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# Nonradial thermal instabilities in the solar core, revisited

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**Abstract.** Mechanisms which produce a mixing in the solar core, based on the original idea of the "solar spoon" (Dilke & Gough 1972), for explaining the neutrino problem have recently been ruled out by Bahcall & Kumar (1993). We therefore revisit the original work of Rosenbluth & Bahcall (1973) on the nonspherical thermal instabilities in the solar core, introducing some improvements. The authors found that their solar model was stable against these perturbations and they concluded that the turbulence in the solar core appeared unlikely to be the explanation of the lack of solar neutrinos. Our analysis is motived by the fact that the updated standard solar models are sensibly different from those used in the 70's, and therefore instabilities of the kind postulated by Rosenbluth & Bahcall might be excited. However our results fully confirm the previous ones that the present Sun is stable against nonradial thermal disturbances, at least in the linear analysis, therefore no solutions for growing thermal modes are possible. The confirmation of the stability of the solar core against the mixing and the impressive agreement of the standard solar models with helioseismic data indicate that the present neutrino problem is not astrophysical in origin but it relies upon non standard neutrino properties.

**Key words:** Sun: interior – Sun: instabilities

## 1. Introduction

A potential instability associated with $^3$He nuclear burning has first been proposed by Dilke & Gough (1972). This was the famous "solar spoon" which had to produce a mixing of the elements with the consequence of smoothing out the central temperature gradient and reducing the predicted $^7$Be and $^8$B neutrino fluxes to those measured by the Homestake experiment, the only neutrino problem existing at that time. These authors found a process, which might lead to an intermittent mixing on a time scale of a few hundred million years, caused by a secular instability of g-modes driven by the gradient of chemical composition.

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Following this initial suggestion, further work in this direction has been done with the aim of finding an astrophysical solution to the solar neutrino problem, beyond the standard solar model (Ulrich 1974; Christensen-Dalsgaard et al. 1974). More recently, the idea of the "solar spoon" has been reconsidered by Roxburgh (1985), Gough (1991), and De Rújula & Glashow (1992), based on the possibility that g-mode nonlinear interactions might give rise to short period large amplitude oscillations which grow on a Kelvin-Helmholtz time scale and drive $^3$He out of equilibrium.

However, the excitation of short period oscillations secularly unstable, as a possible mechanism for generating a mixed core, has recently been ruled out by Bahcall & Kumar (1993), confirming the previous findings of Press (1981), Press & Rybicki (1981) and Spruit (1987).

A different kind of instability on thermal time scales could arise from the dependence of nuclear reaction rates on temperature and density even in the stably stratified central regions of the Sun. This idea was first put forward by Schwarzschild & Härm (1965) who analyzed spherical perturbations which grow on thermal time scales and are produced in a spherical shell of the stellar cores if the energy generation there exceeds the radiative heat loss. Later, in the context of the solar neutrino problem, these authors tested a solar model and verified its stability against this kind of perturbations (Schwarzschild & Härm 1973).

At the same time Rosenbluth & Bahcall (1973) examined the possibility that nonspherical thermal instabilities, of the same kind as those postulated by Schwarzschild & Härm, could produce a material mixing in the Sun's core in order to explain the neutrino problem. They derived a sufficient condition for stability and found that their standard solar model (Bahcall & Ulrich 1971) was stable against these perturbations. Differently from the spherical ones, the nonspherical instabilities require the presence of a molecular weight gradient, as has first been pointed out by Kippenhahn (1967).

Since then, many improvements have been achieved in the description of the physical processes leading to the determination of the equation of state, opacity and nuclear reaction rates. Therefore it may be expected that the behaviour of a modern standard solar model is greatly different from that of a model constructed twentythree years ago. This is the reason why we revisit the original work of Rosenbluth & Bahcall with the aim of verifying the stability of the solar core against nonradial thermal perturbations. Some improvements with respect to their work consist in the inclusion of the PPII branch of the PP chain in the stability condition and in the derivation and solution of the equation for the radial dependence of the temperature perturbation.

## 2. Nonradial thermal instabilities

The dependence of nuclear generation rates on temperature and concentration of the reacting nuclei may cause the growth of nonradial thermal instabilities in the stably stratified radiative cores of low mass stars, as the Sun. This possibility is induced by the radiative equilibrium which yields the set up of molecular weight gradients in the stellar core, due to the nuclear burning during the stellar evolution. In fact the abundance of $^1$H decreases and that of $^4$He increases towards the centre, creating a negative molecular weight gradient. At the same time a peak in the $^3$He abundance distribution is formed at a certain distance from the centre, located in the present Sun at about $0.3\,R_\odot$, due to the secondary character of this element which enters the burning chain through creation and destruction processes. As a consequence of the presence of a molecular weight gradient, circulation currents may be generated which tend to redistribute the material, but slow enough to leave the hydrostatic equilibrium unaffected. In this condition the motions are not accelerated, in analogy to the Kelvin-Helmoltz contraction in quasi hydrostatic equilibrium or to any quasi static work done by a thermodynamic system.

In order to clarify this mechanism let us suppose that in a region of the Sun's core, owing to a perturbation, the energy generation rate increases with respect to that of the surroundings. The excess of heat flux injected in this region will produce both a temperature increase and mechanical work which will cause an expansion of the region and its outward displacement into regions with a lower pressure and density. This disturbance, produced by nuclear energy rate changes, should grow in typical time scales (*thermal energy / nuclear energy*) $\times$ *nuclear time*, as pointed out by Rosenbluth & Bahcall (1973), much longer than the dynamical time scale. In these conditions the expansion work is quasi static and hydrostatic equilibrium can be maintained so as pressure and density in the perturbed region adjust themselves in order to match those of the surroundings. As we shall see, for strictly nonradial perturbations, local pressure and density fluctuations vanish. The instability arises from the fact that the perturbed region carries with it thermal and chemical properties of the medium where the displacement originated and brings them into regions with lower temperature and molecular weight. As a consequence of this inhomogeneity, slow currents may be generated which produce a mixing of the elements.

The growth or decay of the temperature perturbation, namely the condition for the maintenance or damping of the instability, depends on the competitive effects of the increase of the temperature, due to the energy generation rate increase, and the decrease of temperature, due to the radiative diffusion. At constant density, the energy generation rate increases with the increase of the temperature and decreases with the decrease of the mass abundance of the reacting nuclei. A region which moves outwards, as a consequence of a thermal perturbation, has the same density as the surroundings, for the previous considerations, but less $^1$H, more $^4$He, and less, equal or more $^3$He mass abundances, the latter depending on whether or not the moving region penetrates the region of the $^3$He distribution peak, do not pass through it, or escapes from it, and on the temporal $^3$He evolution. The depletion of the energy generation rate in the perturbed region with respect to the surroundings is essentially caused by the $^1$H depletion which is not compensated by the enhancement of the $^4$He abundance, which determines the energy generation through the $^3\mathrm{He}(^4\mathrm{He},\gamma)^7\mathrm{Be}$ reaction. The role played by the $^3$He abundance peak is to deplete the energy generation rate in the perturbed region with respect to that of the surroundings if the perturbed region penetrates the peak region, to leave it unaltered if the perturbed region does not pass through the peak region, and to enhance it if the perturbed region escape from the peak region. The abundance of $^3$He determines the energy generation through the $^3\mathrm{He}(^3\mathrm{He},2\mathrm{p})^4\mathrm{He}$ and $^3\mathrm{He}(^4\mathrm{He},\gamma)^7\mathrm{Be}$ reactions.

Therefore, in a hotter environment with a depletion of $^1$H and $^3$He, there are two competitive processes, the first of which tends to increase the rate of energy generation and the second tends to decrease it with respect to the surrounding medium. If these two conflicting processes give rise to a net increment in the energy production, then the increase of the rate of the energy production determines a further increase of temperature, thus creating a positive reaction loop, but the increase of temperature enhances the temperature gradient in the perturbed region, thus enhancing the radiative losses so as to smooth out the temperature perturbations.

Thus, these nonradial thermal instabilities, which require the presence of a molecular weight gradient, are excited by changes in the energy generation and damped by the radiative diffusion, and their growth time is large with respect to dynamical time scales so that the hydrostatic equilibrium is satisfied throughout.

Since in the present solar core $\mathrm{d}\mu/\mathrm{d}r < 0$, where $\mu$ is the molecular weight and $r$ the radial coordinate, the conditions for the growth of nonradial thermal instabilities may be satisfied.

## 3. Basic equations

The basic equations of the problem are the hydrostatic support, Poisson, energy conservation, and radiative heat flux diffusion equations:

$$\boldsymbol{\nabla} P = -\varrho \boldsymbol{\nabla} \Phi \tag{1}$$

$$\nabla^2 \Phi = 4\pi \varrho G \tag{2}$$

$$\varrho T \frac{\mathrm{d}S}{\mathrm{d}t} = \varrho\varepsilon - \boldsymbol{\nabla}\cdot\boldsymbol{q} \tag{3}$$

$$\boldsymbol{q} = -\chi \boldsymbol{\nabla} T \tag{4}$$

where $P$, $T$, $\varrho$ and $S$ are the gas pressure, temperature, density, and specific entropy respectively, $\Phi$ is the gravitational potential with G the gravitational constant, $\varepsilon$ the energy generation rate, $\boldsymbol{q}$ the heat flow, and $\chi = 16\sigma T^3/3\kappa\varrho$ the thermal conductivity coefficient with $\sigma$ the Stefan-Boltzmann constant and $\kappa$ the radiative opacity.

We shall consider small deviations from the equilibrium state, which is static, stationary and spherically symmetric:

$$\delta f = f' + \delta\boldsymbol{s}\cdot\boldsymbol{\nabla} f_\circ \tag{5}$$

and, in the linear approximation:

$$\frac{\mathrm{d}\delta f}{\mathrm{d}t} = \frac{\partial f'}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla} f_\circ \tag{6}$$

where $\delta f$ and $f'$ stand for the lagrangian and eulerian variations of a generic variable $f$ respectively, $f_\circ$ indicates its equilibrium value, $\delta\boldsymbol{s}$ a small displacement from the equilibrium position, and $\boldsymbol{v} = \partial\delta\boldsymbol{s}/\partial t$. Owing to the characteristics of the equilibrium state, we have also $\boldsymbol{v}_\circ = 0$, $\partial f_\circ/\partial t = 0$, and $f_\circ = f_\circ(r)$.

By introducing eulerian variations in the perturbed quantities and linearizing about the equilibrium state, the solution of Eqs.(1) and (2), for strictly nonradial perturbations, is (Rosenbluth & Bahcall 1973):

$$\Phi' = P' = \varrho' = 0 \tag{7}$$

and consequently from the perfect gas law it follows:

$$\frac{T'}{T_\circ} = \frac{\mu'}{\mu_\circ} \tag{8}$$

Since in the core of the Sun the ionization is complete we can safely assume that $\mu$ depends only on chemical composition, so that during the motion of a fluid element the pressure and temperature changes do not produce ionization changes inside the element itself, and therefore $\delta\mu = 0$, which gives:

$$\mu' = -\delta r\frac{\mathrm{d}\mu_\circ}{\mathrm{d}r} \tag{9}$$

Equations (8) and (9) indicate that nonradial thermal perturbations cannot occurr unless there is a molecular weight gradient, as previously pointed out by Rosenbluth & Bahcall (1973), and that an outward displacement produces an increase of $\mu$ which in turn determines a positive temperature fluctuation. It can also be deduced from Eqs. (8) and (9) that the amplitude of temperature perturbation depends on the inverse of the mean molecular weigth scale height $H_{\mu_\circ} = -\mathrm{d}r/\mathrm{d}\ln\mu_\circ$:

$$T' = T_\circ\frac{\delta r}{H_{\mu_\circ}} \tag{10}$$

The solution (7) indicates that the mechanical problem has been solved separately from the thermal one. Therefore Eqs. (3) and (4) decouple from Eqs. (1) and (2) and can be solved to determine the behaviour of the temperature perturbation. Following Rosenbluth & Bahcall (1973), we assume perturbations of the form:

$$\delta\boldsymbol{s}(\boldsymbol{r},t) = \delta\boldsymbol{s}(r)\mathrm{Y}_\ell^m(\theta,\phi)\exp(\xi t) \tag{11}$$

where $\mathrm{Y}_\ell^m$ is the spherical harmonic of degree $\ell$ and azimuthal order $m$, and $\xi = \lambda - i\omega$ is in general complex with $\lambda$ the growth or decay rate of the perturbation and $\omega$ its angular frequency. On perturbing to the first order Eqs.(3) and (4), and inserting Eq.(4) into Eq.(3), we obtain:

$$\alpha\xi T_r' = \varrho_\circ\varepsilon' - \frac{\chi_\circ\ell(\ell+1)}{r^2}T_r' + \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(G_\circ\frac{\mathrm{d}T_r'}{\mathrm{d}r} + F_\circ T_r'\right) \tag{12}$$

where $T_r'$ is the purely radial part of the temperature perturbation as given from Eq.(11), and:

$$\alpha = \varrho_\circ \mathrm{c}_{\mathrm{v}_\circ} H_{\mu_\circ}\left(\frac{\mathrm{d}\ln P_\circ}{\mathrm{d}r} - \gamma\frac{\mathrm{d}\ln\varrho_\circ}{\mathrm{d}r}\right) \tag{13}$$

$$G_\circ = r^2\chi_\circ \tag{14}$$

$$F_\circ = r^2\chi_\circ\eta_\circ\frac{\mathrm{d}\ln T_\circ}{\mathrm{d}r} \tag{15}$$

$$\eta_\circ = 3 - \left(\frac{\partial\ln\kappa_\circ}{\partial\ln T_\circ}\right)_{\varrho_\circ,\mu_\circ} - \left(\frac{\partial\ln\kappa_\circ}{\partial\ln\mu_\circ}\right)_{\varrho_\circ,T_\circ} \tag{16}$$

where $\mathrm{c}_{\mathrm{v}_\circ} = 3\mathrm{R_G}/2\mu_\circ$ is the specific heat at constant volume with $\mathrm{R_G}$ the gas constant, $\gamma = 5/3$ the adiabatic exponent, and the perfect gas law has been used, which well applies to fully ionized negligibly degenerate stellar cores, as the Sun's core.

In order to derive the stability condition, it is important to note that $\alpha$ is positive in the convectively stable layers where the mean molecular weight decreases outwards, as in the Sun's core case. The second term of the RHS member of Eq.(12) is proportional to the angular part of the temperature laplacian and it represents the diffusion due to nonradial radiative flux, which plays an important role in smoothing out thermal perturbations.

As far as the energy generation is concerned, we consider both the PPI and PPII branches of the PP chain. The PPII branch contributes by about 15% to the total number of terminations (Bahcall 1989), therefore it may be of some relevance in the $^3$He destruction process. The energy production is thus given by the following expression:

$$\varrho\varepsilon = \mathrm{R}_{11}\mathrm{E}_{11} + \mathrm{R}_{33}\mathrm{E}_{33} + \mathrm{R}_{34}\mathrm{E}_{34} \tag{17}$$

where $\mathrm{E}_{11} = 6.67$ MeV, $\mathrm{E}_{33} = 12.86$ MeV, $\mathrm{E}_{34} = 18.93$ MeV are the energies respectively delivered in the processes which create $^3$He, and which lead to the formation of $^4$He through the PPI and PPII branches (Bahcall 1989) and, as usual:

$$\mathrm{R}_{ij} = \frac{\varrho^2 x_i x_j}{(1+\delta_{ij})m_u^2 A_i A_j} < i,j > \tag{18}$$

is the reaction rate of the process between the nuclei $i$ and $j$ (reactions cm$^{-3}$ s$^{-1}$), $x_i$ and $x_j$ are the relative mass abundances of the reacting nuclei with $A_i$ and $A_j$ their atomic masses in units of the atomic mass unit $m_u$, $\delta_{ij}$ is the Kronecker delta which prevents double counting of identical particles, and $<i,j>$ has been approximated as:

$$<i,j>= 2^{5/3}\left(\frac{2}{3M_{ij}}\right)^{1/2} \mathrm{S}_{ij}(\mathrm{E_p})\frac{\mathrm{W}_{ij}^{1/6}}{(kT)^{2/3}}\times \exp\left[-3\left(\frac{\mathrm{W}_{ij}}{4kT}\right)^{1/3}\right] \quad (19)$$

where $M_{ij} = m_u A_i A_j/(A_i + A_j)$ is the reduced mass of the reacting nuclei, $\mathrm{S}_{ij}(\mathrm{E_p})$ the astrophysical factor of the reaction $<i,j>$ at the Gamow-peak energy, $k$ the Boltzmann constant, and:

$$\mathrm{W}_{ij} = \frac{8\pi^4 M_{ij} Z_i^2 Z_j^2 e^4}{h^2} \quad (20)$$

where $Z_i$ and $Z_j$ are the atomic numbers of the reacting nuclei, $e$ the electron charge and $h$ the Planck constant. Equation (17) has been derived by assuming that D, $^7$Be and $^7$Li nuclei are at equilibrium. In order to perturb Eq.(17) we take into account that the rate of destruction of protons through the main reaction p(p, $e^+\nu_e$)D is slow with respect to the growth rate of perturbation so that $\delta$X = $\delta$Y = 0, where X and Y are the relative mass abundances of hydrogen and helium respectively. This leads to $\delta\mu = 0$, in agreement with the previous conclusions, since the heavier element abundance Z is constant and the $^3$He abundance, $\mathrm{Y}_3$, is negligible. We consider the explicit perturbation of the $^3$He abundance and linearize to the first order the evolution equation:

$$\frac{\partial\delta\ln \mathrm{Y}_3}{\partial t} = \frac{\mathrm{R}'_{11} - 2\mathrm{R}'_{33} - \mathrm{R}'_{34}}{\mathrm{n}_3^\circ} \quad (21)$$

where $\mathrm{n}_3^\circ = \varrho \mathrm{Y}_3^\circ/m_u A_3$ and $\mathrm{Y}_3^\circ$ are the $^3$He concentration and relative mass abundance at equilibrium respectively, which satisfy the equilibrium condition $\mathrm{R}_{11}^\circ - 2\mathrm{R}_{33}^\circ - \mathrm{R}_{34}^\circ = 0$. From Eq.(21) and logaritmic differentiation of the equilibrium condition, taking into account the relationship between lagrangian and eulerian variations, we therefore obtain:

$$\frac{\mathrm{Y}'_3}{\mathrm{Y}_3^\circ} = \left(A_\circ - \frac{\xi}{\xi + D_\circ}B_\circ\right)\frac{T'_r}{T_\circ} \quad (22)$$

where the explicit form of the coefficients $A_\circ$, $B_\circ$, and $D_\circ$ is given in the Appendix. The first order perturbation of Eq.(11) gives:

$$\varrho_\circ\varepsilon' = \mathrm{R}'_{11}\mathrm{E}_{11} + \mathrm{R}'_{33}\mathrm{E}_{33} + \mathrm{R}'_{34}\mathrm{E}_{34} \quad (23)$$

From Eqs.(22) and (23) we can eliminate $\mathrm{Y}'_3/\mathrm{Y}_3^\circ$ to obtain:

$$\varrho_\circ\varepsilon' = \left(M_\circ - \frac{\xi}{\xi + D_\circ}N_\circ\right)T'_r \quad (24)$$

where the explicit form of the coefficients $M_\circ$ and $N_\circ$ is given in the Appendix. We insert Eq.(24) into Eq.(12) and obtain:

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(G_\circ\frac{\mathrm{d}T'_r}{\mathrm{d}r} + F_\circ T'_r\right) = \left[\alpha\xi + \frac{\xi}{\xi + D_\circ}N_\circ - M_\circ + \frac{\chi_\circ\ell(\ell+1)}{r^2}\right]T'_r \quad (25)$$

which is a homogeneous $2^{nd}$ order ordinary differential equation, with the coefficients functions of the equilibrium model quantities and the free parameter $\xi$, describing the behaviour of the temperature radial perturbation for any harmonic degree $\ell \geq 1$. With the boundary conditions of the perturbation vanishing at the centre and at the external extreme of the integration interval, Eq.(25) has the trivial solution $T'_r = 0$ everywhere in the interval except for given values of $\xi$ which are the eigenvalues of the problem. Rosenbluth & Bahcall (1973) have demonstrated that, if $\alpha D_\circ + N_\circ > 0$ everywhere in the Sun's core then $\Re(\xi) = \lambda \geq 0$ (instability) implies $\Im(\xi) = \omega = 0$, and a sufficient condition for stability is:

$$M_\circ - \frac{\chi_\circ\ell(\ell+1)}{r^2} < 0 \quad (26)$$

If, on the other hand, $\alpha D_\circ + N_\circ < 0$ in at least a region of the Sun's core then the stability condition splits in two conditions: the first is given by Eq.(26) and it is valid where $N_\circ > 0$; the second is:

$$M_\circ - \frac{\chi_\circ\ell(\ell+1)}{r^2} - N_\circ + \frac{N_\circ D_\circ^2}{D_\circ^2 - \left(N_\circ D_\circ/\alpha - D_\circ^2\right)_{\min}} < 0 \quad (27)$$

and it is valid where $N_\circ < 0$. In this case $\xi$ is in general complex.

## 4. Results and discussion

In order to verify the stability of the solar core we constructed a standard solar model, without elemental gravitational settling, by using the FRANEC (Frascati Raphson Newton Evolutionary Code) code (Chieffi & Straniero 1989) in its most recent version with OPAL opacities (Roger & Iglesias 1992) and MHD equation of state (Mihalas et al. 1988). The Sun has been evolved for $4.6 \times 10^9$ years, reaching the present luminosity ($3.846 \times 10^{33}$ erg s$^{-1}$) and radius ($6.9599 \times 10^{10}$ cm) with an initial helium abundance 0.29 and a mixing-length parameter 1.88. The central temperature, pressure, density, hydrogen abundance and fractional radius of the convection zone base are $T_c = 1.57 \times 10^6$ K, $P_c = 2.34 \times 10^{17}$ dynes cm$^{-2}$, $\varrho_c = 1.52 \times 10^2$ g cm$^{-3}$, $\mathrm{X}_c = 0.33$ and $\mathrm{x}_{cz} = 0.72$ respectively. The neutrino flux of this model for gallium detector is 135 SNU.

The derivatives which appear in the coefficients of the perturbed equations have been obtained analytically by derivation of 9th degree interpolating polynomials of the model quantities. Partial derivatives of opacity, which appear in the coefficient $\eta_\circ$, have been obtained numerically from the opacity tables by taking two variables constant in turn and varying the third by a small amount, so obtaining the variation of opacity with respect to the varying quantity.

We limit our analysis to the region of the Sun from the centre to 0.35 $R_\odot$, both because, above 0.35 $R_\odot$, the energy generation is greatly depleted and molecular weight gradient, which is necessary for the instabilities we are studying, tends to zero

**Table 1.** Results of the stability analysis in some points of the solar core. The values of the relative luminosity $L_r/L_\odot$ are given together with the dimensionless quantities $\Xi_1$ and $\Xi_2$ for the two cases PPI and PPI + PPII respectively. The stability condition (26) is satisfied for $\Xi < \ell(\ell+1)$. For comparison, the results obtained by Rosenbluth & Bahcall (RB) in 1973 for the only PPI case are given.

| $r/R_\odot$ | $L_r/L_\odot$ (RB) | $L_r/L_\odot$ | $\Xi$ (RB) | $\Xi_1$ | $\Xi_2$ |
|---|---|---|---|---|---|
| 0.02 | 0.003 | 0.007 | −0.001 | −0.067 | 0.007 |
| 0.04 | 0.06 | 0.05 | 0.02 | −0.198 | 0.027 |
| 0.10 | 0.38 | 0.45 | 0.05 | −0.060 | 0.015 |
| 0.14 | 0.70 | 0.73 | 0.09 | 0.189 | 0.013 |
| 0.18 | 0.84 | 0.89 | 0.08 | 0.243 | 0.052 |
| 0.25 | – | 0.99 | – | 0.127 | 0.011 |
| 0.30 | – | 1.00 | – | 0.025 | 0.066 |
| 0.35 | – | 1.00 | – | 0.007 | 0.011 |

and consequently $\alpha$ in Eq.(25) [see also the definition (13) of $\alpha$] tends to infinity, so rendering Eq.(25) unstable to numerical integration. On the other hand, to place the outer boundary condition of vanishing temperature perturbation at $0.35\,R_\odot$ is plausible since Eq.(10) indicates that, for $\mathrm{d}\ln\mu_\circ/\mathrm{d}r \to 0$, $H_{\mu_\circ} \to \infty$ and consequently $T' \to 0$. Our region includes the $^3$He peak, which, in the present Sun, extends from $0.21\,R_\odot$ to $0.35\,R_\odot$ with the maximum at $0.28\,R_\odot$.

We found $\alpha D_\circ + N_\circ > 0$ everywhere so that stability condition (26) applies. When applied to the most unstable modes ($\ell = 1$), this condition demonstrates that our standard solar model is stable against nonradial thermal perturbations so reproducing the same results of Rosenbluth & Bahcall (1973) also in the case in which both the PPI and PPII branches have been taken into account. From condition (26) it is easy to see that the second term of LHS member, which represents the diffusion due to the nonradial flux, prevents instability and its stabilizing effect increases with $\ell$.

We can compare our results on the stability directly with those obtained by Rosenbluth & Bahcall (1973) by noting that the quantity listed in the third column of their Table 1(B) is equivalent to the quantity $\Xi = r^2 M_\circ/\chi_\circ$ of our Eq.(26), and it is a measure of the stability when compared with $\ell(\ell+1)$. For the most unstable modes ($\ell = 1$), the stability is satisfied if this dimensionless quantity $\Xi$ is smaller than 2. In Table 1 we give our values of $\Xi$ for both the PPI ($\Xi_1$) and PPI + PPII ($\Xi_2$) cases together with the values of the relative luminosity $L_r/L_\odot$ and compare them with those obtained by Rosenbluth & Bahcall (RB) in 1973, which refer to the PPI case alone. From Table 1 it appears that the Sun's core is certainly stable against nonradial thermal perturbations as all the values of $\Xi$ are much smaller than 2. The differences with the previous results of Rosenbluth & Bahcall (1973) can be understood in terms of differences between the old and modern opacities, which reflect also in the values of $L_r/L_\odot$, and on the different perturbed terms when the PPII branch has been added in the perturbation analysis. On the other hand, since $\Xi \propto \kappa$ it is necessary to increase the opacities by a factor from 10 to 100 to render the solar core unstable.

The fact that unstable solutions do not exist precludes the possiblity that $\lambda \geq 0$ and $\omega = 0$, which means a perturbation growing with time. We limit ourselves to search for stable solutions with $\xi = \lambda < 0$, which represent a perturbation which decays with time. In the stable case we should expect that the decay time is shorter than the growth time of perturbation so that any amplitude of a growing mode is rapidly damped. So doing, we set $\omega = 0$ and exclude possible oscillatory solutions. This is not very important as far as we are concerned with perturbations which decay in time, consistently with the result of the stability analysis. Therefore Eq.(25) becomes an equation where the complex $\xi$ has been replaced by its real part $\lambda$. However, to consider both the real and imaginary parts of Eq.(25), which are linked together, might yield slightly different values of $\lambda$. In general, the temporal behaviour of a mode, whose amplitude is $\Psi_{n\ell}$, can be described as:

$$\frac{\partial \Psi_{n\ell}}{\partial t} = \Psi_{n\ell}\left(\frac{1}{\tau_G} - \frac{1}{\tau_D}\right) + \mathrm{NLT} \qquad (28)$$

where $\tau_G$ and $\tau_D$ are the growth and decay times of the mode respecively, $n$ its radial order, and NLT stands for nonlinear terms, which are neglected in a linear analysis as the present one. The parameter $\lambda$ is a measure of the competitive role of the excitation and damping of thermal perturbations and it is linked to their growth and decay times through the relationship $\lambda = 1/\tau_G - 1/\tau_D$. The growth time of the perturbation can be estimated by considering the heat transfer equation in which the diffusive term has been neglected. This leads to the following expression:

$$\frac{1}{\tau_G} \simeq \frac{1}{\varrho_\circ \mathrm{c}_{\mathrm{v}_\circ} T_\circ} \sum_{ij} \mathrm{E}_{ij} \mathrm{R}^\circ_{ij} \left(\frac{\mathrm{d}\ln <i,j>_\circ}{\mathrm{d}\ln T_\circ}\right)^{-1} \qquad (29)$$

where summation is extended to the three relevant reactions $<1,1>$, $<3,3>$ and $<3,4>$. The decay time can be estimated from the heat diffusion equation, obtaining:

$$\frac{1}{\tau_D} \simeq \frac{1}{\varrho_\circ \mathrm{c}_{\mathrm{v}_\circ} T_\circ} \left| \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \chi_\circ \frac{\mathrm{d}T_\circ}{\mathrm{d}r}\right)\right| \qquad (30)$$

Equations (29) and (30) indicate that the characteristic times of perturbation depend on the local properties of the medium. In Table 2 we show the values of $\tau_G$ and $\tau_D$ with the corresponding local values of $\lambda^{-1}$ for some points in the solar core. From Table 2 it appears that the growth times are larger than the decay times everywhere in the Sun's core, in agreement with the stability condition. The former times increase with the distance from the centre because the nuclear reaction rates decrease with temperature, while the latter times first increase, due to the rapid increase of the opacity above $0.2\,R_\odot$, until they reach a maximum around $0.25\,R_\odot$, then they decrease, due to the decrease of density. The 4th column of Table 2 gives the local damping times of perturbations, which range from $\simeq 10^6$ years in the central regions of the Sun to $\simeq 10^9$ years in the regions where the thermal diffusion has a minimum. We should then expect that the eigensolutions of Eq.(25) give values of $|\,\lambda^{-1}\,|$ in this range.

**Table 2.** Growth ($\tau_G$) and decay ($\tau_D$) times of thermal perturbations with the corresponding local values of $\lambda^{-1}$ in years as functions of the fractional radius in the Sun's core, as estimated from Eqs.(29) and (30). The number in parenthesis indicates the exponent.

| $r/R_\odot$ | $\tau_G$ | $\tau_D$ | $\lambda^{-1}$ |
|---|---|---|---|
| 0.00 | 4.33(6) | 6.95(5) | −7.06(5) |
| 0.05 | 4.85(7) | 5.93(6) | −6.76(6) |
| 0.10 | 7.73(7) | 1.01(7) | −1.16(7) |
| 0.15 | 1.69(8) | 1.54(7) | −1.69(7) |
| 0.20 | 4.71(8) | 2.73(7) | −2.90(7) |
| 0.25 | 1.62(9) | 5.31(8) | −7.90(8) |
| 0.30 | 6.66(9) | 9.89(6) | −9.90(6) |
| 0.35 | 2.25(10) | 2.68(6) | −2.68(6) |

**Table 3.** Eigenvalues $\lambda$ (years$^{-1}$) and the corresponding decay times $\tau = \mid \lambda^{-1} \mid$ (years) for some modes with $\ell = 1$, as obtained by the integration of Eq.(25). The number in parenthesis indicates the exponent.

| $n$ | $\lambda$ | $\tau$ |
|---|---|---|
| 0 | −6.45(−8) | 1.55(7) |
| 1 | −1.69(−7) | 5.92(6) |
| 2 | −3.20(−7) | 3.13(6) |
| 3 | −5.12(−7) | 1.95(6) |
| 4 | −6.97(−7) | 1.43(6) |
| 5 | −9.63(−7) | 1.04(6) |
| 10 | −2.81(−6) | 3.56(5) |

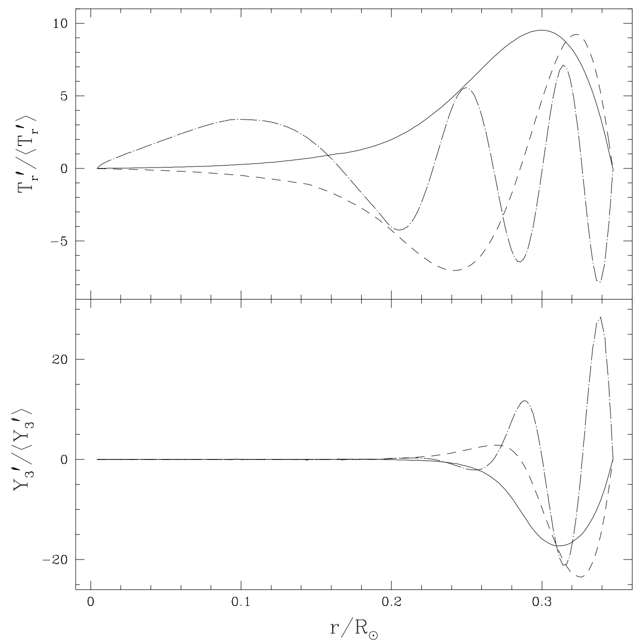


**Fig. 1.** Behaviour of the temperature (top) and $^3$He (bottom) perturbations as functions of the fractional radius in the Sun's core for $\ell = 1$ harmonic degree and different radial orders $n = 0, 1, 5$. The normalized curves represent stable solutions of Eq.(25) with decay time of perturbation $\tau = 1.56 \times 10^7$ years ($n = 0$, continuous), $\tau = 5.90 \times 10^6$ years ($n = 1$, dashed), and $\tau = 1.03 \times 10^6$ years ($n = 5$, dashed-dotted) respectively.

In order to determine the global decay time of perturbations and their radial behaviour we need to integrate Eq.(25) for finding the eigenvalues $\lambda$ which satisfy the equation and the relative eigenfunctions. We integrate Eq.(25) numerically by a 4th order Runge-Kutta method, with boundary conditions $T'_r = 0$ at $r = 0$ and $r = 0.35\, R_\odot$, starting from the centre. A Newton-Raphson iterative method is used in order to determine that $(\mathrm{d}T'_r/\mathrm{d}r)_{r=0}$ which satisfies the boundary condition at the external extreme of the integration interval. The eigensolutions are found by starting from a fixed value of $\lambda$ and incrementing its value successively by a small amount until a solution different from the trivial one is determined. No solutions with $\lambda \geq 0$ have been found in a large interval of $\lambda$, consistently with the fulfillment of the stability condition. The lowest negative value of $\lambda$ is found for $\ell = 1$ and corresponds to an eigenfunction $T'_r(r)$ with no radial nodes $n$ between the two extremes of the integration interval. The immediately higher negative value corresponds to an eigenfunction with one radial node and successively the number of radial nodes increases with increasing the absolute value of $\lambda$. For the eigenfunctions with equal $n$'s, the absolute value of $\lambda$ increases with $\ell$. In Table 3 we give the eigenvalues $\lambda$ and the corresponding decay times $\tau = \mid \lambda^{-1} \mid$ of some modes with $\ell = 1$. In Fig. 1 (top) we show the behaviour of some eigenfunctions corresponding to modes with $\ell = 1$ and $n = 0, 1, 5$. The curves of $T'_r$ have been normalized with $\langle T'_r \rangle = \int_0^{r_c} \mid T'_r \mid \mathrm{d}r/r_c$, where $r_c$ is the external extreme of the integration interval. In the same figure (bottom) we show the behaviour of the $^3$He perturbation, $Y'_3$, obtained by inserting the eigenvalues $\lambda$ with the corresponding eigenfunctions into Eq.(22). The curves have been normalized in the same way adopted for the temperature perturbation.

From Fig. 1 (top) it appears that the temperature perturbations are larger in the outer part of the solar core where the thermal diffusivity is lower and the decay time longer (see Table 2). The amplitude of perturbation in the central regions increases with the increase of the number of radial nodes of the eigenfunction, namely for those modes with higher $\mid \lambda \mid$. This means that any perturbation which is excited in these regions is rapidly damped by the radiative diffusion. Nothing can be said about the value of the perturbation amplitude as Eq.(25) is homogeneous and the value of $T'_r$ remains undetermined. However it is possible to roughly estimate this value from Eq.(10). For a displacement $\delta r$ corresponding to $0.01\, R_\odot$ the relative values of the temperature perturbations range from $10^{-2}$ in the central regions to $10^{-4}$ in the outer regions of the Sun's core.

The perturbation of $^3$He (Fig. 1, bottom) appears to be concentrated in the region of the $^3$He peak, independently of the region in which the temperature perturbation is present, and in phase opposition to the temperature perturbation, as it is expected from the instability mechanism discussed in Sect. 2, in the case the perturbation penetrates the $^3$He peak region from below.

## 5. Conclusions

We have shown that the standard solar model, constructed with the modern data of opacity, equation of state and nuclear reaction cross sections, is stable against nonradial thermal perturbations so confirming Rosenbluth & Bahcall's (1973) results obtained with an old solar model. Also the inclusion of the PPII branch in the perturbation analysis does not alter the previous findings. The reason for thermal stability of the Sun's core stays in the strong efficiency of the radiative heat diffusion owing to the low opacity of the solar core, as can be seen from the stability condition (26) where the second term of LHS member is negative and largely dominant with respect to the first, which, on the other hand, is not positive everywhere in the Sun's core. The second term of LHS member of Eq.(26) depends on the thermal conductivity coefficient which is inversely proportional to the opacity. In order to render the Sun's core unstable to thermal perturbations we need to increase the opacity by a factor from 10 to 100. Even with the uncertainties in the opacity determinations, which are however not very high in the solar core, there is no room for such an error.

The confirmation of the stability of the Sun's core against both the g-mode instabilities (Bahcall & Kumar 1993) and nonradial thermal instabilities (Rosenbluth & Bahcall 1973, present paper) and the impressive agreement of the standard solar models with helioseismic data indicate that the neutrino problem is not astrophysical in origin but it relies upon non standard neutrino properties.

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## Appendix

$$A_\circ = \frac{1}{C_\circ}\left[2\mathrm{R}_{11}^\circ \frac{\mathrm{d}\ln \mathrm{X}_\circ}{\mathrm{d}\ln \mu_\circ} + (2\mathrm{R}_{33}^\circ - \mathrm{R}_{11}^\circ)\frac{\mathrm{d}\ln \mathrm{Y}_\circ}{\mathrm{d}\ln \mu_\circ} + I_\circ\right]$$

$$B_\circ = \frac{I_\circ}{C_\circ}\left(1 - \frac{\mathrm{d}\ln T_\circ/\mathrm{d}r}{\mathrm{d}\ln \mu_\circ/\mathrm{d}r}\right)$$

$$D_\circ = C_\circ/\mathrm{n}_3^\circ$$

$$M_\circ = J_\circ\left(2\frac{\mathrm{d}\ln \mathrm{X}_\circ}{\mathrm{d}\ln \mu_\circ} + \frac{\mathrm{d}\ln <1,1>_\circ}{\mathrm{d}\ln T_\circ}\right) + K_\circ\left[2\left(\frac{\mathrm{d}\ln \mathrm{Y}_\circ}{\mathrm{d}\ln \mu_\circ} + \frac{\mathrm{d}\ln <3,4>_\circ}{\mathrm{d}\ln T_\circ}\right) - \frac{\mathrm{d}\ln <3,3>_\circ}{\mathrm{d}\ln T_\circ}\right]$$

$$N_\circ = L_\circ B_\circ$$

$$C_\circ = \mathrm{R}_{11}^\circ + 2\mathrm{R}_{33}^\circ$$

$$I_\circ = \mathrm{R}_{11}^\circ \frac{\mathrm{d}}{\mathrm{d}\ln T_\circ}\ln\frac{<1,1>_\circ}{<3,4>_\circ} + 2\mathrm{R}_{33}^\circ \frac{\mathrm{d}}{\mathrm{d}\ln T_\circ}\ln\frac{<3,4>_\circ}{<3,3>_\circ}$$

$$J_\circ = \mathrm{R}_{11}^\circ[\mathrm{R}_{11}^\circ(\mathrm{E}_{11} + \mathrm{E}_{34}) + 2\mathrm{R}_{33}^\circ(\mathrm{E}_{11} + \mathrm{E}_{33} - \mathrm{E}_{34})]/T_\circ C_\circ$$

$$K_\circ = \mathrm{R}_{33}^\circ(\mathrm{R}_{11}^\circ - 2\mathrm{R}_{33}^\circ)(2\mathrm{E}_{34} - \mathrm{E}_{33})/T_\circ C_\circ$$

$$L_\circ = 2[\mathrm{R}_{33}^\circ(\mathrm{E}_{33} - \mathrm{E}_{34}) + \mathrm{R}_{11}^\circ \mathrm{E}_{34}]/T_\circ$$

$$\frac{\mathrm{d}\ln \mathrm{X}_\circ}{\mathrm{d}\ln \mu_\circ} = -\frac{1}{1 - \mu_\circ(3 - \mathrm{Z})/4}$$

$$\frac{\mathrm{d}\ln \mathrm{Y}_\circ}{\mathrm{d}\ln \mu_\circ} = -\frac{1}{1 - \mu_\circ(4 - 3\mathrm{Z})/2}$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln T_\circ}\ln\frac{<i,j>_\circ}{<m,n>_\circ} = \left(\frac{\mathrm{W}_{ij}}{4kT_\circ}\right)^{1/3} - \left(\frac{\mathrm{W}_{mn}}{4kT_\circ}\right)^{1/3}$$

$$\frac{\mathrm{d}\ln <i,j>_\circ}{\mathrm{d}\ln T_\circ} = -\frac{2}{3} + \left(\frac{\mathrm{W}_{ij}}{4kT_\circ}\right)^{1/3}$$

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