Astron. Astrophys. 321, 968–984 (1997)

ASTRONOMY AND ASTROPHYSICS

Polarized line formation by resonance scattering

II. Conservative case

V.V. Ivanov, S.I. Grachev, and V.M. Loskutov

Astronomy Department, St. Petersburg University, St. Petersburg 198904, Russia

Received 10 January 1996 / Accepted 7 November 1996

Abstract. We consider multiple resonance scattering with complete frequency redistribution (CFR) in a semi-infinite conservative atmosphere (photon destruction probability $\varepsilon_{\rm I} = 0$) with the sources at infinite depth. The polarization arising in resonance scattering is completely accounted for. The problem we consider is the resonance-scattering counterpart of the Chandrasekhar-Sobolev problem of Rayleigh scattering in the conservative atmosphere. The numerical data on the matrix source function $\mathbf{S}(\tau)$ in the atmosphere with conservative dipole resonance scattering (the depolarization parameter $W = 1$) are presented; we assume Doppler profile. The source matrix is found by a non-iterative numerical solution of the matrix Wiener-Hopf integral equation with the matrix Λ-operator. Depth dependence of the elements of the source matrix $\mathbf{S}(\tau)$ is discussed. Some unexpected peculiarities are revealed in the behavior of its polarization terms. The matrix $\mathbf{I}(z)$ which is the generalization of the Chandrasekhar H-function to the case of polarized resonance scattering is found by the iterative solution of the Chandrasekhar-type nonlinear matrix integral equation. We present high-accuracy (5 s.f.) numerical data on $\mathbf{I}(z)$ for dipole conservative scattering with the Doppler profile. The center-to-limb variation of the degree of polarization in the core of a Doppler broadened resonance line is found. In conservative case, the limiting limb polarization δ_0 in the core of such a line is 9.4430% (for $W = 1$). The dependence of δ_0 on the depolarization parameter W is found. Simple interpolation formula, $\delta_0 = \left(9.443 - 38.05\sqrt{\varepsilon_{\rm I}}\right)\%$, is suggested for the limb polarization of the radiation emerging from an isothermal nearly conservative atmosphere ($\varepsilon_{\rm I} \ll 1, \quad W = 1$). The data on $\mathbf{I}(z)$ are used to find the polarization line profiles and to trace their center-to-limb variation. The asymptotic expansions of $\mathbf{S}(\tau)$ for $\tau \to \infty$ (deep layers) and of $\mathbf{I}(z)$ for $z \to \infty$ (line wings) are found for the case of the Doppler profile. The coefficients of the expansions are determined by recursion relations. The numerical data on the accuracy and the domain of applicability of the asymptotic theory are presented.

Send offprint requests to: V.V. Ivanov

Key words: radiative transfer – line: formation – polarization – scattering – stars: atmospheres – Sun: atmosphere

1. Introduction

The most famous vector transfer problem is this one: to find the radiation field in a semi-infinite atmosphere with conservative Rayleigh scattering and sources at infinite depth. This is the conservative *vector Milne problem* with Rayleigh scattering. Physically, this problem arises if one considers the radiation field in a scattering atmosphere of free electrons, with the sources of radiation deep in the atmosphere (formally, at infinity). Astrophysically, this is an idealization of the atmosphere of an early-type star. The degree of polarization of the radiation emerging from such a free-electron atmosphere monotonically increases from the center to the limb and reaches 11.713% at the limb — the so-called Chandrasekhar-Sobolev polarization limit.

The classical vector Milne problem just outlined, as well as the problem of diffuse reflection of partially polarized radiation from a semi-infinite conservative atmosphere with Rayleigh scattering have been studied in every detail by many authors. Many years ago Chandrasekhar (1950) has shown that, for conservative Rayleigh scattering, the azimuth-averaged Stokes vector of the emergent radiation both in the vector Milne problem and in the problem of diffuse reflection can be expressed in terms of two auxiliary H-functions, $H_l(\mu)$ and $H_r(\mu)$, which obey uncoupled non-linear equations. These results suggest that the $\mathbf{I}$-matrix for conservative Rayleigh scattering is expressible in terms of $H_l(\mu)$ and $H_r(\mu)$; this is indeed the case (Ivanov et al. 1996b). In the sixties, Siewert & Fraley (1967), Bowden & Richardson (1968) and Mullikin (1968, 1969) have shown that for conservative Rayleigh scattering decoupling is possible for the Stokes vector at arbitrary depth in the atmosphere. Siewert & Fraley proceeded from the differential form of the transfer equation and used the Case method of singular eigenfunctions, while Bowden & Richardson and Mullikin used the singular integral equations satisfied by the Stokes vector at arbitrary depth. The Wiener – Hopf integral equations for the vector source function of the conservative Milne problem with Rayleigh scattering were decoupled and solved explicitly by Domke (1971) who used Sobolev's resolvent method. The same problem was solved also by the Wiener – Hopf method (Kuz'mina 1970). For further details, see van de Hulst 1980 (Chap. 16) and the recent review by Loskutov (1994).

The conservative Rayleigh scattering is the only case for which the vector problem can be reduced to auxiliary scalar problems and hence the explicit closed-form solutions of half-space transport problems are available. If photon destruction and/or depolarization are present, decoupling is not possible, and one has to resort to numerical approach. Extensive numerical data on the source functions for the vector Milne problem in case of molecular scattering have been published by Viik (1990). In a recent series of papers (Ivanov 1995, 1996; Ivanov et al. 1995, 1996a) we have reconsidered the whole class of problems of monochromatic molecular scattering, the vector Milne problem included, from a somewhat broader viewpoint than usual. This series of papers will be referred to as GRaS (from Generalized Rayleigh Scattering).

The counterpart of the vector Milne problem for CFR resonance scattering was formulated by Faurobert-Scholl & Frisch (1989) who called it the 'boundary layer solution'. They gave a brief theoretical analysis of this problem, in particular, they attempted to find the asymptotic form of the corresponding vector source function and presented some numerical information on the solutions of the integral vector transfer equation with the matrix Λ-operator corresponding to CFR resonance scattering. A minor addition to their analytical results was made in Ivanov (1990) where the vector version of the $\sqrt{\varepsilon}$-law was formulated.

The aim of the present paper is to present the detailed information, both analytical and numerical, on the vector Milne problem for conservative CFR resonance scattering. It seems that for non-rectangular profiles the reduction of the vector problems to auxiliary scalar ones is not possible. Hence closed-form solutions can hardly be obtained, and the analytical information is rather limited.

In Sect. 2 we reproduce basic equations of the standard problem of CFR resonance scattering discussed in detail in Paper I (Ivanov et al. 1996a). The aim of the Section is not only to formulate the problem to which the present paper is devoted, but also to make it readable without too many references to equations of Paper I. In Sect. 3 we present the detailed numerical data on the matrix source function $\mathbf{S}(\tau)$ in the conservative atmosphere with dipole ($W = 1$) Doppler scattering and discuss somewhat unexpected properties of its polarization terms. In Sect. 4 we give benchmark numerical results on the conservative Stokes matrix $\mathbf{I}(z)$ in terms of which the Stokes vector of the emergent radiation is readily expressed. This $\mathbf{I}(z)$ is the generalization of the Chandrasekhar H-function to the problem at hand. The matrix $\mathbf{I}(z)$ is found by the iterative solution of the alternative form of the non-linear Chandrasekhar-type matrix H-equation. The polarization properties of the emergent radiation are considered in Sect. 5. In particular, we find that for conservative atmosphere with dipole ($W = 1$) CFR Doppler scattering the limb polarization δ_0 is 9.443%; the dependence of δ_0 on photon destruction probability per scattering $\varepsilon_{\rm I}$ in nearly conservative atmospheres ($\varepsilon_{\rm I} \ll 1$) is found explicitly. In Sect. 6 the asymptotic theory of conservative CFR Doppler scattering is developed. The asymptotic expansions of the matrix source function for $\tau \to \infty$ (deep layers) and of $\mathbf{I}(z)$ for $z \to \infty$ (line wings) are found. We discuss the domain of applicability and the accuracy of the asymptotic theory and show that it may be used as a valuable check of the accuracy of numerical solutions of the vector transfer equation. Finally, in Sect. 7 we summarize and briefly discuss our main results.

The notation is identical to that used in Paper I and in GRaS series. Namely, boldface type is used to denote two-component vectors (lowercase Roman characters, e.g., $\mathbf{e}$, $\mathbf{i}$, $\mathbf{s}$ etc.) and 2×2 matrices (uppercase Roman and lowercase and uppercase Greek characters, e.g., $\mathbf{I}$, $\mathbf{S}$, $\boldsymbol{\epsilon}$, $\boldsymbol{\lambda}$, $\boldsymbol{\Psi}$ etc.).

2. Formulation of the problem

2.1. Basic equations

Our basic assumptions on single scattering are, first, complete frequency redistribution (CFR) and second, independence on depth of the photon survival probability per scattering $\lambda_{\rm I}$. In Paper I it was shown that under these assumptions the standard two-level problem of resonance line formation in a semi-infinite homogeneous atmosphere with uniformly distributed embedded sources of partially polarized radiation may be reduced to the matrix Wiener-Hopf integral equation for the 2×2 matrix source function $\mathbf{S}(\tau)$

$$\mathbf{S}(\tau) = \int_0^\infty \mathbf{K}_1(\tau - \tau')\,\mathbf{S}(\tau')\,d\tau' + \boldsymbol{\epsilon}^{1/2}, \tag{1}$$

or, in brief operator notation,

$$\mathbf{S} = \boldsymbol{\Lambda}\,\mathbf{S} + \boldsymbol{\epsilon}^{1/2}. \tag{2}$$

The matrix kernel function $\mathbf{K}_1(\tau)$ is given by

$$\mathbf{K}_1(\tau) = \int_{-\infty}^{\infty} \phi^2(x)\,dx \int_0^1 \mathrm{e}^{-\phi(x)|\tau|/\mu}\,\boldsymbol{\Psi}(\mu)\,d\mu/\mu\,, \tag{3}$$

where $\phi(x)$ is the absorption profile normalized to 1; since in problems of polarized resonance scattering the assumption of CFR applies only to line core, we consider only the most important case of the Doppler profile:

$$\phi(x) = \frac{1}{\sqrt{\pi}}\,\mathrm{e}^{-x^2}; \tag{4}$$

the characteristic matrix $\boldsymbol{\Psi}(\mu)$ is

$$\boldsymbol{\Psi}(\mu) = \frac{\lambda_{\rm I}}{2}\,\mathbf{A}^{\rm T}(\mu)\,\mathbf{A}(\mu)\,, \tag{5}$$

where

$$\mathbf{A}(\mu) = \begin{pmatrix} 1 & \sqrt{\frac{W}{8}}\,(1-3\mu^2) \\ 0 & \sqrt{\frac{W}{8}}\,3\,(1-\mu^2) \end{pmatrix}\,, \tag{6}$$

or explicitly

$$\mathbf{\Psi}(\mu) = \frac{\lambda_\mathrm{I}}{2}\begin{pmatrix} 1 & \sqrt{\frac{W}{8}}\,(1-3\mu^2) \\ \sqrt{\frac{W}{8}}\,(1-3\mu^2) & W\left(\frac{5}{4}-3\mu^2+\frac{9}{4}\mu^4\right) \end{pmatrix}\,. \tag{7}$$

Here λ_I is the albedo of single scattering and the parameter W describes depolarization, so that the phase matrix $\mathbf{P}$ is $\mathbf{P} = (1-W)\mathbf{P}_\mathrm{I} + W\mathbf{P}_\mathrm{R}$, where $\mathbf{P}_\mathrm{R}$ is the Rayleigh phase matrix and $\mathbf{P}_\mathrm{I}$ is the phase matrix of scalar isotropic scatttering. For the *dipole scattering* (transition $j_l = 0 \to j_u = 1$) we have $W = 1$, and the polarization effects are the largest possible. Finally, the primary source matrix in Eq. (1) is

$$\boldsymbol{\epsilon}^{1/2} = \mathrm{diag}\left(\varepsilon_\mathrm{I}^{1/2},\ \varepsilon_\mathrm{Q}^{1/2}\right)\,, \tag{8}$$

where

$$\varepsilon_\mathrm{I} = 1-\lambda_\mathrm{I}, \qquad \varepsilon_\mathrm{Q} = 1-\frac{7}{10}\,W\lambda_\mathrm{I} \equiv 1-\lambda_\mathrm{Q}; \tag{9}$$

the matrix

$$\boldsymbol{\epsilon} = \mathrm{diag}\left(\varepsilon_\mathrm{I},\ \varepsilon_\mathrm{Q}\right) \tag{10}$$

is the matrix generalization of the photon destruction probability.

We note in passing that the basic integral equation (1) for the matrix source function $\mathbf{S}(\tau)$ can be cast into two independent vector integral equations for the first and the second columns of $\mathbf{S}$, respectively. This allows one to treat the two columns independently.

The kernel function $\mathbf{K}_1(\tau)$ is normalized so that

$$\int_{-\infty}^{\infty} \mathbf{K}_1(\tau)\,d\tau = \mathbf{E} - \boldsymbol{\epsilon}\,, \tag{11}$$

while the source function $\mathbf{S}(\tau)$ satisfies the relations

$$\mathbf{S}^\mathrm{T}(0)\,\mathbf{S}(0) = \mathbf{E}\,; \qquad \mathbf{S}(\infty) = \boldsymbol{\epsilon}^{-1/2}, \tag{12}$$

which express the *matrix* $\sqrt{\varepsilon}$*-law*; here $^\mathrm{T}$ means transpose. We note that while the kernel matrix $\mathbf{K}_1(\tau)$ is symmetric, $\mathbf{K}_1^\mathrm{T}(\tau) = \mathbf{K}_1(\tau)$, the source matrix $\mathbf{S}(\tau)$ is *not* symmetric.

The analytical theory developed in this series of papers is based on the fact that the kernel function $\mathbf{K}_1(\tau)$ can be represented as the Laplace-type integral

$$\mathbf{K}_1(\tau) = \int_0^\infty \mathrm{e}^{-|\tau|/z}\,\mathbf{G}(z)\,dz/z\,. \tag{13}$$

The matrix $\mathbf{G}(z)$ is

$$\mathbf{G}(z) = 2\int_{x(z)}^{\infty} \phi^2(y)\,\mathbf{\Psi}\big(z\phi(y)\big)\,dy\,, \tag{14}$$

and in the particular case of the Doppler profile considered here the function $x(z)$ is

$$x(z) = 0,\ \ z < \sqrt{\pi}\,; \qquad x(z) = \sqrt{\ln\frac{z}{\sqrt{\pi}}}\,,\ \ z \ge \sqrt{\pi}\,. \tag{15}$$

The integration variable z appearing in Eq. (13) is related to the physical variables $\phi(x)$ and $\mu \equiv \cos\vartheta$, where ϑ is the angle between the direction of propagation of radiation and the outward normal to the atmospheric layers, as follows:

$$z = \frac{\mu}{\phi(x)}\,. \tag{16}$$

Therefore, large z values correspond to line wings.

2.2. Vector Milne problem

Astrophysically, the most interesting model problem is that of an atmosphere with depth-independent embedded primary sources of *unpolarized* radiation. Apart from the proportionality factor, the corresponding source vector $\mathbf{s}_\mathrm{I}(\tau)$ is the solution of

$$\mathbf{s}_\mathrm{I} = \mathbf{\Lambda}\,\mathbf{s}_\mathrm{I} + \varepsilon_\mathrm{I}^{1/2}\,\mathbf{e}_\mathrm{I}\,, \tag{17}$$

where

$$\mathbf{e}_\mathrm{I} = (1,0)^\mathrm{T}. \tag{18}$$

This vector source function is evidently the first column of the source matrix $\mathbf{S}(\tau)$ satisfying Eq. (1):

$$\mathbf{s}_\mathrm{I}(\tau) = \big(S_{11}(\tau),\ S_{21}(\tau)\big)^\mathrm{T}. \tag{19}$$

The source vector $\mathbf{s}_\mathrm{I}(\tau)$ corresponding to the particular case of conservative scattering ($\varepsilon_\mathrm{I} = 0$) will be denoted $\mathbf{s}_\mathrm{h}(\tau)$. It satisfies the *homogeneous equation*

$$\mathbf{s}_\mathrm{h} = \mathbf{\Lambda}_\mathrm{c}\,\mathbf{s}_\mathrm{h}. \tag{20}$$

The subscript ‘c’ emphasizes the fact that the $\mathbf{\Lambda}$-operator in this equation refers to conservative scattering. In accordance with the first of the relations of Eq. (12), we normalize $\mathbf{s}_\mathrm{h}$ by

$$\mathbf{s}_\mathrm{h}^\mathrm{T}(0)\,\mathbf{s}_\mathrm{h}(0) = 1. \tag{21}$$

Equations (20) and (21) define the *vector Milne problem* for conservative CFR resonance scattering. As we already mentioned, Eq. (20) first appeared in Faurobert-Scholl & Frisch (1989). However neither accurate numerical data nor detailed analytical information on this simplest standard problem of multiple resonance scattering are available. The aim of this paper is to fill in this gap.

As any Milne problem, the vector problem (20) – (21) physically corresponds to purely scattering source-free atmosphere, the radiation field in which is produced by radiation originating at infinite depth. The specific feature of the Milne problem discussed in this paper is that we consider CFR resonance scattering by two-level atoms.

We note that from $\mathbf{S}(\infty) = \boldsymbol{\epsilon}^{-1/2}$ it follows that in conservative case the first component of $\mathbf{s}_\mathrm{h}(\tau)$, i.e., $S_{11}(\tau)$, diverges for $\tau \to \infty$, while the second, $S_{21}(\tau)$, tends to zero. Physically, this means that if there are no losses of photons, the intensity of radiation increases with depth and tends to infinity as $\tau \to \infty$. At the same time, as τ increases, the radiation field becomes more isotropic, and thus polarization decreases. The detailed information on the rates of these changes in deep layers is provided by the asymptotic theory presented in Sect. 6.

2.3. Emergent radiation

We define the *Stokes matrix* of the emergent radiation $\mathbf{I}(z)$ by

$$\mathbf{I}(z) = \int_0^\infty \mathbf{S}(\tau)\, \mathrm{e}^{-\tau/z}\, d\tau/z\,, \quad z \geq 0. \tag{22}$$

The Stokes vector of the emergent radiation $\mathbf{i}(0,\mu,x) = \big(I(0,\mu,x),\ Q(0,\mu,x)\big)^\mathrm{T}$, where I and Q are the usual Stokes parameters, is expressed in terms of $\mathbf{I}(z)$ as follows:

$$\mathbf{i}(0,\mu,x) = \mathbf{A}(\mu)\, \mathbf{I}(\mu/\phi(x))\, \mathbf{e}\,, \quad \mu \geq 0, \tag{23}$$

where $\mathbf{e} = (1,\ 1)^\mathrm{T}$. In case of *unpolarized* primary sources uniformly distributed with depth, instead of Eq. (23) we have

$$\mathbf{i}(0,\mu,x) = \mathbf{A}(\mu)\, \mathbf{I}(\mu/\phi(x))\, \mathbf{e}_\mathrm{I}\,, \quad \mu \geq 0, \tag{24}$$

where $\mathbf{e}_\mathrm{I}$ is given by Eq. (18). In component form Eq. (24) is

$$I(0,\mu,x) = I_{11}\left(\frac{\mu}{\phi(x)}\right) + \sqrt{\frac{W}{8}}\ \left(1-3\mu^2\right)\ I_{21}\left(\frac{\mu}{\phi(x)}\right),$$

$$Q(0,\mu,x) = \sqrt{\frac{W}{8}}\, 3\left(1-\mu^2\right)\ I_{21}\left(\frac{\mu}{\phi(x)}\right). \tag{25}$$

These expressions hold also for the limiting case of conservative scattering, i.e., for the vector Milne problem. Then the matrix $\mathbf{I}$ in Eq. (24) refers to conservative scattering ($\varepsilon_\mathrm{I} = 0$).

The matrix $\mathbf{I}(z)$ satisfies two useful integral equations, one of them non-linear,

$$\mathbf{I}(z)\left(\boldsymbol{\epsilon}^{1/2} + \int_0^\infty \mathbf{I}^\mathrm{T}(z')\, \mathbf{G}(z')\, \frac{z'\,dz'}{z+z'}\right) = \mathbf{E}\,, \tag{26}$$

and the other linear,

$$\mathbf{T}(z)\, \mathbf{I}(z) = \boldsymbol{\epsilon}^{1/2} + \int_0^\infty \mathbf{G}(z')\, \mathbf{I}(z')\, \frac{z'\,dz'}{z'-z}\,, \tag{27}$$

where $\mathbf{T}(z)$ is the *dispersion matrix* defined by (cf. Sect. 5 of Paper I)

$$\mathbf{T}(z) = \mathbf{E} - 2z^2 \int_0^\infty \mathbf{G}(z')\, \frac{dz'}{z^2 - z'^2}\,. \tag{28}$$

The first of these equations is the matrix version of the alternative form of the Chandrasekhar H-equation. Its iterative solution may be used to find $\mathbf{I}(z)$ independently of $\mathbf{S}(\tau)$ (Sect. 4). The linear equation (27) is essentially the Laplace-transformed version of the basic equation (1) for the source matrix. It will be our main tool in the analytical investigation of the asymptotic behavior of $\mathbf{S}(\tau)$ for $\tau \to \infty$, i.e., deep in the atmosphere (Sect. 6).

3. Matrix source functions

In this and the next Section we present benchmark numerical data on, respectively, the source matrix $\mathbf{S}$ and the matrix $\mathbf{I}$ for conservative ($\varepsilon_\mathrm{I} = 0$) dipole ($W = 1$) CFR resonance scattering with the Doppler profile.

First we describe our numerical procedure for the calculation of the $\mathbf{S}$-matrix. It may be used for arbitrary values of $\lambda_\mathrm{I}, \lambda_\mathrm{Q} \in [0,1]$. However, in treating the limiting case of conservative scattering, which is our main concern in the present paper, some minor troubles arise. As we shall see, they may be overcome by using the analytical information on the asymptotic behavior of the solution for $\tau \to \infty$.

To find the elements of the source matrix $\mathbf{S}(\tau)$, we discretize the basic integral equation (1) with the matrix Λ-operator. The τ-discretization points $\tau_1, \tau_2, \ldots, \tau_N$ are distributed uniformly on logarithmic scale; the choice of the deepest point τ_N will be discussed later. This set is supplemented by one more point, $\tau_0 = 0$. The elements of the $\mathbf{S}$-matrix are approximated on this grid by quadratic splines. As a result, we get two separate sets of linear algebraic equations, one for the first, and the other for the second column of the $\mathbf{S}$-matrix. These sets are solved by the standard Gauss elimination. The coefficients of the algebraic equations are expressed in terms of the integrals

$$\int_0^\tau \mathbf{K}_2(t)\, t^i\, dt\,, \qquad i = 0, 1, \tag{29}$$

where $\mathbf{K}_2(\tau)$ is the second kernel function defined by

$$\mathbf{K}_2(\tau) = \int_\tau^\infty \mathbf{K}_1(\tau')\, d\tau'. \tag{30}$$

To evaluate these integrals by Gaussian quadrature, we used the Padé approximations of the elements $K_2^{ij}(\tau),\ i,j = 1,2,$ of $\mathbf{K}_2(\tau)$ (Hummer 1981; Faurobert-Scholl & Frisch 1989). [We note that there is a misprint in one of these coefficients. The coefficient p_8 of the Padé approximation of K_{22}^* given in Table 2b of Faurobert-Scholl & Frisch 1989, namely, $p_8 = -5.958974465 - 5$, is inaccurate; the correct value is $-5.958974649 - 5$. This misprint was found by comparing the published values (10 s.f.) with the list of the coefiicients with 17 s.f. kindly provided to the authors by H.Frisch and M.Faurobert-Scholl.]

After some experimentation, it was found that to get 4 s.f. accuracy in the results, one may set $\tau_1 = 10^{-3}$; up to 30 points per decade are needed. In choosing the deepest node τ_N one has to take into account the following circumstance concerning the evaluation of the integrals $\Lambda_{11}S_{11}$ and $\Lambda_{12}S_{11}$, where Λ_{ij} is the Λ-operator with the kernel $K_1^{ij}(\tau)$. In conservative case, the contribution to these integrals from the 'tails' of the integrands, i.e., from the domain $[\tau_N, \infty)$, is non-negligible even for very large τ_N. The reason is that in conservative case the thermalization depth is infinite, and hence $S_{11}(\tau)$ does not saturate.

Accordingly, in conservative case the value of τ_N was adjusted so that to evaluate this remainder, i.e., the integral over the interval $[\tau_N,\ \infty)$, one might use the values of $S_{11}(\tau)$ found from its asymptotic expansion for $\tau \to \infty$. As we show in Sect. 6, the latter has the form

$$S_{11}(\tau) \sim 4\pi^{-1}\left(\tau\sqrt{T}\right)^{1/2}\sum_{k=0}^{\infty}\frac{s_{11}^k}{T^k}, \tag{31}$$

where $T = \ln(\tau/\sqrt{\pi})$ and $s_{11}^0 = 1$; the coefficients s_{11}^k for $k > 1$ are defined by a recursion relation (cf. Eq. (94) below). The remainders are calculated by the Gauss quadrature.Then they are treated as the primary source terms. In conservative case, we usually set $\tau_N = 10^{10}$.

For non-conservative scattering, the total number of depth discretization points depends on the values of λ_{I} and λ_{Q}, ranging from ~ 50 to ~ 300. To estimate the accuracy of the results, several test cases were recalculated with increased number of nodes, up to ~ 400 (30 points per decade).

We also applied the asymptotic forms of $\mathbf{S}(\tau)$, both conservative and non-conservative, to specify $d\mathbf{S}(\tau)/d\tau$ at $\tau = \tau_N$. We used this specification as the additional condition needed to obtain all the coefficients of the quadratic spline approximation of $\mathbf{S}(\tau)$.

We note in passing that accurate evaluation of the 'tails' becomes crucial when one finds numerically the source matrix for biconservative scattering ($\varepsilon_{\mathrm{I}} = \varepsilon_{\mathrm{Q}} = 0$). This case, which is the direct matrix counterpart of the scalar conservative case (cf. GRaS.II and GRaS.IV), will be considered in a forthcoming paper of the present series. The remedy here is again the analytical information on the asymptotic behavior of the corresponding matrix source function $\mathbf{S}(\tau)$ for $\tau \to \infty$.

In Table 1 we present numerical data on the depth variation of the elements of the **S**-matrix for conservative CFR resonance scattering with the Doppler profile and $W = 1$ (so that $\varepsilon_{\mathrm{Q}} = 0.3$). We remind that $W = 1$ corresponds to the dipole, i.e., $j_l = 0 \to j_u = 1$, transition; in this case the polarization efffects are the largest possible. In Table 1 we give also the values of the scalar conservative source function $S_{\mathrm{h}}(\tau)$. It corresponds to $W = 0$. Comparison of the values of $S_{\mathrm{h}}(\tau)$ and $S_{11}(\tau)$ enables one to estimate the accuracy of the scalar approximation for treating the intensity. The accuracy is very high, definitely higher than the accuracy of the assumption of CFR. [Strangely enough, accurate values of $S_{\mathrm{h}}(\tau)$ have not been published so far, although the properties of this function, in particular, its asymptotic behavior for $\tau \to \infty$, have been established long ago (Ivanov 1973, Sect. 5.6; Frisch & Frisch 1982).]

In Fig. 1 we compare the vector source functions $\mathbf{s}_{\mathrm{h}}(\tau)$ for monochromatic Rayleigh scattering, i.e., for the Chandrasekhar-Sobolev, or **c**oherent **s**cattering (CS) problem (dashed lines), and for CFR **r**esonance **s**cattering (RS) with the Doppler profile (solid curves). At first glance, the behavior of the curves in the two cases is qualitatively the same. Indeed, in both cases the values of $S_{11}(0)$ are very close to unity (0.9944 and 0.9963 for CS and RS, respectively), while the values of $S_{21}(0)$ are close to -0.1 (more accurate numbers are -0.10568 for CS and -0.085993 for RS). In both cases the component S_{11} increases monotonically and tends to infinity as $\tau \to \infty$, with

$$S_{11}(\tau) \sim S_{\mathrm{h}}(\tau), \qquad \tau \to \infty. \tag{32}$$

Although the asymptotic behavior of $S_{\mathrm{h}}(\tau)$ in case of monochromatic scattering and for Doppler CFR dipole scattering is, of course, different, namely,

$$\text{Monochromatic}: \qquad S_{\mathrm{h}}(\tau) \sim \sqrt{3}\,\tau \tag{33}$$

and

$$\text{Doppler}: \qquad S_{\mathrm{h}}(\tau) \sim \frac{4}{\pi}\,\tau^{1/2}\left(\ln\frac{\tau}{\sqrt{\pi}}\right)^{1/4}, \tag{34}$$

respectively, this does not violate the conclusion that $S_{\mathrm{h}}(\tau)$ is monotonic and diverges as $\tau \to \infty$.

Finally, inspection of Fig. 1b gives the impression that for both CS and RS the component $S_{21}(\tau)$ is negative, increases monotonically and tends to zero as $\tau \to \infty$. Somewhat unexpectedly, this observation is erroneous. For CS, $S_{21}(\tau)$ is indeed negative and monotonic for all τ (see Table 6 in GRaS.II and Eq. (64) in GRaS.IV), but for RS with the Doppler profile $S_{21}(\tau)$ changes sign and is *not* monotonic. From the numerical data given in Table 1 it is clear that for τ larger than ~ 6.5 the values of $S_{21}(\tau)$ are positive; there is a maximum at $\tau \sim 14$. [We note in passing that to make the non-monotonicity of $S_{21}(\tau)$ visible, one should not only extend the abscissa to larger values of τ, but also substantially enlarge the scale of Fig. 1b. When we first found that $S_{12}(\tau)$ is positive for large τ, we suspected that it was an artifact caused by inaccuracy of our numerical scheme. Careful analysis has shown, however, that this is not so: the maximum really exists, and $S_{21}(\tau)$ is indeed positive for large τ. The latter fact is clear, e.g., from the asymptotic expansion of $S_{21}(\tau)$ for $\tau \to \infty$, the leading term of which in case of $W = 1$ is

$$S_{21}(\tau) \sim \frac{5\pi}{64\sqrt{2}}\,\tau^{-1/2}\left(\ln\frac{\tau}{\sqrt{\pi}}\right)^{-9/4}, \qquad \tau \to \infty. \tag{35}$$

This result is highly non-trivial. Its proof is given in Sect. 6. We note that the asymptotic form given in Faurobert-Scholl & Frisch 1989, Eq. (49),

$$S_{21}(\tau) = O\left(\tau^{-1/2}\left(\ln\frac{\tau}{\sqrt{\pi}}\right)^{-1/4}\right), \tag{36}$$

is inaccurate. [Their $\bar{P}_0^{BL}(\tau)$ is proportional to our $S_{21}(\tau)$.] To be more precise, Faurobert-Scholl & Frisch have shown that, in principle, the leading term of the expansion might have this form. Unfortunately, they have not attempted to find the numerical coefficient, in which case they would have found that it is zero.

The fact that for CFR scattering with the Doppler profile the polarization component of the source vector $\mathbf{s}_{\mathrm{h}}(\tau)$ of conservative Milne problem changes sign in rather deep layers of the atmosphere is surprising. It implies that in the line wings the polarization of the radiation emerging from such an atmosphere also changes sign. For *non-conservative* atmospheres such a behavior is well known and is easily understandable (see, e.g., Dumont et al. 1977, Rees & Saliba 1982). However, the fact

Table 1. Source matrix $\mathbf{S}(\tau)$ for conservative dipole ($W = 1$) scattering and scalar source function $S_\mathrm{h}(\tau)$

τ	$S_\mathrm{h}(\tau)$	$S_{11}(\tau)$	$S_{21}(\tau)$	$S_{12}(\tau)$	$S_{22}(\tau)$
0.0E+00	1.000E+00	9.963E−01	−8.599E−02	8.599E−02	9.963E−01
1.0E−03	1.002E+00	9.982E−01	−8.565E−02	8.668E−02	9.983E−01
1.0E−02	1.015E+00	1.011E+00	−8.363E−02	9.086E−02	1.011E+00
1.0E−01	1.105E+00	1.100E+00	−7.253E−02	1.158E−01	1.086E+00
2.0E−01	1.186E+00	1.181E+00	−6.451E−02	1.350E−01	1.144E+00
3.0E−01	1.259E+00	1.253E+00	−5.824E−02	1.505E−01	1.190E+00
4.0E−01	1.326E+00	1.321E+00	−5.301E−02	1.638E−01	1.230E+00
5.0E−01	1.390E+00	1.385E+00	−4.852E−02	1.754E−01	1.265E+00
6.0E−01	1.452E+00	1.446E+00	−4.459E−02	1.857E−01	1.296E+00
7.0E−01	1.511E+00	1.505E+00	−4.110E−02	1.950E−01	1.324E+00
8.0E−01	1.568E+00	1.563E+00	−3.798E−02	2.035E−01	1.349E+00
9.0E−01	1.623E+00	1.618E+00	−3.516E−02	2.112E−01	1.373E+00
1.0E+00	1.677E+00	1.673E+00	−3.260E−02	2.183E−01	1.394E+00
1.2E+00	1.782E+00	1.778E+00	−2.813E−02	2.309E−01	1.432E+00
1.4E+00	1.882E+00	1.879E+00	−2.438E−02	2.417E−01	1.465E+00
1.6E+00	1.979E+00	1.976E+00	−2.118E−02	2.511E−01	1.494E+00
1.8E+00	2.073E+00	2.070E+00	−1.844E−02	2.594E−01	1.519E+00
2.0E+00	2.164E+00	2.162E+00	−1.608E−02	2.666E−01	1.542E+00
2.5E+00	2.382E+00	2.381E+00	−1.145E−02	2.814E−01	1.588E+00
3.0E+00	2.589E+00	2.587E+00	−8.141E−03	2.926E−01	1.624E+00
3.5E+00	2.785E+00	2.784E+00	−5.732E−03	3.013E−01	1.652E+00
4.0E+00	2.973E+00	2.972E+00	−3.956E−03	3.081E−01	1.674E+00
4.5E+00	3.154E+00	3.153E+00	−2.633E−03	3.135E−01	1.693E+00
5.0E+00	3.328E+00	3.328E+00	−1.640E−03	3.179E−01	1.708E+00
5.5E+00	3.496E+00	3.497E+00	−8.901E−04	3.215E−01	1.720E+00
6.0E+00	3.660E+00	3.660E+00	−3.201E−04	3.244E−01	1.731E+00
7.0E+00	3.973E+00	3.974E+00	4.480E−04	3.289E−01	1.747E+00
8.0E+00	4.271E+00	4.272E+00	8.989E−04	3.322E−01	1.760E+00
9.0E+00	4.556E+00	4.557E+00	1.163E−03	3.345E−01	1.769E+00
1.0E+01	4.829E+00	4.829E+00	1.314E−03	3.363E−01	1.776E+00
1.2E+01	5.345E+00	5.346E+00	1.434E−03	3.388E−01	1.786E+00
1.4E+01	5.829E+00	5.829E+00	1.440E−03	3.404E−01	1.794E+00
1.6E+01	6.285E+00	6.286E+00	1.400E−03	3.415E−01	1.799E+00
1.8E+01	6.720E+00	6.720E+00	1.343E−03	3.423E−01	1.802E+00
2.0E+01	7.134E+00	7.135E+00	1.282E−03	3.428E−01	1.805E+00
2.5E+01	8.102E+00	8.103E+00	1.135E−03	3.437E−01	1.810E+00
3.0E+01	8.991E+00	8.992E+00	1.011E−03	3.441E−01	1.813E+00
3.5E+01	9.819E+00	9.820E+00	9.085E−04	3.443E−01	1.815E+00
4.0E+01	1.060E+01	1.060E+01	8.255E−04	3.445E−01	1.817E+00
5.0E+01	1.204E+01	1.204E+01	6.942E−04	3.445E−01	1.819E+00
6.0E+01	1.335E+01	1.335E+01	5.998E−04	3.445E−01	1.820E+00
7.0E+01	1.457E+01	1.457E+01	5.291E−04	3.444E−01	1.821E+00
8.0E+01	1.572E+01	1.572E+01	4.739E−04	3.442E−01	1.822E+00
9.0E+01	1.680E+01	1.680E+01	4.298E−04	3.441E−01	1.822E+00
1.0E+02	1.783E+01	1.783E+01	3.932E−04	3.440E−01	1.823E+00
1.5E+02	2.240E+01	2.240E+01	2.788E−04	3.434E−01	1.824E+00
2.0E+02	2.631E+01	2.631E+01	2.178E−04	3.429E−01	1.824E+00
3.0E+02	3.295E+01	3.295E+01	1.544E−04	3.421E−01	1.825E+00
4.0E+02	3.862E+01	3.862E+01	1.213E−04	3.416E−01	1.825E+00
5.0E+02	4.366E+01	4.366E+01	9.965E−05	3.411E−01	1.825E+00
1.0E+03	6.373E+01	6.373E+01	5.602E−05	3.398E−01	1.825E+00
1.0E+04	2.189E+02	2.189E+02	9.211E−06	3.360E−01	1.826E+00

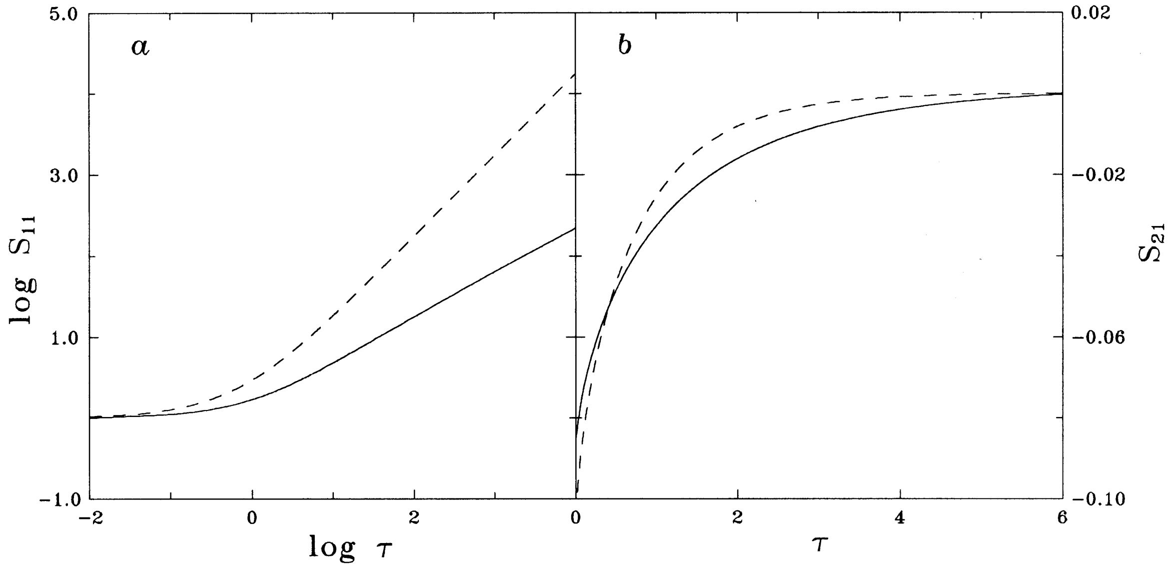

Fig. 1a and b. Depth dependence of the components S_{11} and S_{21} of the vector source function $\mathbf{s}_h$ of the conservative Milne problem for monochromatic (dashed lines) and for CFR scattering with the Doppler profile (solid curves). The data refer to W = 1

that the polarization of the emergent radiation may change sign in *conservative* case is nontrivial and completely unexpected. Of course, from the 'practical', i.e., diagnostic, point of view this result is not interesting since, first, it refers to the frequency domain where CFR approximation is only marginally applicable and, second, the degree of the negative wing polarization is far too low to be observable ($\sim 10^{-4}$%). But there is another, principally more important aspect. This case clearly shows how careful one should be in making any statements on the general features of the polarization patterns arising in multiple scattering. The statements based on 'common sense', and not on careful analysis of the specific problem, may well be erroneous.

4. I-matrices

To find the Stokes vector of the emergent radiation, one should have the matrix $\mathbf{I}(z)$. There are two possibilities. First, it can be found from Eq. (22) by numerical integration, provided the source matrix $\mathbf{S}(\tau)$ is known. Another possibility is to find $\mathbf{I}(z)$ directly, by solving either the linear or the non-linear equation for $\mathbf{I}(z)$. The comparison of the results obtained by any two of these methods is a reliable test of the accuracy of the calculations.

Our most accurate values of $\mathbf{I}(z)$ were found by the iterative solution of the non-linear Eq. (26).We followed the same procedure which we used recently (in GRaS.IV) for monochromatic scattering where it proved its high efficiency (cf. Rooij et al. 1989). The iteration loop was organized as follows. Let $\mathbf{I}_1(z)$ be its input. We calculate

$$\mathbf{I}_{n+1}(z) = \left(\epsilon^{1/2} + \int_0^\infty \mathbf{I}_n^{\mathrm{T}}(z')\,\mathbf{G}(z')\,\frac{z'\,dz'}{z+z'}\right)^{-1} \tag{37}$$

for n = 1 and 2 and form the output

$$\mathbf{I}_1(z) = \big(\mathbf{I}_2(z) + \mathbf{I}_3(z)\big)/2, \tag{38}$$

which is used as the new input. The cycle is stopped when for all the elements ij of the $\mathbf{I}$-matrices, i, j = 1, 2, and all z-discretization points z_α, $\alpha = 1, \ldots, M$, we have

$$|I_3^{ij}(z_\alpha) - I_2^{ij}(z_\alpha)| < \Delta \cdot |I_3^{ij}(z_\alpha)|, \tag{39}$$

where Δ is the desired accuracy (a prescribed number). When the convergence is achieved, Eq. (26) is used to calculate $\mathbf{I}(z)$ on a desired grid.

Typically we set $M \sim 800$ and $\Delta = 5 \cdot 10^{-9}$. We splitted the integral in Eq. (37) in two, from 0 to $\sqrt{\pi}$ and from $\sqrt{\pi}$ to ∞; in the second integral we made the substitution $Z = \ln(z/\sqrt{\pi})$ and then again divided the domain of integration in several parts. Each of the integrals which thus appeared except the last one were calculated by the Gauss quadrature, typically with 128 integration points. To find the last integral, with the large-z tail of the initial integrand, we used the Laguerre quadrature, usually with 64 nodes. To estimate the accuracy, control calculations were done with $M \sim 2000$ and $\Delta = 5 \cdot 10^{-10}$.

The iteration scheme just described provides reasonably fast convergence within the whole 'physical triangle' of the λ-plane, i.e., for all $\lambda_\mathrm{I} \in [0, 1]$ and $W \in [0, 1]$. More than that, the convergence is reasonably fast on the whole λ-plane except in a close vicinity of the point of biconservative scattering ($\lambda_\mathrm{I} = \lambda_\mathrm{Q} = 1$). To give an idea of the convergence rate: 56 iterations are needed for dipole (W = 1) conservative (λ_I = 1) scattering to meet the criterion (39) with $\Delta = 3.5 \cdot 10^{-9}$.

There are several other possibilities to test the accuracy of the numerical data. The first one is to use the matrix $\sqrt{\varepsilon}$ law, Eq. (12). Since according to Eq. (22) with z = 0 we have

$$\mathbf{I}(0) = \mathbf{S}(0), \tag{40}$$

Eq. (12) implies that

$$\mathbf{I}^{\mathrm{T}}(0)\,\mathbf{I}(0) = \mathbf{E}, \tag{41}$$

Table 2. Matrix $\mathbf{I}(z)$ for conservative dipole (W = 1) scattering and scalar functions $I_{\rm I}(z)$ and $I_{\rm Q}(z)$

z	$I_{\rm I}(z)$	$I_{11}(z)$	$I_{21}(z)$	$I_{12}(z)$	$I_{22}(z)$	$I_{\rm Q}(z)$
0.0E+00	1.0000E+00	9.9630E−01	−8.5993E−02	8.5993E−02	9.9630E−01	1.0000E+00
1.0E−01	1.0980E+00	1.0933E+00	−7.4081E−02	1.1270E−01	1.0765E+00	1.0781E+00
2.0E−01	1.1728E+00	1.1678E+00	−6.7301E−02	1.2914E−01	1.1260E+00	1.1260E+00
3.0E−01	1.2398E+00	1.2348E+00	−6.2127E−02	1.4209E−01	1.1650E+00	1.1638E+00
4.0E−01	1.3021E+00	1.2971E+00	−5.7899E−02	1.5292E−01	1.1976E+00	1.1954E+00
5.0E−01	1.3610E+00	1.3560E+00	−5.4321E−02	1.6226E−01	1.2257E+00	1.2226E+00
6.0E−01	1.4172E+00	1.4123E+00	−5.1225E−02	1.7046E−01	1.2505E+00	1.2467E+00
7.0E−01	1.4712E+00	1.4665E+00	−4.8505E−02	1.7776E−01	1.2726E+00	1.2681E+00
8.0E−01	1.5234E+00	1.5188E+00	−4.6086E−02	1.8434E−01	1.2925E+00	1.2875E+00
9.0E−01	1.5740E+00	1.5696E+00	−4.3915E−02	1.9031E−01	1.3107E+00	1.3052E+00
1.0E+00	1.6232E+00	1.6189E+00	−4.1951E−02	1.9577E−01	1.3273E+00	1.3214E+00
1.2E+00	1.7181E+00	1.7140E+00	−3.8528E−02	2.0541E−01	1.3568E+00	1.3502E+00
1.4E+00	1.8089E+00	1.8051E+00	−3.5636E−02	2.1371E−01	1.3822E+00	1.3752E+00
1.6E+00	1.8962E+00	1.8926E+00	−3.3154E−02	2.2094E−01	1.4046E+00	1.3971E+00
1.8E+00	1.9805E+00	1.9772E+00	−3.0995E−02	2.2732E−01	1.4243E+00	1.4166E+00
2.0E+00	2.0622E+00	2.0591E+00	−2.9099E−02	2.3301E−01	1.4420E+00	1.4341E+00
2.5E+00	2.2568E+00	2.2540E+00	−2.5224E−02	2.4487E−01	1.4791E+00	1.4709E+00
3.0E+00	2.4399E+00	2.4374E+00	−2.2238E−02	2.5427E−01	1.5088E+00	1.5004E+00
3.5E+00	2.6136E+00	2.6114E+00	−1.9861E−02	2.6193E−01	1.5332E+00	1.5248E+00
4.0E+00	2.7794E+00	2.7775E+00	−1.7923E−02	2.6832E−01	1.5537E+00	1.5453E+00
4.5E+00	2.9385E+00	2.9368E+00	−1.6312E−02	2.7374E−01	1.5712E+00	1.5629E+00
5.0E+00	3.0917E+00	3.0902E+00	−1.4951E−02	2.7840E−01	1.5863E+00	1.5782E+00
5.5E+00	3.2398E+00	3.2384E+00	−1.3787E−02	2.8246E−01	1.5995E+00	1.5916E+00
6.0E+00	3.3832E+00	3.3820E+00	−1.2778E−02	2.8603E−01	1.6112E+00	1.6035E+00
7.0E+00	3.6579E+00	3.6568E+00	−1.1121E−02	2.9202E−01	1.6310E+00	1.6236E+00
8.0E+00	3.9186E+00	3.9178E+00	−9.8167E−03	2.9686E−01	1.6472E+00	1.6401E+00
9.0E+00	4.1676E+00	4.1669E+00	−8.7637E−03	3.0086E−01	1.6606E+00	1.6538E+00
1.0E+01	4.4063E+00	4.4057E+00	−7.8967E−03	3.0423E−01	1.6720E+00	1.6655E+00
1.2E+01	4.8580E+00	4.8576E+00	−6.5550E−03	3.0958E−01	1.6904E+00	1.6844E+00
1.4E+01	5.2812E+00	5.2810E+00	−5.5673E−03	3.1366E−01	1.7045E+00	1.6989E+00
1.6E+01	5.6811E+00	5.6809E+00	−4.8118E−03	3.1688E−01	1.7158E+00	1.7106E+00
1.8E+01	6.0614E+00	6.0612E+00	−4.2167E−03	3.1948E−01	1.7250E+00	1.7201E+00
2.0E+01	6.4248E+00	6.4248E+00	−3.7365E−03	3.2162E−01	1.7327E+00	1.7281E+00
2.5E+01	7.2732E+00	7.2733E+00	−2.8669E−03	3.2564E−01	1.7474E+00	1.7434E+00
3.0E+01	8.0532E+00	8.0533E+00	−2.2873E−03	3.2844E−01	1.7578E+00	1.7543E+00
3.5E+01	8.7799E+00	8.7800E+00	−1.8762E−03	3.3050E−01	1.7657E+00	1.7625E+00
4.0E+01	9.4633E+00	9.4635E+00	−1.5712E−03	3.3207E−01	1.7718E+00	1.7689E+00
5.0E+01	1.0728E+01	1.0728E+01	−1.1526E−03	3.3431E−01	1.7808E+00	1.7783E+00
6.0E+01	1.1886E+01	1.1886E+01	−8.8201E−04	3.3581E−01	1.7871E+00	1.7849E+00
7.0E+01	1.2962E+01	1.2962E+01	−6.9494E−04	3.3687E−01	1.7918E+00	1.7899E+00
8.0E+01	1.3971E+01	1.3971E+01	−5.5920E−04	3.3767E−01	1.7954E+00	1.7937E+00
9.0E+01	1.4926E+01	1.4926E+01	−4.5706E−04	3.3827E−01	1.7983E+00	1.7967E+00
1.0E+02	1.5834E+01	1.5834E+01	−3.7797E−04	3.3874E−01	1.8007E+00	1.7992E+00
1.5E+02	1.9863E+01	1.9863E+01	−1.5983E−04	3.4005E−01	1.8081E+00	1.8070E+00
2.0E+02	2.3312E+01	2.3313E+01	−6.6418E−05	3.4059E−01	1.8120E+00	1.8112E+00
3.0E+02	2.9183E+01	2.9183E+01	9.8846E−06	3.4092E−01	1.8162E+00	1.8156E+00
4.0E+02	3.4197E+01	3.4197E+01	3.8288E−05	3.4094E−01	1.8183E+00	1.8179E+00
5.0E+02	3.8655E+01	3.8655E+01	5.0733E−05	3.4086E−01	1.8197E+00	1.8193E+00
1.0E+03	5.6419E+01	5.6419E+01	5.9352E−05	3.4025E−01	1.8225E+00	1.8223E+00
2.0E+03	8.2068E+01	8.2068E+01	4.9013E−05	3.3934E−01	1.8240E+00	1.8239E+00
5.0E+03	1.3409E+02	1.3409E+02	3.1635E−05	3.3798E−01	1.8250E+00	1.8250E+00
1.0E+04	1.9387E+02	1.9387E+02	2.1255E−05	3.3697E−01	1.8254E+00	1.8253E+00

whence it follows that $\mathbf{I}(0)$ is a rotation matrix (cf. Sect. 6 of Paper I):

$$\mathbf{I}(0) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \equiv \mathbf{R}. \quad (42)$$

Hence the elements of $\mathbf{I}(0)$ obey three relations, namely,

$$I_{11}^2(0) + I_{21}^2(0) = 1, \qquad I_{12}^2(0) + I_{22}^2(0) = 1 \quad (43)$$

and

$$I_{12}(0) = -I_{21}(0). \quad (44)$$

Incidentally, from Eqs. (42) and (26) with $z = 0$ it follows that

$$\int_0^\infty \mathbf{G}(z)\,\mathbf{I}(z)\,dz = \mathbf{R} - \boldsymbol{\epsilon}^{1/2}. \quad (45)$$

Another possibility to check the accuracy of the data on $\mathbf{I}(z)$ obtained by the numerical solution of Eq. (26) is to compare them with the asymptotic solutions valid for large z; such a comparison is given in Sect. 6.7. Finally, as we already mentioned, one can use the source matrix $\mathbf{S}(\tau)$ as found from the basic integral equation with the Λ-operator to calculate $\mathbf{I}(z)$ by applying the usual source function to emergent radiation mapping, Eq. (22). We used all the three checks.

In Table 2 we give the values of the elements of the matrix $\mathbf{I}(z)$ for conservative scattering with the Doppler profile and $W = 1$. In columns labeled I_I and I_Q we present scalar I-functions corresponding to $\lambda_\mathrm{I} = 1,\ \lambda_\mathrm{Q} = 0$ and to $\lambda_\mathrm{I} = 0,\ \lambda_\mathrm{Q} = 0.7$, respectively. These scalar I-functions were used to form the initial input of our iteration loop:

$$\mathbf{I}_1(z) = \mathrm{diag}\left(I_\mathrm{I}(z),\ I_\mathrm{Q}(z)\right). \quad (46)$$

¿From Table 2 one can see that the qualitative behavior of $I_{11}(z)$ and of $I_\mathrm{I}(z)$ is essentially the same; the only difference that worth mentioning is $I_{11}(0) = \cos\varphi$, whereas $I_\mathrm{I}(0) = 1$. From the numerical data it is evident that $I_{11}(z)$ is asymptotically equal to $I_\mathrm{I}(z)$ for large z; the proof of this fact will be given in Sect. 6.1. The behavior of $I_{21}(z)$ is less trivial. Being negative for not too large z, it increases from $I_{21}(0) = -0.085993$, changes sign at $z \sim 3\cdot 10^2$, reaches maximum at $z \sim 1\cdot 10^3$ and then slowly tends to zero. Although this behavior is qualitatively the same as for $S_{21}(\tau)$ (cf. Sect. 3), the values of the arguments at which the two functions, $S_{21}(\tau)$ and $I_{21}(z)$, change sign, and the position of the maxima are substantially different.

In the physical domain of the λ-plane, i.e., for $\lambda_\mathrm{I} \in [0,1]$ and $W \in [0,1]$, the functions $I_{12}(z)$ and $I_{22}(z)$ vary within rather narrow limits; one can show that for conservative scattering with $W = 1$

$$I_{12}(\infty) = \frac{1}{4}\sqrt{\frac{5}{3}}\,; \quad I_{22}(\infty) = \sqrt{\frac{10}{3}}\,. \quad (47)$$

We note that $I_{12}(z)$ is not monotonic; it has a maximum at $z \sim 4\cdot 10^2$.

5. Polarization of the emergent radiation

In this Section we consider mainly (but not exclusively) the simplest case of the vector Milne problem with resonance scattering. In view of Eq. (25), the degree of polarization of the emergent radiation

$$\delta(\mu, x) \equiv -\frac{Q(0,\mu,x)}{I(0,\mu,x)} \quad (48)$$

is expressed in terms of $I_{11}\left(\mu/\phi(x)\right)$ and $I_{21}\left(\mu/\phi(x)\right)$. [The elements I_{12} and I_{22} of the $\mathbf{I}$-matrix appear only in case of partially polarized primary sources.]

Polarization profiles $\delta(\mu, x)$ are shown in Fig. 2 for several values of μ. The curves 1 to 4 correspond to very small values of μ, namely, $\mu = 0,\ 10^{-4},\ 10^{-3}$ and 10^{-2}, respectively, and hence, from the observer's point of view, are not interesting. For a theoretician, however, they are instructive. In the majority of the publications on resonance scattering (e.g., in Dumont et al. 1977; Rees & Saliba 1982; Faurobert 1988 etc.), the polarization profiles are reproduced only for $\mu = 0.11$, which closely corresponds to our curve 6 ($\mu = 0.1$). According to Fig. 2, at this μ the polarization in the center of the line ($x = 0$) is still much lower than at the limb ($\mu = 0$).

The dependence on μ of the degree of polarization in the center of the line $\delta(\mu, 0)$ is shown in Fig. 3 for three values of W, namely, 1.0 (curve 1), 0.8 (curve 2) and 0.5 (curve 3).

According to Eqs. (25) and (42), the polarization at the limb ($\mu = 0$) is expressed in terms of the polarization angle φ as follows:

$$\delta_0 \equiv \delta(0, x) = 3\sqrt{\frac{W}{8}}\ \sin\varphi\,\left(\cos\varphi - \sqrt{\frac{W}{8}}\,\sin\varphi\right)^{-1}. \quad (49)$$

In Table 3 we give the values of δ_0 as a function of the depolarization parameter W. For $W \geq 0.5$ a reasonable approximation is $\delta_0 = 0.0944W^{5/4}$. For $W = 1$ (dipole scattering) the limb polarization is the largest possible:

$$\delta_0 = 9.4430\%. \quad (50)$$

This limiting polarization is *lower* than the corresponding polarization in case of monochromatic Rayleigh scattering known as the Chandrasekhar-Sobolev polarization limit (11.713%) (cf. Chandrasekhar 1950, Sect. 68; Sobolev 1963, Chap. 5). The reason is clear: frequency redistribution lowers the depth gradients of the radiation field, which results in lowering the anisotropy of the emergent radiation, and this, in turn, decreases the polarization. Somewhat unexpectedly, the influence of frequency redistribution on polarization is rather weak, and the limb polarization remains high.

If the role of redistribution were larger, e.g., if the absorption profile were purely Lorentzian, the value of δ_0 would be lower. On the contrary, for partial frequency redistribution the imprisonment of radiation is stronger than in case of CFR. Therefore, the limb polarization in the center of the line formed by resonance scattering in a conservative atmosphere

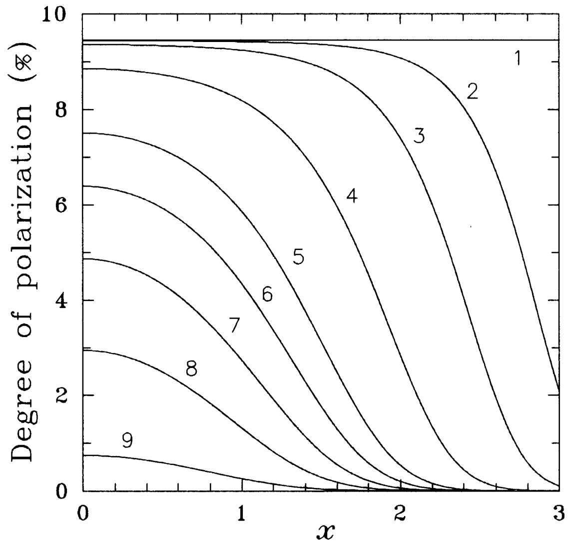

Fig. 2. Polarization profiles of the line formed in conservative atmosphere with $W = 1$. The curves refer to the following values of μ: 1 – 0.0; 2 – 0.0001; 3 – 0.001; 4 – 0.01; 5 – 0.05; 6 – 0.1; 7 – 0.2; 8 – 0.4; 9 – 0.8

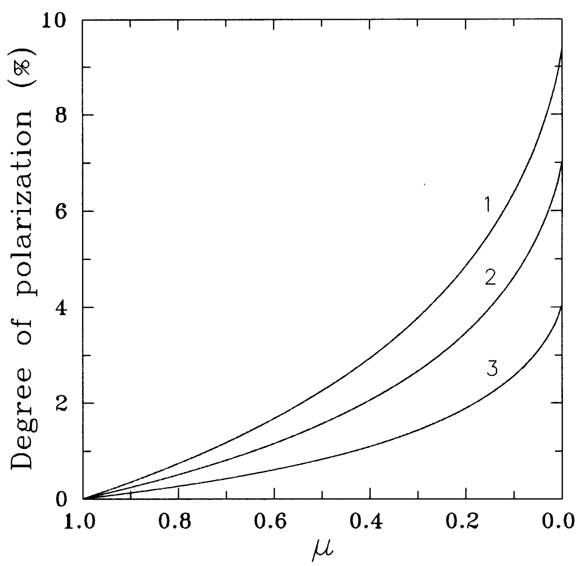

Fig. 3. Center-to-limb variation of the polarization at line center ($x = 0$) (conservative atmosphere). The curves refer to the following values of the depolarization parameter W: 1 – 1.0; 2 – 0.8; 3 – 0.5

with purely Doppler partial frequency redistribution must be (slightly) higher than 9.44%.

All the numerical data given so far refer to the limiting case of conservative scattering ($\varepsilon_{\rm I} = 0$). It seems useful to supplement them with some information on nearly conservative scattering ($\varepsilon_{\rm I} \ll 1$). For dipole scattering the value of $\sin\varphi$ defined by Eq. (42) is

$$\sin\varphi = 0.085993 - 0.333\,\sqrt{\varepsilon_{\rm I}}\,, \quad W = 1. \tag{51}$$

This useful approximation was found from the analysis of our numerical data on nearly conservative scattering. For $\varepsilon_{\rm I} \le 10^{-4}$ it gives $\sin\varphi$ with at least four significant figures. We have not attempted to prove analytically that the dependence on $\varepsilon_{\rm I}$ has indeed this surprisingly simple form. In principle, one might expect an extra factor of the type, say, $(-\ln\varepsilon_{\rm I})^{1/4}$. The numerical data, however, strongly suggest that we have here the 'pure' square root.

Table 3. Limb polarization δ_0 as a function of depolarization parameter W

W	δ_0 (%)	W	δ_0 (%)	W	δ_0 (%)
1.00	9.443	0.8	7.072	0.4	3.193
0.95	8.812	0.7	6.014	0.3	2.344
0.90	8.208	0.6	5.021	0.2	1.531
0.85	7.629	0.5	4.084	0.1	0.751

For an isothermal atmosphere with unpolarized primary source the limb polarization δ_0 is given by the same Eq. (49) as in the case of the vector Milne problem. The only difference is that the **I**-matrix refers to non-conservative case. The dependence of δ_0 on $\varepsilon_{\rm I}$ should have the same functional form as in case of $\sin\varphi$. We found that

$$\delta_0 = \bigl(9.443 - 38.05\sqrt{\varepsilon_{\rm I}}\bigr)\%, \qquad W = 1. \tag{52}$$

This useful approximation reproduces both our extensive numerical data and the result quoted in Bommier & Landi Degl'Innocenti (1996) for $\varepsilon_{\rm I} = 10^{-4}$.

The $\sqrt{\varepsilon_{\rm I}}$ scaling of the correction term of polarization in resonance scattering problems was reported earlier by Faurobert (1988). Her result was obtained assuming monochromatic scattering. The fact the same scaling holds for CFR scattering with the Doppler profile suggests that at the root of the $\sqrt{\varepsilon_{\rm I}}$ appearing in Eqs. (51) and (52) is the translational invariance of our half-space problem.

6. Asymptotic theory

In this Section we develop the matrix generalization of the well-known scalar CFR asymptotic theory of line formation in purely scattering atmospheres (Ivanov 1973, Chap. 5 and 6; Frisch & Frisch 1982; Nagirner 1984; Frisch 1988). In the matrix case, closed-form exact solutions of half-space CFR problems with non-rectangular profiles are unknown (indeed, they hardly exist at all). Hence the asymptotic information should be extracted directly from the basic equations. This is exactly the approach by which the first scalar CFR asymptotics have been found (cf. Ivanov 1973 and references therein).

We consider the asymptotic expansions of $\mathbf{S}(\tau)$ for $\tau \to \infty$, i.e., in deep layers, and of $\mathbf{I}(z)$ for $z = \mu/\phi(x) \to \infty$, i.e. in line wings. As before, we assume Doppler profile and CFR conservative scattering. Special attention is given to the first columns of the matrices $\mathbf{S}(\tau)$ and $\mathbf{I}(z)$, i.e., to conservative vector Milne problem for the case of polarized resonance scattering with Doppler broadening.

We have developed two methods of the asymptotic analysis of the matrix transfer equation. The first method is based directly on the analysis of the basic integral equation for the source matrix. In a sense, this is the most natural approach which makes this method very attractive. Unfortunately, the transformations are extremely long. This approach will be considered elsewhere (Grachev 1996).

The second method is the direct generalization of the technique used earlier in scalar CFR radiative transfer problems (cf., e.g., Ivanov 1973, Sect. 5.6). The point of departure is not the basic linear equation (1) with the Λ-operator for the source matrix $\mathbf{S}(\tau)$, but its Laplace-transformed version, i.e., the linear equation for the matrix $\mathbf{I}(z)$ (Eq. (27)). From this equation one can find the complete asymptotic expansion of $\mathbf{I}(z)$ for $z \to \infty$, which is then used to reconstruct the expansion of $\mathbf{S}(\tau)$ for $\tau \to \infty$. This second approach has serious merits. It is much easier than the first method just described. One follows nearly the same line of reasoning as in the scalar case, where it has proved its efficiency. More than that, by using the relations of the scalar theory, one can circumvent long algebraic manipulations. As a result, the complete asymptotic expansion is obtained — and this is the maximum that one may expect to get analytically in polarized line transfer problems.

In Sect. 6.1 – 6.5 we reproduce the derivation of the asymptotic expansion of the *vector* source function of the conservative CFR Milne problem using the first version of our method II. There are two reasons why we do not present the corresponding results for the second column of the source matrix. First, the elements of the second column vary within narrow limits (in the physical domain of the λ-plane), and hence their asymptotic expansions are not very useful. And second, not only the derivation but even the results themselves are extremely cumbersome. We decided to give (without proof) only a few first terms of the expansions of $S_{12}(\tau)$ and $S_{22}(\tau)$ for $\tau \to \infty$ (Sect. 6.6). However, the numerical values of the coefficients of the first 10 terms of the expansions are presented for all the four elements of the source matrix.

We dare say that the asymptotic results presented in this Section are at the very limits of the possibilities of the analytical radiative transfer (ART). Although practical usefullness of these results is not high, their theoretical value is out of question. They are among the most refined results of ART so far found.

6.1. Basic formula

Let us first write out explicitly the 11 and the 21 components of the matrix equation (27) for $\mathbf{I}(z)$:

$$T(z)\, I_{11}(z) + T_{12}(z)\, I_{21}(z) = \int_0^\infty \left(G(z')\, I_{11}(z') + G_{12}(z')\, I_{21}(z') \right) \frac{z'\, dz'}{z' - z}\,, \tag{53}$$

$$T_{21}(z)\, I_{11}(z) + T_{22}(z)\, I_{21}(z) = \int_0^\infty \left(G_{21}(z')\, I_{11}(z') + G_{22}(z')\, I_{21}(z') \right) \frac{z'\, dz'}{z' - z}\,. \tag{54}$$

Here we have taken into account that in the conservative case ($\lambda_{\mathrm{I}} = 1$) we have $G_{11}(z) = G(z)$ and $T_{11}(z) = T(z)$, where $G(z)$ and $T(z)$ refer to scalar scattering, i.e., to $W = 0$.

In the limiting case of large z these equations assume much simpler form. The radiation with large $z \equiv \mu/\phi(x)$, where μ is the usual direction cosine, comes from deep layers, and the polarization tends to zero as $\tau \to \infty$. Hence, $I_{21}(z) \to 0$ and $I_{11}(z) \to I(z)$ for $z \to \infty$; here $I(z)$ is the scalar I-function. It is related to the H-function used in Ivanov 1973 by $I(z) = H(\phi(0)\, z)$. It is well known that in conservative case $I(z)$ diverges for $z \to \infty$ (Ivanov 1973, Sect. 5.4; cf. Eq. (66) below). As we have shown in Paper I, Eqs. (99) and (105), the *leading terms* of the large-z asymptotics of the matrices $\mathbf{G}(z)$ and $\mathbf{T}(z) - \boldsymbol{\epsilon}$ are given by

$$\mathbf{G}(z) \sim \boldsymbol{\kappa}_1\, G(z) \tag{55}$$

and

$$\mathbf{T}(z) \sim \boldsymbol{\epsilon} + \boldsymbol{\kappa}_1\, T(z), \tag{56}$$

where $\boldsymbol{\kappa}_1$ is a constant matrix without zero elements. In the conservative case the 11 element of $\boldsymbol{\kappa}_1$ is 1. We do not reproduce here the explicit expression for $\boldsymbol{\kappa}_1$, because it is not needed for our immediate purposes. According to Eqs. (55) and (56), the *leading terms* of the asymptotic expansions are given by

$$G_{12}(z) = G_{21}(z) \sim \kappa_{21}\, G(z), \qquad G_{22}(z) \sim \kappa_{22}\, G(z) \tag{57}$$

and

$$T_{12}(z) = T_{21}(z) \sim \kappa_{21}\, T(z), \qquad T_{22}(z) \sim \varepsilon_{\mathrm{Q}}, \tag{58}$$

where κ_{ij} is the element ij of the (symmetric) matrix $\boldsymbol{\kappa}_1$.

Taking into account this information, one can conclude that for large z the second term in the LHS of Eq. (53) is small compared with the first one and thus it can be dropped. As regards the RHS, for large z the main contribution to the integral comes from large z', and in this domain the second term in brackets in the integrand can also be neglected. As a result, for large z Eq. (53) assumes the form

$$T(z)\, I_{11}(z) \sim \int_0^\infty G(z')\, I_{11}(z')\, \frac{z'\, dz'}{z' - z}\,, \qquad z \to \infty. \tag{59}$$

Comparison of this equation with the equation for the scalar conservative I-function (cf. Ivanov 1973, Sect. 5.4),

$$T(z)\, I(z) = \int_0^\infty G(z')\, I(z')\, \frac{z'\, dz'}{z' - z}, \tag{60}$$

reveals that $I_{11}(z)$ is asymptotically equal to $I(z)$:

$$I_{11}(z) \sim I(z), \qquad z \to \infty. \tag{61}$$

It worth mentioning that in case of the Doppler profile this asymptotic equality holds not only for the leading term, but for the whole asymptotic expansion.

Now we apply similar considerations to Eq. (54). Taking into account that $T_{22}(z) \to \varepsilon_{\mathrm{Q}}$ for $z \to \infty$, we obtain

$$I_{21}(z) \sim \varepsilon_{\mathrm{Q}}^{-1} \left(\int_0^\infty G_{21}(z')\, I(z')\, \frac{z'\, dz'}{z' - z} - T_{21}(z)\, I(z) \right). \tag{62}$$

All the functions appearing in the RHS may be considered known. Therefore, this equation gives the explicit expresssion of $I_{21}(z)$ for large z.

The latter formula is the key which unlocks the door to the treasure-house of the matrix asymptotic expansions.

6.2. Leading term: functional form

We denote

$$Z = \ln \frac{z}{\sqrt{\pi}}\,, \qquad T = \ln \frac{\tau}{\sqrt{\pi}}\,. \tag{63}$$

First we prove that

$$I_{21}(z) = O\big(z^{-1/2} Z^{-9/4}\big), \qquad z \to \infty, \tag{64}$$

and hence, according to (22), $S_{21}(\tau) = O\big(\tau^{-1/2} T^{-9/4}\big)$, $\tau \to \infty$. This can be established from Eq. (62) quite easily.

Substituting $G_{21}(z)$ and $T_{21}(z)$ from Eqs. (57) and (58) into Eq. (62), we conclude that the leading term of the expansion of $I_{21}(z)$ is

$$I_{21}(z) \sim \kappa_{21}\, \varepsilon_{\rm Q}^{-1} \left(\int_0^\infty G(z')\, I(z')\, \frac{z'\, dz'}{z' - z} - T(z)\, I(z) \right). \tag{65}$$

But, by virtue of the linear equation (60) for the scalar I-function, the expression in brackets is zero. Since, as it is well known (cf. Ivanov 1973, Sect. 2.7 and 5.6),

$$I(z) \sim O\big(z^{1/2} Z^{1/4}\big), \qquad T(z) \sim O\big(z^{-1} Z^{-3/2}\big), \tag{66}$$

both terms in the brackets in Eq. (65) are $O\big(z^{-1/2} Z^{-5/4}\big)$. These terms cancel, as we have just seen. Therefore, we can conclude that the non-vanishing terms in the brackets in Eq. (62), and hence $I_{21}(z)$, should be of the higher order, i.e., $z^{-1/2} Z^{-9/4}$. In principle, the order might be higher — but this does not happen.

6.3. Asymptotic expansions of $I_{11}(z)$ and $I_{21}(z)$: results

In conservative case ($\varepsilon_{\rm I} = 0$) the asymptotic expansions of the elements of the first column of the matrix $\mathbf{I}(z)$ are given by

$$I_{11}(z) \sim I(z) \sim \frac{2}{\sqrt{\pi}}\, z^{1/2} Z^{1/4} \sum_{k=0}^{\infty} \frac{i_{11}^k}{Z^k}\,, \tag{67}$$

$$I_{21}(z) \sim \frac{15\pi^{3/2}}{64\sqrt{2}}\, \frac{\sqrt{W}}{10 - 7W}\, z^{-1/2} Z^{-9/4} \sum_{k=0}^{\infty} \frac{i_{21}^k}{Z^k}\,, \tag{68}$$

where $i_{ij}^0 = 1$ and i_{11}^k for $k \geq 1$ may be found from the recursion relation

$$i_{11}^k = -\frac{1}{4k} \sum_{j=0}^{k-1} {}_1A_j^k\, i_{11}^j, \tag{69}$$

while i_{21}^k are expressed in terms of $i_{11}^j,\ j \leq k+1$, as follows:

$$i_{21}^k = \frac{4}{3} \sum_{j=0}^{k+1} {}_2A_j^{k+1}\, i_{11}^j. \tag{70}$$

The numbers ${}_1A_j^k$ and ${}_2A_j^k$ appearing in these expressions are given by

$${}_nA_j^k = \sum_{l=0}^{k-j} g_{1n}^l\, f_{k-j-l} \prod_{m=j+l}^{k} (4m+1) - t_{1n}^{k-j}. \tag{71}$$

Here

$$g_{11}^k = (-1)^k \frac{(2k-1)!!}{2^{2k}}\,, \qquad g_{12}^k = \left(3 \cdot 2^{-k} - 2\right) g_{11}^k \tag{72}$$

and

$$f_k = \frac{2^{k+2} - 1}{2^{k-2}(k+2)!}\, \pi^k\, |B_{k+2}|, \tag{73}$$

where B_{k+2} are the Bernoulli numbers ($B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42, \ldots$; for odd k's we have $B_{k+2} = 0$). Finally,

$$t_{11}^k = -4 g_{11}^{k+1} \sum_{l=0}^{[k/2]} 2^{4l}\, f_{2l}, \tag{74}$$

$$t_{12}^k = 4 g_{11}^{k+1} \sum_{l=0}^{[k/2]} \left(2 - 3 \cdot 2^{2l-k}\right) 2^{4l}\, f_{2l}. \tag{75}$$

Here $[k/2]$ is the largest integer not exceeding $k/2$.

The explicit expressions of i_{ij}^1 and i_{ij}^2 are

$$i_{11}^1 = \frac{1}{8}\,, \qquad i_{11}^2 = -\frac{10\pi^2 + 9}{128}\,; \tag{76}$$

$$i_{21}^1 = -\frac{15}{8}\,, \qquad i_{21}^2 = \frac{590\pi^2 + 513}{128}\,. \tag{77}$$

The numerical values of i_{ij}^k are given in Table 4. It worth mentioning that they do *not* depend on the value of W. Therefore, according to Eqs. (67) and (68), in the asymptotic domain ($z \gg 1$) the dependence on W is very simple: $I_{11}(z)$ does not depend on W at all, while $I_{21}(z)$, for a fixed z, is proportional to $\sqrt{W}/(10 - 7W)$.

We note that the numbers g_{ij}^k and t_{ij}^k are the elements of the matrices $\mathbf{G}_k$ and $\mathbf{T}_k$ appearing in the asymptotic expansions of $\mathbf{G}(z)$ and $\mathbf{T}(z)$ for $z \to \infty$ obtained in Paper I, namely,

$$\mathbf{G}(z) \sim \boldsymbol{\kappa}_1 \odot \sum_{k=0}^{\infty} \frac{\mathbf{G}_k}{Z^k}\, \frac{1}{4z^2\sqrt{Z}}\,, \tag{78}$$

$$\mathbf{T}(z) \sim \boldsymbol{\epsilon} - \boldsymbol{\kappa}_1 \odot \sum_{k=1}^{\infty} \frac{\mathbf{T}_k}{Z^k}\, \frac{\pi^2}{16 z Z^{3/2}}\,, \tag{79}$$

Table 4. Coefficients i_{ij}^k of the asymptotic expansion of $\mathbf{I}(z)$ for conservative scattering

k	i_{11}^k	i_{21}^k	i_{12}^k	i_{22}^k
0	1.000000E+00	1.000000E+00	1.000000E+00	1.000000E+00
1	1.250000E−01	−1.875000E+00	3.750000E−01	4.333333E−01
2	−8.413753E−01	4.950052E+01	−3.281250E−01	−5.504696E−01
3	7.615941E−01	−1.696714E+02	5.089268E+00	6.531626E−01
4	−1.037270E+01	5.168654E+03	−1.268569E+01	−2.799760E+00
5	2.021464E+01	−2.569248E+04	2.997698E+02	7.400109E+00
6	−3.580655E+02	9.819290E+05	−1.217286E+03	−4.982447E+01
7	1.066839E+03	−6.410101E+06	4.009191E+04	2.025403E+02
8	−2.285614E+04	3.038041E+08	−2.256817E+05	−1.876099E+03
9	9.139411E+04	−2.459988E+09	9.651090E+06	1.030742E+04
10	−2.296380E+06	1.406517E+11	−6.951539E+07	−1.234333E+05

where

$$\boldsymbol{\kappa}_1 = \lambda_{\rm I} \begin{pmatrix} 1 & -\frac{1}{4}\sqrt{\frac{W}{2}} \\ -\frac{1}{4}\sqrt{\frac{W}{2}} & \frac{W}{2} \end{pmatrix}. \tag{80}$$

In conservative case, of course, $\lambda_{\rm I} = 1$. The symbol $\odot$ means element-by-element matrix multiplication, so that

$$\mathbf{Z} = \mathbf{X} \odot \mathbf{Y} \quad \text{if} \quad z_{ij} = x_{ij} y_{ij}, \;\; i, j = 1, 2. \tag{81}$$

6.4. Asymptotic expansions of $I_{11}(z)$ and $I_{21}(z)$: derivation

The recursion relation (69) for the coefficients of the expansion (67) of $I_{11}(z)$ is a known result of the scalar theory. Its proof may be found in Ivanov 1970; the result is reproduced also in Ivanov 1973, Sect. 5.6. Assuming this relation known, one can obtain Eqs. (68) and (70) surprisingly easily — practically by inspection.

Let us denote ($n = 1, 2$)

$$F_n(z) = \int_0^\infty G_{1n}(z')\, I(z')\, \frac{z'\, dz'}{z' - z} - T_{1n}(z)\, I(z). \tag{82}$$

The asymptotic expansions of $G_{11}(z)$ and of $G_{12}(z)$ have the same dependence on z, and only the coefficients of the two expansions differ. The same is true of the pair $T_{11}(z)$, $T_{12}(z)$. Hence we can conclude that the functional forms of the asymptotic expansions of $F_1(z)$ and $F_2(z)$ should be the same. It can be shown that

$$F_n(z) \sim \frac{A_n}{z^{1/2} Z^{5/4}} \sum_{k=0}^{\infty} \frac{a_n^k}{Z^k}, \tag{83}$$

where

$$a_n^k = \sum_{j=0}^{k} \left(\sum_{l=0}^{k-j} g_{1n}^l\, f_{k-j-l} \prod_{m=j+l}^{k} (4m+1) - t_{1n}^{k-j} \right) i_{11}^j$$

$$\equiv \sum_{j=0}^{k} {}_n A_j^k\, i_{11}^j. \tag{84}$$

Rather lengthy calculation which shows that a_1^k indeed has this form may be found in Ivanov 1970. In view of the above comments it is clear that the coefficients a_2^k are given by the same expression, with the subscript $1n$ replaced by 12. This is the crucial point which, as we shall momentarily see, enables one to find i_{21}^k without any calculations.

Indeed, Eqs. (59), (61) and (62) may be written as

$$F_1(z) \sim 0 \tag{85}$$

and

$$I_{21}(z) \sim \varepsilon_{\rm Q}^{-1} F_2(z), \tag{86}$$

respectively, whence

$$a_1^k = 0, \qquad i_{21}^k = \frac{4}{3}\, a_2^{k+1}, \qquad k = 0, 1, \ldots \tag{87}$$

One can easily see that the first of these relations, $a_1^k = 0$, is equivalent to the recursion relation (69) for the coefficients i_{11}^k. The second relation is the desired result, i.e., Eq. (70); the coefficient 4/3 appears because $a_2^1 = 3/4$, whereas in Eq. (68) it is assumed that $i_{21}^0 = 1$. We note that, according to Eq. (84), we have $a_2^0 = 0$, and that is why $I_{21}(z) = O\left(z^{-1/2} Z^{-9/4}\right)$, and not $O\left(z^{-1/2} Z^{-5/4}\right)$.

6.5. Asymptotic expansion of the vector source function

Now we are ready to obtain the asymptotic expansions of the components $S_{ij}(\tau)$, $ij = 11, 21$, of the source vector $\mathbf{s}_{\rm h}(\tau)$ of the Milne problem.

The expansions (67) and (68) are of the following general form:

$$I_{ij}(z) \sim M_{ij}\, z^{\alpha} Z^{\beta} \sum_{k=0}^{\infty} \frac{i_{ij}^k}{Z^k}, \tag{88}$$

with α and β depending on ij and $i_{ij}^0 = 1$. Since $\alpha \neq -1$, it is natural to seek the expansion of $S_{ij}(\tau)$ in a similar form, namely,

$$S_{ij}(\tau) \sim N_{ij}\, \tau^{\alpha} T^{\beta} \sum_{k=0}^{\infty} \frac{s_{ij}^k}{T^k}, \tag{89}$$

with $s_{ij}^0 = 1$. The coefficients of this expansion one can find from the relation (cf. Eq. (22))

$$I_{ij}(z) = \int_0^\infty S_{ij}(\tau)\,\mathrm{e}^{-\tau/z}\,d\tau/z. \tag{90}$$

For large z, the main contribution to the integral is given by the values of the integrand at large τ. We substitute Eqs. (88) and (89) into Eq. (90) and perform rearrangements similar to those used in Sect. 9 of Paper I in the derivation of the asymptotic expansion of $\mathbf{T}(z)$. The integrals that appear in the RHS after the substitution $y = \tau/z$,

$$\int_0^\infty \left(1 + \frac{\ln y}{Z}\right)^{k+\beta} \mathrm{e}^{-y}\, y^\alpha\, dy, \tag{91}$$

produce series expansions in which the derivatives of the gamma function appear:

$$\Gamma^{(n)}(\alpha+1) = \int_0^\infty (\ln y)^n\, \mathrm{e}^{-y}\, y^\alpha\, dy. \tag{92}$$

As a result, we find that

$$N_{ij} = \frac{M_{ij}}{\alpha\Gamma(\alpha)} \tag{93}$$

and

$$s_{ij}^k = i_{ij}^k + \frac{1}{\alpha\Gamma(\alpha)} \sum_{l=0}^{k-1} \Gamma^{(k-l)}(\alpha+1)\; D_{k-l}\,(-l+\beta)\, s_{ij}^l, \tag{94}$$

where $D_m(p)$ are the binomial coefficients:

$$D_0(p) = 1,$$
$$D_m(p) = \frac{p(p-1)\ldots(p-m+1)}{m!}\,, \qquad m = 1, 2, \ldots. \tag{95}$$

Let us write out explicitly the final result. We have

$$S_{11}(\tau) \sim \frac{4}{\pi}\left(\tau\sqrt{T}\right)^{1/2} \sum_{k=0}^{\infty} \frac{s_{11}^k}{T^k}\,, \tag{96}$$

$$S_{21}(\tau) \sim \frac{15\pi}{64\sqrt{2}}\,\frac{\sqrt{W}}{10-7W}\left(\tau T^{9/2}\right)^{-1/2} \sum_{k=0}^{\infty} \frac{s_{21}^k}{T^k}\,, \tag{97}$$

where $s_{ij}^0 = 1$ and for $k \geq 1$

$$s_{11}^k = i_{11}^k - \frac{2}{\sqrt{\pi}} \sum_{l=0}^{k-1} \Gamma^{(k-l)}\left(3/2\right)\; D_{k-l}\left(-l+1/4\right)\; s_{11}^l, \tag{98}$$

$$s_{21}^k = i_{21}^k - \frac{1}{\sqrt{\pi}} \sum_{l=0}^{k-1} \Gamma^{(k-l)}\left(1/2\right)\; D_{k-l}\left(-l-9/4\right)\; s_{21}^l. \tag{99}$$

In particular,

$$s_{11}^1 = \frac{2\alpha^* - 3}{8}\,,$$
$$s_{11}^2 = -\frac{1}{128}\,(4\pi^2 + 12\alpha^{*2} - 36\alpha^* + 81); \tag{100}$$

$$s_{21}^1 = -\frac{18\alpha^* + 15}{8}\,,$$
$$s_{21}^2 = \frac{1}{32}\left(89\pi^2 + 117\alpha^{*2} + 195\alpha^* + \frac{513}{4}\right). \tag{101}$$

Here

$$\alpha^* = \gamma^* + 2\ln 2\,, \tag{102}$$

where γ^* is the Euler constant, $\gamma^* = 0.577216$.

The derivatives of the gamma function appearing in the recursion relations (98) – (99) may be found as follows. Let $\psi(z)$ be the logarithmic derivative of the gamma function:

$$\psi(z) = \frac{d\ln\Gamma(z)}{dz}. \tag{103}$$

Then

$$\Gamma^{(n+1)}(z) = \frac{d^n}{dz^n}\left[\Gamma(z)\,\psi(z)\right] = \sum_{k=0}^{n} C_n^k\, \psi^{(n-k)}(z)\,\Gamma^{(k)}(z), \tag{104}$$

where C_n^k are the binomial coefficients. For $z = 1/2$ we have (cf. Abramowitz & Stegun 1964, Sect. 6.4)

$$\psi^{(n)}\left(1/2\right) = \left(2^{n+1} - 1\right)\psi^{(n)}(1), \tag{105}$$

while

$$\psi^{(n)}(1) = (-1)^{n+1} n!\,\zeta(n+1), \tag{106}$$

where $\zeta(s)$ is the Riemann ζ-function:

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}\,. \tag{107}$$

Since its values are known (e.g. Abramowitz & Stegun 1964), we can use Eqs. (104) — (106) to find $\Gamma^{(n)}(1/2)$ by recursion. When the values of $\Gamma^{(n)}(1/2)$ are found, the relation

$$\Gamma^{(n)}(z+1) = z\Gamma^{(n)}(z) + n\Gamma^{(n-1)}(z) \tag{108}$$

is used to get $\Gamma^{(n)}(3/2)$.

The numerical values of s_{ij}^k are given in Table 5. They do not depend on W. Hence, according to Eqs. (96) and (97), for large τ the dependence of the vector source function $\mathbf{s}_\mathrm{h}(\tau)$ on W is very simple: S_{11} does not depend on W, while S_{21} is proportional to $\sqrt{W}/(10-7W)$.

6.6. Second columns of the matrices $\mathbf{S}(\tau)$ *and* $\mathbf{I}(z)$

The second column of the source matrix $\mathbf{S}(\tau)$ of the conservative standard problem, Eq. (1), is not so interesting as the first one, and we shall discuss its asymptotic behavior only briefly. For dipole scattering both $S_{12}(\tau)$ and $S_{22}(\tau)$ vary within narrow limits (cf. Table 1). The reason is that the parameter ε_Q, which is the Q-counterpart of the usual photon destruction probability ε_I,

Table 5. Coefficients s_{ij}^k of the asymptotic expansion of $\mathbf{S}(\tau)$ for conservative scattering

k	s_{11}^k	s_{21}^k	s_{12}^k	s_{22}^k
0	1.000000E+00	1.000000E+00	1.000000E+00	1.000000E+00
1	1.158775E−01	−6.292898E+00	3.750000E−01	−7.219412E−01
2	−7.504415E−01	5.751899E+01	−5.445809E−01	1.534691E+00
3	6.822107E−01	−5.445835E+02	4.976157E+00	−4.504304E+00
4	−8.517184E+00	6.273780E+03	−2.011274E+01	1.757279E+01
5	1.743584E+01	−8.110070E+04	2.901721E+02	−8.405594E+01
6	−2.771669E+02	1.207465E+06	−1.907872E+03	4.815415E+02
7	8.935228E+02	−2.012072E+07	3.863797E+04	−3.202225E+03
8	−1.697146E+04	3.749838E+08	−3.527281E+05	2.449477E+04
9	7.492918E+04	−7.708413E+09	9.282869E+06	−2.103224E+05
10	−1.655547E+06	1.737856E+11	−1.085670E+08	2.035131E+06

in case of dipole conservative scattering is not small: $\varepsilon_{\mathrm{Q}} = 0.3$. Hence scattering of the fictitious 'Q-photons' is strongly nonconservative.

One can show that

$$S_{12}(\infty) = \frac{1}{4}\sqrt{\frac{5W}{10-7W}}\,; \quad S_{22}(\infty) = \sqrt{\frac{10}{10-7W}}\,. \tag{109}$$

This is the most crude, but at the same time the most useful asymptotic information on $S_{12}(\tau)$ and $S_{22}(\tau)$. Several first terms of the large-τ expansions in powers of T^{-1} are given by

$$S_{12}(\tau) \sim -\varepsilon_{\mathrm{Q}}^{-1/2}\,\frac{K_2^{12}(\tau)}{K_2^{11}(\tau)} + O\left(T^{-3}\right), \tag{110}$$

$$S_{22}(\tau) \sim \varepsilon_{\mathrm{Q}}^{-1/2} - \varepsilon_{\mathrm{Q}}^{-3/2}\,\frac{\det \mathbf{K}_2(\tau)}{K_2^{11}(\tau)} + O\left(\tau^{-1}T^{-7/2}\right). \tag{111}$$

Here $\mathbf{K}_2(\tau)$ is the second kernel function defined by Eq. (30), and $K_2^{ij}(\tau)$ stands for the element ij of the matrix $\mathbf{K}_2(\tau)$. In Eqs. (110) and (111) it is assumed that we substitute for $\mathbf{K}_2(\tau)$ and $K_2^{ij}(\tau)$ their asymptotic expansions (see Sect. 8 of Paper I) and retain not only the leading, but also the higher-order terms. The asymptotic forms (110) and (111) are really amazing. Quite simple expressions involving only the elements of the (second) kernel matrix $\mathbf{K}_2(\tau)$ provide the asymptotic expansions of $S_{12}(\tau)$ and $S_{22}(\tau) - \varepsilon_{\mathrm{Q}}^{-1/2}$ with *three* terms! The proof of Eqs. (110) and (111) proceeds directly from the coupled integral equations for $S_{12}(\tau)$ and $S_{22}(\tau)$, i.e., is based on the first of the two approaches to the asymptotic analysis briefly outlined at the beginning of Sect. 6. It will be published elsewhere (Grachev 1996).

Using the second approach, i.e., proceeding along the same general lines as in Sects. 6.1 – 6.5, with some minor extra tricks, one can get the complete expansions. More precisely, one can obtain the recursion relations for the coefficients s_{12}^k and s_{22}^k of the asymptotic expansions

$$S_{12}(\tau) \sim \frac{1}{4}\sqrt{\frac{5}{7}}\,\lambda_{\mathrm{Q}}^{1/2}\varepsilon_{\mathrm{Q}}{}^{-1/2}\,\sum_{k=0}^{\infty}\frac{s_{12}^k}{T^k}, \tag{112}$$

$$S_{22}(\tau) \sim \varepsilon_{\mathrm{Q}}^{-1/2} - \frac{75}{448}\lambda_{\mathrm{Q}}\varepsilon_{\mathrm{Q}}^{-3/2}\left(\tau\sqrt{T}\right)^{-1}\sum_{k=0}^{\infty}\frac{s_{22}^k}{T^k}. \tag{113}$$

The numerical values of the coefficients s_{12}^k and s_{22}^k which were found using the recursion relations just mentioned are given in Table 5. We do not reproduce these relations. If one considers the solutions of the matrix transfer equation for the 'physical' values of ε_{Q}, i.e., $\varepsilon_{\mathrm{Q}} \geq 3/10$, the expansions of $S_{12}(\tau)$ and $S_{22}(\tau)$ are of rather limited interest. The reason is that in this domain of the parameter values both $S_{12}(\tau)$ and $S_{22}(\tau)$ vary within narrow limits, and saturation sets in at moderate values of τ.

The large-z expansions of the elements of the second column of the matrix $\mathbf{I}(z)$ in conservative case are

$$I_{12}(z) \sim \frac{1}{4}\sqrt{\frac{5}{7}}\,\lambda_{\mathrm{Q}}^{1/2}\varepsilon_{\mathrm{Q}}^{-1/2}\,\sum_{k=0}^{\infty}\frac{i_{12}^k}{Z^k} \tag{114}$$

and

$$I_{22}(z) \sim \varepsilon_{\mathrm{Q}}^{-1/2} + \frac{\Omega}{z} - \frac{75}{224}\lambda_{\mathrm{Q}}\varepsilon_{\mathrm{Q}}^{-3/2}\,\frac{\sqrt{Z}}{z}\,\sum_{k=0}^{\infty}\frac{i_{22}^k}{Z^k}\,, \tag{115}$$

where the numbers i_{12}^k and i_{22}^k are determined by rather bulky recursive relations. Their numerical values for $k \leq 10$ were given in Sect. 6.3 (Table 4). The asymptotic parameter Ω appearing in the expansion (115) of the element $I_{22}(z)$ of the matrix $\mathbf{I}(z)$ is

$$\Omega = \varepsilon_{\mathrm{Q}}^{-1}\int_0^{\infty}\left[I_{22}(\infty) - I_{22}(z')\right]G_{22}(z')z'\,dz'. \tag{116}$$

It can be expressed also in terms of the 22 element of the source matrix:

$$\Omega = \int_0^{\infty}\left(S_{22}(\tau) - \varepsilon_{\mathrm{Q}}^{-1/2} + \varepsilon_{\mathrm{Q}}^{-3/2}\frac{\det \mathbf{K}_2(\tau)}{K_2^{11}(\tau)}\right)d\tau. \tag{117}$$

The values of Ω are listed in Table 6 for several ε_{Q}.

We note that if one wishes to investigate generalized resonance scattering problems, i.e., if one admits that $\lambda_{\mathrm{I}}, \lambda_{\mathrm{Q}} \in [0,1]$, then the values of ε_{Q} close to 0 are of interest. In this case the asymptotic expansions (112), (113) and (114), (115) become highly useful. We hope to consider these matters in a forthcoming paper.

Table 6. Asymptotic parameter Ω for conservative CFR scattering with the Doppler profile

ε_Q	Ω	ε_Q	Ω
0.01	5.264E+2	0.4	2.342E−1
0.05	2.983E+1	0.5	1.037E−1
0.1	7.589E+0	0.6	4.575E−2
0.2	1.623E+0	0.7	1.872E−2
0.3	5.646E−1	0.8	6.303E−3

6.7. Asymptotic versus numerically exact results

What is the accuracy and the real domain of applicability of our asymptotic theory? Let δ be the fractional error of the asymptotic estimate defined by $\delta = |f_{\rm ex} - f_{\rm as}|/|f_{\rm ex}|$, where $f_{\rm as}$ and $f_{\rm ex}$ are, respectively, the asymptotic estimate and the numerically exact value of a function f. If one retains only the leading term of the expansion of $I_{11}(z)$, then for $z \geq 10^3$ the fractional error is less than 0.5%. In case of $I_{21}(z)$ the situation is much worse: for $z \geq 10^4$ the leading term of the expansion provides accuracy of 13%, and δ = 5% is reached only at $z \sim 3 \cdot 10^8$. We note that all the numerical data in this Section refer to dipole scattering; the 'numerically exact' values are found using the procedures reported in Sects. 3 and 4.

If we follow the standard recipe of truncating the asymptotic series on the term immediately preceding the one which, for a given z, has the smallest absolute value, we get the results shown in Fig. 4. The curves labeled 11 and 21 refer to $I_{11}(z)$ and $I_{21}(z)$, respectively. For $I_{11}(z)$ the accuracy of $\sim$ 1% is reached at $z \sim 100$, which needs retaining three terms of the expansion. The fractional error of the asymptotic estimate of $I_{11}(z)$ decreases to $\sim$ 0.1% at $z \sim 10^3$. To get this result, one has to retain 5 terms of the asymptotic series. For $z \sim 10^9$ we have $\delta < 10^{-8}$. Incidentally, this proves that the accuracy of our numerical data on **I**-matrices is very high.

Reasonably accurate asymptotic estimates of $I_{21}(z)$ can be obtained only for very large z. This is clear from the numerical values of the coefficients i_{12}^k of its asymptotic expansion given in Table 4. For z = 10^4 and 10^5 one has to retain only the leading term, which results in fractional errors δ of 13% and 7.5%, respectively; for 10^6 we retain 3 terms of the series and get $\delta \sim 2\%$, and only for $z = 10^8$, when we should take into account 7 terms of the series, the error drops to $\sim$ 0.2%. Needless to say that at these huge values of z the assumption of CFR breaks down, so that these results have no immediate physical interest.

In Fig. 5 we present data on the accuracy provided by the asymptotic theory for the elements $S_{11}(\tau)$ and $S_{21}(\tau)$ of the **S**-matrix; it is assumed that the asymptotic series are truncated on the term preceding the smallest one (in absolute value). The 'strange' behavior of the curve 11 for $\log \tau > 5$ is caused by insufficient accuracy of our numerical data on the source matrices — only 4 s.f.

As regards $S_{12}(\tau)$ and $S_{22}(\tau)$, the asymptotic formulae (110) and (111) provide rather high accuracy even for not too large τ. If we substitute in these expressions the asymptotic expansions of the kernel functions and retain terms up to $O(\tau^{-1}T^{-5/2})$,

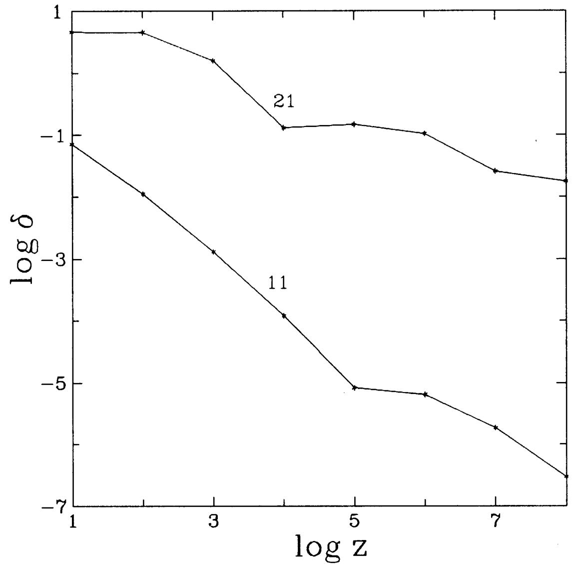

Fig. 4. Accuracy provided by the asymptotic expansions of $I_{11}(z)$ (curve labeled 11) and $I_{21}(z)$ (curve 21)

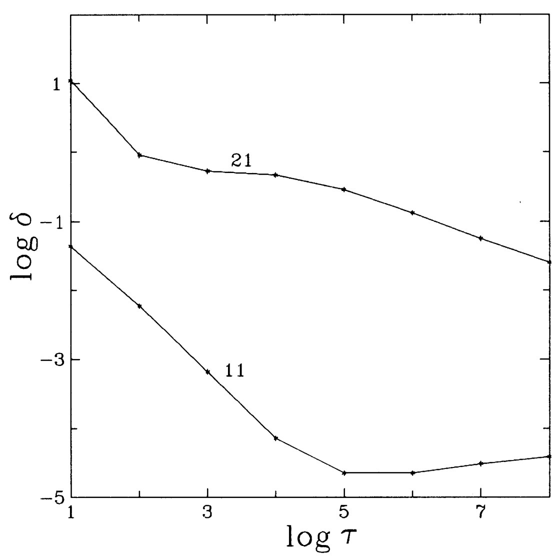

Fig. 5. Accuracy provided by the asymptotic expansions of $S_{11}(\tau)$ (curve labeled 11) and $S_{21}(\tau)$ (curve 21)

we find that at τ = 10 the fractional error is $\sim 5 \cdot 10^{-3}$ for S_{22} and $\sim 7 \cdot 10^{-3}$ for S_{12}.

7. Concluding remarks

1. The main astrophysical result reported in this paper may be formulated as follows: frequency redistribution has rather weak influence on the polarizing properties of conservatively scattering atmospheres. Namely, due to frequency redistribution, the limb polarization in the source-free conservative atmosphere decreases from its classical value of 11.713% for monochromatic scattering to 9.443% for CFR Doppler scattering. The physical reason of the weak influence of frequency redistrubution on

polarization is clear: polarization arises mainly in a rather thin surface layer of optical thickness ~ 1, cf. Fig. 1b. The structure of this layer is dominated by the direct escape of radiation, and hence is not very sensitive to the details of the scattering process.

2. We show that in the deep layers of the source-free conservative atmosphere with dipole scattering ($W = 1$) the two components, S_{11} and S_{21}, of the vector source function $\mathbf{s}_\mathrm{h}$ normalized so that $\mathbf{s}_\mathrm{h}^\mathrm{T}(0)\mathbf{s}_\mathrm{h}(0) = 1$ are asymptotically equal to $S_{11}(\tau) \sim (4/\pi)(\tau\sqrt{T})^{1/2}$ and $S_{21}(\tau) \sim (5\pi\sqrt{2}/128)(\tau T^{9/2})^{-1/2}$, respectively; here $T \equiv \ln(\tau/\sqrt{\pi})$.

3. The complete asymptotic expansion of $\mathbf{s}_\mathrm{h}$ in inverse powers of T is found (Eqs. (96) to (99)). The coefficients of the expansion are determined recursively. We present their numerical values (Table 4) and show that the asymptotic results provide a very informative check of the accuracy of numerical solutions of the vector transfer problems of resonance scattering.

4. A more careful theoretical analysis of nearly conservative CFR scattering is desirable. It might explain, in particular, the surprisingly simple asymptotic dependence (52) of the limb polarization on the photon destruction probability ε_I. We found this relation purely empirically, from the analysis of our numerical data.

5. The numerical data on the **I**-matrices presented in this paper may be used to find the Stokes vector of the line radiation diffusely reflected from a conservative atmosphere. The polarization of this radiation will be analysed in a forthcoming paper of the present series.

6. Since resonance scattering is the degenerate case of a more general problem of scattering in the presence of the magnetic field, our data on the matrix source function $\mathbf{S}(\tau)$ and on $\mathbf{I}(z)$ may be used to test, at least partly, the accuracy of the codes of magnetic line formation.

7. From a more abstract point of view, one of our main results is a contribution to the methods of ART. We have shown that the asymptotic methods of the scalar CFR line formation theory may be adjusted to treat much more difficult vector and matrix transfer problems (Sects. 6.1 to 6.6).

Acknowledgements. The authors are indebted to H.Frisch and M.Faurobert-Scholl for providing the high-accuracy data on the coefficients of the Padé approximations of the kernel matrices. Thanks are due to the referees for making useful suggestions which led to radical restructuring of the presentation. The research described in this publication was made possible in part by Grants #R39000 from the International Science Foundation and #R39300 from the Government of the Russian Federation and the International Science Foundation.

References

Abramowitz M., Stegun I., 1964, Handbook of Mathematical Functions, NBS, Washington
Bommier V., Landi Degl'Innocenti E., 1996, in 'Solar Polarization', eds. Stenflo J.O., Nagendra K.N., Kluwer Publ. Co., Dordrecht
Bowden R.L., Richardson N.R., J. Math. Phys. 9, 1753.
Chandrasekhar S., 1950, Radiative Transfer, Clarendon Press, Oxford
Domke H., 1971, SvA 15, 266
Dumont S., Omont A., Pecker J.C., Rees D., 1977, A&A 54, 675
Faurobert M., 1988, A&A 194, 268
Faurobert-Scholl M., Frisch H., 1989, A&A 219, 338
Frisch H., 1988, in 'Radiation in Moving Gaseous Media', eds. Chmielewsky Y., Lanz T., Geneva Obs.
Frisch H., Frisch U., 1982, JQSRT 28, 361
Grachev S.I., 1996, in preparation
Hulst H.C. van de, 1980, Multiple Light Scattering, Academic Press, New York
Hummer D.G., 1981, JQSRT 26, 187
Ivanov V.V., 1970, JQSRT 10, 665 (in Russian)
Ivanov V.V., 1973, Transfer of Radiation in Spectral Lines, NBS SP #385, Washington
Ivanov V.V., 1990, SvA 34, 621
Ivanov V.V., 1995, A&A 303, 609 (GRaS.I)
Ivanov V.V., 1996, A&A 307, 319 (GRaS.III)
Ivanov V.V., Kasaurov A.M., Loskutov V.M., Viik T., 1995, A&A 303, 621 (GRaS.II)
Ivanov V.V., Grachev S.I., Loskutov V.M., 1996a, A&A (accepted) (Paper I)
Ivanov V.V., Kasaurov A.M., Loskutov V.M., 1996b, A&A 307, 332 (GRaS.IV)
Kuz'mina M.G., 1970, Doklady Acad. Nauk USSR 193, 309 (in Russian)
Loskutov V.M., 1994, Trudy Astron. Obs. St.Petersburg Univ., 44, 154 (in Russian)
Mullikin T.W., 1968, ApJ 145, 886
Mullikin T.W., 1969, Proc. Symp. Applied Math. AMS – SIAM 1, 3
Nagirner D.I., 1984, Astrophys. Space Phys. 3, 255
Rees D.E., Saliba G.J., 1982, A&A 115, 1
Rooij W.A. de, Bosma P.B., van Hooff J.P.C., 1989, A&A 226, 347
Siewert C.E., Fraley S.K., 1967, Ann. Phys. 43, 338
Sobolev V.V., 1963, A Treatise on Radiative Transfer, Van Nostrand, Princeton
Viik T., 1990, Earth, Moon and Planets 49, 163

This article was processed by the author using Springer-Verlag LaTeX A&A style file *L-AA* version 3.