -c_2 \exp(c_3 r + sx)\,\mathrm{Ei}\big(r(x-c_3)\big) \\
&\quad +c_4^* \exp(c_5 r + sx)\,\mathrm{Ei}\big(r(x-c_5)\big) \\
&\quad +c_4 \exp(c_5^* r + sx)\,\mathrm{Ei}\big(r(x-c_5^*)\big) \\
&\quad -c_0 \exp(c_1(r+s))\,\mathrm{Ei}\big((r+s)(x-c_1)\big) \\
&\quad +c_2 \exp(c_3(r+s))\,\mathrm{Ei}\big((r+s)(x-c_3)\big) \\
&\quad -c_4^* \exp(c_5(r+s))\,\mathrm{Ei}\big((r+s)(x-c_5)\big) \\
&\quad -c_4 \exp(c_5^*(r+s))\,\mathrm{Ei}\big((r+s)(x-c_5^*)\big)] \\
&\quad +c_6\,\frac{\exp((r+s)x)}{r(r+s)}
\end{aligned} \tag{B6}$$

where asterisks denote the conjugate complex quantities. The coefficients are

$$\begin{aligned}
c_0 &= 0.247218\,, & c_1 &= 2.00775\,, \\
c_2 &= 1.3703\,, & c_3 &= 1.16175\,, \\
c_4 &= 0.564585 + 0.451446\,i\,, \\
c_5 &= 0.504881 + 0.409859\,i\,, \\
c_6 &= 0.010234\,.
\end{aligned} \tag{B7}$$

The complex exponential-integral function

$$\mathrm{Ei}(z) = -\int_{-z}^{\infty} \frac{\mathrm{e}^{-t}}{t}\,dt \tag{B8}$$

can be evaluated e.g. by the series expansion

$$\mathrm{Ei}(z) = \gamma + \ln z + \sum_{n=1}^{\infty} \frac{z^n}{n n!} \tag{B9}$$

where $\gamma = 0.57721\ldots$ is Euler's constant (Abramowitz & Stegun 1972).

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