# On the role of resonances in nonradial pulsators

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**Abstract.** Resonances or near resonances are ubiquitous among the excited nonradial pulsation modes of variable stars and they must play an important role in determining their pulsational behavior. Here in a first step at *nonlinear asteroseismology* we explore some of the consequences of resonances by means of the amplitude equation formalism.

We show how parity and angular momentum constraints can be used to eliminate many of the possible nonlinear resonant couplings between modes (and multiplets of modes), and how the amplitude equations can thus be simplified. Even when we may not be able, nor wish, to make an *ab initio* computation of the values of the coupling coefficients, it is still possible to obtain *constraints* on the nature of the excited modes if a resonance between observed frequencies can be identified.

Resonances can cause nonlinear frequency locking of modes. This means that the observed frequencies appear in *exact resonance* even though the linear frequencies are only approximately in resonance. The nonlinear frequency lock, when it occurs, it does so over a range of departures from *linear* resonance, and it is accompanied by constant pulsation amplitudes. The locked, nonlinear frequencies can differ noticeably from their nonresonant counterparts which are usually used in seismology. This is particularly true for multiplets of modes split by rotation. Beyond the regime of the frequency lock, *amplitude and frequency modulations* can appear in the pulsations. Far from the resonance condition one recovers the regime of steady pulsations with *nonresonant* frequencies for which the seismological studies, as they are presently carried out, are justified (provided furthermore, of course, that nonlinear frequency shifts are negligible).

Success in identifying a resonance in an observed power spectrum depends on the quality of the data. While keeping this limitation in mind, we discuss the possible existence of peculiar resonances in the pulsations of specific variable white dwarfs and $\delta$ Scuti stars.

## 1. Introduction

Resonances between pulsation frequencies are known to play an important role in the dynamics of classical variable stars such as, for example, Cepheids. (For reviews, see Buchler 1990, 1993.) They have a profound effect on the shape of both light and radial velocity curves.

For nonradial pulsators that exhibit a much denser frequency spectrum—such as variable white dwarfs, $\delta$ Scuti stars, or $\beta$ Cepheids—resonances are likely to occur as well. They can certainly play a role in modal selection and resonant coupling, as has indeed been advocated as the mechanism for saturating the amplitudes of $\delta$ Scuti stars (Dziembowski 1993). But resonances do have other effects that can affect the resulting Fourier spectra. Before proceeding, however, we note that resonances do not exist between all possible modes in a given star. Therefore identifying resonances among observed modes may lead to constraints on the identification of these modes.

Resonances couple (nonlinearly) pulsating modes and therefore play an important role in determining the amplitudes of pulsation. The amplitudes of resonating modes are usually smaller than would be the case were the modes nonresonantly coupled, for instance. In the particular case of a multiplet of modes split by rotation, the resonance between the modes of the multiplet can be responsible for a strong asymmetric pattern in amplitude within a given multiplet (Buchler, Goupil & Serre 1995, hereafter BGS), as is observed in the case of some variable white dwarfs.

Resonances can also strongly affect mode frequencies and the spacings between those frequencies. One way this may happen is by "frequency locking" even when linear eigenfrequencies are not in exact resonance but rather in near-resonance; that is, when the exact resonant relation $\sum_j k_j \omega_j = 0$ ($k_j$ positive or negative integers) between frequencies $\omega_j$ is not exactly satisfied yet nonlinear effects (resonant coupling) can

modify the linear frequencies and cause the *observed* frequencies to be locked into exact resonance. As a consequence, a near-resonance, through frequency locking, causes the observed modes to be equally spaced in frequency. In the case of gravity modes (as seen in white dwarfs) this can result in a departure from the equal period spacing expected from a first order asymptotic analysis of those modes (see, e.g., Sect. 10.3.3 of Hansen and Kawaler 1994). On the other hand, frequency locking within a rotationally split multiplet of a rapidly rotating star could yield equally spaced frequency splitting, which is to be contrasted to the prediction of linear theory where strong departures from equal splitting are expected.

Frequency locking is maintained as long as the modes are linearly in near-resonance. This raises the question of how close to exact resonance must the linear frequencies be for frequency locking to take place.

It is the purpose of the this paper to address the above issues in more detail.

In order to investigate resonant mode coupling one obviously has to go beyond linear effects. Unfortunately, hydrodynamical computations are not yet possible for nonradial pulsators. The only theoretical framework presently available is the Amplitude Equation (AE) formalism, which has been described in a number of reviews (Dziembowski 1993; Buchler 1988, 1990, 1993) and papers (Dziembowski 1982; Buchler & Goupil 1984, hereafter BG84; Goupil & Buchler 1994, hereafter GB94; van Hoolst 1994). For most well-observed nonradial pulsators such as white dwarfs, $\delta$ Scuti stars and $\beta$ Cephei stars, this theory applies rather well because the theoretically derived growth rates of the excited modes are very small compared to their frequencies. Furthermore, the pulsations are generally *mildly nonlinear* in the sense that the nonlinearities are small in the pulsations about the stellar equilibrium configuration. For these two reasons the linear and nonlinear periods are very close, which by the way is also one of the basic assumptions that underlie the current efforts in *linear asteroseismology*.

For details of the derivation of amplitude equations in this context we refer to GB94 who also give explicit expressions for the coefficients of the AEs in an Eulerian formulation. (Van Hoolst 1994 has recently given their Lagrangean equivalent.) Here we intentionally limit the details of the theory but do introduce the notation necessary for further discussion.

The coefficients of the AEs can be computed ab initio, but this presents a formidable coding effort which has only been attempted in a very limited way (Klapp et al. 1985, Dziembowski 1982).

It is also possible to extract the values of the coupling coefficients indirectly from numerical hydrodynamic computations of the nonlinear pulsations. This has been done for RR Lyrae models in Buchler & Kovács (1986a, hereafter BK86a) and for Cepheid models in Kovács & Buchler (1989). It is unfortunate that at the present time numerical hydrodynamics calculations are not yet feasible for nonradial pulsations in this regard.

Thus in this paper we take a phenomenological approach and undertake a general exploration of the resonant couplings that may occur between nonradial modes.

## 1.1. Handling of resonances with amplitude equations

To first order (in amplitude), the velocity pulsations, for example, can be represented by

$$u(\vec{r}, t) = \tfrac{1}{2} \sum_\alpha a_\alpha(t) e^{i\omega_\alpha t} U_\alpha(\vec{r}) \tag{1}$$

where $a_\alpha$ and $U_\alpha(\vec{r})$ represent, respectively, the complex amplitude and the spatial velocity eigenfunction for the vibrational mode $\alpha$. A similar expression is obtained for the luminosity variations, etc. Higher order corrections (BG84, GB94) are quadratic, cubic, etc., in the amplitudes $\{a_\alpha(t)\}$ and thus represent modal interaction terms. (For completeness we give a brief exposition of higher order terms in Sect. 5). The sum is over all the *dominant* modes $\alpha$ (q.v. GB84). We can equivalently count a given vibrational mode and its complex conjugate as separate modes or add a complex conjugate to expressions such as Eq. 1.

If $q$ modes interact to determine the dynamics, then the corresponding AEs have the general form

$$
\begin{aligned}
\tfrac{da_1}{dt} &= (i\omega_1 + \kappa_1)a_1 + f_1(a_1, a_2, \cdots, a_q) \\
\tfrac{da_2}{dt} &= (i\omega_2 + \kappa_2)a_2 + f_2(a_1, a_2, \cdots, a_p) \\
&\vdots \\
\tfrac{da_q}{dt} &= (i\omega_q + \kappa_q)a_q + f_q(a_1, a_2, \cdots, a_q)
\end{aligned}
\tag{2}
$$

where $\omega_\alpha$ and $\kappa_\alpha$ are, respectively, linear eigenfrequencies and growth rates and the functions $f_\alpha$ are strictly nonlinear and depend on which particular resonances are present. The AEs thus form a set of coupled first-order complex nonlinear ordinary differential equations, and they govern the temporal behavior of the amplitudes and phases. In view of the weak nonlinearity of the stellar pulsations we need only consider the lowest relevant powers of amplitudes in the functions $f_\alpha$. In the AEs the $2\ell + 1$ components of multiplets (and their complex conjugates) are counted as individual modes. Here $\ell$ is the angular momentum quantum number of the spherical harmonic $Y_{\ell m}(\theta, \phi)$ and $2\ell + 1$ corresponds to the number of possible values of $m$.

For convenience, one usually converts to real amplitude and phase by setting $a_\alpha = A_\alpha \exp(i\phi_\alpha)$. The solutions of the AEs come in various forms. On one hand, the (real) amplitudes $A_\alpha$ can be constant in time, as so-called *fixed point* (FP) solutions. For example, when the AEs have a FP ($A_\alpha$ constant and $\dot\phi_\alpha$ constant) then

$$u(\vec{r}, t) = \tfrac{1}{2} \sum_\alpha A_\alpha e^{i\tilde\omega_\alpha t} U_\alpha(\vec{r}) \tag{3}$$

where $\tilde\omega_a$ represents the linear frequency corrected for nonlinear effects. The pulsation is described to first order by a multi-periodic Fourier expansion with $q$ dominant peaks in the Fourier spectrum to which should be added "combination frequencies" from higher order corrections.

On the other hand, the amplitudes, $A_\alpha$, and phases, $\phi_\alpha$, may vary on a long time scale. These modulations can further be of several types; namely, periodic, multi-periodic, or irregular (chaotic). The phases can, of course, also undergo corresponding modulations. The periodic or multi-periodic modulations

then give rise to peak splittings in the Fourier spectrum when the observations span a long enough time interval.

Many mechanisms can cause amplitude variations. It is well known, for example, that the fixed points of 2:1 resonant AEs can just be unstable and undergo a Hopf bifurcation leading to a periodic variation of the amplitudes and phase. Further away from the Hopf bifurcation these modulations can also become irregular (see, e.g., Moskalik 1986; BK86a).

Note, for completeness, that on top of this temporal behavior one could also have evolutionary changes generally arising from, for example, the stellar core's nuclear evolution (as in $\delta$ Scuti stars) or from thermal leakage through the star (as in white dwarfs). Such secular changes, however, are extremely slow and we ignore them here.

### 1.2. Applicability

Even though the AEs are easy to write down in general form, we caution the reader that a complete description of the pulsations in terms of these equations is useful from a practical standpoint only when the number of modes is small, or when it is possible to consider certain groups of modes in isolation from others. Thus, for example, BGS have explored the case of a nonradial $\ell = 1$ mode which is rotationally split into three components by making the assumption that such multiplets can be studied independently of each other. An extension to $\ell = 2$ or higher is still straightforward in principle but the number $(2\ell + 1)$ of AEs increases rapidly with $\ell$ as do the number of resonant coupling terms.

Besides the often large number of excited modes the most important puzzle of nonradial stellar pulsations is the frequently observed temporal variation of the amplitudes of the excited modes. The observations are usually not complete enough however to decide whether such variations are periodic, multi-periodic, or chaotic, and whether they have a sizeable stochastic component, or even are stochastically driven. Resonances among excited modes can easily cause amplitude modulations. In contrast, it turns out to be rather difficult to obtain amplitude modulations with nonresonant AEs. For example, Buchler & Kovács (1986b, hereafter BK86b) have shown that in the case of only two nonresonantly interacting modes the amplitudes are necessarily constant. For a larger number of interacting modes, modulations do become possible, but stable FPs (with fewer modes excited) generally coexist with time-dependent solutions, and from evolutionary considerations it appears much more likely to find the system in one of these FPs. (It would take an remarkably strong external kick to make the star jump into such a pulsational state.)

On general grounds we expect the resonant coupling to be simpler in white dwarfs than in pulsating Main Sequence stars ($\delta$ Scuti and $\beta$ Cephei stars). Indeed, for white dwarfs the pulsation frequencies of the $g$–modes (gravity modes) are very well separated from the $p$–modes (pressure modes) by factors of 100. Therefore no simple resonances between $g$- and $p$–modes occur. On the other hand, for the pulsating MS stars the $p$ and $g$ spectra have a considerable overlap and many modes have a

mixed character. The situation here is made even more complicated by the trapping of $g$–modes in the interior. Such $g$–modes may thus be active in a resonance, yet their frequencies do not appear in the spectrum of the light curve (e.g., Dziembowski & Krolikowska 1985).

Confusing statements have appeared in the literature as to the possibility of chaos in stellar pulsators – such as in $\delta$ Scuti stars, for example – and we wish to address this point briefly. One needs to distinguish between two types of chaos. Rapid, erratic amplitude and phase variations, such as those found in R Scuti (e.g., Buchler, Serre, Kollath & Mattei 1995, and Buchler et al. 1996), which one might term "fast chaos", require that the growth rates of the excited modes be of the same order as the frequencies themselves, and thus this regime is clearly beyond the range of validity of AEs. On the other hand, for the majority of nonradial pulsators ($\delta$ Scuti, $\beta$ Cephei, white dwarfs, etc.) the growth rates are relatively very small. Chaos, if it occurs in these stars, must therefore be in the amplitude and phase *modulations* as a sort of "slow chaos." A study of such slow chaos is possible within the amplitude equation framework (e.g., Wersinger et al. 1980; Buchler & Goupil 1988) because $\kappa/\omega \ll 1$ is both the condition for slow chaos and for the validity of the AE formalism.

In Sect. 2 we examine the general phenomenon of frequency locking and its consequences. Interesting, but technical consequences of frequency locking and its disappearance away from the resonance region are given in Sect. 3. In Sect. 4 we consider resonances where angular frequencies are related by near-rational ratios of the form $m\omega_a \approx n\omega_b$ or $m\omega_a + n\omega_b + p\omega_c \approx 0$, where $m$, $n$ and $p$ are (positive or negative) integers. In particular, we shall set down selection rules that can be used to eliminate various combinations of $\ell$ associated with the resonating modes. Sect. 5 is technical and is devoted to proving some useful properties of the coupling coefficients. In Sect. 6, the possible existence of resonances in nonradial pulsators is emphasized with some examples from the literature of variable white dwarfs and $\delta$ Scuti stars. With these examples, we then illustrate what kinds of constraints could be obtained were a resonance to be definitely identified in an observed spectrum. We conclude with a summary discussion in Sect. 7.

The reader who is not interested in technical details is urged to skip Sect. 3 and Sect. 5.

## 2. Frequency locking – a summary

A very general result associated with resonances, discussed briefly earlier, is the nonlinear phenomenon of *frequency locking*, sometimes also called *phase locking*. Frequency-lock between one two or more resonant modes can occur no matter how many additional modes or multiplets may otherwise be interacting.

The reason for this is that for every resonance of the form

$$\sum_i k_i \omega_i = 0$$

(where the $k_i$ are positive or negative integers) we can introduce a phase variable of the form $\Phi_\alpha = \sum k_i \phi_i = 0$, and an AE of the form $d\Phi_\alpha/dt = \text{RHS}_\alpha$, where the $\text{RHS}_\alpha$ are expressible in terms of only the resonant phases $\Phi_\alpha$ (and of course the amplitudes $A$). For the FPs then $d\Phi_\alpha/dt = 0$ or to the frequency lock

$$\sum k_i \tilde{\omega}_i = 0.$$

Each resonance can thus give rise to a separate frequency lock.

### 2.1. Two-mode resonances

We first illustrate frequency locking in the well known case of a 2:1 resonance, $2\omega_a \approx \omega_b$. Here we consider the situation where both modes are linearly unstable, and thus when far from the resonance center neither of them becomes a slave mode. (The complementary case of a linearly unstable mode $a$ and a linearly *stable* mode $b$ was investigated by Klapp et al. 1985, BK86b. In that situation the amplitude of the linearly stable mode gradually vanishes, more precisely this mode becomes just one of the slave modes).

With the introduction of the real amplitudes and phases, the two complex AEs can be rewritten as four coupled AEs in $A$, $B$, $\phi_a$ and $\phi_b$. However, because only the phase combination $\Phi = \phi_b - 2\phi_a$ appears on the right hand sides, these four AEs can further be reduced to the three AEs (cf. Eqs. 18 in Sect. 3)

$$
\begin{aligned}
dA/dt &= \kappa_a + F_a(A, B, \Phi) \\
dB/dt &= \kappa_b + F_b(A, B, \Phi) \\
d\Phi/dt &= \delta\omega + F_\Phi(A, B, \Phi).
\end{aligned}
\tag{4}
$$

The functions $F_a$, $F_b$ and $F_\Phi$ are strictly nonlinear, but they are periodic in $\Phi$ through a $\cos\Phi$ and $\sin\Phi$ dependence. The frequency mismatch or "detuning," which for this resonance is $\delta\omega = \omega_b - 2\omega_a$, appears then in the *slow* phase.

When the amplitudes $A$ and $B$ are constant in time, i.e. for FPs ($dA/dt = dB/dt = 0$), the *nonlinear* frequencies are also constant with

$$
\begin{aligned}
\tilde{\omega}_a &= \omega + d\phi_a/dt \\
\tilde{\omega}_b &= 2\omega + d\phi_b/dt.
\end{aligned}
\tag{5}
$$

Furthermore, $d\Phi/dt = d\phi_b/dt - 2d\phi_a/dt = 0$ and thus $\Phi$ must also be constant. It follows that the *observable* frequencies obey

$$2\tilde{\omega}_a = \tilde{\omega}_b. \tag{6}$$

In other words, for stable FP solutions (assuming that they exist) *the effect of the resonance is then to cause a nonlinear synchronization of the two modes.* A further consequence is that the pulsation exhibits a *single period* of $2\pi/\tilde{\omega}_a$ (e.g., BK86a) so that Eq. (3) reduces to

$$u(\vec{r}, t) = A e^{i\tilde{\omega}_a t} U_a(\vec{r}) + B e^{i\Phi} e^{i2\tilde{\omega}_a t} U_b(\vec{r}) + cc. \tag{7}$$

We stress that *this synchronization occurs because of the nonlinear mode coupling, and despite the fact that the linear resonance condition is only approximately satisfied.*

Generalizing, one can now easily verify that a resonance of the type $n\omega_a \approx m\omega_b$ always introduces a phase

$$\Phi = n\phi_a - m\phi_b \tag{8}$$

and, for FP point solutions (i.e., with $A$ and $B$ constant and nonzero), the resonance leads to a frequency lock of the form

$$n\tilde{\omega}_a = m\tilde{\omega}_b. \tag{9}$$

### 2.2. Three-mode resonances

Three-mode resonances are slightly different. A resonance of the type $m\omega_a \approx n\omega_b + p\omega_c$, for example, introduces the phase

$$\Phi = m\phi_a - n\phi_b - p\phi_c \tag{10}$$

and, when the three amplitudes are constant and nonzero, a frequency lock of the form

$$m\tilde{\omega}_a = n\tilde{\omega}_b + p\tilde{\omega}_c \tag{11}$$

ensues. The general FP solution, instead of involving three independent frequencies, can then be expressed in terms of a two-periodic Fourier expansion with the two basic frequencies $\tilde{\omega}_b$ and $\tilde{\omega}_c$. In lowest order

$$
\begin{aligned}
u(\vec{r}, t) = {} & A e^{i\Phi} e^{i(\tilde{\omega}_b + \tilde{\omega}_c)t} U_a(\vec{r}) + B e^{i\tilde{\omega}_b t} U_b(\vec{r}) \\
& + C e^{i\tilde{\omega}_c t} U_c(\vec{r}) + cc.
\end{aligned}
\tag{12}
$$

Frequency locking can assume yet another form as, for example, in the 1:1:1 resonance within a rotationally-split multiplet considered by BGS. Here, for $\ell = 1$, it is the phase combination

$$\Phi = \phi_+ + \phi_- - 2\phi_0 \tag{13}$$

that appears in the AEs. The subscripts refer to the three $m$ values ($-1$, $0$, $+1$) of the modes. (We note that this case is mathematically equivalent to a resonance of the type $2\omega_0 \approx \omega_+ + \omega_-$.) The nonlinear frequency lock, which again occurs when the amplitudes are constant, therefore leads to the condition

$$\tilde{\omega}_0 = \tfrac{1}{2}(\tilde{\omega}_+ + \tilde{\omega}_-). \tag{14}$$

Or, in other words, the intra-multiplet resonance results in a *symmetric splitting of the multiplet, even when the linear splitting is not symmetric.* Note that there is also a nonlinear shift in the position of the central frequency peak.

For an $\ell = 2$ mode, an additional phase $\Psi = \phi_{+2} + \phi_{-2} - 2\phi_0$ appears. For FPs, with all of $A_0$, $A_{2-}$ and $A_{2+}$ constant and nonzero, this leads to

$$\tilde{\omega}_0 = \tfrac{1}{2}(\tilde{\omega}_{+2} + \tilde{\omega}_{-2}). \tag{15}$$

These two peaks are therefore also organized symmetrically about the center.

Thus, quite generally, whatever the resonance may be, *one always obtains a frequency lock in the vicinity of a resonance provided a stable resonant FP solution exists.*

We end this section with a technical note. The previous very general result follows directly from the existence of a center manifold (see, e.g., Guckenheimer & Holmes 1983; Coullet & Spiegel 1984; BG84). Whenever there are one or more resonances among the modes $\{i\}$ of the center manifold— that is, relations among the linear frequencies of the form $\sum k_i \omega_i \approx 0$, with $k_i$ positive or negative integers—then the resonant terms in the AEs only involve the specific phase combinations $\Phi = \sum k_i \phi_i$. The various phase AEs can then always be combined into an AE for $d\Phi/dt$ which depends on the amplitudes $A_i$ and $\Phi$ only. For FPs, when the amplitudes $A_i$ are constant, a frequency lock of the form $\sum k_i \tilde{\omega}_i = 0$ necessarily results.

## 2.3. Condition for frequency locked pulsations

The above results raise a theoretical question, however. The resonant AEs apply not merely near the resonance center, but also far from resonance. In contrast, the nonresonant AES hold only far from the resonance center. How then are the solutions of the resonant AEs related to those of the nonresonant AEs in that limit?

Phrased in terms of observable frequencies, in the limit of infinitely large departure from exact resonance for the linear eigen-frequencies, one indeed expects to recover a nonresonant coupling behavior in which the frequencies are no longer locked. What happens to the frequency lock as the linear eigenfrequencies move away from the resonance and, more specifically, in what manner, and how far from the resonance center does frequency locking break down?

The question is also of practical interest. First, in view of the possible constraints it can bring up (see next section), one should wonder whether, in stellar conditions, frequency locks can ever occur, and if so, over what range of parameters. A second point concerns the special case of a multiplet of modes split by rotation. With the rotation considered a perturbation, the linear eigenfrequencies of a $\ell = 1$ triplet of modes, for example, are obtained as usual and schematically as:

$$\begin{aligned} \omega_m &= \omega_0 + \Delta\omega_m \\ &= \omega_0(1 + m\frac{\Omega}{\omega_0}\Xi_1 + m^2(\frac{\Omega}{\omega_0})^2\Xi_2 + \ldots) \end{aligned} \tag{16}$$

with $\Omega$ the rotational frequency and $\Xi_1$, $\Xi_2$ factors of order unity depending on the stellar structure and on the nature of the modes. It is then clear that the eigenfrequencies obey the near-resonant relation $2\omega_0 = \omega_+ + \omega_- + \delta\omega$, with a resonance frequency mismatch $\delta\omega/\omega_0 \sim O(\Omega/\omega_0)^2$. Relations (16) are commonly used to infer rotation rates from the observed splittings. *This can be dangerous because nonlinear effects can vitiate this kind of approach.* Indeed, when frequency locking occurs and nonlinear effects are large enough they can cause equal splitting far beyond the small $\Omega$ regime. However, as we will show here, for large enough $\Omega$, the frequency lock is eventually broken. In this regime one expects modulated amplitude and frequency pulsations, and as one moves very far from the resonance condition the amplitude modulations become increasingly shallow and

negligible. Here one recovers the regime of pulsations with non-resonant, nonlocked frequencies, i.e. with those predicted by the nonresonant AEs which differ from the linear frequencies only by nonlinear corrections, i.e. terms of order of the amplitudes squared. *Only in this regime are the current seismological studies justified as they compare observed frequencies with nonresonant ones.* (This statement of course further supposes that the mentioned nonlinear frequency shifts are small).

We now turn to a more detailed discussion of the solutions of the AEs as one moves away from the resonance condition. For the situation to be of interest the parameters must be such that (1) a stable multimode resonant FP exists at the resonance center, and (2) the nonresonant AEs (valid 'infinitely' far from resonance) also have a stable multimode FP (with the same modes). This implies in particular that we assume that the relevant modes are linearly unstable (Otherwise, far from the resonance, the resonant linearly *stable* modes join the multitude of slave modes, i.e. their amplitudes become negligibly small, as for example in the bump Cepheids, cf. BK86a).

A first important result is that, quite generally, a resonant FP solution with frequency locking can exist in the vicinity of a resonance (here we summarize the rather technical Sect. 3); more specifically, for

$$|\delta\omega|/\omega \lesssim O(\kappa/\omega) \tag{17}$$

where $\kappa$ is the growth rate of the mode. This is a rough order-of-magnitude estimate with factors of 10 possible on either side. As is obvious, the nature of the solutions of the AEs depend on the numerical values of the linear and nonlinear coefficients. Note that rel. (17) holds also for splittings, and whether the cause of the splitting is rotational or magnetic.

In turn, does *the absence of frequency locking imply that the amplitudes are time-dependent?* The AEs are too complicated to study analytically, but numerical solutions provide some answers as to the nature of the solutions (see Sect. 3). One can distinguish roughly three regimes when one moves away from the resonance center i.e when $\delta\omega/\kappa$ is increased:

(1) It often happens that near the resonance center ($\delta\omega/\kappa$ small) one has a stable FP for which the quoted frequency lock therefore exists. There, the frequency may substantially differ from their nonresonant frequency counterparts which are used in seismological studies.

(2) As $\delta\omega/\kappa$ is increased this FP eventually becomes unstable, or it disappears. In either case, in this second regime one witnesses a bifurcation to another solution which we find to be of an oscillatory nature in the numerical examples considered, thus giving rise to amplitude modulated pulsations. However, as $\delta\omega/\kappa$ gets large the amplitude of this modulation decreases and eventually evanesces as $1/\delta\omega$. Concomitantly the period of the modulation decreases as $2\pi/\delta\omega$. The satellite-peaks (due to the amplitude modulation) thus gradually move and their amplitude vanishes.

(3) Finally, one thus approaches the third regime, viz. a 'clean' steady spectrum in which the frequencies are given by their (unlocked) nonresonant values. For further details we refer the

reader to Sect. 3, which also shows a few typical Fourier spectra that one may expect.

To summarize, near a resonance one has a frequency locked spectrum, whereas further away from the resonance one obtains peaks at the $\{\tilde{\omega}_i\}$ (which are no longer locked). These peaks may be sharp, or they may be broadenend because of amplitude and frequency modulation.

As an example, consider the effects of rotation. The $\delta$ Scuti stars, for instance, typically have large rotational velocities $v \sim$ 150–200 km/s and stellar radii about twice the solar radius. This leads to a large and asymmetric rotational splitting. This asymmetry thus gives values of the off-resonance parameter $\delta\omega/\omega_0 \sim 10^{-3}-10^{-2}$ in the range of observed frequencies. The growth rates for these stars are small, $\kappa/\omega \sim 10^{-7}$ up to $10^{-4}$ for the most unstable modes in the observed frequency range (Dziembowski 1993). This gives a range of $\delta\omega/\kappa$ of $10 - 10^5$ for $\delta$ Scuti stars.

On the other hand, variable white dwarfs are believed to be slow rotators and $\delta\omega/\omega_0 \sim 10^{-8} - 10^{-6}$. Theoretical computations of growth rates for DAV stars yield $\kappa/\omega \sim 10^{-10} - 10^{-5}$ (Dolez & Vauclair 1981; Lee & Bradley 1993; Bradley & Winget 1994). Here the range of $\delta\omega/\kappa$ is $10^{-3} - 10^{+4}$.

Give or take an order of magnitude, the condition for a frequency lock (Eq. 17) is satisfied and it can occur for the most unstable modes and for modes that predominantly sample the regions of slower rotation. Since the approximate condition for frequency locking depends on linear growth rates which can vary strongly from one part of the power spectrum to another, frequency locking may well occur only for different subsets of modes in a particular star.

Although growth rates remain very uncertain, the above orders of magnitude indicate that one can expect frequency locking in variable white dwarfs and slowly rotating $\delta$ Scuti stars, whereas it is less likely to occur in rapidly rotating $\delta$ Scuti stars.

## 3. The extent of frequency locking – theoretical considerations

The main results of this section were already summarized in Sect. 2. A reader not interested in theoretical fine-points of frequency locking may therefore skip this section without prejudice. Here we examine what happens to the frequency locked solutions of the AEs as one moves away from the resonance center, and we show that these fixed points must eventually disappear, giving rise to amplitude modulations.

### 3.1. 2:1 resonance

We first consider the case of the 2:1 resonance, to be specific. The appropriate resonant AEs are given in Eq. (18)

$$\frac{dA}{dt} = \kappa_a A - \tilde{q}_{aa}^R A^3 - \tilde{q}_{ab}^R AB^2 + R_a AB \cos(\Phi - \delta_a) \quad (18a)$$

$$\frac{dB}{dt} = \kappa_b B - \tilde{q}_{ab}^R A^2 B - \tilde{q}_{bb}^R B^3 + R_b A^2 \cos(\Phi + \delta_b) \quad (18b)$$

$$\frac{d\Phi}{dt} = \delta\omega - (\tilde{q}_{ab}^I A^2 - \tilde{q}_{bb}^I B^2) + 2(\tilde{q}_{aa}^I A^2 + \tilde{q}_{ab}^I B^2)$$

$$+ R_b \sin(\Phi + \delta_b)\frac{A^2}{B} - 2R_a \sin(\Phi - \delta_a)A \quad (18c)$$

where the superscripts R and I refer to the real and imaginary parts, respectively, and where $r_k = R_k \exp(i\delta_k)$, for k=a,b.

Consider the third equation (18c). Since the last two terms are bounded ( $|\sin| \leq 1$, $A$ and $B$ finite by virtue of Eqs. (18a,b) $A \approx O(\sqrt{\kappa/q}) \approx O(\kappa/R)$), one notes that as $\delta\omega$ gets large there can no longer be a DM FP solution after some critical value $\delta\omega_{bif}$. For the resonant terms to play a role the quadratic and cubic terms must contribute to the same order, hence we have $\kappa \approx qA^2 \approx RA$. It is therefore easy to see that one has to have $\delta\omega_{bif}/\omega = O(\kappa/\omega)$. This is of course an important practical result. However, we need to stress that this is an order of magnitude result since we have assumed here that the real and imaginary parts of the $\tilde{q}$ and $r$ are of the same order.

We wish to add a technical comment here. Prima facie, the conclusions of the previous paragraph appear at odds with those of Buchler & Kovács (1986a) who, in the context of the bump Cepheid resonance, found the DM FP solution to exist for all $\delta\omega \to -\infty$, with $B \to 0$. The reason is that different assumptions were made. In Buchler & Kovács it was assumed that $\kappa_b < 0$ so that in the NR limit only a SM FP exists. Far from the resonance mode $b$ becomes just another slave mode. Here both $\kappa > 0$, and both amplitudes stay of the same order as $\delta\omega$ is increased.

When we are far from resonance then the nonresonant (NR) AEs should apply. For the case of two modes they are given by

$$\frac{dA}{dt} = \kappa_a A - (q_{aa}^R A^2 + q_{ab}^R B^2)A \quad (19a)$$

$$\frac{dB}{dt} = \kappa_b B - (q_{ab}^R A^2 + q_{bb}^R B^2)B \quad (19b)$$

Strictly speaking the standard cubic coefficients $\tilde{q}_{jk}$ in Eqs. (18) and $q_{jk}$ in (19) differ by terms which are $O(1/\delta\omega)$ (BK86a, Eqs. 12), and thus vanish in the limit of large $\delta\omega$. The question again arises as to how the solutions of the resonant and nonresonant AEs compare in the limit of large $\delta\omega$).

The resonant AEs cannot be solved analytically. For the purpose of illustration we consider a numerical example. We stress that the results we obtain depend on the values of the coefficients, both quantitatively, but also qualitatively. Thus for example the nature of the bifurcations can be affected.

In order to put ourselves into the situation where a double-mode FP exists in the nonresonant limit ( i.e. $A$ and $B$ different from zero) we have to assume that $both$ $\kappa_a$ and $\kappa_b$ are positive, and that the discriminant $D = q_{aa}^R q_{bb}^R - q_{ab}^R q_{ba}^R > 0$ (BK 86b). (The case where the $a$ mode is linearly stable and the $b$ mode is unstable has been considered in BK86a, and the reverse case in Moskalik & Buchler 1990).

In this numerical example we choose the parameters values ($\kappa_a = 1.0$, $\kappa_b = 2.0$, $\tilde{q}_{aa}^R = 1.0$, $\tilde{q}_{ab}^R = 0.5$, $\tilde{q}_{ba}^R = 0.5$, $\tilde{q}_{bb}^R = 4.0$, $R_a = 15.0$, $R_b = 3.0$, $\delta_a = \delta_b = 0$, and all $\tilde{q}_{ij}^I = 0$. Note that with this choice of $\delta$ phases all $\tilde{q}_{jk}^R = q_{jk}^R$). For convenience and without loss of generality we have effectively rescaled the time in the AES by setting $\kappa_a = 1$.
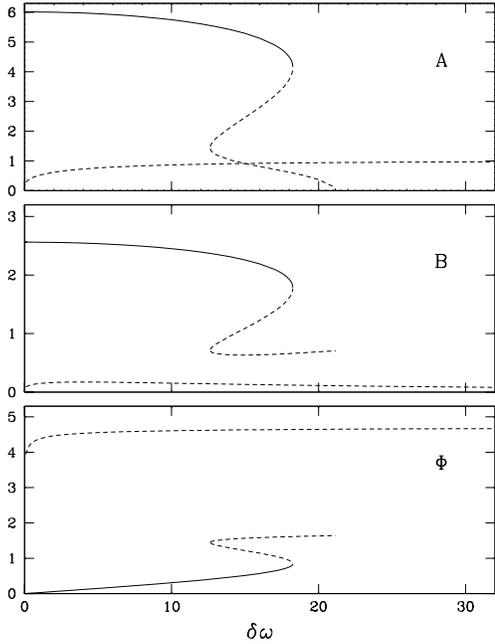
**Fig. 1.** 2:1 Resonance: Amplitudes and phase as a function of the resonance detuning parameter $\delta\omega$ in units of $\kappa_a$; *solid line*: stable and *dashed*: unstable frequency locked solution; *dotted line*: nonresonant FP.

The FPs of the NR AEs and their stability can easily be found ( cf. Buchler& Kovács 1986). There are 2 unstable SM FPs with respective amplitudes, $A = 1.0, B = 0.0$ and $B = 0.7071, A = 0.0$. Further they have a stable DM FP with amplitudes $A = 0.8944$ and $B = 0.6325$.

In Fig. 1 we present a bifurcation diagram for the FPs of the resonant AEs Eqs. 18. Because of our choice of parameters the bifurcation diagram is symmetric around $\delta\omega = 0$. From the top down we show the amplitudes $A$ and $B$ and the phase $\Phi$. The solid lines denote the stable FPs, and the broken lines correspond to other *unstable* FPs. One observes that a stable FP point exists *only* for a range $|\delta\omega| < \delta\omega_{bif} \approx 18.24$. In the domain $12.64 < \delta\omega < 18.24$ four FPs are seen to coexist, but 3 of them are unstable. One of the unstable branches (long dashes) exists for all $\delta\omega$. Far from the resonance this unstable solution approaches the unstable single-mode NR FP $A = 1, B = 0$. For $\delta\omega \approx 21.16$ the $A$ amplitude of one of the unstable FPs (short dashes) vanishes, and interestingly at this point the system bifurcates to the unstable NR FP $A = 0, B = 0.707$.

In the region $\delta\omega > \delta\omega_{bif}$ where no stable FPs exist, a numerical integration of the AEs leads to an oscillatory solution (limit cycle). In the star this corresponds then to periodically modulated pulsations.

A sample for several values of $\delta\omega$ is shown in Fig. 2. In these examples the $A$ amplitude is greater than $B$. The oscillations are seen to occur about the double-mode FP of the NR AEs, viz. $A = 0.8944, B = 0.6325$ (shown as thin horizontal lines). For large $\delta\omega$ the the amplitude modulations become sinusoidal and their amplitudes are evanescent, scaling with $1/\delta\omega$. The modu-

lation frequency, on the other hand, becomes very large and approaches $\delta\omega$. As $\delta\omega$ is decreased the amplitude modulations first reach a plateau at $\delta\omega \approx 15.5$, and then in a very short interval their period gets large and they become very distorted. We have not found any oscillatory solutions below $\delta\omega \approx 15.495$. (However, they can be hard to find and a more thorough search might uncover them). Note that in the region $15.495 \lesssim \delta\omega \lesssim 18.25$ the oscillatory solution coexists with the stable phase-locked FP.

This coexistence of two possible types of pulsational states also raises the question of how the star selects its pulsational state. This clearly depends on its past history, and on fluctuations that are present and can make the star jump from one branch to another.

In this example we have illustrated the correspondence between the solutions of the resonant and the NR AEs. The two AEs agree in the limit $\delta\omega \to \infty$ where they should, but, short of the limit, the solution of the resonant AEs undergoes extremely rapid oscillations with vanishing amplitude. Such small differences are expected in the reduction to normal forms which extract the essence of the pulsations and discard finer details.

### 3.2. Rotationally-split multiplet – 1:1:1 resonance

We now turn to the 1:1:1 resonance which was discussed in connection with a rotationally induced splitting in an $\ell = 1$ multiplet (BGS). Because of the cancellation of many of the resonant terms (due to angular momentum constraints) only the combination phase $\Phi$ of Eq. 13 appears. Rather than being those of two 1:1 resonances, mathematically the AEs are the same as one would obtain for a three-mode resonance of the type $2\omega_1 \approx \omega_2 + \omega_3$. This is the reason why only the parameter $\delta\omega = \omega_+ + \omega_- - 2\omega_0$ rather than the separate frequency differences $\omega_+ - \omega_0$ and $\omega_0 - \omega_-$ appear.

For a triplet $\ell = 1$ of modes split by rotation of respective frequencies $\omega_-, \omega_0, \omega_+$, the corresponding amplitudes $A_-, A_+, A_0$ and phases $\phi_-, \phi_0, \phi_+$ then obey the *resonant* AEs (BGS):

$$\dot{A}_- = \kappa_- A_- + R_- A_0^2 A_+ \cos(\Phi - \delta_-)$$
$$- A_-(q_{11}A_-^2 + q_{12}A_0^2 + q_{13}A_+^2) \quad (20a)$$
$$\dot{A}_0 = \kappa_0 A_0 + R_0 A_0 A_+ A_- \cos(\Phi + \delta_0)$$
$$- A_0(q_{21}A_-^2 + q_{22}A_0^2 + q_{23}A_+^2) \quad (20b)$$
$$\dot{A}_+ = \kappa_+ A_+ + R_+ A_0^2 A_- \cos(\Phi - \delta_+)$$
$$- A_+(q_{31}A_-^2 + q_{32}A_0^2 + q_{33}A_+^2) \quad (20c)$$
$$\dot{\Phi} = \delta\omega - 2R_0 A_- A_+ \sin(\Phi + \delta_0)$$
$$+ A_0\left(R_-\frac{A_+}{A_-}\sin(\Phi - \delta_-) + R_+\frac{A_-}{A_+}\sin(\Phi - \delta_+)\right) \quad (20d)$$

The quantity $\delta\omega = \omega_+ + \omega_- - 2\omega_0$ measures the departure from exact resonance which, here, is also the departure from equal frequency splitting. For simplicity, we have set equal to zero the imaginary parts of the NR cubic nonlinear coefficients in the appropriate resonant AEs.

The FP solution of (20), i.e. with $\dot{A}_m = 0, m = -, 0, +; \dot{\Phi} = 0$ with all 3 amplitudes constant and nonzero, when it exists,
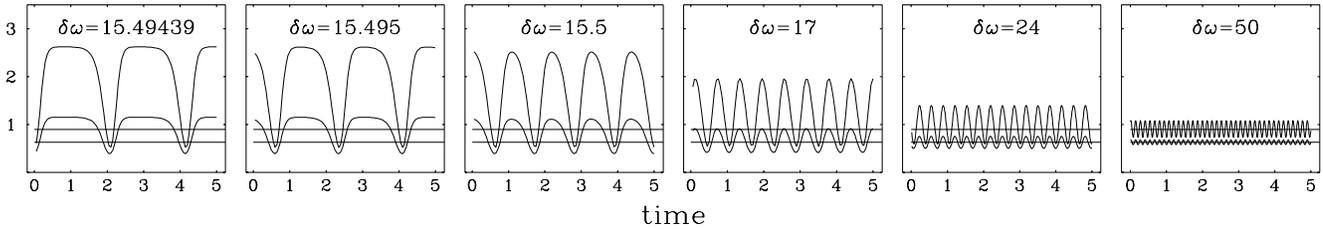
**Fig. 2.** 2:1 Resonance: Amplitude modulations, as a function of the resonance detuning parameter $\delta\omega$ in units of $\kappa_0$; the large and small amplitudes correspond to modes $a$ and $b$, resp.

corresponds to a frequency-locked triplet. In addition to such a triple mode (TM) there can also be single mode (SM) and double mode (DM) FPs. Finally we recall that the resonant AEs can also have time-dependent solutions corresponding to amplitude modulated pulsations (BGS).

In this section we explore what happens to a TM (frequency locked) FP as $|\delta\omega/\kappa_0|$ increases. This is of interest in a given star because of a variation of the growth rates among the excited multiplets, or, when we consider a collection of stars, because of the variety of rotation rates.

As far as the corresponding NR AEs are concerned, there is no need to write them down, because they are simply obtained by setting the $r_m$'s equal to zero in Eqs. 20a,b and c. It is easy to show that the $\tilde{q}$ and $q$ are identical for this type of resonance (this can readily be shown following Buchler & Kovács 1986a). Depending on the values of the parameters, the NR FPs in general can also come as SM, DM, TM, and may exist or not, be stable or unstable, depending on the coefficients. It is important to note the absence of the fourth equation for the phase, i.e. one no longer has an equation $\dot{\Phi} = 0$ as in the resonant case. This has as an important consequence that *the NR triplet is necessarily unlocked.*

As in the previous section one can show that the frequency locked FP solutions cannot exist when $\delta\omega/\kappa_0$ is sufficiently large. What happens to these frequency locked solutions away from the resonance center?

It should be obvious that a general parameter study, neither analytical nor numerical, is not possible and that we have to resort to numerical methods to provide illustrations of possible behavior.

*First example*

In a first such illustrative example we set the phases $\delta_m$ equal to zero for simplicity. Further, the coefficients in the AEs are chosen so that a stable resonant TM exists at $\delta\omega = 0$, on the one hand, and so that the NR AES have a stable TM as well.

Our parameter values are: $q_{11} = 0.326$, $q_{12} = 0.2062$, $q_{13} = 0.19$, $q_{21} = 0.2122$, $q_{22} = 0.3425$, $q_{23} = 0.2139$, $q_{31} = 0.2351$, $q_{32} = 0.2132$, $q_{33} = 0.3194$, $r_- = 0.51$, $r_0 = 0.018$, $r_+ = 0.6$. For the growth rates: $\kappa_- = (1 + 6. \times 10^{-3})$ and $\kappa_+ = (1 - 5. \times 10^{-2})$. We have again rescaled the time in the AEs (Eqs. 20), by setting $\kappa_0 = 1$.
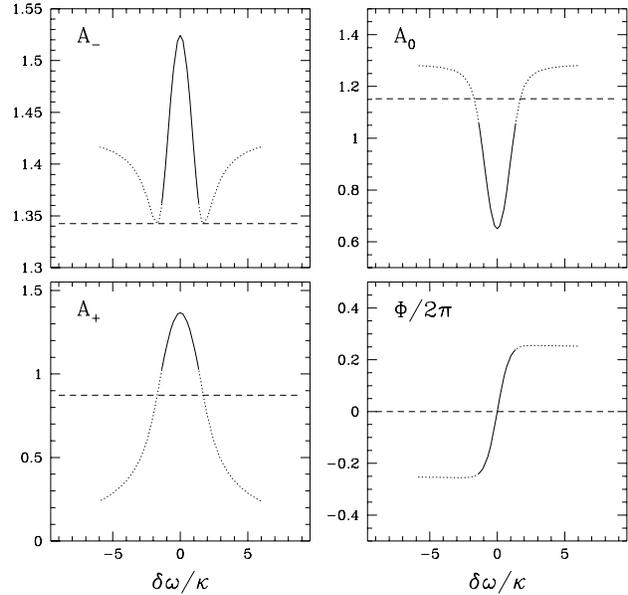


**Fig. 3.** Amplitudes and phase $\Phi$ of a resonant triplet $\ell = 1$ as the resonance detuning parameter $\delta\omega/\kappa_0$ is increased; *solid line: stable, dashed:* unstable solution; *Dotted* lines: amplitudes of the NR triplet FP.

In this simplified case, it can easily be checked that the resonant TM FP satisfies the symmetry relation $\Phi \rightarrow -\Phi, \delta\omega \rightarrow -\delta\omega$, so that we can limit ourselves to positive $\delta\omega$.

With the chosen parameter values, one has a stable TM (frequency lock) right at the resonance center ($\delta\omega = 0$) with $A_- = 1.524$, $A_0 = 0.651$, $A_+ = 1.367$, $\Phi = 0.00$.

On the other hand, the NR AEs, that apply for large $\delta\omega/\kappa_0$, have the following solutions. A TM $A_- = 1.343$, $A_0 = 1.152$, $A_+ = 0.873$. (Note the different asymmetries in the amplitudes of the resonant TM and NR TM FPs). The question arises how the system evolves from the resonant to the NR TM as $\delta\omega/\kappa_0$ is increased.

Fig. 3 shows the results. The resonant TM FP remains stable (solid lines) as long as $\delta\omega/\kappa_0 \leq 1.367$ above which it becomes unstable (dashed lines). For comparison, the amplitudes of the NR TM is indicated as horizontal, dotted lines. For large $\delta\omega/\kappa_0$ the story is similar to the one described for the 2:1 resonance in the previous section, namely the amplitudes oscillate around

the NR TM values with a period $\approx 2\pi/\delta\omega$. The magnitudes of the amplitude modulations decrease rapidly with $|\delta\omega/\kappa_0|$. Ultimately, for large $\delta\omega/\kappa_0$, the magnitudes of the modulation are so small that the star would be seen in practice to pulsate with constant amplitudes with the latter given by the NR AEs.

What happens in the vicinity of the transition $\delta\omega/\kappa_0 \gtrsim 1.367$ does not seem generic, and depends on the chosen parameter values. In this example one obtains first a Hopf bifurcation with oscillations about the now unstable resonant TM, but further away the oscillations occur about the NR TM, as just mentioned. The reason for this smooth transition from the constant amplitude, frequency locked pulsations to amplitude and frequency modulated pulsations seems to be that the resonant TM FP amplitudes nearly coincide with the NR TM ones at $|\delta\omega/\kappa_0| \sim 1.6$ just after the onset of instability of the resonant TM solution.

The transition from constant amplitude and frequency locked pulsations to modulated pulsations can be confirmed with a simple system of 3 coupled pulsators i.e.

$$\ddot{x} + \omega_-^2 x = 2\kappa_- \dot{x} + 8R_- y\dot{y}z - 8q_{11}\dot{x}x^2$$
$$-4q_{12}(y\dot{y}x + y^2\dot{x}) - 4q_{13}(z\dot{z}x + z^2\dot{x}) \qquad (21a)$$

$$\ddot{y} + \omega_0^2 y = 2\kappa_0\dot{y} + 4R_0 y(\dot{x}z + x\dot{z}) - 4q_{21}(\dot{y}x^2 + y\dot{x}x)$$
$$-2q_{22}\dot{y}y^2 - 4q_{23}(\dot{y}z^2 + y\dot{z}z) \qquad (21b)$$

$$\ddot{z} + \omega_+^2 z = 2\kappa_+\dot{z} + 8R_+ y\dot{y}x - 4q_{31}(x\dot{x}z + x^2\dot{z})$$
$$-4q_{32}(y\dot{y}z + y^2\dot{z}) - 8q_{33}\dot{z}z^2 \qquad (21c)$$

which have been chosen so that the amplitude equation formalism applied to the system 21 yields the AEs of Eq. 20.

To mimic for instance a rotationally induced splitting, the side frequencies are chosen as: $\omega_\pm = \omega_0(1 \pm 0.5\sqrt{\delta\omega/\omega_0} + 0.5\delta\omega/\omega_0)$ where $\delta\omega/\omega_0 \propto (\Omega/\omega)^2$. The same values as above are taken for the nonlinear coefficients $q$ and $r$. The time in (21) has been scaled with $\omega_0 = 1$. For our numerical experiments, two values of $\kappa_0/\omega_0$ are considered below, viz. $10^{-2}$ and $10^{-3}$.

Fig. 4a shows the behavior of the component $z$ with time for $\delta\omega/\kappa_0 = 0.5, 4.0$ and $50.0$. For better visualization we have taken $\kappa_0/\omega_0 = 1.5 \times 10^{-2}$ on the top. Transients have been removed and the time origin is arbitrary. At $\delta\omega/\kappa_0 = 0.5$, $z$ pulsates with a frequency $\tilde{\omega}_+$ with a constant (resonant) amplitude $A_+$. At $\delta\omega/\kappa_0 = 50$, $z$ pulsates with an almost constant NR amplitude. At $\delta\omega/\kappa_0 = 4$, the pulsation is amplitude modulated with a modulation period of $2\pi/\delta\omega$.

The bottom figures show the Fourier amplitude spectra of the signal $x+y+z$ built over a time interval of 240 basic periods $P_0 = 2\pi/\omega_0$ with $\delta\omega/\kappa_0 = 0.5$. Here $\kappa_0/\omega_0 = 10^{-3}$.

A Fourier spectrum of this signal (Fig. 4b) shows how the triplet goes from a frequency-locked to a frequency-unlocked pattern when the ratio of the departure from exact resonance (i.e. from equal splitting) to the growth rate increases, everything else kept constant. Note that at $\delta\omega/\kappa_0 = 4$ in the intermediate regime, the time interval of the signal in Fig. 4 corresponds to about 1 period of modulation. It can be easily checked that for a longer time interval i.e. covering several periods of modulation, additional side frequencies appear in the Fourier spectrum.
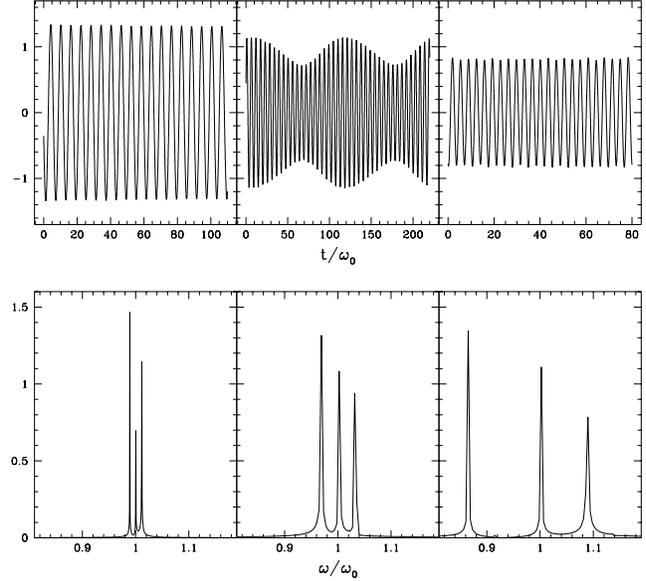


**Fig. 4.** *top from left to right:* Temporal behavior of the component $z(t)$ (Eq.21c) for $\delta\omega/\kappa_0 = 0.5, 4, 50$ with $\kappa_0/\omega_0 = 10^{-2}$. *bottom:* Corresponding Fourier amplitude spectra, see text.

As far as observations are concerned, with typically $\delta\omega/\omega_0 \sim (\Omega/\omega_0)^2 \sim 10^{-6}$, $\Omega$ the rotational frequency, and a basic pulsation $P_0 = 2\pi/\omega_0 \sim 200s$, the period of modulation for a variable white dwarf is of the order $P_{mod} \sim 10^6 P_0 \sim 6-7$ years. For a delta Scuti star, with typically $\delta\omega/\omega_0 \sim 10^{-4}$ and $P_0 \sim 90$ mn, $P_{mod} \sim 10^4 P_0 \sim 1 - 2$ years. Therefore, with the lengths of current continuous observation runs, if the the star were in the intermediate regime, only short phases of the modulation would be captured and the amplitude would barely vary over a same run. Of course, one would observe a pattern in amplitude which would vary from one run to depending at which phase of the modulation they are recorded.

*Second example*

Turning to the full case, that is with complex nonlinear coefficients entering the AEs (BGS), leads to similar conclusions. The set of possible types of behavior for the full system, however, can be more complex. The symmetry $\Phi \rightarrow -\Phi, \delta\omega \rightarrow -\delta\omega$ no longer exist and the system can behave quite differently for $\delta\omega$ and $-\delta\omega$. In addition, several types of solution can coexist.

Fig. 5 shows the evolution of the amplitudes of the resonant TM solution as one moves away from the exact resonance for the set of parameters

$q_{11} = (0.326 - i\,1.413); q_{12} = (0.2062 - i\,1.394);$
$q_{13} = (0.19 - i\,3.368); q_{21} = (0.2122 - i\,1.351);$
$q_{22} = (0.3425 - i\,2.521); q_{23} = (0.2139 - i\,1.412);$
$q_{31} = (0.2351 - i\,3.26); q_{32} = (0.2132 - i\,1.419);$
$q_{33} = (0.3194 - i\,1.451);$
$r_- = (-0.018 - i\,0.51) = 0.5103\exp(1.5355\,i);$
$r_0 = (0.024 - i\,1.05) = 1.0503\exp(-1.5479\,i);$
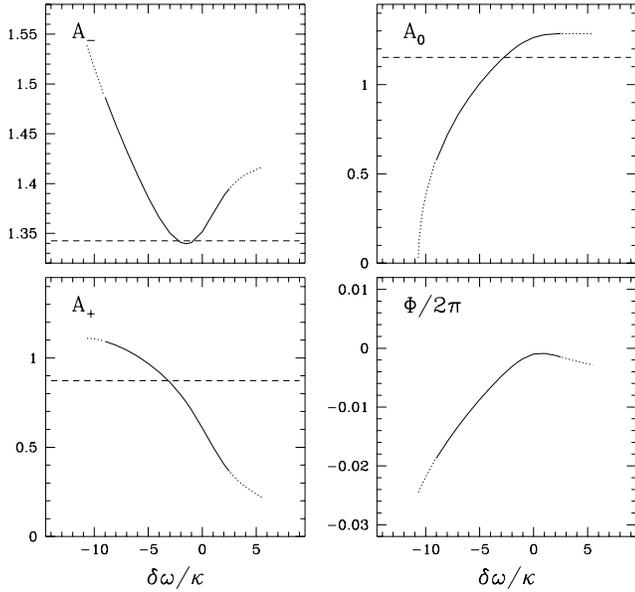$r_+ = (-0.021 - i\,0.6) = 0.6004\exp(1.5358\,i).$

**Fig. 5.** Amplitudes and phase of a resonant triplet $\ell = 1$ as the ratio $|\delta\omega/\kappa_0|$ increases – example 2; (*solid line*: stable and *dashed lines*: unstable triplet); *dotted line*: amplitudes of the constant amplitude, nonresonant triplet which exists and is stable far away from the resonance.

In the resonance center ($\delta\omega = 0$) the resonant AEs have a TM (frequency locked) FP. Here the solutions and the stability diagram are no longer symmetric about the resonance center. As in the previous two examples the frequency locked FP becomes unstable for a sufficiently large deviation from resonance, i.e. for $\delta\omega/\kappa_0 < -9.033$ and $\delta\omega/\kappa_0 > 2.53$, respectively, as Fig. 5 shows. For these numerical coefficients the system bifurcates abruptly from the frequency locked FP to an amplitude modulated pulsation. The amplitude modulations occur about the NRT FP solution, at first with large excursions which later scale as $1/\delta\omega$ and whose period scales also as $1/\delta\omega$. However, as for the 2:1 resonance example above, there is hysteresis, i.e. a narrow regime of $\delta\omega/\kappa_0$ in which the frequency locked and the amplitude modulated solutions coexist.

## 4. Resonant multiplets

We assume now that *two* excited multiplets can interact resonantly, but that they are decoupled from other possible excited multiplets. In the following we label the two interacting multiplets $a \equiv \{a_m, m = -\ell_a, \cdots, \ell_a\}$ and $b \equiv \{b_m, m = -\ell_b, \cdots, \ell_b\}$.

Among all possible two-mode resonances, three of the types we consider below are especially important because they can affect the linear stability of a pulsation. These are: integer resonances (Sect. 3.1), direct resonances (Sect. 3.2), and half-integer resonances (Sect. 3.3).

### 4.1. Integer resonances: $\qquad n\omega_a \approx \omega_b$

For didactic purposes we first consider the simplest integer resonance; that is, the 2:1 resonance. For simplicity of notation we will temporarily use the symbols $a$ and $b$ to stand generically for any component of the associated multiplet.

** <u>2:1 resonance</u> $\quad - \quad 2\omega_a \approx \omega_b$

The appropriate AEs are easily seen to be (e.g., Dziembowski & Kovács 1984, BK86a)

$$
\begin{aligned}
\dot{a} &= \sigma_a a - q_{aa}aa^*a - q_{ab}bb^*a \quad + r_1 ba^* \\
\dot{b} &= \sigma_b b - q_{ba}aa^*b - q_{bb}bb^*b \quad + r_2 a^2
\end{aligned}
\tag{22}
$$

where an asterisk denotes complex conjugate. The four cubic $q_{\alpha\beta}$ coupling terms are the important amplitude saturation terms that always arise, whether there is a resonance or not and whose expressions are given in BG84 and GB94. In all our applications of AEs to radial pulsators we have found that the real parts of the $q_{\alpha\beta}$ coefficients are always positive (The physical reasons are addressed in Buchler 1993). In some of the following discussion we shall assume, for specificity, that this also holds true for nonradial pulsators.

The expressions for the resonant coupling coefficients $r_1$ and $r_2$ are given by the integrals over space and solid angle

$$
\begin{aligned}
r_1 &\propto r_2^* \propto \langle \mathbf{a}|\mathbf{N_2}|\mathbf{a^*b}\rangle \\
&\propto \int dr\rho r^2 \int d\Omega \, \xi_a^*(r) Y_{\ell_a}^* \, \mathbf{N_2} \, \xi_a^*(r)\xi_b(r) Y_{\ell_a}^* Y_{\ell_b}.
\end{aligned}
\tag{23}
$$

where $Y_\ell^*$ is the usual spherical harmonic and $\mathbf{N_2}$ is an operator acting on $\xi$ (the r-dependent part of the eigenfunctions) which comes from the second order Taylor expansion of the hydrodynamical stellar equations (BG84; GB94, and see Sect. 5). Because of the assumed slow rotation and accompanying spherical symmetry of the equilibrium stellar model, the stellar structure operators $\mathscr{L}$ (the linear pulsation operator), $\mathbf{N_2}$, $\mathbf{N_3}$, $\cdots$, have even parity and azimuthal symmetry. The angular integrals thus restrict the types of couplings that are allowed (Dziembowski 1984, GB94). Therefore the angular part of (23) simplifies to

$$
r_1 \propto r_2^* \propto \int Y_{\ell_a}^* Y_{\ell_a}^* Y_{\ell_b} d\Omega.
\tag{24}
$$

Parity considerations (see Sect. 5) show that the coupling terms $r_1$ and $r_2$ vanish identically

> (a) when $\ell_b$ is odd and $\forall\ell_a$, or
> (b) when $\ell_a = 0$ and $\ell_b \neq 0$.

Of course, additional constraints on the $m_a$ and $m_b$ values arise from angular momentum ($L_z$) constraints. Here, $m_b = 2m_a$ must be satisfied.

It is of interest to note the consequences of the respective linear stability or instability of the two modes. If mode $a$ is linearly unstable ($\kappa_a > 0$) and $a \neq 0$, then mode b will necessarily be nonzero, even when it is linearly stable. (This is the situation that occurs in the bump Cepheids, BK86.) In other words, mode $a$ always entrains mode $b$.

When mode b is unstable with $B \neq 0$, then $A = 0$ is always a solution. One can easily show that when this FP solution is unstable then a solution with $A \neq 0$ and $B \neq 0$ exists. In that

**Table 1a.** Resonance $2\omega_a \approx \omega_b$

| $\ell_a$ | $\ell_b$ | allowed | allowed $(m_a,m_b)$ |
|---|---|---|---|
| 0 | 1 | no | |
| 0 | 2 | no | |
| 1 | 0 | yes | $(0,0)$ |
| 2 | 0 | yes | $(0,0)$ |
| 1 | 1 | no | |
| 2 | 1 | no | |
| 1 | 2 | yes | $(0,0)$;  $(\pm 1, \pm 2)$ |
| 2 | 2 | yes | $(0,0)$;  $(\pm 1, \pm 2)$ |

**Table 1b.** Resonance $3\omega_a \approx \omega_b$

| $\ell_a$ | $\ell_b$ | allowed | allowed $(m_a,m_b)$ |
|---|---|---|---|
| 0 | 1 | no | |
| 0 | 2 | no | |
| 1 | 0 | no | |
| 2 | 0 | yes | $(0,0)$ |
| 1 | 1 | yes | $(0,0)$ |
| 2 | 1 | no | |
| 1 | 2 | no | |
| 2 | 2 | yes | $(0,0)$ |

case one says that mode $b$ parametrically excites mode $a$ (cf. below under 1:2 resonance).

In Table 1a we have summarized the situation for the combinations of the lowest $\ell_a$ and $\ell_b$ values that are of practical interest.

In the realm of low $\ell$ nonradial modes condition (a) shows that the *higher frequency mode* thus needs to be an $\ell_b = 2$ (even) mode, but that the *lower frequency one* could be $\ell_a = 1$ or $\ell_b = 2$.

Condition (b) shows that *a radial mode cannot excite a nonradial mode through this type of resonance* unless the radial mode is the high frequency partner in the resonance. Thus, for variable white dwarfs for instance, $g$–modes might be excited this way, but never $p$–modes.

** **n:1 resonance** $\quad - \quad n\omega_a \approx \omega_b, \quad n > 1$

We now go on to the general case

$$\dot{a} = \cdots + r_1 b a^{* \, n-1}$$
$$\dot{b} = \cdots + r_2 a^n. \tag{25}$$

where, here and from now on, we omit the usual linear and cubic saturation terms.

The angular integrals show that the "direct" coupling terms (due to $\mathbf{N}_n$) vanish when

(a) $n\ell_a + \ell_b$ is odd,

(b) $\ell_a = 0$ and $\ell_b \neq 0$.

For $n > 2$, there are also "indirect" contributions to $r_1$ and $r_2$. For example, in the 3:1 case where the direct contribution is $\langle \alpha | \mathbf{N}_3 | \beta \gamma \delta \rangle$, the indirect contributions all contain terms of the general form

$$\sum_k \langle \alpha | \mathbf{N}_2 | \beta k \rangle \frac{1}{D_k} \langle k | \mathbf{N}_2 | \gamma \delta \rangle \tag{26}$$

where the denominators $D_k$ represent sums and differences of linear frequencies (e.g. BG84, GB94, BGS). Typically these indirect terms are of the same size as the direct term. For $n = 3$ one can still obtain relatively simple explicit expressions for the cubic resonant coefficients (as for the 1:1 direct resonance, BGS). For higher order ($n > 3$) resonances, however, the indirect terms are quite complicated.

But, fortunately, we can take advantage of a general result, established in Sect. 5.

*When the direct coupling term vanishes because of parity and angular momentum $(L_z)$ considerations then each of the possible indirect coupling terms also vanishes, and conversely.*

As a consequence, whatever the resonance is *it is sufficient to study the simple direct term to determine whether a specific resonant term should appear in the AEs or not.*

This enables us to obtain the allowed n:1 resonant couplings as listed in Table 1b for the example of a 3:1 resonance and for couplings involving $l = 0$ to $l = 2$ modes.

The above constraints on $\ell$ values given in Tables 1a and 1b show, for example, that the Blazhko effect in RR Lyrae stars *cannot be explained* as the coupling of a radial mode with a nonradial $p$–mode of higher frequency through a 2:1 or 3:1 resonance nor, for that matter, through a $n$:1 ($n > 1$) resonance in general.

### 4.2. Direct or 1:1 resonance: $\qquad \omega_a \approx \omega_b$

A special situation arises for a direct 1:1 resonance which, like the 3:1 resonance, also contributes cubic terms.

Denoting the amplitudes again generically by $a$ and $b$ we obtain the AEs

$$\dot{a} = \ldots + r_1 b^2 a^* + s_1 b a a^* + t_1 a^2 b^* + u_1 b b b^*$$
$$\dot{b} = \ldots + r_2 a^2 b^* + s_2 a b b^* + t_2 b^2 a^* + u_2 a a a^* \tag{27}$$

Here there are four possible resonant terms in each AE. As we have just seen, on the basis of the theorem of Sect. 5, we only need to consider the direct terms to determine which of these resonant terms survive parity and angular momentum constraints. For these direct terms not to vanish, the constraints on $l$ and $m$ are the same as those given by the product of the four appropriate spherical harmonics. Thus the conditions for the existence of $r_1$ and $r_2^*$ can be obtained from

$$\int \mathrm{Y}_{\ell_a}^* \mathrm{Y}_{\ell_a}^* \mathrm{Y}_{\ell_b} \mathrm{Y}_{\ell_b} d\Omega. \tag{28}$$

This implies that $r_1$ and $r_2^*$ are different from zero independently of the values of $\ell_a$ and $\ell_b$.

Similarly,

$$\int \mathrm{Y}_{\ell_a}^* \mathrm{Y}_{\ell_a} \mathrm{Y}_{\ell_a} \mathrm{Y}_{\ell_b}^* d\Omega \tag{29}$$

implies that $s_1$, $t_1$, and $u_2$ vanish when $\ell_a + \ell_b$ is odd, or when $\ell_a = 0$, $\forall \ell_b$. Finally, the integral

$$\int \mathrm{Y}_{\ell_a}^* \mathrm{Y}_{\ell_b} \mathrm{Y}_{\ell_b} \mathrm{Y}_{\ell_b}^* d\Omega \tag{30}$$

**Table 2.** Direct Resonance $\omega_a \approx \omega_b$

| $\ell_a$ | $\ell_b$ | $r_1, r_2$ | $s_1, t_1, u_2$ | $u_1, s_2, t_2$ |
|---|---|---|---|---|
| 0 | 1 | yes | no | no |
| 0 | 2 | yes | no | yes |
| 1 | 0 | yes | no | no |
| 1 | 1 | yes | yes | yes |
| 1 | 2 | yes | no | no |
| 2 | 0 | yes | yes | no |
| 2 | 1 | yes | no | no |
| 2 | 2 | yes | yes | yes |

**Table 3.** Resonance $(2n+1)\omega_a \approx 2\omega_b$, $n > 0$ (e.g. $3\omega_a \approx 2\omega_b$, $5\omega_a \approx 2\omega_b$)

| $\ell_a$ | $\ell_b$ | allowed |
|---|---|---|
| 0 | 1 | yes |
| 0 | 2 | yes |
| 1 | 0 | no |
| 1 | 1 | no |
| 1 | 2 | no |
| 2 | 0 | yes |
| 2 | 1 | yes |
| 2 | 2 | yes |

The 2:(2n+1) case is obtained by inverting the roles of $a$ and $b$

yields the conditions that $u_1$, $s_2$, and $t_2$ vanish when either $\ell_a + \ell_b$ is odd, or when $\ell_b = 0$, $\forall \ell_a$.

The $m$-constraint is the same for all coefficients; i.e., $m_a = m_b$.

In Table 2 we examine the combinations of low $\ell$ 1:1 multiplets of potential interest in nonradial stellar pulsators.

If two multiplets are of the same type and one is a $\ell = 1$ and the other $\ell = 2$, then Table 2 shows that all the $s$, $t$ and $u$ vanish and there remains only one type of resonant term. This same simplification occurs when an $\ell = 1$ $p$– or $g$–mode is in direct resonance with a radial mode.

### 4.3. Half-integer resonances: $(2n+1)\omega_a \approx 2\omega_b$

One notes that this simplest 1:2 ($n = 0$) case is essentially the same as a 2:1 resonance provided that we invert the roles of $a$ and $b$. The coupling terms thus vanish when (a) $\ell_a$ is odd and $\forall \ell_b$, or (b) when $\ell_b = 0$ and $\ell_a \neq 0$.

The higher frequency mode ($a$) must therefore have even $\ell_a$ to be able to parametrically excite the linearly stable lower frequency mode $b$ (which can have any $\ell_b$ value). Generally, a radial mode could thus excite any resonant $g$–mode, even when the latter is linearly stable. This situation is very similar to the three-mode resonance $\omega_a \approx \omega_b + \omega_c$ considered by Dziembowski (1982).

The relevant terms in the AEs are

$$
\begin{aligned}
\dot{a} &= \quad \cdots \quad + r_1 b^2 a^{*2n} \\
\dot{b} &= \quad \cdots \quad + r_2 a^{2n+1} b^*.
\end{aligned}
\tag{31}
$$

The direct contribution to $r_1$ and $r_2$ is easily seen to vanish when $\ell_a$ is odd. This resonance can couple only central modes of the multiplets ($m_a = m_b = 0$).

Eqs. 24 show that depending on the linear stability and the values of the nonlinear coupling coefficients three types of solutions may exist.

When mode $a$ is linearly unstable ($\kappa_a > 0$), but mode $b$ is linearly stable, and if furthermore the FP solution ($A \neq 0$, $B = 0$) would be stable in the absence of the resonance (i.e., if $\kappa_b + q_{ba}^R A^2 < 0$), then mode $b$ gets excited parametrically when (cf. Moskalik & Buchler 1990) the Floquet exponent

$$
\kappa_b - q_{ba}^R A^2 + \sqrt{|r_2|^2 A^{2(2n+1)} - (\delta\omega - q_{ba}^I A^2)^2}
\tag{32}
$$

is positive. The superscripts $R$ and $I$ refer to real and imaginary parts, respectively, of the saturation coefficients. If, however, mode $b$ is also linearly unstable, then of course a double-mode solution occurs.

On the other hand, when mode $b$ is linearly unstable, but mode $a$ is linearly stable, one easily sees from Eq. 31 that mode $b$ cannot excite mode $a$.

In Table 3 we summarize the situation for the $(2n+1):2$, $n > 0$, resonance. One notes that a radial $a$ mode can couple to and parametrically excite a higher frequency $\ell_b = 1$ or $\ell_b = 2$ mode, in contrast to the $n:1$ resonance.

### 4.4. Other potentially important resonance, 4:3    $4\omega_a \approx 3\omega_b$

The AEs in this case contain the resonant terms of the form

$$
\begin{aligned}
\dot{a} &= \quad \cdots \quad + r_1 b^3 a^{*3} \\
\dot{b} &= \quad \cdots \quad + r_2 a^4 b^{*2}.
\end{aligned}
\tag{33}
$$

Angular momentum constraints make the direct contribution to the $r$ coefficients vanish only when mode $b$ has an odd $\ell$. In addition, the coupling is possible only between modes with $m = 0$. Parametric entrainment of one of the modes by the other is not possible here because this resonance does not affect the linear stability of either single mode FP. It is only when both modes are excited that the resonance term is allowed to play a role.

### 4.5. Three-multiplet couplings

AEs can also be useful in the case of three-mode or multiplet couplings. Dziembowski (1982), for example, considered the coupling of three modes through a resonance of the type $\omega_a \approx \omega_b + \omega_c$, which is well known in plasma physics under the name of resonant wave interaction. In later work Dziembowski & Krolikowska (1985) suggested that it is this type of resonance that limits the amplitudes of $\delta$ Scuti type stars to their relatively small observed values.

Also we have already noted that the situation of a rotationally split multiplet is mathematically equivalent to a resonance of the type $2\omega_a \approx \omega_b + \omega_c$.

## 5. Properties of the coupling coefficients

This section is very technical and can be skipped in a first reading. We prove here a very useful property of the nonlinear coupling coefficients. We assume that the equilibrium model is spherically symmetric (slow rotation, no magnetic fields). In zeroth order in the rotation rate $\Omega$ the linear eigenvectors are then represented by spherical harmonics $Y_{\ell m}$ which have parity $(-)^{\ell}$; i.e., they obey $\Pi Y_{\ell m} = (-)^{\ell} Y_{\ell m}$, and $L_z Y_{\ell m} = m Y_{\ell m}$ (e.g., Blatt & Weisskopf 1952). Because of the spherical symmetry the nonlinear structure operators have positive parity as well as $m = 0$.

To briefly introduce what follows, we first write down the pulsation equations in a compact form using the convenient Dirac notion where $\mathbf{z}$ represents, for example, the density or other stellar variable in an Eulerian representation (as in BG84 or GB94):

$$\frac{\partial |\mathbf{z}\rangle}{\partial t} = \mathscr{L}|\mathbf{z}\rangle + \mathbf{N}_2[|\mathbf{z}\rangle] + \mathbf{N}_3[|\mathbf{z}\rangle] + \cdots \qquad (34)$$

where $\mathscr{L}$ is the linear pulsation operator and the operators $\mathbf{N}_2$, $\mathbf{N}_3$, etc., are quadratic, cubic, etc., in the displacement vector $\mathbf{z}$.

The AE formalism then provides a sytematic expansion of $\mathbf{z}$ as

$$|\mathbf{z(t)}\rangle = |\mathbf{z_0(t)}\rangle + |\mathbf{z_1(t)}\rangle + |\mathbf{z_2(t)}\rangle + \cdots \qquad (35)$$

where

$$|\mathbf{z_0(t)}\rangle = \sum_{\alpha} a_{\alpha} e^{i\omega_{\alpha} t} |\alpha\rangle \qquad (36)$$

and $|\mathbf{z_1(t)}\rangle$, $|\mathbf{z_2(t)}\rangle$, etc., are quadratic, cubic, etc., in the amplitudes $\{a_{\alpha}(t)\}$, and thus represent modal interaction terms ("combination frequencies"). The sum is over all dominant modes $\alpha$.

The resonant coupling coefficients, which measure the strength of the coupling between modes, are given by such integrals as in Eq. 23 and Eq. 26 and are broken up into "direct" and "indirect" contributions, as indicated by the discussion leading to Eq. 26.

<u>Theorem</u>: *If, generically, the direct coupling term is zero because of parity and angular momentum considerations, this implies that each of the possible indirect terms is also zero, and conversely.*

*Parity*: Consider first the lowest—that is, the *cubic coupling term*—for which there is a direct and indirect contribution. The *direct term* is of the form

$$r_{dir} \propto \langle \mathbf{a}|\mathbf{N}_3|\mathbf{bcd}\rangle. \qquad (\mathbf{N_3z_0^3}) \qquad (37)$$

The expression in parentheses indicates the perturbative origin of this term; cf. expansions (34) and (35). The parity of the RHS is $\Pi(\mathbf{bcd}) = \Pi(\mathbf{b})\Pi(\mathbf{c})\Pi(\mathbf{d})$, and the parity of the stellar structure operator is $\Pi(\mathbf{N}_3)=1$. The direct term thus vanishes unless $\Pi(\mathbf{a}) = \Pi(\mathbf{c})\Pi(\mathbf{c})\Pi(\mathbf{d})$. The indirect term (Eq. 26) contains terms of the form (e.g. BG84, GB94)

$$r_{ind} \approx \sum_{k} \langle \mathbf{a}|\mathbf{N}_2|\mathbf{bk}\rangle\langle \mathbf{k}|\mathbf{N}_3|\mathbf{cd}\rangle. \qquad (\mathbf{N_2z_0z_1}) \qquad (38)$$

For the purpose of clarity we have ignored denominators (sums and differences of frequencies) and factors of unity. The sum runs over *all eigenstates* of the system. This is one of the reasons why it is so difficult to make an *ab initio* computation of these terms. It is understood that $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ represent both $\mathbf{bcd}$ *and* their complex conjugates, and that the appropriate permutations of ($\mathbf{bcd}$) are included. The last factor in (38) vanishes unless $\Pi(\mathbf{k}) = \Pi(\mathbf{c})\Pi(\mathbf{d})$, and the first one vanishes unless $\Pi(\mathbf{a}) = \Pi(\mathbf{b})\Pi(\mathbf{k}) = \Pi(\mathbf{b})\Pi(\mathbf{c})\Pi(\mathbf{d})$; i.e., the same condition as for the direct term.

Consider now the *quartic coefficient*. The direct term

$$r_{dir} \propto \langle \mathbf{a}|\mathbf{N}_4|\mathbf{bcde}\rangle \qquad (\mathbf{N_4z_0^4}) \qquad (39)$$

vanishes unless $\Pi(\mathbf{a}) = \Pi(\mathbf{b})\Pi(\mathbf{c})\Pi(\mathbf{d})\Pi(\mathbf{e})$.

The indirect terms are more complicated now. They are given by

$$r_{ind} \propto \sum_{k} \langle \mathbf{a}|\mathbf{N}_3|\mathbf{bck}\rangle\langle \mathbf{k}|\mathbf{N}_3|\mathbf{de}\rangle \qquad (\mathbf{N_3z_0^2z_1})$$

$$+ \sum_{k} \langle \mathbf{a}|\mathbf{N}_2|\mathbf{bk}\rangle\langle \mathbf{k}|\mathbf{N}_3|\mathbf{cde}\rangle \qquad (\mathbf{N_2z_0z_2})$$

$$+ \sum_{k}\sum_{k'} \langle \mathbf{a}|\mathbf{N}_2|\mathbf{bk}\rangle\langle \mathbf{k}|\mathbf{N}_2|\mathbf{ck'}\rangle\langle \mathbf{k'}|\mathbf{N}_2|\mathbf{de}\rangle.$$

$$(\mathbf{N_2z_0z_2}) \qquad (40)$$

It is easy to see again that all three contributions vanish unless $\Pi(\mathbf{a}) = \Pi(\mathbf{b})\Pi(\mathbf{c})\Pi(\mathbf{d})\Pi(\mathbf{e})$. In fact one sees that each nonvanishing matrix element successively "passes through" the parity from the left side to the right side. This holds true for all matrix elements, whatever their order. Since all higher order indirect terms are built up in a manner similar to Eq. 40 we conclude that the parity part of the theorem is valid for all orders of coupling. *Angular momentum*: The nonvanishing of a given matrix element requires that the $L_z$ component on the right match the $L_z$ component on the left; i.e., $m_{\mathrm{lhs}} = \sum m_{\mathrm{rhs}}$ (because the $\mathbf{N}_k$ all have $m = 0$).

Referring to the previous section on parity, *mutatis mutandis* we see that this sum is passed through the nonvanishing matrix elements. Each of the indirect contributions therefore vanishes generically when the direct term vanishes and *vice versa*. QED.

The theorem thus implies that if one is interested in determining which of the normal coupling terms survive in the AEs it is sufficient to consider the expression of the simple, direct term.

## 6. Possible resonances in observed power spectra

In this section we briefly review some examples taken from the observational literature which, in many cases, strongly suggest that resonant interactions are indeed present in nonradially pulsating stars. For the most compelling examples we also outline some consequences of the suspected interaction between modes.

For some of our examples we must caution the reader that the quoted observational results depend on data that may be misleading because of insufficient time coverage of the star being

discussed. For these stars, and others of course, this section then serves to draw attention on the one hand to the need for and on the other to the potential benefits of more extensive observations directed to the study of these resonances.

### 6.1. Integer resonances:    $n\omega_a \approx \omega_b$

The ZZ Ceti star GD 385 (Kepler 1984) provides a perfect example that can be explained by a 2:1 resonance. The Fourier spectrum of GD 385 has three well-defined peaks located at $f_{1-}=3.9043$, $f_{1+}=3.9012$ and $f_2=7.8055$ mHz. The observations show that

$$f_2 = f_{1-} + f_{1+} \tag{41}$$

to a relative accuracy of $10^{-6}$. In the following we use the symbol $f$ for observed frequencies which are to be interpreted as *nonlinear* frequencies ($f = \tilde{\omega}/(2\pi)$).

Let us assume, as suggested by Kepler (1984), that the two low-frequency peaks are modes $\pm m$ belonging to the same multiplet of given $\ell$ and radial order. The central component of this multiplet is frequency locked to $f_2$ to within one part in $10^{-6}$. The linear eigenfrequencies of the two interacting multiplets then are in a near 2:1 resonance with $\omega_2 \approx 2\omega_{10} \approx \omega_{1-} + \omega_{1+}$. The $m = 0$ central peak $f_{10}$ is not seen, perhaps due to inclination effects, or due to nonlinear effects: the $m = 0$ mode indeed may or may not be involved in the pulsation. Table 1a shows that the higher frequency multiplet containing $f_2$ must therefore be even and probably $\ell_2 = 2$. (An $\ell$ of three or higher has never yet been identified in a variable white dwarf) and $m_2 = 0$. The lower frequency doublet, on the other hand, can be of odd or even angular momentum and the likely possibilities are $\ell_1 = 1$ or 2. In either case, $|m_1|$ must be equal to 1.

One might think that $f_2$ is just a simple harmonic of $f_{10}$, with the $f_{10}$ peak presumably not observed because the inclination angle is close to $\pi/2$. We believe this to be unlikely because the harmonics of $f_{1\pm}$ should then have similar power, but they are not visible.

One can easily show that it is the phase $\Phi = \phi_2 - \phi_{1-} - \phi_{1+}$ which enters the apposite AEs. If the pulsations correspond to fixed points of the AEs then, as we have seen in Sect. 2, this leads to the nonlinear frequency lock of Eq. 41. However, some amplitude variations have been reported by Kepler (1984) that occur on the very long time scale of years. Would this be confirmed, it would imply that it is *not* a fixed point solution, but rather a time-dependent one. Such time-dependent behavior, if generated by the resonance, would of course not vitiate the angular momentum attributions of the mode.

But because the observed modulation time scale is very long, the pulsations may still appear to be in a frequency lock when observed over shorter scales. Indeed, as the AEs show (Sect. 3), depending on the values of the coefficients, a resonant coupling can generate modulations of amplitude that are quite different in size from the period (or phase) modulations. In the present case, the magnitude of the amplitude modulation would be much larger than that of the phase modulation.

### 6.2. Direct resonances:    $\omega_a \approx \omega_b$

During the course of evolution of a star, pulsation frequencies change and, in some circumstances, the frequency of one mode may closely approach that of another mode of the same $\ell$. In this case, called *avoided crossing*, the two linear eigenfrequencies never become exactly equal to each other. Near the resonance, the two modes take on a mixed character having some of the properties of both modes. This avoided crossing, which is a purely linear effect, is discussed in a stellar context by, for example, Unno et al. (1989, Sect. 15.3.1). Nonlinear effects can however cause the two frequencies to lock, i.e. for the two frequencies to coincide *exactly*.

Possible examples of direct resonances may be present in the variable white dwarf PG1159–035 where only high-order $g$–modes of $\ell = 1, 2$ are observed. With a rich spectrum of modes (125 were resolved in PG1159 by Winget et al. 1991) there may be an occasional overlap in frequency between an $\ell = 1$ and an $\ell = 2$ in the Fourier spectrum. This is especially true at low frequencies where, for $g$–modes, the frequency spectrum becomes particularly dense. Examples of overlap are reported in Winget et al. (1991, Table 3) where an $\ell$ value is assigned as either $\ell = 1$ or 2. Reference to our Table 2 shows that a direct resonance is indeed possible here (with only the $r_1$ and $r_2$ terms contributing) which may distort what is normally expected to be seen in the two overlapping multiplets. To disentangle the effects here is both an observational and theoretical challenge.

Again, depending on the physical situation, and thus on the values of the coupling coefficients, *a direct resonance can lead to unsteady behavior* (amplitude and phase modulations). If the modulation amplitude is small and sinusoidal, a single pulsation then appears in a power spectrum as if it were governed by three or more closely spaced frequencies.

Finally we note that a direct resonance between a radial and a nonradial mode has recently been suggested as a possible cause of the Blazhko effect in RR Lyrae stars (Kovács 1994, 1995; van Hoolst 1995).

### 6.3. Half-integer resonances: $(2n+1)\omega_a \approx 2\omega_b$

#### 6.3.1.  Variable white dwarfs

##### PG1351+489

The light curve of the DBV star PG1351+489 described by Winget et al. (1987) is dominated by a single period of 489.5 s corresponding to a frequency of $f_0 \approx 2.05$ mHz. Strong harmonics are also seen at $2f_0, 3f_0$, etc. Of interest to us, however, are lower amplitude signals seen in most of the observations made in the late 1980's at (approximately) $1.47f_0, 2.47f_0$, and $3.47f_0$, in order of increasing frequency and decreasing amplitude. Goupil et al. (1988) have claimed that power is also present in the data at $0.47f_0$ in two observations reported by Winget et al. (1987). The situation is murky, however, because red noise due to low-frequency perturbations in the atmosphere can mimic or even cancel out the signal we seek. This is especially serious if, as in the Winget et al. (1987) observations, data are taken from a single site with the inevitable one day, or longer, gaps.

To circumvent this, as yet unreported observations utilizing the "Whole Earth Telescope" (WET) were made in early 1995. (For a description of WET see Nather et al. 1990.) With the extended global coverage capabilities and improved resolution of WET, a signal at $0.46 f_0$ is seen very clearly as are ones at $1.46 f_0$, etc. The amplitude of the peak at $0.46 f_0$ is only 4% of $f_0$ itself, which illustrates the virtue of extended coverage in resolving peaks of low amplitude. Although the 1995 data have not yet been completely reduced and analyzed, the peak at $f_0$ shows evidence for splitting (or unsteadiness) whereas, to the limits of the observations, neither $1.46 f_0$ nor $2.46 f_0$ seem to have multi-plet structure. (There are other low amplitude signals in the data but they appear to be independent modes not directly associated in a resonant fashion with the main peaks.)

It seems reasonable to interpret the Fourier spectrum as generated two modes, at $f_0$ and at $f_1 = 1.46 f_0$ (or perhaps $f_1 = 2.46 f_0$). All other peaks would then be harmonics of $f_0$ and combination peaks of $f_0$ and $f_1$.

There is no absolutely compelling reason why these could not be two independent, nonresonant modes. It is difficult to understand though why there would be just two excited modes, with a rather tight frequency ratio of 1.46. A coincidence seems unlikely because there are two additional white dwarfs known with a similar Fourier spectrum and a frequency ratios of 1.52 and 1.54, respectively (see below). However, the observed be-havior appears quite natural if we assume that a half integer resonance is active in this star (3:2, or perhaps 5:2).

It might be objected that the observed intermediate frequen-cies do not appear at 1.5 (or 2.5), but that they always fall short of those values. In a resonance situation one would expect non-linear effects to lead to either of two situations, viz. either exact frequency-lock, or fluctuations about the resonance condition i.e. in both directions. The spectrum of PG1351 can however be explained if the observed second frequency ($f_1 = 1.46$ or $2.46$ $f_0$) is one of the $m \neq 0$ components of the multiplet, and is the only one that is large enough to be observed. The work of BGS has shown that large amplitude asymmetries within a given mul-tiplet can easily happen. In the present case, this would yield a rotational splitting $\pm \sim 60 - 80$ $\mu$Hz of the $1.5 f_0$ peak. This indicates a rotation that is probably too fast if the $f_1$ peak cor-responds to the $m = 1$ component (part of the split could be explained by nonlinear effects). The size of the split therefore favors again an $\ell = 2, m = 2$ (or higher $\ell, m$) component. In either case, if we are dealing strictly with nonradial modes of low order — and the observed periods $do$ rule out $\ell = 0$ modes – then Table 3 suggests that the mode at $f_0$ is an $\ell = 2$ mode. (Note that $\ell \geq 3$ modes have not yet been identified in white dwarf light curves.)

With either of the two resonant possibilities mentioned above, Table 3 restricts the mode at $f_0$ to be an $\ell = 2$ mode. (The observed periods definitely rule out $\ell = 0$). The discussion surrounding Table 3 also states that a 3:2 or 5:2 resonance can couple only $m = 0$ modes of a possible multiplet.

If $f_0$, with the aid of future observations, turns out to have $\ell = 2$ then that would be a strong argument for our resonant identification.

### GD154

Another curious example is the DA variable white dwarf GD154 first reported on by Robinson et al. (1978). This 1978 pa-per describes a star that was a virtual twin of PG1351 discussed above except that the ratio of near 3:2 (or 5:2) was $1.52 f_0$, etc., rather than (in the latest observations) $1.46 f_0$, etc. With this information alone, we would assign $\ell = 2$ to the mode at $f_0$ (where, in this case, $f_0 = 0.842$ mHz).

If, however, we only had the more recent WET observations of Pfeiffer et al. (1993) we would tell a different story: they observe $no$ half-integer signals at all and we would have no reason to assign $\ell = 2$ to $f_0$. The true story, we believe, is that the assignment $\ell = 2$ may still be valid but — and being true to the whole story we tell here — nonlinear effects of the 3:2 (or 5:2) resonance force the half-integer mode to essentially appear and disappear on time-scales that are as yet unknown. Pfeiffer et al. may just have caught the star at an inopportune time.

### BPM31594

This is a white dwarf to have been observed with simi-lar Fourier spectral features (O'Donoghue, Warner & Cropper, 1992). Here the second frequency appears at $1.54 f_0$, and har-monics at 2.54, 3.54.

### 6.3.2. $\delta$ Scuti stars

Another potential candidate for a 3:2 resonance is afforded by the $\delta$ Sct star DL Eri (HR1225) reported on by Poretti (1989). The Fourier spectrum seems to be dominated by three fre-quencies, $f_1 = 6.41$, $f_{2-} = 8.98$ and $f_{2+} = 10.26$ c/d. One notes that $3 f_1 = f_{2-} + f_{2+}$ to within the claimed accuracy of the frequen-cies; in other words, the low frequency is in exact resonance with the center of the high frequency doublet. Table 3 indicates that the low frequency mode must have even $\ell$ (and could also be radial); there are no $\ell$ constraints on the high frequency mode. It remains to see whether these conclusions would survive an increase in resolution and signal-to-noise ratio as provided by a multisite observation campaign.

### 6.4. 1:1:1 resonance: rotational splitting

#### 6.4.1. The ZZ Ceti star G226–29

This star is observed to have a Fourier spectrum composed of a triplet of modes which is split almost exactly evenly in fre-quency (Kepler, Robinson & Nather 1983). No amplitude vari-ations are mentioned. More recent observations using WET by Kepler et al. (1995) provide updated information. The main peak appears at $P = 2\pi/\omega = 109.3$s and has an average splitting of $16.15 \mu$Hz. It appears that the frequencies in the triplet have re-mained constant to within about one part in $10^4$ (as the limit of their measurements) and there is only marginal evidence that one of the amplitudes has changed by perhaps 20%. From these considerations it seems that the theoretical connection of am-plitude constancy and frequency locking is indeed verified for this star.

Since white dwarfs are slow rotators and since the linear deviation from equidistance is second order in the rotation fre-

quency (Eq. 17 and following paragraph) nonlinear effects need not be strong to bring about frequency locking. Thus, in this case, the splitting gives a good estimate of the rotation frequency (Kepler et al. 1995), from which we infer $\delta\omega/\omega \approx 1.2 \times 10^{-5}$. The approximate condition for frequency locking (Eq. 17) requires a growth rate to frequency ratio $\kappa/\omega$ of the order or larger than $\sim 10^{-5} - 10^{-6}$. This figure is consistent with what is currently obtained for low overtone $g$ – modes of DAV models (Bradley 1994).

### 6.4.2. DOV White dwarfs

Two additional variable white dwarfs are of interest for this discussion and these point to the need for more detailed studies of 1:1:1 resonances. These stars are PG1159-035 (Winget et al. 1991) and PG2131+066 (Kawaler et al. 1995), which have been studied extensively with WET. They are similar with respect to luminosity and effective temperature (luminous and hot) and both show conclusive evidence for a large number of $\ell = 1$ nonradial triplet modes. In PG1159 the triplet of highest amplitude has undergone a modulation in amplitude (of unknown time scale) among members of the triplet, while some triplets in PG2131 have shown curious changes in relative frequencies within a triplet. In PG2131 the low frequency peaks appear sharp, whereas the central and especially the high frequency peaks show a broader structure. The observed broadening of the lines in the Fourier spectrum could be caused by amplitude modulations which can have different depths for the sub-peaks of a given multiplet (BGS).

The Fourier spectra of PG1159 and PG2131 are dominated by just a few modes, but without any apparent resonances. All the other peaks have much lower power. According to Kawaler et al. (1995) these low-amplitude modes are observed to come and go on time scales of months to years. This is the kind of time scale for amplitude modulations that one may expect on the basis of linear growth rates in many circumstances (Lee & Bradley 1993). Could these low-amplitude intermittent modes be stochastically excited by, for example, turbulence associated with convection?

We mention the following as a possible scenario for the observed spectra. One or a few modes (multiplets) are linearly unstable and grow to saturation amplitudes given by

$$A^2 \approx \frac{1}{2q}(\kappa \pm \sqrt{\kappa^2 - 4q\mathscr{S}}) \approx \frac{\kappa}{q} \qquad (42)$$

where $\kappa$ and $q$ are a typical growth rate and cubic saturation coefficient (see, e.g. Buchler, Goupil & Kovács 1993). $\mathscr{S}$ is the noise intensity that drives the mode and has its origin in the convective and turbulent motions. The intra-multiplet amplitudes primarily owe their variations to nonlinear effects that are inherent in the 1:1:1 resonance (BGS) of the multiplet with itself.

The modes on either side are increasingly stable ($\kappa < 0$). If they are stable but stochastically excited, their amplitudes are given by $A^2 \approx \mathscr{S}/|\kappa|$. (This is the usual linear result valid when $q\mathscr{S}/\kappa^2 \ll 1$). Thus assuming that the noise intensity is

more or less the same for neighboring multiplets, one expects the amplitudes to taper off gradually on either side of the linearly unstable modes. It was shown theoretically (cf. Fig. 1 of BGK) that the amplitude fluctuations are largest for modes near $\kappa = 0$. If the above picture is correct then one expects the modes of lower amplitudes to undergo larger amplitude fluctuations than the dominant modes of large amplitude, with a concomitant correlation with broad and narrow peaks in a Fourier spectrum, features which are amenable to observational verification.

Superposed on this scenario would be the effects of trapping on the modes. Indeed one expects the growth rates to "oscillate" as a function of modal overtone. Thus there could be one or more further enhancements of the amplitudes on one or both sides of the dominant peaks. The spectrum of the DB variable white dwarf GD358 (Winget et al. 1994) shows amplitude variations between successive modes that might be an expression of this scenario.

### 6.4.3. δ Scuti stars

The observed frequencies $f_1 = 7.217139$, $f_2 = 7.346146$, $f_3 = 7.455268$ c/d of the $\delta$ Scuti star 1 Mon are locked, i.e $f_1 + f_3 = 2f_2$, to a relative accuracy of $10^{-5}$. The frequencies $f_1$ and $f_3$ have been identified as the $m = -1$ and $m = +1$ components of a $\ell = 1$ triplet split by rotation. The rotation rate is then $\Omega = 0.131$ revolution per day (Balona and Stobie, 1980). That $f_2$ is a radial mode as it has been identified or the centroid mode of the $l = 1$ triplet is still questioned (Balona and Stobie, 1980; Breger, 1992).

Whether the observed frequency-lock is in agreement with Eq. 17 depends on the eigenfrequencies *and* on the growth rates of the modes. If $f_2$ is the frequency of the $m = 0$ component of the triplet, the frequency lock ought to exist for a large enough growth rate i.e. $\kappa/\omega > (\Omega/\omega)^2 \approx 3.2 10^{-4}$ ($\omega \approx 0.5(f_1 + f_3)$), again within a factor 10 due to the involved, but discarded linear and nonlinear coefficients. This is the order of magnitude expected for the strongest linearly excited modes of $\delta$ Scuti stars. This would explain why the triplet appears equally spaced when a linear framework predicts, with the above rotation rate, a significant departure from equidistance.

On the other hand, the selection rules show that a locked radial mode, associated with $f_2$, is also possible. Of course, this requires that the $\ell = 0$ and the (unseen) $m = 0, \ell = 1$ modes are close enough for the resonance to be effective in locking the frequencies, a closeness that is encountered relatively often in models of $\delta$ Scuti stars. On a practical side, this second possibility shows that an observed multiplet of locked frequencies is not necessarily associated with modes of same degree $\ell$ but may well correspond to modes of different degrees.

## 7. Conclusion

We have discussed the constraints imposed by resonances and have considered the observational material to determine where such resonances may play a role, and where new information

may be gained from applying the amplitude equation formalism to nonradially pulsating stars.

We have assumed that individual or perhaps two or three multiplets can be studied in isolation from other observed multiplets. We would like to point out that this assumption may be even better justified when the *additional* observed multiplets, rather than being excited because of positive driving (by the $\kappa$-mechanism, for example), are merely stochastically excited.

With these assumptions we have shown that it is possible to put specific constraints on the nature of the interacting modes in a couple of variable white dwarfs.

We have discussed in some detail the relation between the frequency locks that can be introduced by resonances, and the constancy of the associated modal amplitudes. Conversely we have also discussed the amplitude and frequency modulations that can appear beyond the immediate vicinity of the resonances.

In the case of a 2:1 resonance we have shown on the basis of a numerical example the following scenario. When the star evolves toward a resonance condition it first undergoes pulsations with rapid, but small amplitude and frequency modulations. These modulations increase in size and decrease in frequency as the resonance is approached. In the vicinity of the resonance the star suddenly jumps into constant frequency locked pulsations.

For rotationally split multiplets the presence of frequency lock depends on the ratio of $(\Omega/\omega)^2$ being small (in order of magnitude) compared to $\kappa/\omega$ (Eqs. 16 and 17). When $\Omega$ is sufficiently large or $\kappa$ is sufficiently small one obtains amplitude and frequency modulated pulsations.

The general question of 'mode selection' also arises when there are possible several coexisting but mutually exclusive stable pulsational states. Which one does the star choose to pulsate in? In general, as evolution proceeds, the choice of pulsational state is unique. The simplest example is given by the nonresonant interaction of two modes ( cf. Buchler & Kovács 1986). At the instability blue edge the star starts to pulsate in the mode which has just become linearly unstable (more precisely by mode we mean the nonlinear mode which is an outgrowth of the unstable linear mode). It will then stay in this mode until this mode itself becomes unstable or bifurcates into another one. There can be hysteresis. During the rightward evolution in the HR diagram the pulsational state may differ from the one a leftward evolution would give. For completeness we recall that the coexisting stable pulsational states which can only be reached through a finite perturbation away from a stable pulsational state in which the system finds itself. Whether intrinsic turbulent or convective fluctuations could be large enough to effect such a "hard" bifurcation is unknown.

The nonlinear asteroseismology that we have discussed here is still in its infancy. We have shown how resonances can be used to help in the identification of modes. A few interesting results can be obtained from a phenomenological confrontation between the predictions of AEs and the observations. The major quantitative progress that we foresee, though, will come with the wedding of *ab initio* computations of the coupling coefficients to more comprehensive observations. With respect to the latter, great strides have been made with multi-site observations (WET, STEPHI, etc.). These collaborative efforts have been mostly geared toward linear asteroseismology and it would be very useful if they could be more attuned to specifically nonlinear considerations; in particular concentrate on those white dwarfs that have simple spectra, with ostensive resonances, both those with apparently constant amplitudes and those with modulations.

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