Astron. Astrophys. 322, L5–L8 (1997)

ASTRONOMY
AND
ASTROPHYSICS

LETTER

Letter to the Editor

Equation of state and helioseismic inversions

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Received 20 February 1997 / Accepted 8 April 1997

Abstract. Inversions to determine the squared isothermal sound speed and density within the Sun often use the helium abundance Y as the second parameter. This requires the explicit use of the equation of state (EOS), thus potentially leading to systematic errors in the results if the equations of state of the reference model and the Sun are not the same. We demonstrate how this potential error can be suppressed. We also show that it is possible to invert for the intrinsic difference in the adiabatic exponent Γ_1 between two equations of state. When applied to solar data such inversion rules out the EFF equation of state completely, while with existing data it is difficult to distinguish between other equations of state.

Key words: Sun: oscillations — Sun: interior — Equation of state

1. Introduction

Solar oscillation frequencies can be inverted to determine the internal structure of the Sun. This is generally done by relating the frequency differences between a solar model and the Sun to differences in the structure by linearising the oscillation equations under the assumption of hydrostatic equilibrium. If, for example, we express the frequencies in terms of the squared adiabatic sound speed c^2 and the density ρ, the result can be written

$$\frac{\delta\omega_i}{\omega_i} = \int K^i_{c^2,\rho}\frac{\delta c^2}{c^2}\mathrm{d}r + \int K^i_{\rho,c^2}\frac{\delta\rho}{\rho}\mathrm{d}r + \frac{F_\mathrm{s}(\omega_i)}{E_i}\ . \quad (1)$$

(e.g. Dziembowski et al. 1990). Here $\delta\omega_i$ is the difference in the frequency ω_i of the ith mode between the solar data and a reference model. The kernels $K^i_{c^2,\rho}$ and K^i_{ρ,c^2} are known functions of the reference model which relate the changes in frequency to the changes in c^2 and ρ respectively. The term in F_s results from the near-surface errors in the model, such as the assumption of adiabatic oscillations; E_i is the inertia of the mode, normalised by the photospheric amplitude of the displacement.

The kernels for the (c^2, ρ) combination can be easily converted to kernels for others pairs of variables like (Γ_1, ρ), (u, Γ_1) with no extra assumptions (cf. Gough 1993); here $u \equiv p/\rho$ is the squared isothermal sound speed, p being pressure. However, very often in addition to the oscillation equations it is assumed that the equation of state is known; this may be used to construct the kernels for u or ρ and Y. The implicit assumption that the equations of state in the Sun and reference model are the same leads to potential systematic errors in the inversion for u or ρ.

2. Formulation of the inverse problem

The conversion of the kernels for (c^2, ρ) to those for (u, Y) uses $\delta \ln c^2 = \delta \ln \Gamma_1 + \delta \ln u$. Had the equation of state been known, $\delta\Gamma_1$ could have been determined from $\Gamma_1 = \Gamma_1(p, \rho, \{X_i\})$, where $\{X_i\}$ is the composition. In fact, $\delta\Gamma_1$ contains an additional term, *viz.* the intrinsic difference $(\delta\Gamma_1/\Gamma_1)_\mathrm{int}$ at fixed $(p, \rho, \{X_i\})$ between the true and the model equations of state. Characterizing the composition by the abundances by mass Y and Z of helium and heavy elements and assuming an unchanged heavy-element abundance Z, we therefore get

$$\frac{\delta\Gamma_1}{\Gamma_1} = \left(\frac{\partial\ln\Gamma_1}{\partial Y}\right)_{p,\rho}\delta Y + \left(\frac{\partial\ln\Gamma_1}{\partial\ln p}\right)_{\rho,Y}\frac{\delta p}{p} + \left(\frac{\partial\ln\Gamma_1}{\partial\ln\rho}\right)_{p,Y}\frac{\delta\rho}{\rho} + \left(\frac{\delta\Gamma_1}{\Gamma_1}\right)_\mathrm{int}. \quad (2)$$

Note that differences in Z, or in the relative composition of the heavy elements, will also appear in $(\delta\Gamma_1/\Gamma_1)_\mathrm{int}$.

Equation 2 can now be used to rewrite Eq. 1 in terms of $\delta u/u$ and δY, by expressing the terms in $\delta\rho/\rho$ and $\delta p/p$ in terms of $\delta u/u$ by means of the equation of hydrostatic support. This expression contains a contribution from the intrinsic difference in Γ_1 weighted by the kernel for c^2 at constant ρ. Thus, the full

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equation is:

$$\frac{\delta\omega_i}{\omega_i} = \int K^i_{u,Y}\frac{\delta u}{u}\mathrm{d}r + \int K^i_{u,Y}\delta Y\mathrm{d}r + \int K^i_{c^2,\rho}\left(\frac{\delta\Gamma_1}{\Gamma_1}\right)_{\mathrm{int}}\mathrm{d}r + \frac{F_{\mathrm{s}}(\omega_i)}{E_i} . \quad (3)$$

In the expression normally used to invert for (u, Y) the term in $\left(\delta\Gamma_1/\Gamma_1\right)_{\mathrm{int}}$ is ignored; this clearly introduces a systematic error, if the equations of state are in fact different. A similar argument also holds for density inversions using the pair (ρ, Y). Here too the contribution from $\left(\delta\Gamma_1/\Gamma_1\right)_{\mathrm{int}}$ should be taken into account. We also note that using Eq. 3, one may directly invert for $\left(\delta\Gamma_1/\Gamma_1\right)_{\mathrm{int}}$.

3. Inversion technique

We have used the Subtractive Optimally Localised Averages method of Pijpers & Thompson (1992), adapted to inversion for structure differences (e.g. Basu et al. 1996). The principle of the inversion technique is to form linear combinations of Eqs 3 with weights $c_i(r_0)$ chosen such as to obtain an average of, for example, $\delta u/u$ localised near $r = r_0$ while suppressing the contributions from the remaining terms in Eqs 3, including the near-surface errors. In addition, the statistical errors in the combination must be constrained.

To invert for $\delta u/u$ the coefficients c_i are chosen to minimise

$$\int\left(\sum_i c_i K^i_{u,Y} - \mathscr{T}\right)^2\mathrm{d}r + \beta_1\int\left(\sum_i c_i w(r)K^i_{Y,u}\right)^2\mathrm{d}r + \beta_2\int\left(\sum_i c_i w(r)K^i_{c^2,\rho}\right)^2\mathrm{d}r + \mu\sum_{i,j}c_i c_j E_{ij} , \quad (4)$$

with the constraint that the averaging kernel be unimodular, i.e.,

$$\sum_i c_i(r_0)\int_0^R K^i_{u,Y}(r)\mathrm{d}r = 1 . \quad (5)$$

Here, $\mathscr{T}(r_0, r)$ is a target averaging kernel, chosen to be a Gaussian of unit area centred at r_0. E_{ij} is the covariance matrix of errors in the data. The parameters β_1 and β_2 control the contributions of δY and $(\delta\Gamma_1/\Gamma_1)_{\mathrm{int}}$, respectively, and μ is a trade-off parameter which controls the effect of data noise. The function $w(r)$ is a suitably chosen, increasing function of radius, which ensures that the contributions from the second and third terms from the surface layers are suppressed properly.

To reduce the influence of near-surface uncertainties we apply the additional constraints that

$$\sum_i c_i(r_0)E_i^{-1}\Phi_\lambda(\omega_i) = 0 , \quad \lambda = 0, \ldots, \Lambda , \quad (6)$$

where the Φ_λ are B-Splines with a suitably scaled argument (cf. Däppen et al. 1991).

To carry out an inversion for $(\delta\Gamma_1/\Gamma_1)_{\mathrm{int}}$ one minimises instead

$$\int\left(\sum_i c_i K^i_{c^2,\rho} - \mathscr{T}\right)^2\mathrm{d}r + \beta_1\int\left(\sum_i c_i w(r)K^i_{u,Y}\right)^2\mathrm{d}r + \beta_2\int\left(\sum_i c_i w(r)K^i_{Y,u}\right)^2\mathrm{d}r + \mu\sum_{i,j}c_i c_j E_{ij} . \quad (7)$$

We have used four solar models for this work. All models have been constructed with OPAL opacities (Iglesias, Rogers & Wilson 1992) at temperatures higher than 10^4 K and Kurucz tables (Kurucz 1991) at lower temperatures. The models have been constructed with different equations of state — Livermore (OPAL) (Rogers, Swenson & Iglesias, 1996) MHD (e.g. Mihalas, Däppen & Hummer 1988), EFF (Eggleton, Faulkner & Flannery 1993) and CEFF (cf. Christensen-Dalsgaard & Däppen 1992). The properties of the models, identified by the EOS, are summarised in Table 1. Model OPAL is Model S of Christensen-Dalsgaard et al. (1996).

Table 1. Solar models used; d_{CZ} is the depth of the convection zone, and T_{c} and ρ_{c} are central temperature and density

EOS	Z/X	d_{CZ}/R	T_{c} (10^6 K)	ρ_{c} (g/cm^3)
OPAL	0.0245	0.2885	15.67	154.2
MHD	0.0245	0.2876	15.67	154.5
CEFF	0.0248	0.2863	15.68	155.0
EFF	0.0248	0.2852	15.74	157.2

We use solar oscillation data obtained by the LOWL instrument during the first year of data collection (Tomczyk et al. 1995, Schou and Tomczyk, in preparation). The dataset consists of modes of degrees 0 to 99 in the frequency range 1 to 3.5 mHz. The observed modeset and errors were also used in tests for solar models, in order to get realistic properties of the inversion.

4. Results

In Fig. 1 we show inversion results, without and with suppression of $\left(\delta\Gamma_1/\Gamma_1\right)_{\mathrm{int}}$, for u in Model MHD with Model OPAL as the reference model. The resolution of the inversion is the same in both cases. Note that when the contribution from $\left(\delta\Gamma_1/\Gamma_1\right)_{\mathrm{int}}$ is not constrained, the results are not very accurate. The results improve dramatically, particularly in the core, when the intrinsic difference is taken into account. However, the price paid for increased accuracy is decreased precision, as reflected in the increased error-bars. Indeed, it is evident that the use of the data to suppress the possible error in the EOS reduces the amount of information available for the determination of $\delta u/u$ and hence results in larger errors if the resolution is kept approximately the same. The results for density inversion are shown in Fig. 2. We note that the errors in the inversion corrected for a possible

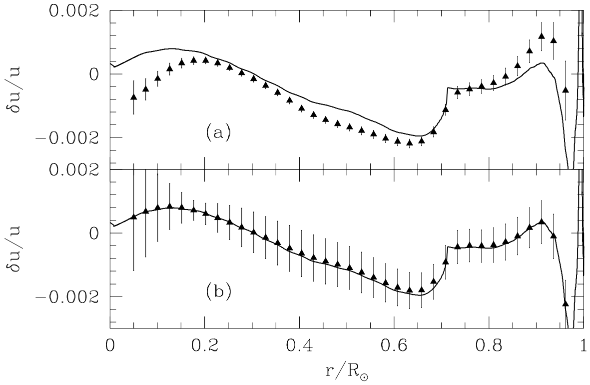

Fig. 1. The inversion for the squared isothermal sound speed (u) difference between model MHD and model OPAL. The solid line is the exact difference and the points are the difference obtained by inverting the frequency differences between the models. **a** Inversion results when the intrinsic difference in Γ_1 between the OPAL and MHD equations of state is ignored. **b** Inversion results when the intrinsic difference is taken into account. The vertical error-bars are 1σ propagated errors

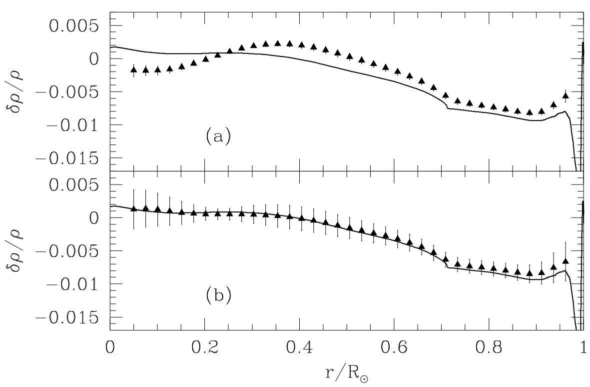

Fig. 2. The same as Fig. 1, but for the density difference between models MHD and OPAL

inconsistency in the equation of state are much reduced if accurate data on higher-degree modes are available, as is the case, e.g., for the frequencies obtained by the SOI/MDI experiment on SOHO (cf. Kosovichev et al. 1997).

Although it is useful to be able to suppress the effects of errors in the equation of state when inverting for u, it is evidently of greater interest to obtain a localized measure of these errors, i.e., to invert for the intrinsic $(\delta\Gamma_1/\Gamma_1)_{\rm int}$ between the equations of state of the Sun and the model. To illustrate our ability to achieve such localization, Fig. 3 shows averaging kernels for the inversions for $(\delta\Gamma_1/\Gamma_1)_{\rm int}$ through minimization of expression 7. Note that for $r_0 \gtrsim 0.5 R_\odot$ the averaging kernels are quite well localized, indicating that reliable inversion is in fact possible. The averaging kernels are not as small near the surface as one would hope for; inclusion of higher-degree modes would substantially improve the behaviour in this region.

Fig. 4 shows the inversion for the intrinsic differences in Γ_1, using the Models MHD, CEFF and EFF as test models and Model OPAL as reference. For comparison are shown exact

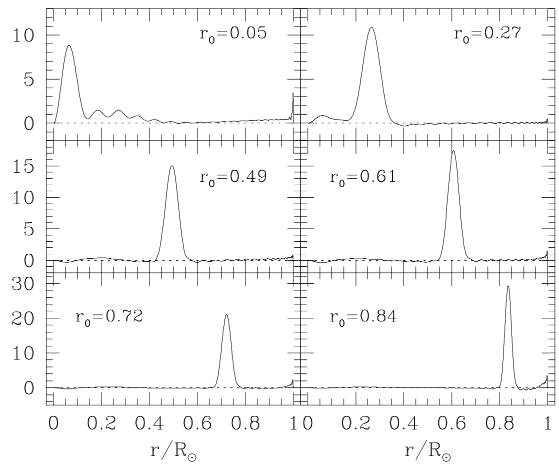

Fig. 3. Averaging kernels for the inversion for the intrinsic $(\delta\Gamma_1/\Gamma_1)_{\rm int}$ with the LOWL mode set and the OPAL model as the reference model

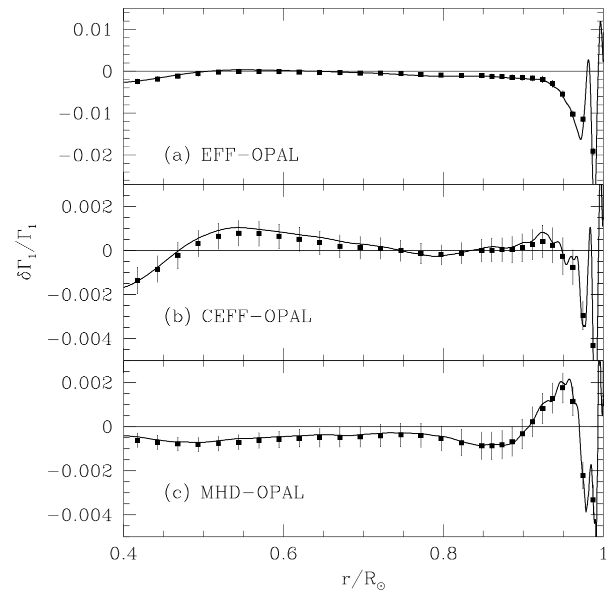

Fig. 4. The results of inversion for the intrinsic Γ_1 difference between **a** EFF, **b** CEFF and **c** MHD models and Model OPAL. The solid line is the exact difference and the points are the difference obtained by inverting the frequency differences between the models. Note that the scale in Panel **a** is much larger than that in Panels **b** and **c**

differences resulting from differences in the equation of state, evaluated at fixed p, ρ, and Y in Model OPAL. The inversion of the frequencies clearly successfully reproduces even the subtle intrinsic EOS differences between the MHD and OPAL formulations, although the statistical errors are fairly substantial compared with these differences, at least beneath the dominant ionization zones.

Given the success of this test on artificial data, we may consider differences between the solar and the model equations of state, as obtained from analysis of the observed frequencies. Fig. 5 shows the resulting intrinsic differences in Γ_1 between

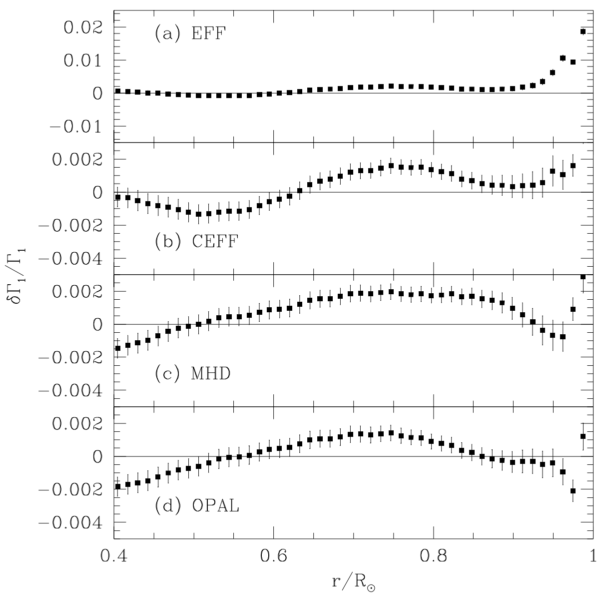

Fig. 5. The intrinsic Γ_1 difference between the Sun and the EFF, CEFF, MHD and OPAL models obtained by inversion of LOWL Year-1 data. Note the difference in scale between panel **a** and the other panels

the Sun and the four models of Table 1. It is evident that the EFF equation of state is inconsistent with the data. With the current level of errors, it is difficult to distinguish between the other three equations of state. Our ability to do so would be greatly improved by analysis of higher-degree data, since we may expect that the dominant differences in the equations of state are close to the surface of the Sun.

5. Conclusions

We have shown that inversions for the squared isothermal sound speed u and the density ρ may suffer from systematic errors when based on the common implicit assumption that the equations of state in the Sun and the reference model are the same. These errors can be removed by suppressing the contribution from the intrinsic difference in Γ_1 to the frequency difference. However, this is achieved at the price of an increase in the propagated errors.

We also show that we can successfully invert for the intrinsic difference in Γ_1 between the currently available equations of state. This differs from the analysis by Elliott (1996) who investigated the EOS in terms of the total difference between the solar and the model Γ_1. Inversions of solar oscillation frequencies show that the EFF equation of state can be ruled out by direct inversions. With the current level of data errors, it is difficult to judge the significance of the differences between the solar equation of state and the CEFF, MHD and OPAL equations of state. We hope, however, that as more precise data and data on high-degree modes become available, this method can be used as a direct test of the solar equation of state.

Acknowledgements. We thank M. J. Thompson for useful comments. This work was supported by the Danish National Research Foundation through its establishment of the Theoretical Astrophysics Center

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