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ASTRONOMY AND ASTROPHYSICS

The opposition effect of 51 Nemausa*

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Abstract. From three nights of uvby-observations of 51 Nemausa in September 1994, covering solar phase angle 0°.64 to 1°.37, the phase factor as a function of rotational phase was determined to vary in the range 0.05 to 0.13 mag/deg. The large values are due to the opposition effect. An ephemeris for physical observations facilitated the reductions and allowed a certain identification of minor details with the 1983 and 1990 lightcurves. The difficulties connected with the development of lightcurves in Fourier series are discussed. The advantage of a statistical analysis in terms of uncorrelated normal points is demonstrated.

Key words: asteroids – methods: data analysis – methods: individual: 51 Nemausa

1. Introduction

The asteroid 51 Nemausa attained the small phase angle 0°.64 on 1994 September 24.90 and was observed at La Silla on September 25, 26 and 27 in the phase angle interval 0°.64 to 1°.37. In this small interval magnitudes are linear functions of the solar phase angle β. The phase factors are the coefficients on β and are denoted c with an index u, v, b or y for color. Phase factors are in general considered functions of β but not of the rotational phase P. Our aim here is to prove that phase factors depend on the rotational phase. This confirms the phase factor variation observed in 1990 and 1991 (Kristensen & Gammelgaard (1993)). Due to the opposition effect large values were obtained for the phase factors in 1994, for instance c_b = 0.136±0.004 mag/deg at rotational phase P = 0.87 rev. The first observed opposition of this object was in 1858 and the 1994 opposition is numbered 100. It is, however, the very first opposition where an ephemeris for physical observations is available. This is a very great advantage for the reduction and interpretation of photometric observations.

2. Observations

The observations reported here were done in the period 1994 September 25.0 to 27.4 U.T. with the 6-channel spectrophotometer (Florentin Nielsen 1983) attached to the Danish 1.54 m telescope at La Silla. The photomultipliers in four of the channels independently and simultaneously measure the magnitudes in the Strömgren uvby system.

The observation procedure was similar to that described in Kristensen & Gammelgaard (1993). The four comparison stars were observed approx. every 16 minutes during the night, and the extinction coefficients could therefore be accurately determined by Bouguer's method. Before and after the uninterrupted lightcurve observations photometric uvby-standard stars were observed and by correction with the above extinctions the uvby mean differences between comparison stars and standard stars could be determined. Table 1 gives uvby values of the photometric standard stars and Table 2 the mean uvby values of the comparison stars which were determined by adding the above mean differences to the values of Table 1 for HD 16031 and HD 22054.

When HD213985 was observed it turned out that the standard errors of the mean uvby differences to the comparison stars were about twice the standard errors of the corresponding differences between the other two photometric standards (HD16031, HD 22054) and the comparison stars. A search in the Strasbourg data base showed HD213985 as a star varying in V between 9.2 and 10.0.

The differential observations of the planet are given relative to the average magnitude of the comparison stars with equal weights (Kristensen & Gammelgaard (1993)) as given above. The magnitudes of this mean star is given in Table 2. The airmass is eliminated by star observations before and after the observations of the planet. This method gives high accuracy and no discontinuities as one never shifts to other stars. To eliminate second order, color dependent extinction the comparison stars were chosen to match the color of the planet. We also relied on the small bandwith of the uvby-system. However, in the future we should use weights of stars α_i such that

$$(B-V)_{planet} = \sum_{i=1}^{N} \alpha_i (B-V)_i \quad (1)$$

and minimize $\sum_{i=1}^{N} \alpha_i^2$ with the additional condition $\sum_{i=1}^{N} \alpha_i = 1$.

* Based on observations made with the Danish 1.54m telescope at the European Southern Observatory, La Silla, Chile

Table 1. Extinction Standard Stars. The Strömgren uvby values have been derived from the V, b-y, m1, c1 values in Gray & Olsen (1991) and Schuster & Nissen (1988) by the transformation:
y = V - 0.026 (b-y)
b = y + b-y
v = m1 + 2b - y
u = c1 + 2v - b

HD	u	v	b	y
16031	11.169	10.479	10.093	9.769
22054	9.862	8.876	8.488	8.233
213985*	10.917	9.207	8.982	8.850

*Variable star (cf. text)

Table 2. Comparison Stars. Mean Strömgren uvby values observed

SAO	u	v	b	y
109073	12.558 ±13	11.201 ±10	10.183 ±9	9.567 ±8
128552	11.744 ±12	10.141 ±10	8.818 ±8	8.060 ±8
128576	11.426 ±12	10.029 ±9	8.929 ±8	8.276 ±7
128659	11.445 ±10	10.122 ±8	9.120 ±8	8.496 ±7
mean star	11.793 ±6	10.373 ±5	9.263 ±4	8.600 ±4

Table 3. The table is a small extract of the physical ephemeris and gives the solar phase angle β, the equation of time E, the aspect θ and Julian Date. The ephemeris separates the long period orbital terms from the short period rotation and give the rotational phase P or central meridian as:
P = 3.08366928 · $(T^* - 2444591.4675)$ + E/360° revolutions.

1994 E.T.	β	E	θ	JD
Sept. 15	4°868	6°4	109°0	9610.5
Sept. 25	0.635	3.3	108.6	9620.5
Oct. 5	4.962	0.2	108.2	9630.5

3. The ephemeris for physical observations

From a certain level of precision it is necessary to know which side of the body is being observed so an ephemeris for physical observations is indispensable.

A convenient form of ephemerides for physical observations, which separates long and short period terms, was proposed by Kristensen (1991). An excerpt of this ephemeris based on the rotational elements by Kristensen (1993) is given in Table 3.

The (solar) phase angle is denoted β, the aspect θ and E is the "equation of time". The latter is defined analogous to the familiar concept. It is the planetocentric right ascension of the bisector between Sun and Earth **minus** the right ascension of a fictitious mean Sun moving uniformly in the equatorial plane of the planet. The main inequalities in E are 2e $\sim$ 7°6 due to the orbital eccentricity, $\tan^2(\epsilon/2) \sim$ 3°2 due to the obliquity (26°6) and - in addition to these familiar terms - we have the angle $\beta/2$ from the Sun to the bisector. Extreme values of E due to phase angle occur at quadratures and are +12°5 before and -12°5 after opposition, - the signs being determined by the rotation being **retrograde**. The rotational phase P is given by

$$P = 3.08366928(T^* - 2444591.4675) + E/360°(\text{rev}) \quad (2)$$

where T* is E.T. corrected for light-time. The expression (2) **defines** the central meridian. The great advantage of using the **synodic**, rather than the sidereal rotation period, is that we can obtain the time of lightcurve features within a quarter of an hour by simply ignoring E in (2).

All quantities in the physical ephemeris depend on the slow orbital motions and may be tabulated and interpolated at large intervals. Due to the here nearly stationary value of β interpolation should be in β^2 and to second order in time.

To illustrate the use of the physical ephemeris let us compute the rotational phase t days from the epoch September 26.336 E.T.= 244 9621.836. Interpolation gives, in units of revolutions,

$$E = 0.0080 - 0.00086 \cdot t\,\text{rev}. \quad (3)$$

which inserted into (2) gives

$$P(t) = 0.0008 + t \cdot 24/7.78511\,\text{rev}. \quad (4)$$

The rotational phase is practically zero at the adopted epoch. As Δ^2 E = 0 the apparent rotation is very uniform and the observations can be reduced by a constant period 7.78511 hours. It is a very great advantage that we do not need to solve for the rotational period.

The aspect at the epoch is θ = 108°5. Table 1 in Kristensen (1993) gives the aspects of earlier oppositions. Aspects are θ = 112°7 in 1983 and θ = 116°9 in 1990 so these oppositions should be directly comparable with the present one. Opposite lightcurves have aspect $180° - \theta$ = 71° and occurred in 1989 (θ = 69°4) and 1991 (θ = 74°9).

The 1983 lightcurve (Fig.4 in AN 306(1985)) has sharp minima at 0.24, 0.59 and 0.82 but the origo adopted in this figure has phase +3°11. The same minima should then occur in 1994 at phase 0.249, 0.599 and 0.829, in good accordance with 0.25, 0.59 and 0.83, in the present Fig. 1. The good consistency may be due to the similar values of β, respectively 1°85 and 0°94. Important is also that the small (0.01 mag.) peak (the fourth maximum) around 0.20 is confirmed. Fig.2 p.347 in Kristensen & Gammelgaard (1993) gives minima at phase 0.26 and 0.87 in 1990 at β = 13°01. With the correction -7°47 in phase this corresponds to 0.239 and 0.849, or -0.011 and +0.019 rev. relative to the small phase angles above. This may be compared with the bisector angle $\beta/2$ = 0.018 rev. and may be regarded as a confirmation of the usefulness of the bisector. Phases over 11 years are thus consistent with mean errors of order ±0.01 revolutions.

4. Fourier coefficients

In Russell's expansion of the lightcurves in spherical harmonics the term $Y_{2n}(\theta, \phi)$ has a factor

$$(-)^{n+1}\frac{(2n-3)!!}{2^n(n+1)!} \sim \frac{1}{\sqrt{4\pi}} n^{-\frac{5}{2}} \quad (5)$$

with (2n-3)!! = (2n-3) ×(2n-5)... 5 · 3 · 1. This term is 0.001 for n = 10 but it would be erroneous to conclude that a 20th-degree fit with 40 harmonic coefficients would reproduce lightcurves to ±0.001 mag. It is often noted, for particular objects, that lightcurves have sharp angles at minima; this was for instance the case for 51 Nemausa in 1983. The discontinuity in the first derivative is related to Gibbs' phenomena (Carslaw p.289-310) and gives a slow convergence. This may be illustrated by a rotating, flat plate with the lightcurve

$$|\sin x| \simeq \frac{2}{\pi} - \frac{1}{\pi}\sum_{n=1}^{N} \frac{\cos 2nx}{n^2 - \frac{1}{4}} \; ; \; x = \omega t \ldots \quad (6)$$

The maximum errors occur at the minima $x = \ldots -\pi, 0, +\pi \ldots, 0$, and may be expressed in terms of the asymptotic expression for the digamma-function (Abramowitz 6.3.18)

$$\frac{1}{\pi}\left(\psi(N+\frac{3}{2}) - \psi(N+\frac{1}{2})\right) \sim \frac{1}{\pi N} - \frac{1}{2\pi N^2} + \ldots \quad (7)$$

which illustrates the very slow convergence at the singular points.

Most published lightcurves are affected by discontinuities at the beginning and end of the nights caused by a change of comparison stars and errors in the correction for phase angle and extinction. Let a small error ϵ be introduced into the composite lightcurve from phase P_1 to P_2. The Fourier expansion will then contain terms like

$$\epsilon\left(\theta(x-x_1) - \theta(x-x_2)\right) =$$

$$\sum \frac{8\epsilon}{\pi} \frac{\sin(2n+1)(\frac{x_2-x_1}{2}) \times \cos(2n+1)(x - \frac{x_1+x_2}{2})}{(2n+1)}$$

$$; x = 2\pi P \ldots \quad (8)$$

where $\theta(x)$ is the unit step function

$$\theta(x) = \begin{cases} +1 & \text{for } 0 < x < \pi \\ \;\;0 & \text{for } -\pi < x < 0 \end{cases} \quad (9)$$

The decrease of observed Fourier coefficients to the limit given by observational errors and the convergence of the series are thus in practice less than expected by equation (5). It is difficult in practice to decide how many harmonics should be used to represent lightcurves. In retrospect, it was an error to use only 10th degree for the 1990 lightcurve. Figure 5 in Kristensen & Gammelgaard (1993) shows that this number of harmonics does not reproduce the 0.009 mag. hump (a fourth maximum) at P = 0.23 which is so clearly shown in 1983.

Another drawback of a Fourier expansion is connected to the statistical analysis of errors. If the weights of the observations are not uniformly distributed in phase the gaps will induce large correlations between the parameters. This can only be handled if we know a priori that the formula used to represent the observations is practically exact. The statistical difficulties are connected with the simultaneous treatment of all observations.

Table 4. Fourier coefficients for y in units of $10^{-}4$ mag. The formal mean error is of order $\pm 0\overset{m}{.}0004$ to $\pm 0\overset{m}{.}0006$. The epoch of zero phase is 1994 Sept. 26.336 ET corrected for light-time. The position of the bisector is $0^h05\overset{m}{.}9 - 0°50'$ (1950).

n	c_n	s_n	n	c_n	s_n
1	–80	+115	11	+4	–9
2	–235	+98	12	+22	–19
3	–92	–175	13	+11	+3
4	–113	+139	14	–11	+15
5	+50	+6	15	–6	–16
6	–1	–41	16	–5	+17
7	+32	–43	17	–0	+3
8	+12	+0	18	–7	–2
9	–38	+18	19	–6	+5
10	–1	–14	20	–11	–17

In the next section we shall avoid this by analyzing small groups of observations independently.

The Fourier coefficients are, however, useful for the pole determination (Kristensen (1993)) and Table 4 gives these coefficients to degree n=20 in the lightcurve and degree 2 in the phase factor curve. The 41+5=46 unknowns are solved for rigorously by the method of least squares. Weights were given to the individual observations on the basis of their internal scatter.

At airmass X>1.5 the comparison stars indicated so irregular fluctuations in u and v that the observations had to be rejected. Due to the approximate 3.0 rev/day rotation frequency this rejection occured around the same rotational phase and resulted in gaps in coverage. This again gave large correlations between the Fourier coefficients in u and v. The numbers of observations retained were 310, 324, 327 and 327 respectively in uvby and among the $\binom{46}{2}$ = 1035 correlation coefficients the number exceeding 0.50 were respectively 219 (the largest being +0.88 and -0.75), 21, 8 and 8.

In y and b the 8 correlations occured between the harmonics n = 13 to 15. This is connected with the accidental grouping of the observations around the 28 nearly equidistant phases as apparent in Fig. 1.

In b the coefficient to sin(20P) is −0.0025±0.0006 which is – formally – significant. A fast decrease of the higher harmonics is a delusion. The reason to include harmonics to order 20 was to represent the sharp minimun at P = 0.25; to degree n=10 a much smoother lightcurve was obtained. It is important, however, that the Fourier coefficients of low order to be used for pole determination were identical within ±0.0010 mag.

5. Normal points

The observations of Nemausa were given in series of 5 planet observations interrupted by an approximately constant time interval for the star observations. By chance these groups happened to have nearly the same phase on the 3 nights and this is the reason for the grouping of the observations in phase as is visible on Fig. 1. This is illustrated by group No. 25 in Table 6 which has 5 observations from each night from phase 0.859 to

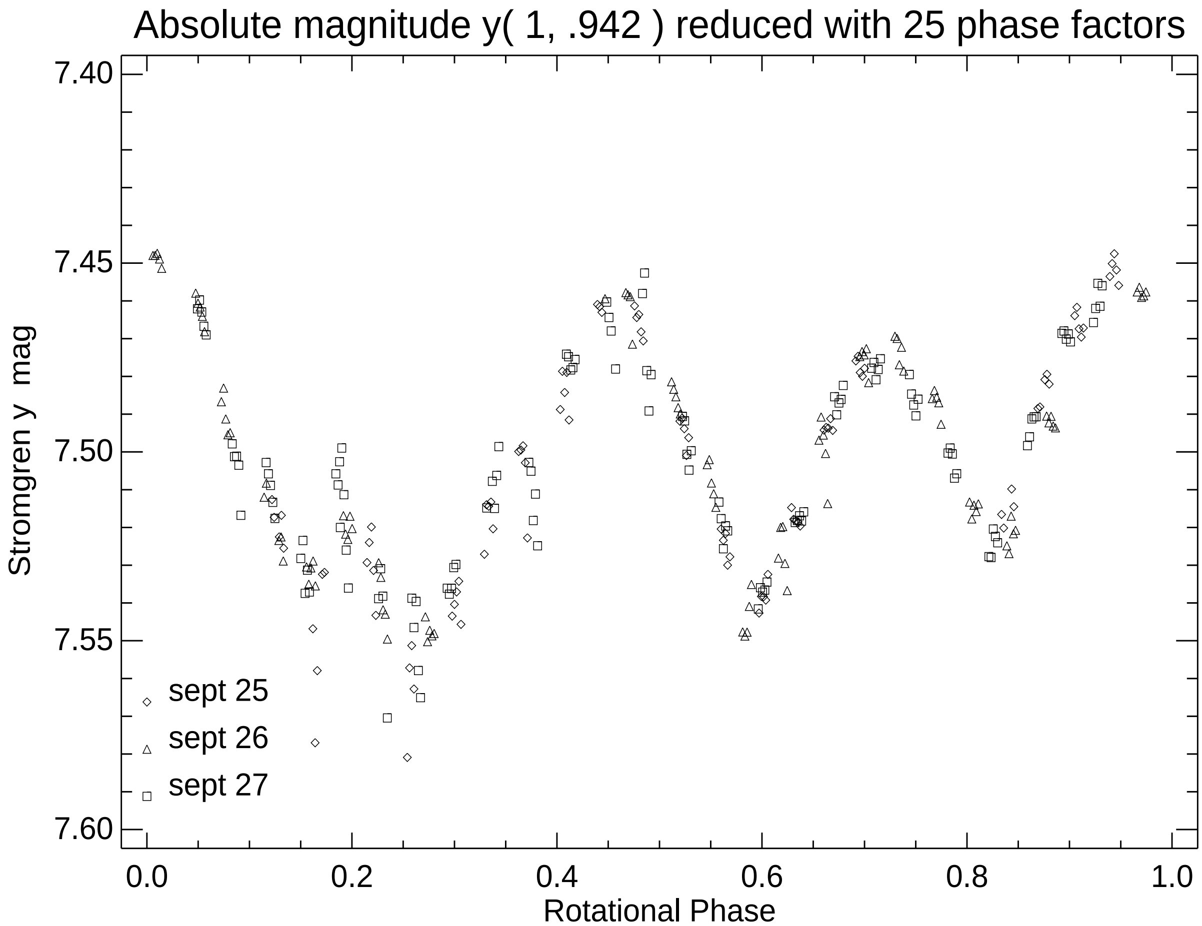

Fig. 1. Plot of all y observations reduced to absolute y magnitude with 25 phase factors to $\beta = 0.^\circ942$ corresponding to September 26.3.

0.886. Before and after this group are gaps in phase from 0.847 to 0.859 and from 0.886 to 0.893.

In the small phase intervals of each group we can assume that the lightcurve can be approximated by a straight line and that the phase factors (c_y or c_b) are constants. The observational equations in the y magnitude are then

$$y_0 + y_0' \cdot P + c_y \cdot (\beta - \beta_0) = y \tag{10}$$

which should be solved for the 3 parameters y_0, y_0' and c_y by least squares. It is only possible to use the fundamental equation (10) because the rotational phase P is regarded known as a function of time and the rotation period need not be solved for. Due to the phase factor variation the concept of a lightcurve makes only sense when referring to a definite value of β_0. The groups Nos. 1, 23 and 28 have only 5 observations from the third night and can not be reduced to other phase angles. The mean $\beta_0 = 0.942$ of the respective phases 0.952, 0.928 and 0.947 is adopted as standard phase.

The observations of each individual group are best represented as a normal point $y_0 + y_0' P_0$ at phase P_0. The epoch P_0 is determined such that the mean error of the magnitude is a minimum. If Q_{ij} is the covariance (reciprocal) matrix corresponding to the normal equations the most accurate magnitude is obtained for

$$P_0 = -Q_{12}/Q_{22} = \frac{(P)(\beta\beta) - (\beta P)(\beta)}{(1)(\beta\beta) - (\beta)(\beta)} \tag{11}$$

Due to the dependence on the phase angle, P_0 differs a little from the mean value of the epoch. These deviations were in the present cases very small. Introducing (11) into the normal equations and solving, we obtain a simple expression for the best magnitude

$$\frac{(o)(\beta\beta) - (\beta o)(\beta)}{(1)(\beta\beta) - (\beta)(\beta)} \tag{12}$$

where the observations of uvby are denoted o, β is substituted for $\beta - \beta_0$ and the brackets ( ) is Gauss' notation for weighed sums. We note that the expression (12) is independent of the distribution of the phases (does not contain sums like (Po), (PP) etc.) The mean error σ of the best magnitude is given by

$$\sigma^2 = \sigma_0^2 \frac{(\beta\beta)}{(1)(\beta\beta) - (\beta)^2} \tag{13}$$

where σ_0 is the observational mean error of unit of weight. The highest accuracy is obtained if $(\beta) = 0$, that is at the mean value of the phase angle.

Table 5. Contains normal points from 28 groups in rotational phase P. A line gives group number, P, and in order phase factor with mean error, magnitude with mean error. All multiplied by 10^4. The lightcurve refers to 0.942 deg. The epoch of zero phase is 1994 Sept. 26.336 ET corrected for light-time. The position of the bisector is $0^{h}05^{m}.9 - 0°50'$ (1950).

No.	P	c_y		y		c_b		b	
1	101			–562	± 14			–572	± 16
2	519	848	± 55	–426	16	757	± 71	–497	20
3	789	678	107	–143	15	673	127	–145	15
4	1246	852	40	93	11	811	40	86	10
5	1595	909	74	281	14	1177	93	339	14
6	1949	1131	89	148	14	1274	137	181	15
7	2294	850	121	344	18	970	160	429	24
8	2721	619	80	418	13	772	104	430	14
9	2997	647	52	276	13	822	77	281	17
10	3359	795	63	59	16	899	75	65	17
11	3717	435	121	–45	19	600	115	–48	18
12	4103	614	60	–298	15	630	66	–284	14
13	4468	610	109	–457	14	795	116	–406	14
14	4776	1000	61	–387	15	1117	72	–368	16
15	5216	950	41	–134	9	959	46	–140	10
16	5590	846	40	106	10	844	41	121	10
17	5973	908	35	341	9	874	35	333	9
18	6316	983	32	147	9	939	37	148	9
19	6677	893	47	–141	9	902	47	–139	9
20	7019	816	75	–297	9	794	76	–296	8
21	7348	621	156	–316	17	668	141	–310	13
22	7711	810	156	–175	15	925	173	–171	15
23	8068			99	15			110	18
24	8376	1154	68	171	12	1044	62	138	10
25	8744	1231	41	–128	8	1357	44	–193	8
26	9042	1276	72	–325	11	1437	84	–457	11
27	9372	1134	101	–469	12	1310	110	–570	12
28	9703			–471	14			–540	15
No.	P	c_v		v		c_u		u	
8	2721	942	±121	386	± 16	1304	±270	249	±21
9	2997	773	75	264	18	1082	154	168	28
10	3359	850	67	40	17	808	160	–80	31
11	3717	415	117	–93	19	194	248	–232	30
12	4103	602	65	–327	16	628	111	–444	18
13	4468	824	114	–423	14	947	184	–456	21
14	4776	1036	67	–443	15	955	114	–479	22
15	5216	939	36	–164	9	962	68	–287	11
16	5590	802	32	96	8	890	48	–34	9
17	5973	801	32	312	9	897	68	191	13
18	6316	882	32	136	8	935	58	32	12
19	6677	865	46	–155	9	970	78	–269	12
20	7019	944	70	–318	8	717	109	–396	9

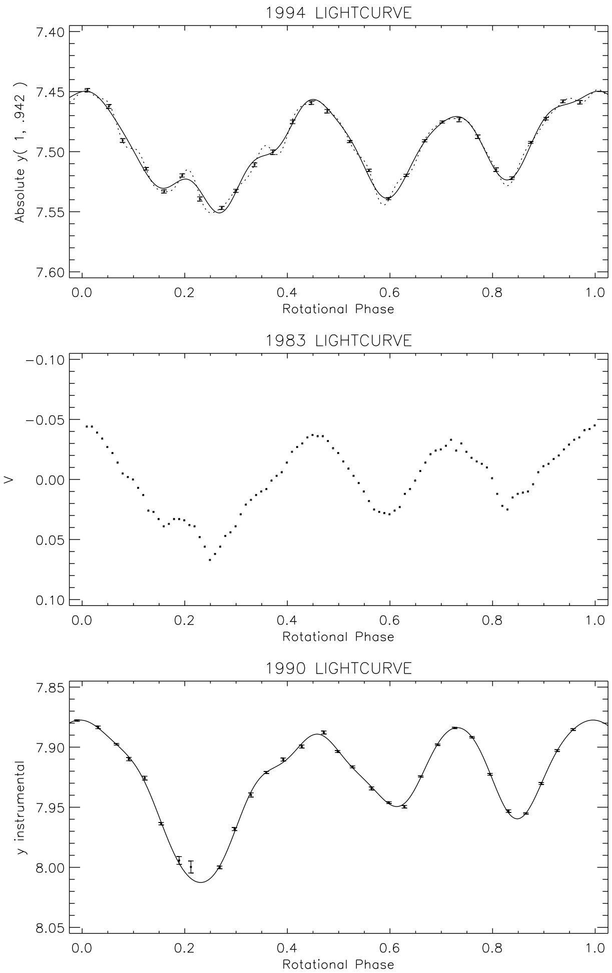

Fig. 2. Plot of the 28 normal points in Strömgren y absolute magnitude reduced with 25 phase factors to β = 0°.942 corresponding to September 26.3. The solid curve corresponds to a Fourier series of order 10 and the dotted to a Fourier series of order 20. The 1983 and 1990 lightcurves with the same aspect are given for comparison. The 1983 lightcurve at β = 1°.85 is from Kristensen & Gammelgaard (1985) but with the rotational phase corrected by +0.009 rev. The 1990 lightcurve at β = 13°.01 from Kristensen & Gammelgaard (1993) has rotational phase corrected by −0.021 rev. The curve is given by 30 normal points and a Fourier series fit to order 10. Note that the small peak (a fourth maximum) at $P \sim 0.21$ rev. can not be reproduced by Fourier terms to order 10. The 1990 and 1994 curves may be different because of the large (13°.01 − 0°.94) phase difference. The 1983 and 1994 oppositions were within 9° in the sky, but the equality of the 1990 and 1994 aspects is derived from the adopted pole solution.

Formulae (11) and (12) define weighted averages of P and o, respectively. This average of 1 and β is 1 and 0, respectively. Averaging the observational equations (10) we obtain the normal point $y_0 + y_0' P_0$ at phase P_0 with the smallest possible variance (13) or the highest possible accuracy.

Table 6 shows the O-C residuals for the 15 y and b observations in group 25. The most accurate value is y = -0.0128 ± 0.0008 at phase P = 0.8745 for the here adopted phase angle: β_0 = 0°.942. The phase factor is $c_y = 0.1231 \pm 0.0041$. This large value can only be due to the small phase angle. The observational mean error per unit of weight is ±0.0055 by f = 15 - 3 = 12 degrees of freedom. On basis of all 327 observations in 25 groups this mean error is accurately determined to ±0.0059 by f = 327 - 3 · 25 = 252 degrees of freedom and is used in the error estimates of y and c_y above. The average weight of the observations in group 25 is 3.4, giving the small observational error ± 0.0032 mag.

Table 6. The 15 observations in group 25 from rotational phase 0.8590 to 0.8863. The columns are: rotational phase P, night number from 1 to 3, solar phase angle in units of degrees, airmass X, observed differential magnitude b, adoped weight w_b, $(O-C)_b$ residual, observed differential magnitude y, adopted weight w_y, and $(O-C)_y$ residual. Equation (10) with $\beta_0 = 0\overset{\circ}{.}942$ is solved by least squares. The most accurate normal point is obtained by (11) at $P_0 = 0.8744$ and gives $b = -0\overset{m}{.}0193 \pm 0\overset{m}{.}0008$ and $y = -0\overset{m}{.}0128 \pm 0\overset{m}{.}0008$. The phase factors are $c_b = \frac{db}{d\beta} = +0.1357 \pm 0.0094$ mag/deg and $c_y = \frac{dy}{d\beta} = +0.1231 \pm 0.0041$. These values are transferred to Table 5.

P	Night	β	X	b	w_b	$(O-C)_b$	y	w_y	$(O-C)_y$
0.8590	3	1.327	1.321	0.0340	3.51	0.0040	0.0407	3.26	0.0040
0.8611	3	1.327	1.324	0.0344	3.51	0.0019	0.0384	3.26	0.0019
0.8633	3	1.327	1.328	0.0330	3.51	−0.0025	0.0337	3.26	−0.0025
0.8654	3	1.328	1.331	0.0311	3.51	−0.0027	0.0332	3.26	−0.0027
0.8675	3	1.328	1.335	0.0277	3.51	−0.0025	0.0332	3.26	−0.0025
0.8691	1	0.667	1.733	−0.0510	3.01	0.0032	−0.0427	3.01	0.0032
0.8712	1	0.667	1.742	−0.0526	3.01	0.0031	−0.0431	3.01	0.0031
0.8759	1	0.667	1.761	−0.0617	3.01	−0.0034	−0.0503	3.01	−0.0034
0.8777	2	0.937	1.516	−0.0166	3.56	−0.0010	−0.0150	3.62	−0.0010
0.8780	1	0.667	1.771	−0.0637	3.01	−0.0046	−0.0517	3.01	−0.0046
0.8799	2	0.937	1.522	−0.0159	3.56	0.0010	−0.0132	3.62	0.0010
0.8802	1	0.667	1.780	−0.0622	3.01	−0.0017	−0.0491	3.01	−0.0017
0.8820	2	0.937	1.528	−0.0175	3.56	−0.0004	−0.0149	3.62	−0.0004
0.8841	2	0.937	1.534	−0.0203	3.56	0.0025	−0.0123	3.62	0.0025
0.8863	2	0.938	1.540	−0.0180	3.56	0.0033	−0.0118	3.62	0.0033

Table 7. Mean values of y and b magnitudes observed only one night. The rotational phase (P) and the phase angle (β) are averaged with the same weights as y and b. The variation of β a single night is too small for a determination of the phase factor. Instead the value c = 0.09 ± 0.04 mag/deg was adopted and used for the correction (corr) to phase angle 0.942 deg. The resulting normalpoint is transferred to Table 5.

	P	β	y	σ	b	σ	corr
1:	0.0101	0.952	−0.0553	±0.0014	−0.0563	±0.0016	−0.0009
23:	0.8068	0.928	+0.0086	0.0015	+0.0097	±0.0018	+0.0013
28:	0.9703	0.948	-0.0466	0.0014	-0.0535	±0.0015	-0.0005
Mean:		0.942					

The approximation involved by regarding the lightcurve as linear in the interval 0.8590< P < 0.8863 can easily be estimated by the use of the analytical form of the y-curve given in Table 4 . The y-curve is linear within ± 0.001 mag., which is accceptable in view of the observational errors. The non-linearity is mainly due to the higher harmonics (n=11 to 20 incl.) and the interpolation error is

$$(\pi\Delta P)^2\sqrt{\sum_{n=1}^{N} n^4(s_n^2+c_n^2)} \qquad (14)$$

where the root-square-sum of n^2a is 0.75 mag. For $2n\Delta P <$ 1 the non-linearity is given by $n^2(2\pi\Delta P)^2\sqrt{s_n^2+c_n^2}/16$, for $2n\Delta P \sim 1$ it is given simply by $\sqrt{s_n^2+c_n^2}$.

If two nights with different phase angles are available then the constant difference between these two nights and hence the phase factor can be determined. The mean error in the determination of the constant difference between two series of respectively n and n' equally good observations around mean phases $\bar{P}$ and $\bar{P}'$ is given by

$$\sigma_0\sqrt{\frac{1}{n}+\frac{1}{n'}+Q_{33}(\bar{P}-\bar{P}')^2} \qquad (15)$$

Here Q_{33} is the component of the covariance matrix corresponding to the phase factor c_y.

This error may be large if the distance between the two groups is large compared to the number of observations and their scatter around their mean phases. However, if only a single observation from one night is situated between two observations from the other night then the critical, phase dependent, term in the radical in (15) has the upper limit 2. Hence, in this case it will always be possible to derive an approximate phase factor if the difference between the phase angles (β) is large enough. In the present situation there were 7 groups (viz: 2, 3, 11, 21, 22, 26 and 27) which had observations from two nights only and which did not overlap. These groups were, however, near enough to give useful results. Three groups (1, 23 and 28) contained only

Table 8. The maxima of c_y (P) for four different phase factor curves. For comparison, the maxima of the 3P-component in the lightcurves are given and denoted Max. 3P. The aspect dependent 3P feature dominates the variation in shape of the lightcurves.

Source	M_1	M_2	M_3
1993	0.20	0.50	0.90
1983-1994	0.23	0.52	0.75
1990	0.20	0.55	0.95
1991	0.10	?	?
Mean:	0.18	0.52	0.87
Max. 3P	0.22	0.56	0.89

observations from a single night and no phase factor could be derived. In such case the normal points on the lightcurve can not strictly be reduced to a standard phase angle. To minimize this inconvenience the adopted phase angle is chosen to be the $\beta_0 = 0\overset{\circ}{.}942$ which is mean of the 3 phase angles. In this case we take the average values given in Table 7.

With c in the range 0.05-0.13 mag/deg we can estimate the correction to the standard phase β_o. For group 23 the correction is of order: $(0.942 - 0.928) \cdot (0.09 \pm 0.04) = +0.0013 \pm 0.0006$ mag. The last column in the above table gives these corrections.

The observational mean errors of unit of weight were also obtained for the individual groups and were consistent with the following adopted mean values for uvby:

0.0080 0.0058 0.0064 0.0059

6. The phase factors

Table 5 gives our main results which are the normal points for all 28 groups. The table gives the color (index 1..4 is uvby), the phase factor and its mean error, the phase P_{best} and the magnitude and its mean error. For groups 1, 23 and 28 the phase factor could not be derived directly so for color y the estimates from the table above were used.

A statistical "analysis of variance" shows that the values in Table 5 can **not** be regarded as an observation of the **same** phase factor. The variance **between** groups is significantly larger than the variance inside groups. The variance ratio is: F = 9.10, with 24 degrees of freedom between groups and 51 inside groups; the 1% limit is 2.17 . This highly significant variation between groups is the phase factor variation found earlier in the 1990 and 1991 oppositions (Kristensen & Gammelgaard (1993)) and shown on Fig. 3 . The correlation coefficient between c_y and c_b is r = 0.90 and is highly significant; the 1% limit is 0.505.

Fig. 3 shows the phase factors c_y and c_b from Table 5. The quite independent values in y and b obtained by two different photomultipliers seem to support each other and indicate a structure with three maxima. The observed phase factors are too large to be explained by an effect of small errors in rotational phases in the subtracted magnitudes (dy/dP=1.0 mag/rev at max., with errors $\pm$ 0.01 in P it only may explain 10% of the phase factors).

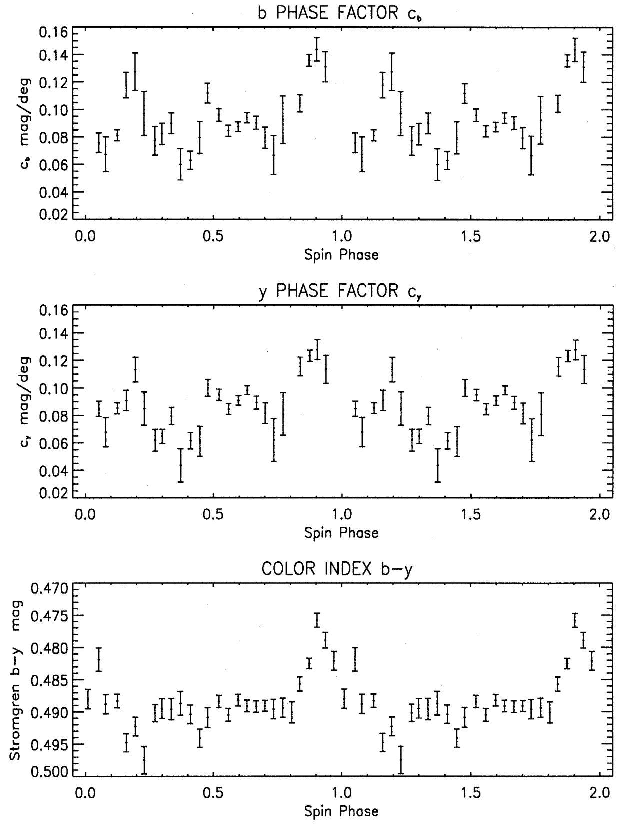

Fig. 3. The 25 phase factors c_b and c_y plotted over two revolutions in order to see better the 3 maxima. The third plot is the 28 differences between b and y normal points.

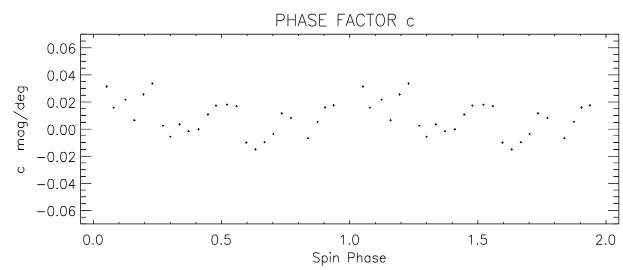

Fig. 4. Phase factors computed as c=(V(1983)−y(1994))/$0\overset{\circ}{.}898$. V(1983) is observed at $\beta = 1\overset{\circ}{.}84$ and y(1994) is observed at $\beta = 0\overset{\circ}{.}942$ at nearly same aspect.

In Fig. 4 the 1983 lightcurve at phase angle β =1.84 is compared with the 1994 curve at phase $0\overset{\circ}{.}942$ at nearly the same aspect. The V magnitudes in 1983 are interpolated from table 8 in Astron. Nachr. 306 (1985) 246 and the figure gives c=(V(1983)−y(1994))/$0\overset{\circ}{.}908$. Due to undetermined constants in V and y (resulting in values $c < 0$) the phase factor can only be determined apart from a constant. We note, however, the three maxima around 0.2, 0.5 and 0.8 and the two minima 0.4 and 0.65. The phase factor variation found earlier is confirmed and the more numerous data now available seem to indicate the presence of three maxima in the phase factor curve. Let us assume that the non-linear phase curve y(G,β) depends on a slope parameter G(P) which is a function of rotational phase P and assume that for all G_1 and G_2 the difference $y(G_1,\beta)-y(G_2,\beta)$ is a monotonous function of β. Then all phase factor curves dy/dβ are similar in the sense that if for a given β_1 we have

$$dy(G_1,\beta_1)/d\beta > dy(G_2,\beta_1)/d\beta \qquad (16)$$

then the same relation should be valid for all values of β

$$dy(G_1,\beta)/d\beta > dy(G_2,\beta)/d\beta \qquad (17)$$

This means that the phase factor curves should have extrema at the same rotational phases.

We note from figs. 3 & 4 that local minima occur at 0.35 and 0.75, maxima at 0.55 and 0.9. The two minima are, however switched, that is primary and secondary are interchanged when compared with Fig. 3 in Kristensen & Gammelgaard (1993). In both y and b minimum occur at phase 0.37 rev and maximum at 0.91. The average value of c_y = 0.0811 differs from the constant =0.0909 mag/deg found by the Fourier series in Table 4. The two quantities refer to different averages, one integrates all phase and the other one only a part where the higher values around phase 0 is excluded.

7. Color variations

Outside the interval $0.25 < P < 0.70$ u and v were rejected. In this interval there seems to be a minimum around 0.4 and a variation in accordance with the color curves obtained in 1990 and 1991. Unfortunately, the present data do not allow an investigation of the phase factor as a function of color.

8. Conclusion

The 1994 observations confirm the variation of the phase factor as a function of rotational phase. The statistical analysis is performed in independent parts each being a direct one with few magnitudes near the same phase P directly compared. A Fourier analysis would require that a great number of interrelated unknowns are solve for in a single solution involving all observations simultaneously, with a more uncertain interpretation as a result.

Errors in the computed rotational phases can not account for the large values 0.05 to 0.13 mag/deg of the phase factors. The largest slope is dy/dP=-1.4 mag/deg at P=0.57. The 10 period determinations from single oppositions (AN **312** p.215) gives errors $<<$ 0.001 rev/day. In the 3 days observation interval with a phase change of 0.6 deg the max. error in c_y is thus less than $\pm 0.001 \cdot 3 \cdot 1.4$ mag/0.6 deg = ± 0.007 mag/deg. In sec.3 is shown that the comparison 1983-94 gives errors of order ± 0.01 rev, corresponding to $\pm 0.01 \cdot 1.4/0.6 = \pm\ 0.023$ in c_y. The effect found is thus not caused by a possible "alias" error in the rotational period.

The correlation coefficient between c_y and dy/dP is r=-0.205 with f=23 degrees of freedom. This is insignificant as the 5% limit is 0.396. On the other hand the present data does not prove that there is no "alias" error, only that if present it is not the cause of the large values of the phase factor.

The "amplitude-phase relationship" (Zappala et al. 1990) implies different phase factors at maxima and minima. For reasons of continuity we should therefore expect a correlation between the magnitude, say y, and the phase factor c_y. This correlation coefficient is r =-0.050 and the 5% limit is 0.396. The present data do not indicate a connection between the slope of the phase curve and the magnitude itself. The amplitude is here 0.10 mag. which according to Zappala et al. should give a phase factor variation of order 0.100·0.015=0.0015 mag/deg which can not be observed. The present data does not contradict Zappala et al.

The separation of the lightcurve from the phase curve requires a uniform distribution of observations in both rotational and solar phase. With the ephemeris for physical observations now available it would be possible – given enough telescope time – to determine individual phase curves for the different sides of the body.

So far 51 Nemausa is the only object for which phase factor variation has been detected. The development of asteroid photometry requires that other objects must be investigated.

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References

Abramowitz M., Stegun I., 1965, Handbook of mathematical functions. Dover Publ.
Carslaw H.S., 1930, Theory of Fourier's series and integrals. 3. ed. Dover Publ.
Florentin Nielsen R., 1983, Inst. Theoret. Astrophys. Oslo, 59, 141
Gammelgaard P., Kristensen L.K., 1991, A&A 244, 544
Gray R.O., Olsen E.H., 1991, A&AS 87, 541
Kristensen L.K., 1991, Astron. Nachr. 312, 209
Kristensen L.K., 1993, Astron. Nachr. 314, 381
Kristensen L.K., Gammelgaard P., 1985, Astron. Nachr. 306, 241
Kristensen L.K., Gammelgaard P., 1993, A&A 272, 345
Schuster W.J., Nissen P.E., 1988, A&AS 73, 225
Zappala V., Cellino A., Barucci A.M., Fulchignoni M., Lupishko D.F., 1990, A&A 231, 548

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