Astron. Astrophys. 322, 352–358 (1997)

ASTRONOMY AND ASTROPHYSICS

Theoretical study of the partial derivatives produced by numerical integration of satellite orbits

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Received 5 August 1996 / Accepted 30 September 1996

Abstract. For the two-body system Saturn-Mimas and the theoretical three-body non-resonant system Saturn-Mimas-Tethys we present a theoretical analysis of the behaviour of the partial derivatives of the satellites' coordinates with respect to the parameters of the system, namely the satellites' initial conditions and their mass-ratios over Saturn. With the use of Floquet theory for the stability of periodic orbits we prove that all the partial derivatives have amplitudes that increase linearly with time. Their motion is a combination of periodic motions the periods of which can also be accurately predicted by the theory. This theoretical model can be used for checking the accuracy of the results of the different numerical integration methods used on satellite systems with the purpose of fitting the results to observations or analytical theories. On this basis, in the last part of the paper we extend the investigation of Hadjifotinou & Harper (1995) on the stability and efficience of the 10^{th}-order Gauss-Jackson backward difference and the Runge-Kutta-Nyström RKN12(10)17M methods by now applying them to the above mentioned three-body system.

Key words: methods: numerical – celestial mechanics – ephemerides – planets and satellites

1. Introduction

The numerical integration of the system of equations of motion and variational equations is widely used in the study of natural satellite systems, since the values of the partial derivatives obtained in this way are necessary for fitting the results of the integration to observations or analytical theories and thus improving the values of some parameters of the satellite system. In this work we are dealing with the Saturnian satellite system and we attempt a theoretical analysis of the behaviour of the partial derivatives of the satellites' coordinates with respect to the satellites' initial conditions and their mass-ratios over Saturn. These partial derivatives are in fact solutions of the variational equations of the system.

Hadjifotinou & Harper (1995) studied the behaviour of two numerical integration methods, the 10^{th}-order Gauss-Jackson backward difference and the Runge-Kutta-Nyström RKN12(10)17M, using as a test problem the system of the equations of motion and variational equations of the two-body problem Saturn-Mimas. For testing the values of the partial derivatives obtained by the numerical methods, they found two approximate bounds for the partials with regard to the satellite's initial position and velocity: $3nt$ and $3t$ respectively, where n is the satellite's mean motion, and t the time since the start of the integration.

The main objective of this paper is to actually prove the linear growth of all the partial derivatives and monitor their exact behaviour by use of the well-known Floquet theory on the stability of periodic orbits. We first refer to the relatively simple two-body case and then present a full analysis of the three-body case, using a theoretical non-resonant Saturn - Mimas - Tethys system as our test problem. Finally, knowing the exact behaviour of the partials in the three-body case, we extend the comparison of the Gauss-Jackson and RKN12(10)17M numerical integration methods using this more complex test problem, and we reestimate the efficiency of these methods.

2. The two-body case

The variational equations of the two-body problem comprise a linear system of the form:

$$\dot{\boldsymbol{\xi}} = P(t)\boldsymbol{\xi} \qquad (1)$$

where $\boldsymbol{\xi}$ is a column of the partials with regard to any coordinate of the satellite's initial position or velocity and

$$P(t) = \Omega \nabla^2 H(t)$$

where H is the Hamiltonian of the two-body problem,

$$\Omega = \begin{pmatrix} O & I_3 \\ -I_3 & O \end{pmatrix}$$

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is the standard symplectic structure, and $\nabla^2 H$ is the Hessian of H with respect to the canonical variables. In the two-body problem, any bounded orbit is periodic, therefore $P(t)$ is a T-periodic matrix, where T is the period of the orbit. As a result, we can apply symplectic Floquet theory (e.g. Hadjidemetriou 1988) to monitor the behaviour of the solution $\boldsymbol{\xi}(t)$.

Let $\Delta(t)$ be the fundamental matrix of solutions of Eq. (1), such that $\Delta(0) = I_6$. Then, the properties of $\boldsymbol{\xi}(t)$ are fully determined by the eigenvalues and eigenvectors of the *monodromy matrix* $\Delta(T)$. The three-dimensional two-body problem is maximally superintegrable, i.e. it possesses 5 independent integrals of motion (Goldstein 1980, p. 104) and, according to a theorem of Poincaré (Poincaré 1892, Chapt. 4), the monodromy matrix has at least 5 unit eigenvalues and 5 independent eigenvectors. However, since $\Delta(T)$ is symplectic (e.g. Yakubovich & Starzhinski 1975, p. 116), its eigenvalues are in reciprocal pairs and therefore it has 6 unit eigenvalues. It can be proved that $\Delta(T)$ has exactly five independent eigenvectors. Since all the eigenvalues ($\lambda_j,\ j = 1,..,6$) are equal to unity, all the corresponding *characteristic exponents*, defined by the relation:

$$a_j = \frac{1}{T}\ln\lambda_j \mod i\frac{2\pi}{T}, \quad j = 1,..,6$$

are zero, and, according to Floquet theory, the solutions of Eq. (1) are of the form:

$$\boldsymbol{\xi}(t) = c_2\boldsymbol{f}_2(t) + c_1 t\boldsymbol{f}_1(t)$$

where $\boldsymbol{f}_1(t), \boldsymbol{f}_2(t)$ are T-periodic vectors.

In fact, since our system (i.e. the two-body problem) is an autonomous Hamiltonian system, it is $\boldsymbol{f}_1(t) = \dot{\boldsymbol{X}}(t)$, where $\boldsymbol{X}(t)$ is the T-periodic solution, around which, the variational equations (1) are obtained.

Thus, we have proved that in the two-body case, the motion of the partials with regard to the satellite's initial conditions is a combination of a T-periodic and a linear motion, that is, the amplitudes of the partial derivatives increase linearly with time.

For the partials with regard to the satellite's mass-ratio over Saturn, the system is non-homogeneous:

$$\dot{\boldsymbol{\xi}} = P\boldsymbol{\xi} + \boldsymbol{b} \tag{2}$$

where $\boldsymbol{b}$ is a T-periodic vector whose explicit form is of no interest here. The solution of this system is (e.g. Jordan & Smith 1979, p. 225):

$$\boldsymbol{\xi}(t) = \Delta(t)\boldsymbol{\xi}(0) + \Delta(t)\int_0^t \Delta^{-1}(t)\boldsymbol{b}(t)\,dt \tag{3}$$

where $\Delta(t)$ is the fundamental matrix of solutions of Eq. (1). Since $\Delta(t)$ is symplectic, $\Delta^{-1}(t) = -\Omega\Delta^T(t)\Omega$. Also, for $\boldsymbol{\xi}(0) = \boldsymbol{0}$, Eq. (3) becomes:

$$\boldsymbol{\xi}(t) = -\Delta(t)\int_0^t \Omega\Delta^T(t)\Omega\boldsymbol{b}(t)\,dt. \tag{4}$$

We already know by the solution of Eq. (1) that there is a linear secular term in $\Delta(t)$. Therefore, from Eq. (4) one should expect that the secular term in the solution $\boldsymbol{\xi}(t)$ is of order t^3. However, it is proved (the proof, for the three-body case, is illustrated in Sect. 3.2) that, also for the partials with regard to the mass-ratio, the secular term is only linear in t and therefore these partials behave in exactly the same way as the partials with regard to the satellite's initial conditions.

Table 1. Initial conditions and mass-ratios of the original orbit in the Saturnicentric frame

	Mimas	Tethys
	Position, in A.U.	
X_0	0.0011714610	-0.0005603899
Y_0	0.0003336770	-0.0018889781
Z_0	-0.0000008043	-0.0000031526
	Velocity, in A.U. per day	
$\dot{X}_0$	-0.0023556375	0.0062823156
$\dot{Y}_0$	0.0080778103	-0.0018633650
$\dot{Z}_0$	-0.0002370600	-0.0001236929
	Mass-ratios, satellite/Saturn	
	0.0634×10^{-6}	1.060×10^{-6}

3. The general three-body case

In order to apply Floquet theory also in this case, our three-body orbit has to be periodic, or at least quasiperiodic, close to a periodic one (Jorba *et al.* 1995). It is known (Szebehely 1969) that, in the inertial frame, periodic orbits are isolated and therefore one should not try to find an exact periodic orbit in this frame. However, in a suitably defined rotating reference frame, the periodic orbits are dense according to the Poincaré conjecture, and we can look for a periodic orbit there.

The test problem that we use for the three-body case is the system Saturn - Mimas - Tethys. Since in this case all bodies are considered as point masses, our system is simplified by omitting Saturn's oblateness terms, and as a result, also the Mimas-Tethys 4:2 resonance. The initial conditions for this system were provided by Harper (p.c.) by the method of fitting a numerical integration to the Vienne & Duriez (1995) analytical theories. These initial conditions, together with the mass-ratios of the satellites over Saturn which are the ones of Vienne & Duriez (1995), are given in Table 1. They refer to the epoch 2441400.5 and their reference frame is the Saturnicentric frame used in Sinclair & Taylor (1985).

In order to find whether the orbit corresponding to these initial conditions is periodic, we transformed the initial conditions to a rotating reference frame introduced by Michalodimitrakis (1979), with Saturn and Tethys as the primaries and Mimas as the third body. The origin of this system is the Saturn-Tethys centre of mass, its z-axis is kept parallel to the constant angular momentum vector and its xy-plane is rotating in such a way that the primaries are always on the xz-plane. The equations

Table 2. The original orbit in the rotating frame at $t = t_0$

	Mimas	Tethys
	Position, in A.U.	
x	0.0012442715	-0.0019709054
y	0.0	0.0
z	-0.0000500888	-0.0000037646
	Velocity, in A.U. per day	
$\dot{x}$	0.0001762602	-0.0000037477
$\dot{y}$	0.0040969612	0.0
$\dot{z}$	0.0000064754	0.0000005053

Table 3. The symmetric periodic orbit in the rotating frame at $t = t_0$

	Mimas	Tethys
	Position, in A.U.	
x	0.0012084658	-0.0019701589
y	0.0	0.0
z	-0.0000594298	-0.0000045924
	Velocity, in A.U. per day	
$\dot{x}$	0.0	0.0
$\dot{y}$	0.0044481321	0.0
$\dot{z}$	0.0	0.0

of motion as well as the variational equations of the three-body problem in this reference frame are given by Katopodis (1979). We transformed the initial conditions of Table 1 to this rotating frame and integrated numerically the equations of motion up to the first time that the third body crosses the xz-plane. By interpolation we found exactly the time of this crossing: $t_0 = 0.287286$ days after the initial epoch. The positions and velocities of Mimas and Tethys in the rotating frame at $t = t_0$ are given in Table 2. The coordinates of the other primary, Saturn, can be easily calculated from the ones of Tethys since the origin of the reference frame is the Saturn-Tethys centre of mass.

Integrating further the orbit of Table 2, we found that the third body crosses for the second time after t_0 the xz-plane at $t = t_0 + 1.89798$ days. This time period is, as expected, close to the orbital period of Tethys. Also, plots of this orbit in the rotating frame showed that it might be periodic with multiplicity 1 and, furthermore, that it is nearly symmetric with regard to the xz-plane. However, it was found that this orbit is not periodic with multiplicity 1, since, the maximum relative difference of its initial conditions to the final positions and velocities in the rotating frame taken after the possible period 1.89798 days is of order $O(10^{-1})$.

By the method of differential correction to the initial conditions, we found that close to this orbit lies an exact periodic one which is symmetric with regard to the xz-plane. The plots of this orbit are almost identical to the ones of the original orbit and its period is $T \simeq 1.8919251$ days. The initial conditions of the symmetric periodic orbit in the rotating reference frame for $t = t_0$ are given in Table 3.

By applying Floquet theory to this periodic orbit we can describe sufficiently the behaviour of the partial derivatives of the nearby quasiperiodic orbits such as our original orbit (Jorba *et al.* 1995).

A three-body orbit in the rotating frame is described by 8 independent state coordinates:

$$x_3, \quad y_3, \quad z_3, \quad x_1, \quad \dot{x}_3, \quad \dot{y}_3, \quad \dot{z}_3, \quad \dot{x}_1$$

where the indices 3 and 1 refer to the third body and the first primary respectively. The rest of the coordinates, as well as the angular velocity $\dot{\theta}$ of the rotating frame, can be calculated from the above coordinates with the help of the angular momentum integral (Michalodimitrakis 1979; Katopodis 1979).

The monodromy matrix of the symmetric periodic orbit of Table 3 is an 8x8 matrix and was found to have 2 unit eigenvalues (originated from the existence of the energy integral) and 6 complex eigenvalues of modulus 1 which form conjugate and reciprocal pairs. This shows that the symmetric periodic orbit under study is stable. The characteristic exponents corresponding to the eigenvalues of the monodromy matrix are:

$$0, 0, \pm i\alpha_1, \pm i\alpha_2, \pm i\alpha_3,$$

where

$$\alpha_1 = 0.010104, \alpha_2 = 0.001502, \alpha_3 = 0.001473.$$

As a result, according to Floquet theory, the solutions of the variational equations of this periodic orbit in the rotating reference frame, will be of the form:

$$\begin{aligned}\boldsymbol{\xi}(t) = \; & c_1\boldsymbol{f}_1(t) + c_2(\boldsymbol{f}_2(t) + t\boldsymbol{f}_1(t)) + \\ & c_3\boldsymbol{f}_3(t)e^{i\alpha_1 t} + c_3^*\boldsymbol{f}_3^*(t)e^{-i\alpha_1 t} + \\ & c_5\boldsymbol{f}_5(t)e^{i\alpha_2 t} + c_5^*\boldsymbol{f}_5^*(t)e^{-i\alpha_2 t} + \\ & c_7\boldsymbol{f}_7(t)e^{i\alpha_3 t} + c_7^*\boldsymbol{f}_7^*(t)e^{-i\alpha_3 t}\end{aligned} \tag{5}$$

where $\boldsymbol{f}_j(t)$ and their complex conjugate $\boldsymbol{f}_j^*(t)$ are T-periodic vectors, and c_j are arbitrary constants. Also, as in the two-body case, $\boldsymbol{f}_1(t) = \dot{\boldsymbol{X}}(t)$, where $\boldsymbol{X}(t)$ is the periodic orbit.

The three pairs of imaginary exponents in Eq. (5) introduce three additional periods in the solution of the variational equations:

$$T_1 = 621.8595 \text{ days}$$
$$T_2 = 4182.8512 \text{ days}$$
$$T_3 = 4265.6955 \text{ days}.$$

3.1. Study of the partial derivatives with regard to the satellites' initial conditions

By the above results we monitor the behaviour of the solutions of the variational equations of the periodic orbit in the rotating

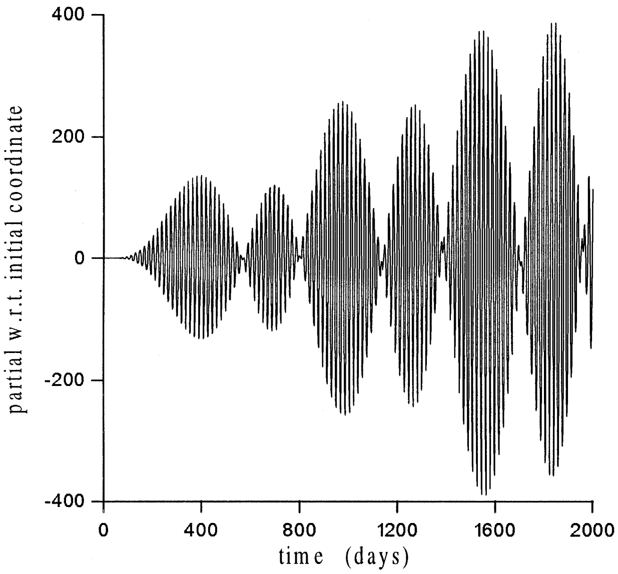

Fig. 1. The partial $\partial X_1/\partial Y_{20}$ versus time in days. The indices 1 and 2 stand for Mimas and Tethys respectively

reference frame. However, examining the equations of transformation from the rotating to the original Saturnicentric frame, it is easy to see that the behaviour of the partials in the Saturnicentric frame is identical to that of the partials in the rotating, but for an additional frequency due to its rotation. This in practice corresponds to the orbital period of Tethys' motion around the Saturn-Tethys centre of mass and therefore it does not affect the long period motions enforced by T_1, T_2 and T_3.

Thus, the evolution of all the partial derivatives of the satellites' coordinates with respect to the satellites' initial conditions of our original orbit in the Saturnicentric frame, is well described by equations of the form of Eq. (5), and is the result of the composition of a linear increase with time and four periodic motions with periods T, T_1, T_2 and T_3. For each partial derivative however, the dominant among the above four periods, depend on the actual values of c_j and $\boldsymbol{f}_j(t)$ that appear in its own Eq. (5).

The above behaviour is reflected to the plots of all the partial derivatives with regard to the initial positions and velocities of the original orbit in the Saturnicentric frame. For instance, in Fig. 1 we can clearly see the linear increase in the amplitude and, furthermore, we can trace, apart from the short period T, a period around 580 days which is close enough to T_1, if one considers that the orbit we are examining is not the periodic but is a quasiperiodic one near to it.

3.2. Study of the partial derivatives with regard to the satellites' mass-ratios over Saturn

In the three-body problem, the equations of the partials with regard to the satellites' mass-ratios over Saturn, as well as their solutions, are given, as in the two-body case, by Eqs. (2) and (3) respectively. However, since now we are working in the rotating frame, $\Delta(t)$ is the 8x8 fundamental matrix of solutions of the variational equations of the periodic orbit in the rotating frame. The columns of $\Delta(t)$ are the vectors in Eq. (5). If we choose $\Delta(t)$ in this way, then $\Delta(0) \neq I_8$, and instead of the symplectic property, $\Delta(t)$ satisfies the equation: $\Delta^T(t)\Omega\Delta(t) = E$, where now

$$\Omega = \begin{pmatrix} O & I_4 \\ -I_4 & O \end{pmatrix},$$

E is a constant matrix which has the form:

$$E = \begin{pmatrix} 0 & -\gamma & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\delta & 0 & 0 & 0 & 0 \\ 0 & 0 & i\delta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i\varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & i\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i\zeta \\ 0 & 0 & 0 & 0 & 0 & 0 & i\zeta & 0 \end{pmatrix}$$

and $\gamma, \delta, \varepsilon$ and ζ are constants depending on the elements of the periodic solution (see for example Hadjidemetriou (1973, 1975)). Then, $\Delta^{-1}(t) = E^{-1}\Delta^T(t)\Omega$, and since $\boldsymbol{\xi}(0) = \mathbf{0}$, Eq. (3) becomes:

$$\boldsymbol{\xi}(t) = \Delta(t)E^{-1}\int_0^t \Delta^T(t)\Omega\boldsymbol{b}(t)\,dt, \qquad (6)$$

where

$$\begin{aligned}\Delta(t)E^{-1} = (\quad & -\frac{1}{\gamma}(\boldsymbol{f}_2(t) + t\boldsymbol{f}_1(t)), \frac{1}{\gamma}\boldsymbol{f}_1(t), \\ & \frac{i}{\delta}\boldsymbol{f}_3^*(t)e^{-i\alpha_1 t}, -\frac{i}{\delta}\boldsymbol{f}_3(t)e^{i\alpha_1 t}, \\ & \frac{i}{\varepsilon}\boldsymbol{f}_5^*(t)e^{-i\alpha_2 t}, -\frac{i}{\varepsilon}\boldsymbol{f}_5(t)e^{i\alpha_2 t}, \\ & \frac{i}{\zeta}\boldsymbol{f}_7^*(t)e^{-i\alpha_3 t}, -\frac{i}{\zeta}\boldsymbol{f}_7(t)e^{i\alpha_3 t}).\end{aligned}$$

As mentioned above,

$$\boldsymbol{f}_1(t) = \dot{\boldsymbol{X}}(t) = (\dot{x}_3, \dot{y}_3, \dot{z}_3, \dot{x}_1, \ddot{x}_3, \ddot{y}_3, \ddot{z}_3, \ddot{x}_1)^T.$$

Since the periodic orbit on study is symmetric with regard to the xz-plane, the coordinates x_3, z_3 and x_1 are even functions of time, while y_3 is an odd function of time. So, we have

$$\begin{array}{cccc} x_3^c & y_3^s & z_3^c & x_1^c \\ \dot{x}_3^s & \dot{y}_3^c & \dot{z}_3^s & \dot{x}_1^s \\ \ddot{x}_3^c & \ddot{y}_3^s & \ddot{z}_3^c & \ddot{x}_1^c \end{array}$$

where upperscripts c, s stand for an even or an odd function of time respectively. Then,

$$\boldsymbol{f}_1(t) = (f_{11}^s, f_{21}^c, f_{31}^s, f_{41}^s, f_{51}^c, f_{61}^s, f_{71}^c, f_{81}^c)^T$$

and expressing the equations of the partials with regard to the mass-ratios in the rotating frame, we find that, for each mass-ratio, the non-homogeneous term $\boldsymbol{b}$ of Eq. (2) is of the form:

$$\boldsymbol{b}(t) = (0, 0, 0, 0, \Phi_1^c(t), \Phi_2^s(t), \Phi_3^c(t), \Phi_4^c(t))^T$$

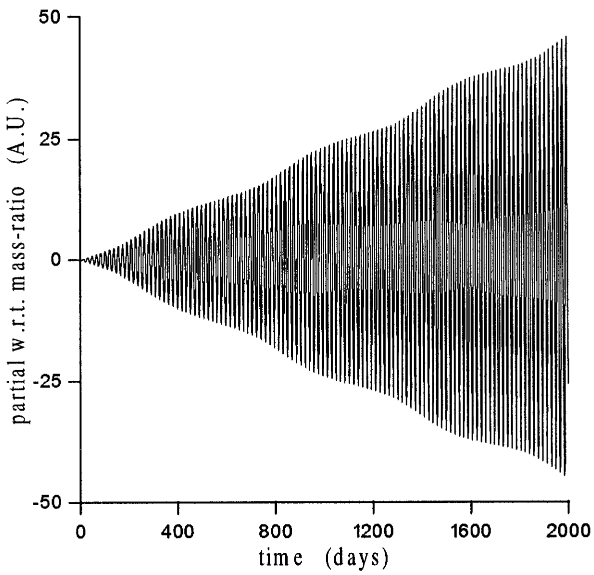

Fig. 2. The partial $\partial X_1/\partial m_1$ in A.U. versus time in days. The index 1 stands for Mimas and m_1 is the mass-ratio Mimas-Saturn

where $\Phi_j(t)$ are also T-periodic functions of time.

As a result, the integrand in Eq. (6) is the vector:

$$\Delta^T(t)\Omega \boldsymbol{b}(t) = \begin{pmatrix} A_1^s \\ A_2 + tA_1^s \\ A_3 e^{i\alpha_1 t} \\ A_3^* e^{-i\alpha_1 t} \\ A_5 e^{i\alpha_2 t} \\ A_5^* e^{-i\alpha_2 t} \\ A_7 e^{i\alpha_3 t} \\ A_7^* e^{-i\alpha_3 t} \end{pmatrix}$$

where

$$\begin{aligned} A_1^s &= f_{11}^s\Phi_1^c + f_{21}^c\Phi_2^s + f_{31}^s\Phi_3^c + f_{41}^s\Phi_4^c, \\ A_2 &= f_{12}\Phi_1^c + f_{22}\Phi_2^s + f_{32}\Phi_3^c + f_{42}\Phi_4^c, \\ A_3 &= f_{13}\Phi_1^c + f_{23}\Phi_2^s + f_{33}\Phi_3^c + f_{43}\Phi_4^c, \\ A_5 &= f_{15}\Phi_1^c + f_{25}\Phi_2^s + f_{35}\Phi_3^c + f_{45}\Phi_4^c, \\ A_7 &= f_{17}\Phi_1^c + f_{27}\Phi_2^s + f_{37}\Phi_3^c + f_{47}\Phi_4^c, \end{aligned}$$

f_{ij} the corresponding elements of the vectors $\boldsymbol{f}_j$, and A_j^* the complex conjugate of A_j. Finally, Eq. (6) becomes:

$$\begin{aligned} \boldsymbol{\xi}(t) = &-\frac{1}{\gamma}(\boldsymbol{f}_2 + t\boldsymbol{f}_1)\int_0^t A_1^s\,dt + \frac{1}{\gamma}\boldsymbol{f}_1\int_0^t (A_2 + tA_1^s)\,dt \\ &+\frac{i}{\delta}\boldsymbol{f}_3^* e^{-i\alpha_1 t}\int_0^t A_3 e^{i\alpha_1 t}\,dt - \frac{i}{\delta}\boldsymbol{f}_3 e^{i\alpha_1 t}\int_0^t A_3^* e^{-i\alpha_1 t}\,dt \\ &+\frac{i}{\varepsilon}\boldsymbol{f}_5^* e^{-i\alpha_2 t}\int_0^t A_5 e^{i\alpha_2 t}\,dt - \frac{i}{\varepsilon}\boldsymbol{f}_5 e^{i\alpha_2 t}\int_0^t A_5^* e^{-i\alpha_2 t}\,dt \\ &+\frac{i}{\zeta}\boldsymbol{f}_7^* e^{-i\alpha_3 t}\int_0^t A_7 e^{i\alpha_3 t}\,dt - \frac{i}{\zeta}\boldsymbol{f}_7 e^{i\alpha_3 t}\int_0^t A_7^* e^{-i\alpha_3 t}\,dt. \end{aligned} \tag{7}$$

From Eq. (7) we see that each element of the solution $\boldsymbol{\xi}(t)$ of Eq. (2) has a secular term originated from the first two terms of the sum and its motion is again a combination of this secular motion and the four periodic motions T, T_1, T_2 and T_3, that characterise the motion of the partials with regard to the initial conditions.

Examining the secular term more closely, we see that, although, due to the existence of the integrals, one could expect it to be of order t^2, in fact it is only of order t. This is justified as follows:

Since A_1^s is a sine Fourier series, it does not have a constant term in its expansion. As a result, the integral $\int_0^t A_1^s\,dt$ does not give a secular term, but only a constant one. Therefore, the first term of the sum: $-\frac{1}{\gamma}(\boldsymbol{f}_2 + t\boldsymbol{f}_1)\int_0^t A_1^s\,dt$ is only of order t. Also,

$$\int_0^t (A_2 + tA_1^s)dt = \int_0^t A_2 dt + t\int_0^t A_1^s dt - \int_0^t \left(\int_0^t A_1^s dt\right)dt.$$

The term $\int_0^t A_2\,dt$ is at most t-secular, and, since $\int_0^t A_1^s\,dt$ gives only a constant term, the other two terms in the above expression are also t-secular. As a result, the second term of the sum in Eq. (7): $\frac{1}{\gamma}\boldsymbol{f}_1\int_0^t (A_2 + tA_1^s)\,dt$, like the first term, is of order t.

This proves that, in the rotating frame, the amplitudes of the partials with regard to the satellites' mass-ratios increase linearly with time, and therefore these partials behave in exactly the same way as the rest of the partials. In the Saturnicentric frame, the behaviour of the partials is the same, but for the additional high frequency of the rotating frame mentioned in the previous section. Plots of various partials with regard to the mass-ratio of Mimas or Tethys over Saturn in the Saturnicentric frame (e.g. Fig. 2), verify the linear growth in the amplitudes as well as the above mentioned periodic motions.

4. Numerical investigation of the GJ and RKN methods in the three-body case

Knowing the exact behaviour of all the partial derivatives in the general three-body case, we can extend the comparison of the 10^{th}-order Gauss-Jackson backward difference numerical integration method described by Herrick (1972) and Merson (1974), and the Runge-Kutta-Nyström RKN12(10)17M of Dormand *et al.* (1987), that was done by Hadjifotinou & Harper (1995) for the two-body case.

It was found, that, for the numerical integration of the three-body system Saturn - Mimas - Tethys and the corresponding variational equations, the step-size of $\frac{1}{76}$ of the orbital period of the innermost satellite, Mimas, is still the critical step-size for the stability of the Gauss-Jackson method when the predictor only cycle is used for the integration of the partial derivatives. Increasing the step-size beyond this limit, caused exponential growth in the amplitudes of the partial derivatives. However, using this relatively small step-size (0.0124 days) and integrating the whole system for 1 million steps, that is for a time-span of 12400 days, we found that the Gauss-Jackson predictor-only method gave very accurate results, since, the relative error in the energy integral at the end of the integration was 2×10^{-13}. The

CPU-time needed to integrate this time-span on a VAX-9000 machine was about 36 minutes.

Introducing the corrector-cycle at the integration of the partials, as expected, allowed the use of larger step-sizes. We used the Gauss-Jackson corrector with the step-size of $\frac{1}{64}T_{Mimas}$ = 0.0147 days and integrated the same system for 840000 steps in order to cover the same time- span as with the Gauss-Jackson predictor. The CPU-time needed was about 32 minutes, but, at the end of the integration, the relative error in the energy integral was 7×10^{-12}.

Finally, for the integration of the same system, we tested also the RKN12(10)17M method. We used the same implementation as Hadjifotinou & Harper (1995) and integrated for 10000 intervals of 1.2 days each. For tolerance 10^{-12}, the relative error in the energy integral at the end of the time-span was 1×10^{-12} and the CPU-time needed for the integration on the VAX-9000 machine was about 50 minutes, although the step-sizes used by the routine were between $\frac{1}{67}$ and $\frac{1}{15}$ of T_{Mimas}. The larger CPU-time required by the RKN method has its reason on the 17 function evaluations per step performed by the routine instead of only two needed by the Gauss-Jackson. Therefore, the more complex the system of equations is, the larger will be the difference of the two methods in integration time.

5. Comments and conclusions

In this work we examine the behaviour of the partial derivatives that are used in the fitting process of the numerical integration results to observations or analytical theories with the purpose of improving the parameters of satellite orbital systems. If the initial conditions used for the numerical integration correspond to a periodic orbit or at least a quasiperiodic one near a periodic orbit of the satellite system, then Floquet theory can fully predict the behaviour of the partial derivatives of the satellites' coordinates with regard to all the parameters of the system.

Here we apply this theory i) to any bounded orbit of the two-body problem and ii) to an orbit of the three-body system Saturn-Mimas-Tethys when this is simplified by considering all bodies as point-masses. This orbit, in the rotating frame, was found to be quasiperiodic near a stable symmetric periodic one.

For both problems, we examine the behaviour of the partial derivatives of the satellites' coordinates with regard to the satellites' initial conditions and their mass-ratios over Saturn. We show that in both cases the amplitudes of all the partial derivatives increase linearly with time. In more detail:

i) In the two-body case, the motion of all the partials is a combination of a linear and a T-periodic motion, where T the satellite's orbital period.
ii) In the three-body case, for the given initial conditions, the motion of all the partials is a combination of a linear and four periodic motions.

It is worth mentioning that the linear growth in the amplitude of the partial derivatives with regard to the satellites' initial conditions can be proved, following the method of Sect. 3, not only for the specific Saturn-Mimas-Tethys orbit, but for any three-body orbit that in the rotating frame is quasiperiodic and close to a stable periodic orbit. For the proof concerning the partials with regard to the satellites' mass-ratios, the nearby stable periodic orbit needs additionally to be symmetric. As a result, this property of the partial derivatives can be used to test the accuracy of various numerical methods used to integrate the variational equations together with the equations of motion of any satellite orbit that satisfies the above conditions.

Here, extending the numerical investigation of Hadjifotinou & Harper (1995) in the Saturn-Mimas-Tethys problem, we verified the instability that the 10^{th}-order backward difference Gauss-Jackson method with the predictor only cycle for the integration of the partials presents when used with step-sizes larger than the critical. However, comparing the accuracy and speed of this method when used with the largest acceptable step-size ($\frac{1}{76}T$), to those of the Gauss-Jackson corrector method and the RKN12(10)17M, one can see that, despite the small step-size, the Gauss-Jackson predictor is very efficient. Therefore, it should not be rejected due to its instability, but only used with step-sizes smaller than the critical step-size.

Finally, focusing on the study of the Saturnian satellite system, our knowledge of the behaviour of the partial derivatives would be more complete if Saturn's oblateness perturbations were included in the equations of motion and the variational equations of our two-satellite system. Then, the motion of all the partial derivatives (including the partials with regard to the oblateness coefficients of Saturn) near a stable symmetric periodic orbit is still predicted by Floquet theory to be a combination of a linear and periodic motions. However, the difficulty in this case lies in actually defining a suitable rotating reference frame and proving the existence of families of periodic orbits there. As is pointed out by Duriez (p.c.), the families of periodic orbits near the real motion of Mimas-Tethys could be researched by using at first the resonant behaviour of the system, in spite of the long period of the resulting libration.

Acknowledgements. We are very grateful to Dr. D. Harper for his provision of initial conditions for the Saturn-Mimas-Tethys system and for his valuable comments. We would also like to thank Prof. J. Hadjidemetriou, Prof. M. Gousidou-Koutita, Dr. E. Meletlidou and the referee Dr. L. Duriez for their helpful suggestions. K.G.H. is financially supported by the Hellenic Scholarship Foundation (I.K.Y.).

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