Astron. Astrophys. 321, 960–967 (1997)

ASTRONOMY AND ASTROPHYSICS

A deeper look at resonance trapping

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Received 5 August 1996 / Accepted 30 October 1996

Abstract. The problem of resonance trapping is further analyzed by the consideration of a quasi Hamiltonian model, which is the classical one-degree-of-freedom Hamiltonian model, modified by the inclusion of a nonconservative term for the derivative of the momentum. A new formulation to compute capture probabilities is deduced, which is a function of the particular nonconservative force involved. The new results can be quite different from those associated with the classical model, when the extra term for the variation of the momentum is not negligible in comparison with the variation of the resonance detuning parameter. This is the case of resonance trappings induced by gas drag. In this new formulation, we study all possible cases of resonance trappings as combinations of converging and diverging orbits with interior and exterior resonances. We find the possibility of resonance trapping for diverging orbits, although the cases for which they occur have not been found to be associated to any real Solar System force.

Key words: celestial mechanics – Solar System: general

1. Introduction

The association of nonconservative forces with resonant gravitational effects is a rich source of dynamical phenomena, which may be important both in a solely theoretical view or with applications to the Solar System. It has been widely accepted that the averaged equations of motion in a canonical formulation is a most appropriate formulation to study the process of trapping and evolution in resonance induced by nonconservative forces. In this respect, resonance trapping of natural satellites provoked by tidal forces has been studied by Dermott et al (1988), Malhotra (1988, 1990), Peale (1986), just to quote a few. The canonical equations can be parameterized into a simpler set of equations, more suitable for analytical deductions involving resonance trapping probabilities (Henrard 1982; Lemaitre 1984). One of the assumptions made for the successful application of this Hamiltonian averaged model with a varying parameter is the adiabatic variation of this parameter. That is why this model is suitable for problem involving tides which induces adiabatic variations of the parameter associated to the nonconservative force. The application of the same model to other nonconservative problems like those induced by Poynting-Robertson drag is not so natural, as this drag may usually lead to fast secular variation of the orbital elements (Beaugé & Ferraz-Mello 1984; Gomes 1995a). Even so a slightly modified model proved suitable to establish some results about stability and instability of resonance locks associated with nonconservative forces, even for the case of relatively fast dissipation (Gomes 1995b). This quasi-Hamiltonian model, henceforth referred to as QHM, is the same one-degree of freedom Hamiltonian system, now including a nonconservative variation of the momentum. Our purpose is to apply this model for the problem of resonance trapping.

One of the main results obtained by the pure Hamiltonian model is the impossibility of trapping for diverging orbits. However, numerical simulations show that these trappings, although infrequent and short-lived, are not at all impossible. This happens both for adiabatic processes (Dermott et al 1988; Afonso et al. 1994) and nonadiabatic processes (Marzari & Vanzani 1994). We investigate the question of resonance trapping for diverging orbits using the QHM. We find a modified expression for capture probability, for the low eccentricity case, which depends also on the value of the nonconservative variation of the momentum. Assuming that the QHM well represents the planar restricted three-body problem with dissipation, we also find that for some kinds of nonconservative forces, resonance trapping for diverging orbits is possible. However, for the cases of real Solar System forces, as quoted above, the QHM cannot explain the phenomenon. This may have another explanation as close approaches to the perturber or a chaotic regime (Gomes 1996), which prevents an explanation by a simple model. Nevertheless, the QHM gives a better theoretical view of the problem of resonance trapping than the traditional Hamiltonian model. It is important to add here that all references made to resonance trapping in this paper refer to libration resonance. There are also examples of resonance trapping for diverging orbits in the case of corotational resonances (Kary and Lissauer, 1995), but this cannot be studied by the traditional Hamiltonian model or the QHM. In their paper, a heuristic explanation for this kind of trapping is given.

In Sect. 2, we present the model, through which a generalization of the expression for capture probability is deduced. In Sect. 3, resonance trapping is reviewed under the QHM. Although all practical results are obtained for the low-eccentricity case, we give , in this section, a generic view of resonance trapping for the QHM. We present some ideas about stability and instability of a resonance lock in Sect. 4. For low eccentricities, we present a more detailed analysis of resonance trapping with the QHM in Sect. 5, and this topic is further developed in Sect. 6, where we link capture probabilities with generic nonconservative forces. Finally, we discuss our results in Sect. 7.

2. The quasi Hamiltonian model

We restrict ourselves to disturbed massless bodies near an eccentricity-type resonance. For this case, it is sufficient to consider a planar restricted 3-body problem. We start with the canonical Delaunay variables $L = (\mu a)^{1/2}$, $P = L[1-(1-e^2)^{1/2}]$, λ and $g = -\varpi$, where $\mu = n^2a^3$ is a constant, a is the semimajor axis, e is the eccentricity, λ is the mean longitude and ϖ is longitude of the perihelion, all referring to the perturbed body. The equations for these variables are:

$$\dot{L} = -\frac{\partial H}{\partial \lambda} + S_L \qquad \dot{\lambda} = \frac{\partial H}{\partial L} + S_\lambda$$
$$\dot{P} = -\frac{\partial H}{\partial g} + S_P \qquad \dot{g} = \frac{\partial H}{\partial P} + S_g$$

with:

$$H = -\frac{\mu^2}{2L^2} + \mathscr{R}(L, P, \lambda, g)$$

S_L, S_P, S_λ and S_g are generic non-canonical terms coming from a particular nonconservative force and $\mathscr{R}$ is the disturbing function.

Near a resonance, we may assume a one-degree-of-freedom quasi Hamiltonian system, which is obtained from the above one by convenient canonical transformations (Gomes 1995b):

$$\dot{P} = -\frac{\partial H}{\partial \phi} + S_P \qquad \dot{\phi} = \frac{\partial H}{\partial P} \tag{1}$$

with:

$$H = -\frac{\mu^2}{2(C+\frac{j+k}{k}P)^2} - \frac{j}{k}n_p P + \mathscr{R}(P, \phi; C) \tag{2}$$

$\mathscr{R}$ is the resonant disturbing function, $\phi = (1+j/k)\ \lambda - (j/k)\ \lambda_p + g$, where j and k are integers and λ_p is the disturbing body's mean longitude. $C = L - (1+j/k)P$ is a constant in the conservative problem. Here, the parameter C has a secular variation given by $\dot{C} = S_L - (1+j/k)S_P$. The expressions for S_P and $\dot{C}$ in terms of S_n and S_e, the secular variation for the mean motion and the eccentricity, are (Gomes 1995b):

$$S_P = -\frac{na^2}{3}\{(1-\sqrt{1-e^2})\,S_n/n - \frac{3e^2}{\sqrt{1-e^2}}S_e/e\}$$
$$\dot{C} = \mathscr{E}\{(\frac{-j}{j+k} + \sqrt{1-e^2})\,S_n/n + \frac{3e^2}{\sqrt{1-e^2}}S_e/e\} \tag{3}$$

where

$$\mathscr{E} = -\frac{na^2(j+k)}{3k}$$

3. Resonance trapping for the QHM

With $\dot{C} = 0$ and $S_P = 0$, Eq.(1) defines (P, ϕ) curves that either circulate or librate. For low eccentricities, $\mathscr{R}$ can be approximated by a single term (Lemaitre 1984). For high eccentricities, a development of $\mathscr{R}$ in higher order terms is necessary. However, for a real configuration of the phase space defined by Eqs.(1) and (2), one must resort to a numerical computation of $\mathscr{R}$. This is done in (Beaugé 1994) for the exterior resonances 2:1, 3:2 and 3:1 showing that the phase space structure is quite different from their corresponding interior resonances (exception made for the 3:2 resonance), with the appearance of asymmetric libration points. However, in spite of the complication that arises in the phase space defined by Eqs.(1) and (2) for high eccentricities, the existence of an exterior and an interior separatrix, which divide the phase space into a resonance region and two circulation zones, is always observed (let us restrict ourselves to a non-colliding phase space). This fact is fundamental for the computation of probability of capture into resonance, which is given by the relative variation of H with respect to H^* (the Hamiltonian associated to the separatrices) when the system makes a last circulation very close to one of the separatrices and a first circulation in the opposite direction very close to the other separatrix. In this way, the quantity B_i is defined by (Peale 1986):

$$B_i = \oint_{C_i} \frac{dN}{dt}dt$$

where $N = H - H^*$. Here, C_i stands for the exterior separatrix when $i = 1$ and for the interior separatrix, when $i = 2$.

In a pure Hamiltonian system dN/dt varies only through $\dot{C}$, and B_i is given by(Henrard 1982; Peale 1986):

$$B_i = \oint_{C_i}\left(\frac{\partial H}{\partial C} - \frac{\partial H^*}{\partial C}\right)\dot{C}dt = -\oint_{C_i}\frac{\partial \tilde{P}}{\partial C}\dot{C}d\phi$$

where $\tilde{P}$ is the momentum associated with the separatrices. If $|B_i| > |B_j|$ where i stands for the first crossed separatrix, capture may take place and its probability is (Henrard 1982; Peale 1986):

$$P_c = \frac{B_i + B_j}{B_i}$$

Now, there is nothing that prevents the use of the same reasoning that led to the deductions of B_i and P_c, for the quasi Hamiltonian system(1). In this case, the time derivative of H must have another term due to the secular variation of P. This yields the following expression for B_i:

$$B_i = -\oint_{C_i}\frac{\partial \tilde{P}}{\partial C}\dot{C}d\phi + \oint_{C_i} S_P d\phi \tag{4}$$

This new expression must be considered for the computation of capture probabilities associated with Eq. (1). It must be noted that Eq. (4) differs from the its classical expression basically by the second term that is absent for the pure Hamiltonian systems. But there is another subtle difference. It is the placement of $\dot{C}$ inside the integral. For the same reason as, in the pure Hamiltonian model, S_P is neglected in presence of $\dot{C}$ to compute capture probabilities, the variation of $\dot{C}$ as (P, ϕ) varies on the separatrices is also neglected, but here this must also be considered, together with S_P, so $\dot{C}$ inside the integral is necessary and another basic difference in respect with the expressions of B_i given by the pure Hamiltonian approach.

4. Some comments on stability

Another noteworthy point to mention here is the stability of a resonance after a capture. A necessary condition for an aspect of stability is that we have $\dot{C} = 0$ for a given resonance and some value of e. For this analysis, we consider four cases which are the combinations of dissipative and antidissipative regimes with interior and exterior resonances. Eq. (3) is basic for this analysis. Dissipative regimes are associated with $S_n/n > 0$ and antidissipative regimes with $S_n/n < 0$. We also assume $S_e/e < 0$ for dissipative regimes and $S_e/e > 0$ for antidissipative regimes [1]. It is not difficult to show from Eq. (3) that only for exterior resonances ($k > 0$), there is a e_0, $0 < e_0 < 1$, such that $\dot{C}(e) = 0$. $\dot{C} \neq 0$ for every e unavoidably leads to a rupture of the resonance. That is the case of interior resonances for which, even if trapping occurs, it is never stable. For the case of exterior resonance associated to a dissipative regime, we have $\dot{C} < 0$, for $e < e_0$ and $\dot{C} > 0$ for $e > e_0$. Thus $\dot{C} = 0$ defines an attractor point, because if a trapping occurs for $e < e_0$, $\dot{C} < 0$ implies $\dot{e} > 0$ and for $e > e_0$, $\dot{C} > 0$ implies $\dot{e} < 0$ (this comes from the expression linking C,P and L and the condition $< \dot{L} >= 0$, after a capture), leading to $e = e_0$ and $\dot{C} = 0$ after some time. On the other hand, for antidissipative regimes, the same reasoning yields $\dot{e} < 0$ for $e < e_0$ and $\dot{e} > 0$ for $e > e_0$, thus disallowing $e = e_0$, $\dot{C} = 0$. So we are left with the case of dissipative regime associated with exterior resonance, for which trapping may be probable and stability possible. It is important to note however that $\dot{C} = 0$ is just a necessary condition for stability. After capture and after evolution to $\dot{C} = 0$, the libration width may increase continuously until resonance is broken, thus no stability is reached. This view of stability is highly dependent on the particular form of the nonconservative force considered (Gomes 1995b).

5. The case of low eccentricities

Let us now turn back to the problem of trapping probabilities. For the case of low eccentricities, the Hamiltonian can be simplified into a parameterized form (Henrard 1982; Lemaitre 1984; Peale 1986). The quasi Hamiltonian equations become:

$$R' = -\frac{\partial K}{\partial \phi} + S_R$$
$$\phi' = \frac{\partial K}{\partial R} \tag{5}$$

where K is a Hamiltonian given by:

$$K = -\Delta R + R^2 - 2(2R)^{|k|/2} \cos \phi$$

Here R is the parameterized momentum, ϕ is the resonant angle and K is the parameterized Hamiltonian. Δ is a parameter (related with C) which defines whether System (5) is in resonance or not. The prime after a variable means the derivative with respect to a parameterized time τ. Like Eq. (1), Eq. (5) is also a quasi Hamiltonian model including the nonconservative term S_R for the variation of the momentum R. This is usually neglected in models where Δ' is assumed constant. For coherence, we must consider here also a linear term for Δ'. Thus we can write:

$$S_R = \mathscr{C} R$$
$$\Delta' = \mathscr{A} + \mathscr{B} R \tag{6}$$

where $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ are constants. The expressions for these constants can be deduced as a function of the particular nonconservative force considered and they are mutually related (see appendix), but for now let us assume them independent. For the pure Hamiltonian system, trapping probabilities have been deduced (Peale, 1986) and are given by:

$$P_c = \frac{2}{1 + \pi/(2\nu)} \tag{7}$$

where

$$\nu = \arcsin \frac{R_{max} + R_{min} - 2R_c}{R_{max} - R_{min}}$$

R_{max}, R_{min} and R_c are respectively the maximum value, the minimum value and the value associated with the unstable equilibrium point in an instantaneous separatrix. They depend only on Δ and the specific resonance considered (Gomes 1995a). When we assume the expressions (6) for S_R and Δ', Eq.(7) must be modified. We start with the expression for B_i (Peale 1986):

$$B_i = \oint_{C_i} \frac{dN}{d\tau} d\tau$$

where

$$N = K - K^*$$

[1] These last assumptions are valid for known real Solar System forces, although we can define fictitious forces for which this is not true.

and K^* is the Hamiltonian computed on the separatrices. Thus we get:

$$\frac{dK}{d\tau} = \Delta' R - \frac{\partial K}{\partial R} S_R$$

$$\frac{dK^*}{d\tau} = \Delta' R_c$$

This yields:

$$B_i = \oint_{C_i} \Delta'(R - R_c) d\tau - \oint_{C_i} S_R \phi' d\tau$$

Considering Eq.(6), we arrive at the following expressions for B_1 and B_2.

$$B_1 = B_{10} + B_{11} + B_{12}$$

$$B_2 = B_{20} + B_{21} + B_{22}$$

where

$$B_{10} = 2\mathscr{A}(\pi/2 + \nu)$$

$$B_{11} = 2\mathscr{B}[D_1 + \frac{R_{max} + R_{min}}{2}(\pi/2 + \nu)]$$

$$B_{12} = -\mathscr{C}(4 - |k|)[D_1 + D(\pi/2 + \nu)]$$

$$B_{20} = 2\mathscr{A}(-\pi/2 + \nu)$$

$$B_{21} = 2\mathscr{B}[D_1 + \frac{R_{max} + R_{min}}{2}(-\pi/2 + \nu)]$$

$$B_{22} = -\mathscr{C}(4 - |k|)[D_1 + D(-\pi/2 + \nu)]$$

$$D = \frac{|k| - 2}{4 - |k|}\Delta + R_c + \frac{R_{max} + Rmin}{2}$$

$$D_1 = \sqrt{(R_c - R_{min})(R_{max} - R_c)}$$

The expression for capture probability is thus deduced to be:

$$P_c = \frac{2}{1 \pm \frac{\pi}{2(\nu+\xi)}} \tag{8}$$

where

$$\xi = \frac{[2\mathscr{B}/\mathscr{A} - (4 - |k|)\mathscr{C}/\mathscr{A}]D_1}{2 + [R_{max} + R_{min}].\mathscr{B}/\mathscr{A} - (4 - |k|)(\mathscr{C}/\mathscr{A})D} \tag{9}$$

In the above probability formula, the plus sign stands for the cases where the exterior separatrix is crossed first (converging orbits), whereas the minus sign stands for the cases where the interior separatrix is crossed first (diverging orbits). We can notice that for the classical case ($\mathscr{B} = \mathscr{C} = 0$), we get $\xi = 0$ and the old probability formula is obtained. Because ν is always smaller than $\pi/2$, diverging orbits in the classical model is associated to a negative P_c, which means no possibility of trapping. In this modified model, $\nu + \xi$ may be larger than $\pi/2$, which may yield a possible positive trapping probability. Thus the trapping probability is somewhat modified by the presence of the extra term ξ, which is a function of $\mathscr{B}$ and $\mathscr{C}$, thus a function of the secular variation of the momentum and the first order (in R) variation of Δ. Given a resonance order (k) and the ratios $\mathscr{B}/\mathscr{A}$ and $\mathscr{C}/\mathscr{A}$, we get a function of probability through Eqs.(8) and (9) with Δ. Fig. 1 shows, by the continuous line, one such variation of capture probability with Δ, for $k = 1$, $\mathscr{C}/\mathscr{A} = 0.04$ and $\mathscr{B}/\mathscr{A} = -0.0001$. The dots are probabilities computed through the numerical solution of Eqs.(5), with $\mathscr{A} = -0.01$ and other parameters equal to those which yielded the analytical probabilities given by the continuous line. This graph checks the probabilities given by the new formula (8) with number of trappings given by the integration of Eqs.(5), from where Eqs.(8) and (9) were deduced. The numerical capture probability is computed as follows. For a given Δ, we start a numerical integration of Eqs.(5) with initial coordinates on the X-axis very close to the unstable equilibrium point, but slightly shifted to the right. This integration shows a capture. When we shift the initial point rightward, there will be a limit point, from the right of which initial points are now associated to noncaptured orbits. If we go further rightward we find another point to the right of which, all integrations starting on it show ϕ turning more than 360° before we get $\dot{\phi} = 0$ for the first time. With each of these 3 points there is an associated Hamiltonian value, denoted respectively by K_1, K_2 and K_3, where K_1 corresponds to the leftmost point and K_3 to the rightmost point. Trapping probability is thus computed through $P_c = (K_1 - K_2)/(K_1 - K_3)$,

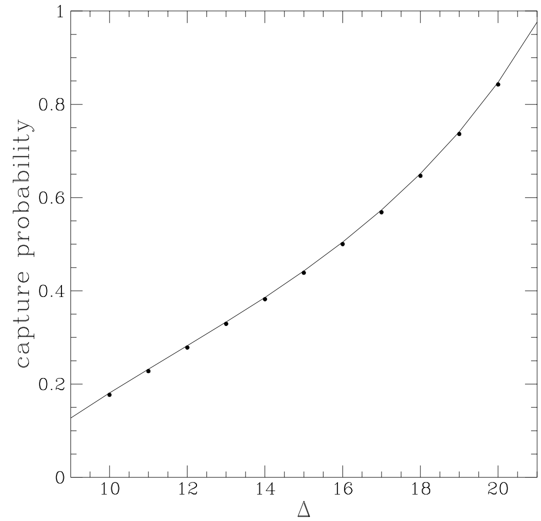

Fig. 1. Comparison of capture probabilities given by Eq. (8) with probabilities computed by the numerical solution of Eq.(5)

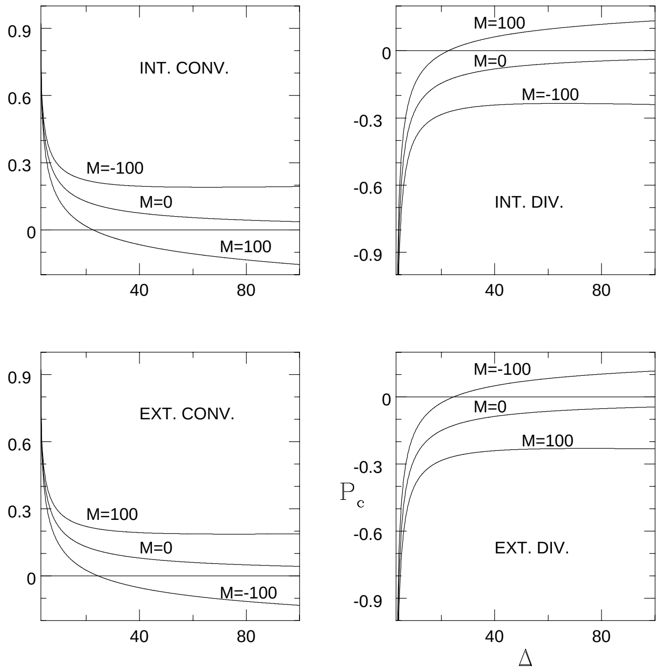

Fig. 2. Capture probabilities for the QHM, with the perturber's mass $m = 1/330000$

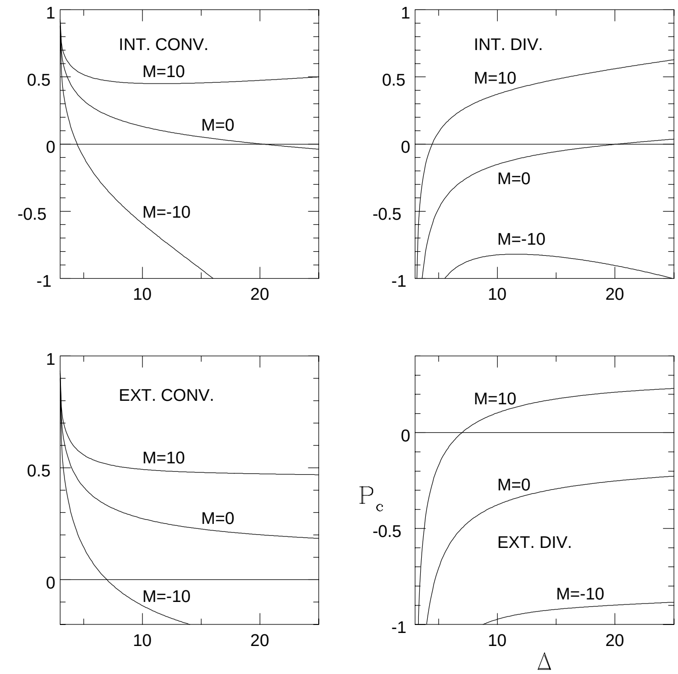

Fig. 3. Capture probabilities for the QHM, with the perturber's mass $m = 1/1047$

which gives the dots in Fig 1. The sign of $\mathscr{A}$ indicates that this is an example of a diverging orbit.

6. Capture probability for a generic nonconservative force

In the previous section, a new trapping probability formula was deduced as a function of 3 parameters ($\mathscr{A}, \mathscr{B}, \mathscr{C}$), which define the first order variation (with R) of Δ' and S_R. Starting with the planar restricted 3-body problem, we can find $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ as functions of $M = \frac{S_e/e}{S_n/n}$ (see appendix for deductions), which characterizes a specific nonconservative force. In a first order resonance, we find:

$$\frac{\mathscr{C}}{\mathscr{A}} = JM$$

$$\frac{\mathscr{B}}{\mathscr{A}} = J(1 + 2M)$$

where

$$J = 3k(j + k)\left[\frac{m^2 f^2}{9(j + k)^4}\right]^{\frac{1}{3}}$$

with $k = 1$, for exterior resonance, and $k = -1$, for interior resonances. m is the perturber's mass relative to the primary and f is a function of the Laplace coefficients (see appendix for details)

Replacing the expressions above into Eq. (9), we get:

$$\xi = \frac{JD_1(2 + M)}{2\{1 + J[\Delta - R_c + (\Delta - 2R_c)M]\}}$$

Considering the complete expression for capture probability (Eq. 9), we can plot P_c as a function of Δ , given m and M. Fig. 2 shows $P_c(\Delta)$ for four possible cases as combinations of diverging and converging orbits with interior and exterior resonances. The resonances considered are the 3:4 and 4:3. The perturber's mass is $m = 1/330000$ and three values are assumed for M. Let us first examine the case EXT-CONV which is the one most associated to real Solar System problems. For a more realistic check, refer to Fig. 4, where the dependence of e^* (the eccentricity associated to the unstable equilibrium point) with Δ is given for two values of the perturber's mass. The curve [2] for $M = 0$ is not very different from the one obtained with the classical formulation (in fact, $M = -2$ makes P_c equal to the classical expression). High positive M improves trapping probabilities, whereas high negative M makes trapping probabilities lower. High negative M is associated with gas drag ($M = -100$ is approximately the value for Stokes drag (Gomes 1995b)). What we see in Fig. 2 is that capture probabilities are lower for the Stokes drag in comparison with Poynting-Robertson drag, for instance ($M \cong -1$). This fact should not however be overemphasized since gas drag induces an important decrease of the eccentricity and, for low eccentricities, trapping probabilities are high anyway, as shown in Fig. 2. Moreover we must also consider the

[2] We opt for plotting continuous $P_c(\Delta)$ curves so as to make it clearer the variation of P_c with other parameters like M. Of course, a negative P_c means just a null probability

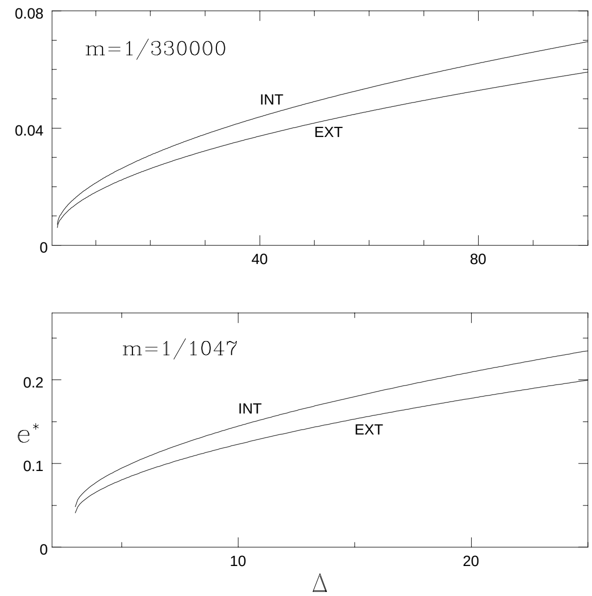

Fig. 4. Variation of the unstable equilibrium eccentricity (e^*) with Δ

nonadiabaticity which is usually associated with gas drag, and that a good fraction of resonance trappings induced by gas drag are corotational resonances (Beaugé & Ferraz-Mello 1993) and this is not studied by Hamiltonian models (or QHM). Now let us check the two cases of diverging orbits. We see that trapping is possible for high positive M in the case of interior resonances and for high negative M in the case of exterior resonances. Since the pair EXT-DIV means an antidissipative effect, this cannot be associated with gas drag although we have positive trapping probabilities for a high negative M. Fig. 3 is similar to Fig. 2, now assuming a perturber's mass $m = 1/1047$. The interpretation of the graphs is not too different here. We see that for higher perturber's mass, the probability function varies faster with Δ. On the other hand, as Fig. 4 shows, $e*$ also varies faster with Δ, so that there is not much difference in the interpretation of both figures. For instance, for the case INT-DIV in Fig. 3, one might argue that for low negative M we may have a positive P_c. But this happens for a Δ associated with a relatively high eccentricity (note that for the resonances considered, we already have intersecting orbits for $e \cong 0.2$) and our model cannot be applied.

We close this section giving a numerical example of a resonance trapping shown possible by Fig. 2. We choose the case of diverging orbits in an interior resonance with a large positive M. Although we do not know any real force that satisfies these conditions, we can simulate such a force. Motivated by the Stokes drag force model, we define a generic force which yields the following acceleration:

$$\boldsymbol{a} = \mathscr{K}(\boldsymbol{v} - \alpha \boldsymbol{v}_k)$$

where $\mathscr{K}$ and α are constants, $\boldsymbol{v}$ is the perturbed body's velocity and $\boldsymbol{v}_k$ is the Kepler velocity at the body's position. This expression yields Stokes drag for $\alpha = 0.995$ and a negative $\mathscr{K}$. If we do not fix α and also allow for positive $\mathscr{K}$, we can simulate a dissipative force that yields a high positive M. Fig. 5 shows the dependence of M with α for two values of the eccentricity (see (Gomes 1995b) for more details). Based on this figure, we started a numerical integration for a planar restricted 3-body problem, including a dissipative force given by the above formula with $K = 10^{-6} AU^{-1}$ and $\alpha = 1.002$. The perturber's mass is $m = 1/330000$ in a circular orbit with unitary semimajor axis. The massless body starts with $a_0 = 0.828$ and $e_0 = 0.03$. Fig. 6 shows a trapping into the 3:4 interior resonance. The eccentricity keeps increasing and resonance is eventually disrupted. Other examples also showed resonance trappings for cases predicted by Figs. 2 and 3.

7. Conclusions

We considered a more accurate model to study resonance trapping. This method, denoted by a quasi hamiltonian model (QMH), is a modification of the classical one-degree-of-freedom Hamiltonian model, where we include a nonconservative term in the variation of the momentum. This term is usually not considered in the solution of the problem of resonance trapping probability. In fact, for Poynting-Robertson drag, it can be shown that Δ'/S_R is proportional to the square of the eccentricity (Gomes 1995a) so that, for low eccentricity, S_R has no effective influence in resonance capture. When we consider a generic nonconservative force, we find that Δ'/S_R is also proportional to $M = \frac{S_e/e}{S_n/n}$. In this way, for forces that induce a high absolute value for M, one must consider the complete probability expression (Eqs. 9 and 10). For the case of converging orbits associated with exterior resonances, forces that induce a high negative M (gas drag) produce lower trapping probabilities than explained by the classical Hamiltonian model. Also, a natural consequence of this revised theory is the possibility of trapping for diverging orbits. This is achieved for cases where $|M|$ is high. However, the cases that show this possibility are associated only to fictitious forces.

Although the present revised theory is a step forward on the understanding of the process of capture into resonance, it is also disappointing that it does not explain some real cases of resonance trapping for diverging orbits. It is a well known fact that numerical simulations involving Poynting-Robertson drag show resonance trappings of the orbits of test particles into interior resonances with planets (Marzari and Vanzani 1994). Figure 2 does not however indicate any possibility of trapping for diverging orbits in the case of Poynting-Robertson drag (this is also true for higher j resonances). These cases are not anyway good examples to be explained by average theories because trappings usually occur for moderate eccentricities, so that there is usually a close approach to the perturber (Gomes 1996).

Appendix

We deduce the expressions for $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ as functions of M. We start with Eqs. (1) and (2). For low eccentricities, we

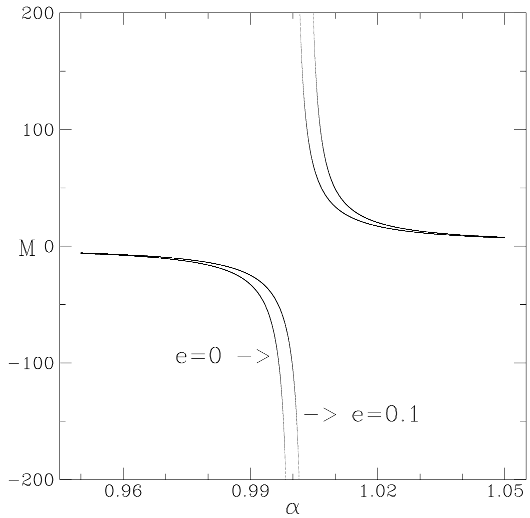

Fig. 5. Variation of $M = \frac{S_e/e}{S_n/n}$ with α for a generic force whose acceleration is given by $\boldsymbol{a} = K(\boldsymbol{v} - \alpha \boldsymbol{v}_k)$

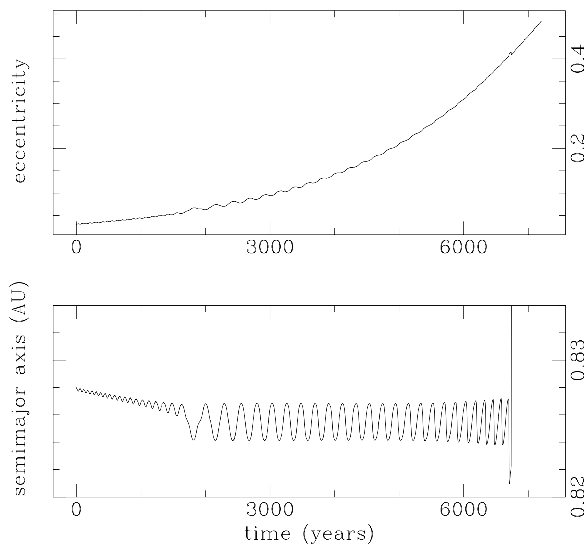

Fig. 6. Example of resonance trapping for a case of diverging orbits

can develop H in powers of P/C and, at second order in P, we arrive at:

$$H = \left[\frac{\mu^2(j+k)}{kC^3} - \frac{j}{k}n_p\right]P - \frac{3\mu^2(j+k)^2}{2k^2C^4}P^2 + \mathscr{R}(P,\phi;C)$$

where the disturbing function $\mathscr{R}$ can be given by its low order expression:

$$\mathscr{R} = m\ n^2a^2e^{|k|}f(\alpha)cos\phi$$

where $\alpha < 1$ is the ratio of semimajor axes.

Also, for low eccentricities, $P = (L/2)I$, with $I = e^2$, and the first derivative of P can be expressed by the sole variation of I. So we get the following low order quasi Hamiltonian system:

$$\dot{I} = -\frac{\partial H}{\partial \phi} + S_I$$

$$\dot{\phi} = \frac{\partial H}{\partial I}$$

$$H = -A_1 I - A_2 I^2 - A_3 I^{\frac{|k|}{2}} cos\phi$$

where

$$A_1 = -\frac{j+k}{k}n\left[1 + \frac{3}{2}\frac{(j+k}{k}e^2\right] + \frac{j}{k}n_p$$

$$A_2 = \frac{3}{4}\frac{(j+k)^2}{k^2}n$$

$$A_3 = -2\ m\ n\ f$$

This system can be applied for the problem of resonance trapping (Malhotra 1988). In this case, A_2 and A_3 are considered constant and A_1 has a secular variation due to S_n and S_I. This is justified due to the fact that, near a resonance, $A_1 \cong 0$ so that the evolution of the orbits defined by the above system can be accurately explained solely by the variation of A_1.

A better way to study resonance trapping is achieved by considering a transformed parameterized quasi Hamiltonian system, given by (Lemaitre, 1984; Peale, 1986; Gomes, 1995a):

$$\frac{dR}{d\tau} = -\frac{\partial K}{\partial \Phi} + S_R$$

$$\frac{d\Phi}{d\tau} = \frac{\partial H}{\partial R}$$

$$K = -\Delta R + R^2 - 2(2R)^{\frac{|k|}{2}} cos\Phi$$

where

$\Phi = \phi - \pi$ if $f < 0$, $\Phi = \phi$ if $f > 0$

$R = L_1 I$, $\tau = L_2 t$, $\Delta = -A_1/L_2$

L_1 and L_2 are computed through:

$L_1 L_2 = A_2$, $L_1^{\frac{4-|k|}{2}} = -2\ sign(f)\ 2^{\frac{|k|}{2}} A_2/A_3$

Now we are ready to compute Δ' and S_R as linear functions of R. From $\Delta' = -A_1'/L_2 = -\dot{A}_1/L_2^2$, we get:

$$\Delta' = \frac{j+k}{k}\frac{1}{L_2}S_n\left[1 + \frac{3}{2}\frac{j+k}{k}\frac{1}{L_1}(1+2M)R\right]$$

Also, from $S_R = L_1 I' = (L_1/L_2)\dot{I}$, we get:

$$S_R = \frac{2}{L_2} S_n / n \; M \; R$$

From these last results, we finally get the following ratios between the parameters $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$, where $\Delta' = \mathscr{A} + \mathscr{B} R$ and $S_R = \mathscr{C} R$:

$$\frac{\mathscr{C}}{\mathscr{A}} = \frac{3}{2} \frac{j+k}{k} \frac{1}{L_1} M$$

$$\frac{\mathscr{B}}{\mathscr{A}} = \frac{3}{2} \frac{j+k}{k} \frac{1}{L_1} (1 + 2M)$$

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