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An eigenfunction method for the comptonisation problem. Angular distribution and spectral index of radiation from a disk

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Abstract. We present a semi-analytic approach to solving the Boltzmann equation describing the comptonisation of low frequency input photons by a thermal distribution of electrons in the Thomson limit. Our work is based on the formulation of the problem by Titarchuk & Lyubarskij (1995), but extends their treatment by accommodating an arbitrary anisotropy of the source function. To achieve this, we expand the eigenfunctions of the integro/differential eigenvalue problem defining the spectral index of comptonised radiation in terms of Legendre polynomials and Chebyshev polynomials. The resulting algebraic eigenvalue problem is then solved by numerical means, yielding the spectral index and the full angular and spatial dependence of the specific intensity of radiation. For a thin ($\tau_0 < 1$) plasma disk, the radiation is strongly collimated along the disk surface – for an optical thickness of $\tau_0 = 0.05$, the radiation intensity along the surface is roughly ten times that along the direction of the normal, and varies only slightly with the electron temperature. Our results for the spectral index confirm those of Titarchuk & Lyubarskij over a wide range of electron temperature and optical depth; the largest difference we find is roughly 10% and occurs at low optical depth.

Key words: radiative transfer – methods: analytical – galaxies: active – X-rays: galaxies – X-rays: stars

1. Introduction

The process of Comptonisation, in which soft photons increase their energy by scattering in a gas of hot electrons is thought to be the main mechanism responsible for the formation of non-thermal continuum spectrum in a range of astrophysical objects. Much of the early work on this subject assumed that the optical depth of the scattering cloud was fairly large, and that the photon frequency ν and electron temperature T_{e} were both small: $x \equiv h\nu/m_{\mathrm{e}}c^2 \ll 1, \Theta \equiv k_{\mathrm{B}}T_{\mathrm{e}}/m_{\mathrm{e}}c^2 \ll 1$. The transport of a photon in both configuration space and in energy space can then be approximated by a Fokker-Planck equation and the computation of the spectrum can be split into two parts – the evaluation of the distribution of the number of scatterings undergone by a photon, and the convolution of this with the solution for the evolution of the spectrum in an infinite homogeneous medium (Sunyaev & Titarchuk 1980).

In the wake of this important result, many authors have investigated methods of extending the range of applicability of the calculations, e.g., by calculating corrections to the diffusion coefficient in energy space to higher order in the parameters Θ and x (Prasad et al. 1988; see also Cooper 1971). Of particular interest is the generalisation to scattering media of optical depth $\tau \sim 1$, since such conditions are indicated in many applications (e.g., AGN: Haardt et al. 1994; Zdziarski et al. 1995 and galactic black hole candidates Sunyaev & Trümper 1979; Ebisawa et al. 1996). Thus, Sunyaev & Titarchuk (1985) presented a numerical solution to the spatial transport of photons in slab geometry. Assuming the problem remains separable, this can be combined with the solution for diffusion in energy space to yield the emergent spectrum. These and other generalisations are summarised by Titarchuk (1994), who also pointed out that the angular distribution of comptonised radiation emerging from an optically thin disk forms a ‘knife-blade’ pattern, collimated almost parallel to the surface of the disk. In addition to analytic work, comptonisation has been investigated using numerical methods (Katz 1976; Poutanen & Svensson 1996), in particular the Monte-Carlo simulation technique (Pozdnyakov et al. 1983; Zdziarski 1986; Hua & Titarchuk 1995; Stern et al. 1995a, 1995b).

In a recent paper, Titarchuk & Lyubarskij (1995) have attacked the problem using the Boltzmann equation, without recourse to a Fokker-Planck approximation in either configuration or energy space. As well as confirming the result that power-law spectra are produced when low frequency photons are scattered in the Thomson regime (i.e, when the dimensionless photon energy x' measured in the electron rest frame satisfies $x' \ll 1$), they demonstrate explicitly the separation of the problem into its configuration and energy space parts, provided that the *source function* can be considered isotropic. In this case, the problem reduces to an eigenvalue equation for the source function.

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Titarchuk & Lyubarskij also provide analytic approximations for the computation of the power-law index α over a very large range of optical depth and temperature.

In this paper we generalise the approach of Titarchuk & Lyubarskij (1995) by solving the Boltzmann equation without assuming isotropy of either the radiation or the source function. The approach we adopt is to formulate the equation determining the power-law index as an integral eigenvalue problem. To find the eigenfunctions, we expand the angular dependence in a series of Legendre polynomials, and the spatial dependence in a series of Chebyshev polynomials. Our results confirm the accuracy of the formulae presented by Titarchuk & Lyubarskij (1995) and also give the full spatial and angular dependences of the comptonised radiation. As predicted by Titarchuk (1994), the radiation from an optically thin slab is strongly collimated along the surface of the slab.

The paper is laid out as follows: in Sect. 2 we present the equations leading to the formulation of the integral eigenvalue problem for the power-law index α. Sect. 3 (and in more detail the appendix) presents the method of solution. The phase function is first of all expanded in terms of Legendre polynomials of the scattering angle and the integration over electron velocity is performed in the case of low temperature ($\Theta \ll 1$) and high temperature ($\Theta \gg 1$). Then, using an expansion in Legendre polynomials which are complete over the half-range $0 \leq \mu \leq 1$ of the cosine of the angle between the photon direction and the normal to the slab, the problem is converted to a system of differential equations in the spatial coordinate normal to the disk surface. This is finally reduced to an algebraic eigenvalue problem by expanding the spatial dependence in a series of Chebyshev polynomials. Singular value decomposition is then used to find the eigenvalues and eigenfunctions. Our results are presented in Sect. 4. These consist of plots of the spectral index as a function of temperature Θ and half-thickness τ_0 of the slab, the angular dependence of the specific intensity of radiation, and its spatial distribution. We compare these to the formulae for α presented by Titarchuk & Lyubarskij (1995). Sect. 5 contains a summary of our conclusions, and a short discussion of their range of applicability in astrophysical sources.

2. Formulation of the basic equations

Let us first define the geometry of the problem. We consider an infinite disk of thickness $2z_0$ containing a uniform, non-degenerate gas of free-electrons of temperature T_e and number density n_e. The only process of importance for the transport of photons in this disk is Compton scattering. The optical half-thickness of the disk is defined as $\tau_0 = \sigma_\mathrm{T} n_\mathrm{e} z_0$, where $\sigma_\mathrm{T} = 6.65 \times 10^{-25}\,\mathrm{cm}^2$ is the Thomson cross-section. Let the spatial coordinate normal to the disk be z, with $z = 0$ on the mid-plane of the disk, and define the optical depth variable $\tau = n_\mathrm{e}\sigma_\mathrm{T} z$ which is bounded by $-\tau_0 \leq \tau \leq \tau_0$.

We denote the cosine of the angle between the normal to the disk and the direction of a photon after scattering by μ. In the same sense, the direction before a scattering event is denoted by μ_1. Let η be the cosine of the angle between these directions. For an isotropic electron distribution, the phase function, which describes the change in photon direction due to scattering, depends only on η. To calculate this phase function, we have to perform an integral over the electron velocity. Because of the isotropy of the electron distribution, the electron direction does not refer to the disk normal. It is therefore convenient to switch to a coordinate system, in which the electron direction defines the z-axis. The cosines of the photon directions with respect to the electron direction are then denoted by $\tilde{\mu}$ (after scattering), and $\tilde{\mu}_1$ (before scattering). An analogous notation is used for the azimuthal angles.

We are interested in a situation in which low frequency radiation is injected into the disk, is scattered by the electrons, and forms a power-law spectrum at high frequency, as shown by Shapiro et al. (1976) and Sunyaev & Titarchuk (1980), and seen in numerical studies of Katz (1976). In the power-law regime, there is no source of radiation in the disk, and no radiation which enters from the outside. The time independent, polarisation averaged, equation of transfer for the specific (up-scattered) intensity $I(\nu, \mu, \tau)$ is then given by

$$\mu \frac{\partial}{\partial\tau} I(\nu,\mu,\tau) = \frac{1}{n_\mathrm{e}\sigma_\mathrm{T}} \int_0^\infty \mathrm{d}\nu_1 \int_{4\pi} \mathrm{d}\Omega_1 \left[\frac{\nu}{\nu_1}\sigma_\mathrm{s}(\nu_1 \to \nu, \eta) I(\nu_1, \mu_1, \tau) - \sigma_\mathrm{s}(\nu \to \nu_1, \eta) I(\nu, \mu, \tau)\right] \tag{1}$$

(see e.g., Pomraning 1973). This transport equation is a special case of the linearised Boltzmann equation and we shall simply refer to it as the Boltzmann equation.

The power-law part of the spectrum we describe occurs at frequencies lower than that of the Wien cut-off ($h\nu < k_\mathrm{B}T_\mathrm{e}$), so that the energy change of the photon due to the recoil of the electron, can be neglected in comparison with the Doppler shift of the photon. In the electron rest frame (in which quantities are adorned with a prime), the classical Thomson scattering kernel is then given by

$$\sigma'_\mathrm{s}(\nu' \to \nu'_1, \eta') = \frac{3}{16\pi} n'_\mathrm{e}\sigma_\mathrm{T} \left\{1 + (\eta')^2\right\} \delta(\nu' - \nu'_1)\,. \tag{2}$$

The electrons are distributed isotropically. They are described by a relativistic Maxwell distribution, which is defined as

$$f(v) = \frac{\gamma^5 \exp(-\gamma/\Theta)}{4\pi\,\Theta\,\mathrm{K}_2(1/\Theta)}\,, \tag{3}$$

where K_2 is the modified Bessel function of the second kind of order two, and $\gamma = (1 - v^2)^{-1/2}$, (here and in the following we set $c = 1$). This distribution is normalised according to

$$\int \mathrm{d}^3 v\, f(v) = 4\pi \int \mathrm{d}v\, v^2\, f(v) = 1\,. \tag{4}$$

The scattering kernel in the disk system can be calculated by performing a Lorentz boost of σ'_s, multiplying it by $f(v)$ and integrating over v. Then the scattering kernel is given by[1]

$$\sigma_s(\nu \to \nu_1, \eta, \Theta) = \frac{3}{16\pi} \frac{n_e \sigma_T}{\nu x} \int d^3 v \, \frac{f(v)}{\gamma} \cdot \left\{ 1 + \left(1 - \frac{1-\eta}{\gamma^2 D D_1} \right)^2 \right\} \cdot \delta \left[\gamma \left(\frac{D}{x_1} - \frac{D_1}{x} \right) \right], \quad (5)$$

with the definitions

$$D := 1 - \tilde{\mu}\, v, \qquad D_1 := 1 - \tilde{\mu}_1\, v. \quad (6)$$

We now look for a solution of the Boltzmann equation of the form

$$I(\nu, \mu, \tau) = J(\mu, \tau)\, x^{-\alpha}. \quad (7)$$

Inserting this, and Eq. (5), into Eq. (1), and performing the trivial integral over ν_1 using the δ- function, we find

$$\mu \frac{\partial J(\mu, \tau)}{\partial \tau} = -J(\mu, \tau) + B(\mu, \tau), \quad (8)$$

where the source function $B(\mu, \tau)$ is defined as

$$B(\mu, \tau) = \frac{1}{4\pi} \int_{-1}^{1} d\mu_1 \int_{0}^{2\pi} d\phi \, R(\eta)\, J(\mu_1, \tau). \quad (9)$$

The phase function $R(\eta)$ is given by

$$R(\eta) = \frac{3}{4} \int d^3 v \, \frac{f(v)}{\gamma^2} \left(\frac{D_1}{D} \right)^{\alpha+2} \frac{1}{D_1} \left\{ 1 + (\eta')^2 \right\}. \quad (10)$$

The last three equations (8, 9 and 10), together with the boundary condition

$$J(\mu < 0, \tau = \tau_0) \equiv 0 \equiv J(\mu > 0, \tau = -\tau_0), \quad (11)$$

define an integral eigenvalue problem. Our aim is to reduce this to an algebraic eigenvalue equation by expanding the intensity $J(\mu, \tau)$ into a polynomial series, as mentioned above and shown in the Sects. A.3 and A.4. We further perform the phase function integral using an expansion for $\Theta \ll 1$ and $\Theta \gg 1$. This is shown in the Sects. A.1 and A.2.

3. Method of solution

To reduce the integral eigenvalue problem described above to an algebraic eigenvalue equation, we first expand the phase function in a series of Legendre polynomials:

$$R(\eta) = \sum_{i=0}^{M} \omega_i(\alpha, \Theta) P_i(\eta). \quad (12)$$

The details of this and the following steps are shown in the appendix. The coefficients $\omega_i(\alpha, \Theta)$ can be written, in the case of small electron temperature ($\Theta \ll 1$), as polynomials in α, with coefficients in terms of moments of the Maxwell distribution, which can easily be calculated numerically. In the case of high electron temperature ($\Theta \gg 1$), the coefficients $\omega_i(\alpha, \Theta)$ can be expressed using the incomplete Γ-function.

The essential step is to expand the specific intensity $J(\mu, \tau)$ into a series of Legendre and Chebyshev polynomials. For the half-space $0 \le \mu \le 1$, we write (compare Eq. (A23))

$$J(\mu, \tau)\Big|_{\mu \ge 0} = \sum_{n=0}^{N} \frac{2n+1}{2} P_n(2\mu - 1) Q_n^+(\tau). \quad (13)$$

We now expand each of these N+1 expansion coefficients $Q_n^+(\tau)$ into a series of Chebyshev polynomials (T_i). In vector-form this reads (compare Eq. (A45))

$$\boldsymbol{Q}^+(\tau/\tau_0) = \boldsymbol{q}_0 + \sum_{i=1}^{K} \left[1 - T_i(-\tau/\tau_0) \right] \boldsymbol{q}_i. \quad (14)$$

All expansion coefficients can be represented by a common $(K+1) \cdot (N+1)$ dimensional vector:

$$\underline{\boldsymbol{q}} := \begin{pmatrix} \boldsymbol{q}_0 \\ \boldsymbol{q}_1 \\ \vdots \\ \boldsymbol{q}_K \end{pmatrix}. \quad (15)$$

As shown in detail in the appendix, the eigenvalue problem (Eqs. 8-10) becomes with these expansions a set of homogeneous linear equations for the vector $\underline{\boldsymbol{q}}$, which is very easy to treat numerically. The set of equations can be written as a matrix equation (see Eq. (A58)):

$$\underline{\underline{\mathsf{F}}}(\alpha, \Theta, \tau_0)\, \underline{\boldsymbol{q}} = 0. \quad (16)$$

The first step is to calculate the matrix $\underline{\underline{\mathsf{F}}}(\alpha, \Theta, \tau_0)$ for given values of τ_0 and Θ. The solution for the spectral index α can be found by solving the following equation for the determinant of $\underline{\underline{\mathsf{F}}}(\alpha)$:

$$\det \left[\underline{\underline{\mathsf{F}}}(\alpha) \right] \stackrel{!}{=} 0. \quad (17)$$

We used *Mathematica* to find the roots of this equation, which give the spectral index α. For an expansion of the spatial and angular dependence of $J(\mu, \tau)$ to (e.g.) 9th order, $\underline{\boldsymbol{q}}$ is a 100 dimensional vector and $\underline{\underline{\mathsf{F}}}(\alpha)$ is a 100×100 matrix. A solution of Eq. (17) for this dimension takes less than one minute with a *Pentium* PC in both the relativistic, and the non-relativistic cases.

For each value of α, which satisfies Eq. (17), the expansion coefficients, and therefore $J(\mu, \tau)$, can be found by solving the equation

$$\underline{\underline{\mathsf{F}}}\, \underline{\boldsymbol{q}} = 0, \quad (18)$$

where $\underline{\underline{\mathsf{F}}}$ is now a known singular square matrix. This matrix can be decomposed using a singular value decomposition routine (see e.g. Press et al. 1986; Wolfram 1991), which yields

[1] See also Titarchuk & Lyubarskij (1995) Eq. (2) after correction of a minor typographical error.

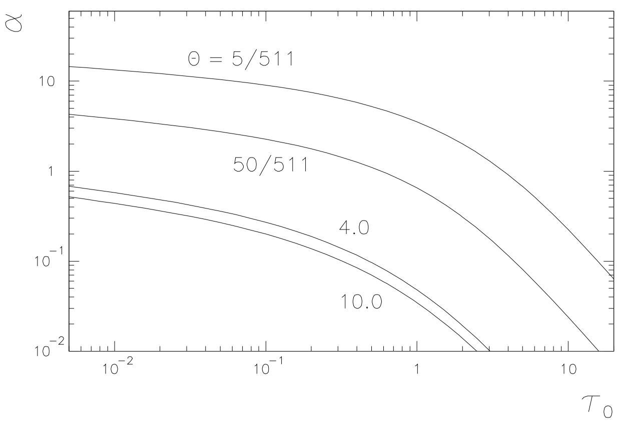

Fig. 1. Spectral index α vs. Thomson optical half thickness τ_0, for non-relativistic and relativistic electron plasma temperatures $\Theta = k_B T_e/m_e \approx k_B T_e/(511\,\mathrm{keV})$.

the vector null-space. This vector can immediately inserted into Eq. (14), which gives $\boldsymbol{Q}^{\pm}(\tau)$. Inserting this into Eq. (13), we end up with the intensity $J(\mu, \tau)$. The resulting values for the spectral index α and the shape of the intensity in μ and τ are described in the next section. Use of the singular value decomposition technique has the advantage that an automatic check for the presence of repeated roots of Eq. (17) is provided. More importantly, however, it immediately provides the null-space and the associated specific intensity of radiation, which must be physically realistic in the sense that $J(\mu, \tau)$ must be positive definite. This important condition enables one to identify the physically relevant power-law index α, which is the smallest positive root of the nonlinear Eq. (17).

4. Results

4.1. Spectral index

The spectral index α of the comptonised radiation, for a given temperature ($\Theta \ll 1$ or $\Theta \gg 1$), and *arbitrary* optical depth $2\tau_0$, can be found by solving Eq. (17).

Note that the spectral index is the exponent of the v-dependent functions D and D_1 in the phase function. For non-relativistic plasma temperatures, the spectral index can be well above 1. In this case, accuracy is achieved only if the phase function is expanded to high order in a Taylor series in v – we use an expansion up to 16th. order ($L = 16$ in Eq. (A5)). This sum is represented by an expansion to 10th. order in η ($M = 10$ in Eq. (A5)). In the relativistic temperature regime, on the other hand, the spectral index α is much lower. Therefore, we take into account only the leading order of $\omega_i(\alpha, \Theta)$ in the small parameter $1/\gamma$. The first four coefficients of Eq. (A4) are then given by the Eqs. (A17), together with Eq. (A20). The spectral index is not sensitive to the expansion of the μ and τ dependence of $J(\mu, \tau)$. It is more than sufficient to choose $N = 10$ in Eq. (A23), and $K = 10$ in Eq. (A45). Results are given for an electron plasma temperature $k_B T_e = \Theta m_e$ of 5 keV and 50 keV in the non-relativistic regime. In the relativistic regime we choose $\Theta = 4.0$ and $\Theta = 10.0$, corresponding roughly to 2 and 5 MeV respectively. Fig. 1 shows the spectral index α versus the Thomson optical half thickness τ_0, using the expansion parameters given above (L, M, N and K).

Table 1. Spectral index α for two plasma temperatures and different expansion parameters M of the source function $B(\mu, \tau)$, and for Thomson optical half thickness τ_0 = 0.05, 1.0 and 3.0 .

$\Theta = \frac{k_B T_e}{m_e}$	τ_0	M	α
50/511	0.05	0	2.83
50/511	0.05	1	2.80
50/511	0.05	2	2.75
50/511	0.05	3	2.74
50/511	0.05	4	2.74
50/511	1.0	0	0.678
50/511	1.0	1	0.656
50/511	1.0	2	0.652
50/511	1.0	3	0.652
50/511	3.0	0	0.186
50/511	3.0	1	0.179
50/511	3.0	2	0.179
4.0	0.05	0	0.366
4.0	0.05	1	0.361
4.0	0.05	2	0.359
4.0	0.05	3	0.359
4.0	1.0	0	0.0538
4.0	1.0	1	0.0480
4.0	1.0	2	0.0480
4.0	3.0	0	0.0125
4.0	3.0	1	0.0103
4.0	3.0	2	0.0103

These values of α are in good agreement with those given by Titarchuk & Lyubarskij (1995). These authors assumed an isotropic source function $B(\mu, \tau)$ (Eq. 9), which is certainly a good approximation for $\tau_0 \gg 1$. To relax this restriction, at least the first three expansion coefficients of the source function must be taken into account ($M = 0, 1, 2$), because of the intrinsic $1 + (\eta')^2$ dependence of the Thomson scattering kernel. Especially when the intensity $J(\mu, \tau)$ is highly anisotropic (which is the case for $\tau_0 \ll 1$, as shown in the next section) the source function $B(\mu, \tau)$ depends on the higher expansion coefficients of the phase function, which leads to an anisotropy of $B(\mu, \tau)$. In fact for $\tau_0 = 0.05$ the anisotropy of the source function at the disk surface for $\Theta = 50/511$ becomes

$$\frac{B(\mu = 0, \tau_0 = 0.05)}{B(\mu = 1, \tau_0 = 0.05)} = 2.3 \quad . \tag{19}$$

The sensitivity of α to the angular expansion of the source function is, however, not very strong, and it is well approximated by taking into account the first four Legendre polynomials of the expansion ($M = 3$ in Eq. (A5)). This is shown for $\tau_0 = 0.05$, 1.0 and 3.0, and two different plasma temperatures in Table 1.

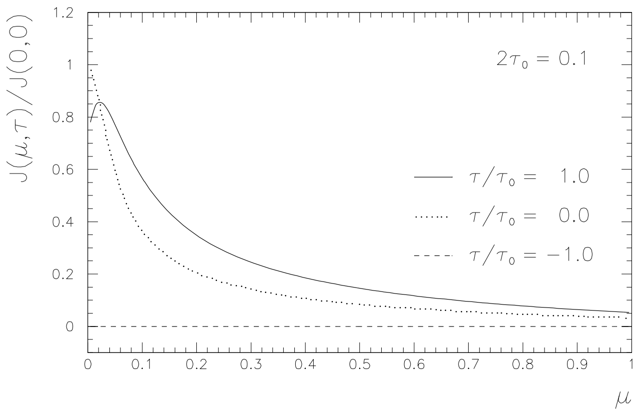

Fig. 2. Angular dependence of the specific intensity $J(\mu, \tau)$ for a Thomson optical thickness of $2\tau_0 = 0.1$. The intensity at the surface of the disk is given by $\tau/\tau_0 = 1.0$. The anisotropy of the integrated intensity is $A(\tau_0) \equiv (J_{\|} - J_{\perp})/(J_{\|} + J_{\perp}) = 0.82 \pm 0.03$ for a wide temperature range. (Expansion up to $N = 16$ and $K = 10$).

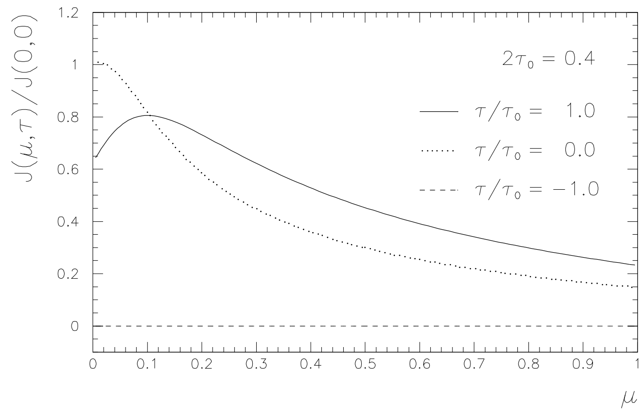

Fig. 3. Angular dependence of the specific intensity $J(\mu, \tau)$ for a Thomson optical thickness of $2\tau_0 = 0.4$. The intensity at the surface of the disk is given by $\tau/\tau_0 = 1.0$. The anisotropy of the integrated intensity is $A(\tau_0) \equiv (J_{\|} - J_{\perp})/(J_{\|} + J_{\perp}) = 0.47 \pm 0.03$ for a wide temperature range. (Expansion up to $N = 12$ and $K = 14$).

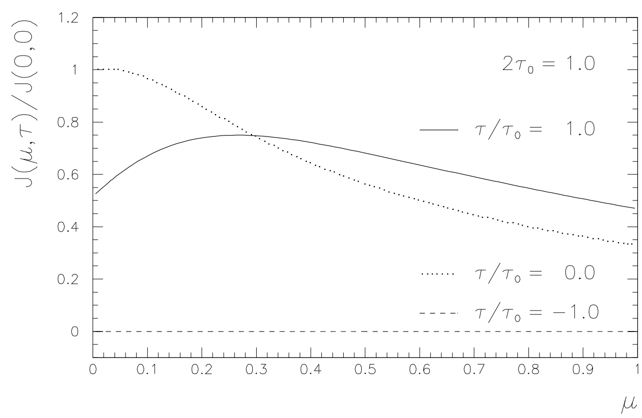

Fig. 4. Angular dependence of the specific intensity $J(\mu, \tau)$ for a Thomson optical thickness of $2\tau_0 = 1.0$. The intensity at the surface of the disk is given by $\tau/\tau_0 = 1.0$. The anisotropy of the integrated intensity is $A(\tau_0) \equiv (J_{\|} - J_{\perp})/(J_{\|} + J_{\perp}) = 0.08 \pm 0.02$ for a wide temperature range. (Expansion up to $N = 8$ and $K = 16$).

Note that even for a value of $\alpha = 2.74$ it is necessary to choose a Taylor expansion of sufficiently high order (here: $L = 16$), for *any* value of M.

Titarchuk & Lyubarskij (1995) suggested interpolation formulas (see Eq. (17) and Eq. (21) therein, and also Eq. (27) in Titarchuk 1994), which are valid for all optical depths and all plasma temperatures. The values of the spectral index given by these formulas, differ by less then 1% for $\tau_0 \gg 1$ from those given in Fig. 1; the largest discrepancy is 10% at smaller values of τ (see also Table 1).

4.2. Angular distribution

As described at the end of Sect. 3, a singular value decomposition of the matrix $\underline{\underline{\mathsf{F}}}$ gives $J(\mu, \tau)$ in the form of a polynomial of order K in τ/τ_0, and of order N in μ (for any set of parameters Θ, τ_0 and α). We choose $L = 16$ and $M = 6$ (see Eq. (A5)) for the phase function representation, and $\Theta = 5/511$ for the plots discussed below. The minimum expansion for the angular and spatial dependence (K and N) have to be chosen differently for each optical depth (see figure captions).

Because we are interested especially in *non*-isotropic intensities, we define a measure of the anisotropy of $J(\mu, \tau)$. First, let $J_{\|}(\tau)$ and $J_{\perp}(\tau)$ be the integrated intensity parallel and perpendicular to the disk, over an interval of $\Delta\mu = 0.1$:

$$J_{\|}(\tau) := \int_0^{0.1} \mathrm{d}\mu\, J(\mu, \tau) \ ,$$

$$J_{\perp}(\tau) := \int_{0.9}^{1} \mathrm{d}\mu\, J(\mu, \tau) \ . \qquad (20)$$

From this we define an asymmetry[2] A, which is 0 for isotropic intensity, and 1 for extremely focussed intensity along the disk surface:

$$A(\tau_0) := \frac{J_{\|}(\tau_0) - J_{\perp}(\tau_0)}{J_{\|}(\tau_0) + J_{\perp}(\tau_0)} \ . \qquad (21)$$

At an optical thickness of order unity ($2\tau_0 \simeq 1$), this anisotropy is about zero (see Fig. 4). For smaller values of τ_0, $A(\tau_0)$ can become very close to 1, as shown in Fig. 2, where $J(\mu, \tau)$ is plotted for an optical depth of $2\tau_0 = 0.1$, normalised to the intensity in the middle of the disk, parallel to the surface. The boundary condition gives $J(\mu, \tau = -\tau_0) \equiv 0$ for $0 < \mu \leq 1$ (no radiation enters the disk from outside, see dashed line). The dotted line shows the intensity in the middle of the disk ($\tau = 0$),

[2] In analogy to the forward-backward asymmetry of the electro-weak interaction.

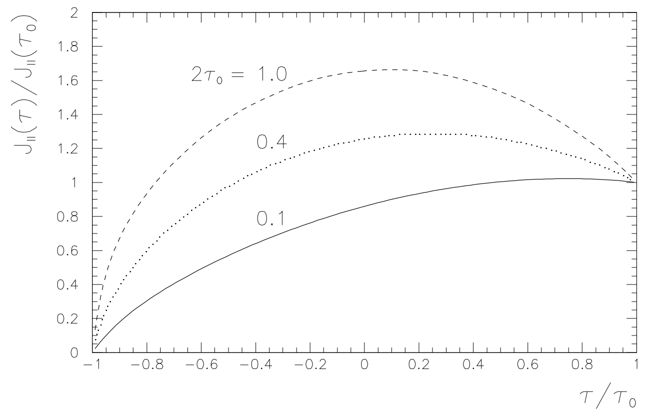

Fig. 5. Spatial distribution of the intensity parallel to the disk (as defined in Eq. (20)), normalised to the parallel intensity at the disk surface. Note that this plot shows the contributions of the half-space $0 \leq \mu \leq 1$.

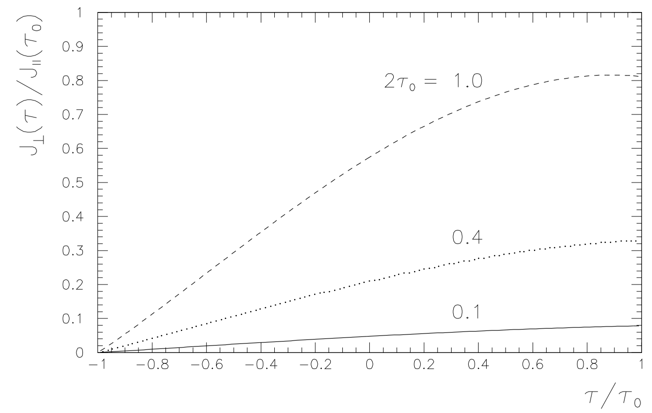

Fig. 6. Spatial distribution of the intensity perpendicular to the disk (as defined in Eq. (20)), normalised to the parallel intensity at the disk surface. Note that this plot shows the contributions of the half-space $0 \leq \mu \leq 1$.

whereas the solid line shows the intensity at the surface, given by $J(\mu, \tau = \tau_0)$. The normalised specific intensity of the radiation is approximately independent of the plasma temperature. As shown by Titarchuk & Lyubarskij (1995) the electron energy and photon spatial variables are completely decoupled if the source function is exactly isotropic. However, as discussed in the preceding section, the source function has a weak angular dependence, which leads to a coupling of the electron energy and photon spatial variables. This, in turn, implies that the angular dependence is a function of the plasma temperature. The range of variation over the temperature range considered is expressed in form of an error of the anisotropy. The upper bound of the anisotropy is valid for $\Theta = 5/511$, whereas the lower bound was calculated for $\Theta = 100$.

Even for a moderately small optical depth of $2\tau_0 = 0.1$ (see Fig.2) the anisotropy is $A(\tau_0) = 0.82 \pm 0.03$, which means that the radiation in the interval $0 \leq \mu \leq 0.1$ parallel to the disk surface, is a factor of about 10 more intense than the radiation in the interval $0.9 \leq \mu \leq 1$ perpendicular to the disk. At increased angular resolution (smaller $\Delta\mu$) this factor is even larger.

The reason for the high anisotropy at small optical depth is the following: photons which contribute to the power-law part of the spectrum have to undergo a number of scatterings on electrons to gain the required energy. The energy gain has a maximum for back-scattering of the photons ($\Delta\theta = 180°$). Thus, those photons most effectively boosted in energy and least likely to escape the disk are those which move almost parallel to the surface. This leads to a strong collimation in the disk plane for an optically thin disk.

4.3. Spatial distribution

The intensity $J(\mu, \tau)$ provides of course not only the angular distribution, but also the spatial distribution. The solution discussed above gives $J(\mu, \tau)$ for the half space $0 \leq \mu \leq 1$. A full angle integrated intensity at any space point is then the sum of the two half space intensities, which are symmetric with respect to the middle of the disk. To compare the spatial distribution to the previous figures, we show $J(\mu, \tau)$ also for the half space $0 \leq \mu \leq 1$, and used the same set of parameters as in Sect. 4.2.

Fig. 5 shows the intensity $J_{\|}$ parallel to the disk, as defined in Eq. (20) for various optical depths, normalised to the parallel intensity at the disk surface. At $\tau = -\tau_0$ the intensity has to be 0, because we take into account only the half space $0 \leq \mu \leq 1$, for which the boundary condition is $J(\mu, \tau = -\tau_0) \equiv 0$. Fig. 6 shows the intensity $J_{\perp}$ perpendicular to the disk, normalised to the parallel intensity $J_{\|}$ at the disk surface. Note the difference in scale between this plot and Fig. 5. A comparison of Fig. 5 and Fig. 6 provides the anisotropy $A(\tau)$ for all values of τ. Because the perpendicular intensity $J_{\perp}$ drops faster for smaller values of τ/τ_0 than the parallel intensity $J_{\|}$, the anisotropy $A(\tau < \tau_0)$ is even larger than $A(\tau_0)$, which is given in the captions of the Figs. 2 – 4.

5. Discussion

In this paper we have presented a new, semi-analytic method of obtaining solutions to the comptonisation problem, and used it to find the power-law index of photons scattered in the Thomson regime, neglecting the recoil of the scattered electron. Because these photons undergo a large number of scatterings before emerging from the plasma, they have 'forgotten' the details (spatial and angular dependences) of their injection at low frequency. Once in the power-law region, which exists even for optically thin plasmas, the spatial and angular dependence of the specific intensity of radiation ceases to be a function of frequency – it is determined by a particular eigenfunction of the reduced transfer equation and depends only on the optical depth and temperature of the plasma. We compute this eigenfunction. This distinguishes our approach from other numerical techniques in the literature. Haardt (1993), for example, uses an approximation method in which anisotropy is accounted for only in the first scattering undergone by a photon, so that his results are accurate only fairly close to the frequency of injection. Poutanen & Svensson (1996) have developed a comprehensive code based on the iterative scattering method, which treats the anisotropy 'exactly' (on a discrete grid) and accounts for several processes which we neglect. This method converges rapidly provided only a few photon scatterings are important, i.e., at small optical depth and high temperature. However, it would need a large number of grid points in order to resolve the sharp dependence of the radiation intensity on angle such as displayed in Fig. 2. In principle, it should be possible to extend our technique to solve 'inhomogeneous' problems, where the emergent spectrum depends on the input radiation. However, in this case it would be necessary to compute several eigenfunctions. Further extensions of the method to geometries other than slab, or to arbitrary electron distributions are straightforward.

To apply our results to observations of astrophysical objects we have to consider the range of spectral indices of order 3 or less, because higher values are probably too steep to be observable. A spectral index in this range can be achieved by a non-relativistic plasma with optical thickness of $\tau_0 \gtrsim 1$, or by a relativistic plasma with optical thickness of $\tau_0 \lesssim 1$. The assumption of Thomson scattering (as opposed to Klein-Nishina) restricts the relevant frequency range to $x < 1/\langle\gamma\rangle$ (where $\langle\gamma\rangle$ is the averaged Lorentz factor according to Eq. (A6)), i.e. to X-rays for the highest temperatures considered here. We also require the dominant source of soft photon input to be at a frequency which is low enough to require more than a single scattering before X-ray frequencies are achieved, otherwise the pure power-law spectrum is not achieved. The average frequency change on scattering is given by $\langle\Delta x\rangle/x \simeq 4\Theta$ in the non-relativistic regime, and $\langle\Delta x\rangle/x \simeq (4\Theta)^2$ for relativistic temperatures (see Pozdnyakov et al. 1983).

As an example, consider a plasma with temperature $k_{\rm B}T_{\rm e}$ = 50 keV and input photons with energy 5 eV. After 20 scatterings, which removes all information of their initial distribution, they achieve an energy of roughly 4 keV. The resulting normalised photon energy is then $x = 0.008$, which is well below $1/\langle\gamma\rangle = 0.86$. At an optical depth of $2\tau_0 = 0.4$ the intensity at the disk surface, parallel to it ($J_{\|}$) becomes three times the intensity perpendicular to the surface ($J_{\perp}$), ($A(\tau_0) = 0.49$, compare Fig. 3). The spectral index for these values of temperature and optical depth is $\alpha = 1.77$. For relativistic temperatures, where spectral indices between 0 and 1 can be produced in a thin plasma disk ($\tau_0 \ll 1$), which leads to a very high anisotropy $A(\tau_0)$, the input photon energy has to be very much lower than 5 eV, in order to have more than 1 scattering which shifts the photon energy into the X-ray regime.

In particular, our results concerning the degree of anisotropy (which is only weakly dependent on temperature) are relevant to the case of Seyfert galaxies, where plasma temperatures of the order of 100 keV have been suggested (Titarchuk & Mastichiadis 1994; Zdziarski et al. 1995). In these objects, the ratio of optical to X-ray luminosity should be much smaller for objects seen 'edge-on' than for those seen 'face-on' (an effect predicted by Haardt & Maraschi 1993). It may, however, be difficult to disentangle this effect from that of the increased absorption of optical radiation expected in edge-on sources.

Quite apart from application to astrophysically important objects, the results of our computations should be helpful as a check on other numerical methods of solution, in particular Monte-Carlo simulations. Here it is worth mentioning that we have used a polarisation averaged treatment of the transport. All current Monte-Carlo codes also use this same approximation, so that the results they obtain are directly comparable to ours. Generalisation to polarisation dependent transfer is in principle possible.

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Appendix A: reduction of the Boltzmann equation to an algebraic eigenvalue equation

To express the Boltzmann equation in the form of an algebraic eigenvalue equation, we express the angular and spatial dependence of the source function $R(\eta)$ and the specific intensity $J(\mu,\tau)$ in terms of Legendre and Chebyshev polynomials.

Let us first write the source function as a series

$$R(\eta) = \sum_{i=0}^{M} \omega_i(\alpha,\Theta) P_i(\eta)\,, \tag{A1}$$

where the angular dependence of $R(\eta)$ is expressed in form of Legendre polynomials, with the normalisation:

$$\int_{-1}^{1} P_i(\mu)P_j(\mu)\mathrm{d}\mu = \frac{2}{2i+1}\ \delta_{ij}\,. \tag{A2}$$

These polynomials obey an addition theorem (see e.g., Landau & Lifschitz 1988 Eq. (c.10)):

$$\begin{aligned} P_l(\eta) \ &= \ P_l(\mu)P_l(\mu_1) \\ &\quad + \sum_{m=1}^{l} 2\frac{(l-m)!}{(l+m)!} P_l^m(\mu)P_l^m(\mu_1)\cos(m(\phi-\phi_1))\,, \end{aligned} \tag{A3}$$

where ϕ, $\mu \equiv \cos\theta$ and ϕ_1, $\mu_1 \equiv \cos\theta_1$ define two directions with $\gamma \equiv \arccos\eta$ the angle between these directions. Eq. (A1) can then be written:

$$\frac{1}{2\pi}\int R(\eta)\mathrm{d}\phi = \sum_{i=0}^{M}\omega_i(\alpha,\Theta)P_i(\mu)P_i(\mu_1) =: K(\mu,\mu_1)\,. \tag{A4}$$

To calculate the coefficients $\omega_i(\alpha,\Theta)$ for given values of the temperature Θ, we have to perform the integral over the electron velocity v, which is tractable in the limit of high or low electron plasma temperature. In these limits we can expand the integrand into a series in v, as described in Sect. A.1 for the non-relativistic case, and in $1/\gamma$, as described in Sect. A.2 for the relativistic case.

The expansion of the angular and spatial dependence of $J(\mu,\tau)$ due to Legendre and Chebyshev polynomials is described in Sects. A.3 and A.4.

A.1. Expansion of the phase function in the limit $\Theta \ll 1$

For small plasma temperatures the main contribution to the integral over electron velocity arises from the region of small velocities. Therefore, the integrand (more precisely the factor multiplying $f(v)$) can be expanded into a Taylor series at $v = 0$ with expansion coefficients a_i:

$$\begin{aligned} R(\eta) &= \frac{3}{4}\int \mathrm{d}^3v f(v)\frac{1}{\gamma^2}\frac{(1-\tilde{\mu}_1 v)^{\alpha+1}}{(1-\tilde{\mu}v)^{\alpha+2}} \\ &\quad \cdot\left\{1+\left(1-\frac{1-\eta}{\gamma^2(1-\tilde{\mu}v)(1-\tilde{\mu}_1 v)}\right)^2\right\} \\ &= \int v^2\mathrm{d}v\,\mathrm{d}\tilde{\mu}_1\,\mathrm{d}\tilde{\phi}_\mathrm{e}\, f(v)\sum_{i=0}^{L} a_i(\alpha,\tilde{\mu}(\tilde{\mu}_1,\tilde{\phi}_\mathrm{e},\eta),\tilde{\mu}_1,\eta)\,v^i \\ &= \sum_{i=0}^{M}\omega_i(\alpha,\Theta)P_i(\eta)\,. \end{aligned} \tag{A5}$$

We choose the direction of the outgoing photon as z-axis. The cosine of the polar angle of the electron is then $\tilde{\mu}_1$ and the azimuth of the electron direction is denoted by $\tilde{\phi}_\mathrm{e}$. With this choice of coordinate system, the integration over electron direction ($\mathrm{d}\tilde{\mu}_1\,\mathrm{d}\tilde{\phi}_\mathrm{e}$) becomes trivial. The remaining electron velocity integral can be expressed in the form of moments, which can easily be calculated numerically:

$$\langle v^k\rangle := 4\pi\int_0^1 f(v)v^{k+2}\,\mathrm{d}v\,. \tag{A6}$$

The normalisation of $f(v)$ gives $\langle v^0\rangle = 1$, and $\langle v^2\rangle = 3\Theta$ for non-relativistic electron plasma temperatures. We used *Mathematica* (see e.g. Wolfram 1991) to expand the integrand of $R(\eta)$ up to 16th. order in v. Therefore $\langle v^k\rangle$ has to be calculated for $k = 0, 2, 4, \ldots, 16$. The odd moments vanish due to the angular integration. This expansion in v results in a polynomial series of order $M = L/2 + 2$ in η. (Note, however, that one may still choose to truncate at $M \le L/2+2$). As an example we give the expansion up to 4th. order in v. The coefficients are then given by:

$$\begin{aligned} \omega_0(\alpha,\Theta) &= 1+\frac{\langle v^2\rangle}{3}(\alpha^2+3\alpha) \\ &\quad +\frac{\langle v^4\rangle}{150}(\alpha^2+3\alpha)(7\alpha^2+21\alpha+22)\,, \\ \omega_1(\alpha,\Theta) &= -\frac{2}{5}\langle v^2\rangle(\alpha^2+3\alpha+1) \\ &\quad -\frac{\langle v^4\rangle}{25}(2\alpha^4+12\alpha^3+21\alpha^2+9\alpha+6)\,, \\ \omega_2(\alpha,\Theta) &= \frac{1}{2}+\frac{\langle v^2\rangle}{6}(\alpha^2+3\alpha-6) \\ &\quad +\frac{\langle v^4\rangle}{210}(10\alpha^4+60\alpha^3+55\alpha^2-105\alpha+78)\,, \\ \omega_3(\alpha,\Theta) &= -\frac{\langle v^2\rangle}{10}(\alpha-1)(\alpha+4) \\ &\quad -\frac{\langle v^4\rangle}{50}(\alpha-1)(\alpha+4)(\alpha^2+3\alpha-7)\,, \\ \omega_4(\alpha,\Theta) &= \frac{\langle v^4\rangle}{175}(\alpha-1)(\alpha-2)(\alpha+4)(\alpha+5)\,. \end{aligned} \tag{A7}$$

For very small electron temperatures *and* sufficiently small values of α, the sum of Eq. (A5) converges very quickly. If we truncate it at (e.g.) $L = 2$, the coefficient of the isotropic part of Eq. (A4) is

$$\omega_0(\alpha,\Theta) = 1+\Theta\,(\alpha^2+3\alpha)\,, \tag{A8}$$

in agreement with Eq. (18) of Titarchuk & Lyubarskij (1995)[3].

A.2. Expansion of the phase function in the limit $\Theta \gg 1$

In the relativistic limit, the expansion has to be done in a slightly different way, because the phase function contains singular parts at $v = 1$. Using the orthogonality of the Legendre polynomials, the coefficients of Eq. (A1) can be written:

$$\omega_i(\alpha,\Theta) = \frac{2i+1}{4\pi}\int R(\eta)P_i(\eta)\,\mathrm{d}\tilde{\Omega}\,, \tag{A9}$$

where $\mathrm{d}\tilde{\Omega} = \mathrm{d}\eta\,\mathrm{d}\tilde{\phi}$ with $\tilde{\phi}$ the azimuth of the incoming photon with respect to the outgoing one, which defines the z-axis. In this reference frame, the electron volume-element can be written $\mathrm{d}^3v = v^2\mathrm{d}v\,\mathrm{d}\tilde{\Omega}_\mathrm{e}$. Using this relation, and the definition of the phase function, the expansion coefficients can be written as

$$\omega_i(\alpha,\Theta) = 3\pi\int_0^1 \mathrm{d}v\,v^2\frac{f(v)}{\gamma^2}\,\hat{\omega}_i(\alpha,\gamma)\,, \tag{A10}$$

with the temperature independent kernel

$$\begin{aligned} \hat{\omega}_i(\alpha,\gamma) &= \frac{2i+1}{(4\pi)^2}\int\left(\frac{D_1}{D}\right)^{\alpha+2}\frac{1}{D_1} \\ &\quad\cdot\left\{1+(\eta')^2\right\}P_i(\eta)\,\mathrm{d}\tilde{\Omega}\,\mathrm{d}\tilde{\Omega}_\mathrm{e}\,. \end{aligned} \tag{A11}$$

The axis of integration in this reference frame can be changed, so that the electron direction becomes the z-axis. This is expressed by $\mathrm{d}\tilde{\Omega}\,\mathrm{d}\tilde{\Omega}_\mathrm{e} = \mathrm{d}\tilde{\Omega}\,\mathrm{d}\tilde{\Omega}_1$. With this choice of coordinates, it is easy to perform a Lorentz boost to the electron rest frame. The azimuths of the in- and outgoing photons do not change, and the angle between the photon direction and the boost direction changes according to

$$\tilde{\mu} = \frac{\tilde{\mu}'+v}{1+v\tilde{\mu}'}\,. \tag{A12}$$

The differential solid angle transforms as

$$\mathrm{d}\tilde{\Omega} = \frac{\mathrm{d}\tilde{\Omega}'}{\gamma^2(1+v\tilde{\mu}')^2}\,. \tag{A13}$$

[3] Eq. (A.16) should read: $\hat{C}_0 = \frac{4}{3\gamma^2}\left\{1+\frac{v^2}{3}\left[\alpha(\alpha+3)+6\right]\right\}$.

These transformations lead to

$$\hat{\omega}_i(\alpha,\gamma) = \frac{2i+1}{(4\pi)^2\gamma^2}\int \mathrm{d}\tilde{\mu}'\,\mathrm{d}\tilde{\mu}_1'\,\mathrm{d}\tilde{\phi}\,\mathrm{d}\tilde{\phi}_1' \cdot \frac{(1+v\tilde{\mu}')^\alpha}{(1+v\tilde{\mu}_1')^{\alpha+3}}\left\{1+(\eta')^2\right\} P_i(\eta)\,, \qquad \text{(A14)}$$

where the argument of the Legendre polynomials has to be inserted as

$$\eta = 1 - \frac{1}{\gamma^2}\frac{1-\eta'}{(1+v\tilde{\mu}')(1+v\tilde{\mu}_1')}\,. \qquad \text{(A15)}$$

To solve the above integrals one might use integral tables, or computer programs. We used *Mathematica* to find analytic solutions for $i = 0,1,2,3$ (see Titarchuk & Lyubarskij 1995, Eqs. (A10)-(A15), for $i = 0$). In the limit $v \to 1$ these solutions diverge. Therefore we separated the leading order terms. Extracting a γ-dependent factor according to:

$$\hat{\omega}_i(\alpha,\gamma) = (2\gamma)^{2\alpha+2}\,\hat{\omega}_i(\alpha)\,, \qquad \text{(A16)}$$

these solutions are:

$$\hat{\omega}_0(\alpha) = \frac{\alpha(\alpha+3)+4}{(\alpha+1)(\alpha+2)^2(\alpha+3)}\,, \qquad \text{(A17)}$$

$$\hat{\omega}_1(\alpha) = -3\,\frac{\alpha(\alpha+3)+4}{(\alpha+2)^2(\alpha+3)^2}\,,$$

$$\hat{\omega}_2(\alpha) = 5\,\alpha\,\frac{\alpha(\alpha+3)+4}{(\alpha+2)^2(\alpha+3)^2(\alpha+4)}\,,$$

$$\hat{\omega}_3(\alpha) = 7\,\alpha(1-\alpha)\,\frac{\alpha(\alpha+3)+4}{(\alpha+2)^2(\alpha+3)^2(\alpha+4)(\alpha+5)}\,.$$

Again, the isotropic part, $\hat{\omega}_0(\alpha,\gamma)$, is in agreement with Eq. (A17) of Titarchuk & Lyubarskij (1995).
For high plasma temperatures one can expand the integrand of Eq. (A10) into a series. Using the following relation:

$$v^2\,\mathrm{d}v = \frac{1}{\gamma^3}\sqrt{1-\frac{1}{\gamma^2}}\,\mathrm{d}\gamma = \frac{1}{\gamma^3}\left(1-\frac{1}{2\gamma^2}+\ldots\right)\mathrm{d}\gamma \qquad \text{(A18)}$$

Eq. (A10) becomes:

$$\omega_i(\alpha,\Theta) = \frac{3}{4}\,\frac{1}{\Theta\,\mathrm{K}_2(1/\Theta)}\,\hat{\omega}_i(\alpha) \cdot \int_1^\infty \mathrm{d}\gamma\left(1-\frac{1}{2\gamma^2}\right)(2\gamma)^{2\alpha+2}\exp\left(-\frac{\gamma}{\Theta}\right). \qquad \text{(A19)}$$

The above integration can be expressed in terms of the incomplete Γ-function[4]. Then the coefficients $\omega_i(\alpha,\Theta)$ can be written:

$$\omega_i(\alpha,\Theta) = 3\frac{(2\Theta)^{2\alpha}}{\mathrm{K}_2(1/\Theta)}\,\hat{\omega}_i(\alpha) \cdot\left[\Theta^2\Gamma\left(2\alpha+3,\frac{1}{\Theta}\right)-\frac{1}{2}\Gamma\left(2\alpha+1,\frac{1}{\Theta}\right)\right]. \qquad \text{(A20)}$$

Together with Eqs. (A17) this defines the expansion coefficients (for $i = 0,1,2,3$) of the phase function for a high electron plasma temperature.

[4] $\Gamma(z,a) := \int_a^\infty t^{z-1}e^{-t}\mathrm{d}t$

A.3. The angular dependence of $J(\mu,\tau)$

The angular dependence of the intensity $J(\mu,\tau)$ can be expressed in the form of a series of Legendre polynomials with τ-dependent coefficients. Because of the symmetry of the boundary conditions, it is necessary only to calculate $J(\mu,\tau)$ for one half-space: $0 \le \mu \le 1$ and $-\tau_0 \le \tau \le \tau_0$. The intensity in the second half-space is then given by

$$J(-\mu,\tau) = J(\mu,-\tau)\,. \qquad \text{(A21)}$$

Having in mind the orthogonality relation for Legendre polynomials on the interval (e.g.) $0 \le \mu \le 1$:

$$\int_0^1 P_n(2\mu-1)P_k(2\mu-1)\mathrm{d}\mu = \frac{1}{2n+1}\delta_{nk}\,, \qquad \text{(A22)}$$

we expand the intensity as follows

$$J(\mu,\tau)\big|_{\mu\le 0} = \sum_{n=0}^{N}\frac{2n+1}{2}P_n(2\mu+1)Q_n^-(\tau)\,,$$

$$J(\mu,\tau)\big|_{\mu\ge 0} = \sum_{n=0}^{N}\frac{2n+1}{2}P_n(2\mu-1)Q_n^+(\tau)\,. \qquad \text{(A23)}$$

Defining the new variables ξ and ζ, according to:

$$\zeta := 2\mu+1\big|_{\mu<0}\,,$$

$$\xi := 2\mu-1\big|_{\mu>0}\,, \qquad \text{(A24)}$$

the inversion of Eq. (A23) can be written as

$$Q_n^-(\tau) = \int_{-1}^{1} J\left(\frac{\zeta-1}{2},\tau\right)P_n(\zeta)\mathrm{d}\zeta\,,$$

$$Q_n^+(\tau) = \int_{-1}^{1} J\left(\frac{\xi+1}{2},\tau\right)P_n(\xi)\mathrm{d}\xi\,. \qquad \text{(A25)}$$

Noting that $P_n(\mu) = (-1)^n P_n(-\mu)$, the symmetry condition (Eq. A21) gives

$$Q_n^-(\tau) = (-1)^n Q_n^+(-\tau)\,. \qquad \text{(A26)}$$

It is useful to rewrite the kernel $K(\mu,\mu_1)$ using a transformation of variables and to split it into two parts according to $\mu_1 < 0$ and $\mu_1 \ge 0$. This is done in the following way: written in the form of a scalar product, the right-hand side of Eq. (A4) reads:

$$K(\mu,\mu_1) = \boldsymbol{P}(\mu)\,\omega(\alpha,\Theta)\,\boldsymbol{P}(\mu_1) \qquad \text{(A27)}$$

with a diagonal, $M+1$–dimensional matrix $\omega(\alpha,\Theta)$ and vectors $\boldsymbol{P}$. Let W be the matrix, which transforms the Legendre polynomial as[5]

$$\boldsymbol{P}(2\mu_1-1) = \mathsf{W}\;\boldsymbol{P}(\mu_1)\,. \qquad \text{(A28)}$$

[5] The matrix W can be calculated from Eq. (A28) by expressing the vectors $\boldsymbol{P}(\mu_1)$ and $\boldsymbol{P}(2\mu_1-1)$ in the common basis $(1,\mu,\mu^2,\ldots,\mu^M)$ and inverting the matrix of coefficients of $\boldsymbol{P}(\mu_1)$.

Then, for $\mu_1 \geq 0$, one can write:

$$\begin{aligned} \omega(\alpha,\Theta)\,\boldsymbol{P}(\mu_1) &= \omega(\alpha,\Theta)\,\mathsf{W}^{-1}\,\mathsf{W}\,\boldsymbol{P}(\mu_1) \\ &=: \omega^+(\alpha,\Theta)\,\boldsymbol{P}(2\mu_1-1)\,, \end{aligned} \tag{A29}$$

with an $M+1$–dimensional, *non*-diagonal, matrix $\omega^+(\alpha,\Theta)$. An analogous transformation in the range $\mu_1 < 0$ leads to

$$\begin{aligned} K(\mu,\mu_1) = \boldsymbol{P}(\mu)\Big[\,&\omega^+(\alpha,\Theta)\,\boldsymbol{P}(2\mu_1-1)\,\mathrm{H}(\mu_1) \\ &+\omega^-(\alpha,\Theta)\,\boldsymbol{P}(2\mu_1+1)\,\mathrm{H}(-\mu_1)\Big]\,, \end{aligned} \tag{A30}$$

with the Heaviside function:

$$\mathrm{H}(\mu_1) = \begin{cases} 1 & : \quad \mu_1 \geq 0 \\ 0 & : \quad \mu_1 < 0 \end{cases}\;. \tag{A31}$$

The source function $B(\mu,\tau)$ (Eq. 9) becomes with this (and the use of Eq. (A25), and suppressing the dependence of $\omega^{\pm}$ on α and Θ):

$$B(\mu,\tau) = \frac{1}{4}\Big[\boldsymbol{P}(\mu)\,\omega^+\,\boldsymbol{Q}^+(\tau) + \boldsymbol{P}(\mu)\,\omega^-\,\boldsymbol{Q}^-(\tau)\Big]\,. \tag{A32}$$

Inserting this into the Boltzmann equation (Eq. 8), substituting $\xi = 2\mu - 1$ for $\mu \geq 0$, multiplying by $P_k(\xi)$, and integrating ξ from -1 to 1, we obtain

$$\begin{aligned} &\frac{\partial}{\partial\tau}\int_{-1}^{1} \mathrm{d}\xi\,\frac{\xi+1}{2}\,P_k(\xi)J\Big(\frac{\xi+1}{2},\tau\Big) = \\ &\quad\frac{1}{4}\sum_{i,j=0}^{N}\int_{-1}^{1}\mathrm{d}\xi\;P_k(\xi)P_i\Big(\frac{\xi+1}{2}\Big)\,\Big[\omega_{ij}^+\,Q_j^+(\tau) + \omega_{ij}^-\,Q_j^-(\tau)\Big] \\ &\quad-\int_{-1}^{1}\mathrm{d}\xi\,P_k(\xi)\,J\Big(\frac{\xi+1}{2},\tau\Big)\,. \end{aligned} \tag{A33}$$

The indices i,j,k run from 0 to N. If $N > M$ the matrices ω_{ij}^+ and ω_{ij}^- can be defined with all elements equal to 0 for $i,j > M$. If $N < M$ only the elements with $i,j \leq N$ are taken into account, so that in both cases the matrices formally become $N+1$–dimensional. Using the recurrence relation for Legendre polynomials,

$$\xi\,P_k(\xi) = \frac{k}{2k+1}\,P_{k-1}(\xi) + \frac{k+1}{2k+1}\,P_{k+1}(\xi)\,, \tag{A34}$$

the left hand side of Eq. (A33) can be written in terms of a tridiagonal matrix, whose elements are defined as:

$$Z_{kl} := \frac{k}{2k+1}\,\delta_{k,l+1} + \delta_{kl} + \frac{k+1}{2k+1}\,\delta_{k,l-1}\,. \tag{A35}$$

We further define a matrix with the elements:

$$S_{ki} := \frac{1}{2}\int_{-1}^{1}\mathrm{d}\xi\,P_k(\xi)P_i\Big(\frac{\xi+1}{2}\Big)\,. \tag{A36}$$

Using these definitions, and again Eq. (A25), Eq. (A33) becomes

$$\begin{aligned} \sum_{l=0}^{N} Z_{kl}\,\frac{\partial Q_l^+(\tau)}{\partial\tau} = &\sum_{i,j=0}^{N} S_{ki}\,\Big[\,\omega_{ij}^+ Q_j^+(\tau) + \omega_{ij}^- Q_j^-(\tau)\Big] \\ &-2\,Q_k^+(\tau)\,. \end{aligned} \tag{A37}$$

Having in mind the symmetry relation, Eq. (A26), and defining a diagonal matrix according to

$$E_{jm} := (-1)^j\,\delta_{jm}\,, \tag{A38}$$

Eq. (A37) reads in matrix form

$$\begin{aligned} \mathsf{Z}\,\frac{\partial\boldsymbol{Q}^+(\tau)}{\partial\tau} = &\,\mathsf{S}\,\Big[\,\omega^+\,\boldsymbol{Q}^+(\tau) + \omega^-\,\mathsf{E}\,\boldsymbol{Q}^+(-\tau)\Big] \\ &-2\,\boldsymbol{Q}^+(\tau)\,. \end{aligned} \tag{A39}$$

This can be expressed in somewhat shorter way, using the definitions

$$\begin{aligned} \mathsf{M} &:= -2\,\mathbb{1} + \mathsf{S}\,\omega^+\,, \\ \tilde{\mathsf{M}} &:= \mathsf{S}\,\omega^-\,\mathsf{E}\,. \end{aligned} \tag{A40}$$

Now the Boltzmann equation can be written as

$$\mathsf{Z}\,\frac{\partial\boldsymbol{Q}^+(\tau)}{\partial\tau} = \mathsf{M}\,\boldsymbol{Q}^+(\tau) + \tilde{\mathsf{M}}\,\boldsymbol{Q}^+(-\tau)\,, \tag{A41}$$

and represents a system of $N+1$ coupled differential equations.

For a given temperature in the non-relativistic regime ($\Theta \ll 1$), the moments $\langle v^k\rangle$ are to be calculated up to $k = L$. The matrix elements of M and $\tilde{\mathsf{M}}$ are then polynomials in α of maximal order L.

In case of high plasma temperature ($\Theta \gg 1$), the matrix elements of M and $\tilde{\mathsf{M}}$ depend on the incomplete Γ-function (see Eq. (A20)).

A.4. The spatial dependence of $J(\mu,\tau)$

Eq. (A41) is a system of coupled differential equations for the $N+1$ expansion coefficients of $J(\mu,\tau)$, which contain the τ-dependence of the intensity. In expanding these coefficients themselves into a series, the problem is reduced to an algebraic eigenvalue problem. A difference arises compared to the above treatment of the angular dependence in that we have to take into account the boundary condition, which is that no radiation enter the disk from outside:

$$J(\mu,\tau_0)\big|_{\mu<0} \equiv 0 \equiv J(\mu,-\tau_0)\big|_{\mu>0}\,. \tag{A42}$$

We introduce at this point a new variable according to

$$y(\tau) = \frac{\tau}{\tau_0}\,. \tag{A43}$$

Using this transformation and Eq. (A23), the boundary condition reads for all expansion coefficients:

$$\boldsymbol{Q}^+(-1) = 0 = \boldsymbol{Q}^-(1)\,. \tag{A44}$$

We chose an expansion of $\boldsymbol{Q}^{+}$ in Chebyshev polynomials according to

$$\boldsymbol{Q}^{+}(y) = \boldsymbol{q}_0 + \sum_{i=1}^{K} \left[1 - T_i(-y)\right] \boldsymbol{q}_i \,. \tag{A45}$$

with $\boldsymbol{Q}^{-}(y)$ given by the symmetry condition Eq. (A26). The boundary condition then gives $\boldsymbol{q}_0 = 0$. Inserting this into Eq. (A41) one gets

$$\frac{1}{\tau_0} \sum_{i=1}^{K} (-1)^{i+1} \frac{\partial T_i(y)}{\partial y} \mathsf{Z}\, \boldsymbol{q}_i = \sum_{i=1}^{K} T_i(y) \left[(-1)^{i+1} \mathsf{M} - \tilde{\mathsf{M}}\right] \boldsymbol{q}_i + \sum_{i=1}^{K} \left[\mathsf{M} + \tilde{\mathsf{M}}\right] \boldsymbol{q}_i \,. \tag{A46}$$

Multiplying this equation by $T_j(y)/\sqrt{1-y^2}$ and integrating y from -1 to 1, it becomes

$$\frac{1}{\tau_0} \sum_{i=1}^{K} (-1)^{i+1} \int_{-1}^{1} \mathrm{d}y \, \frac{T_j(y) \frac{\partial}{\partial y} T_i(y)}{\sqrt{1-y^2}} \mathsf{Z}\, \boldsymbol{q}_i = \sum_{i=1}^{K} \int_{-1}^{1} \mathrm{d}y \, \frac{T_j(y)\, T_i(y)}{\sqrt{1-y^2}} \left[(-1)^{i+1} \mathsf{M} - \tilde{\mathsf{M}}\right] \boldsymbol{q}_i + \sum_{i=1}^{K} \int_{-1}^{1} \mathrm{d}y \, \frac{T_j(y)}{\sqrt{1-y^2}} \left[\mathsf{M} + \tilde{\mathsf{M}}\right] \boldsymbol{q}_i \,. \tag{A47}$$

This is a vector equation with $K \times K$ matrices, defined by the integrals over Chebyshev polynomials. We denote these matrices by

$$(\mathsf{T}_\mathrm{d})_{ji} := \int_{-1}^{1} \mathrm{d}y \, \frac{T_j(y) \frac{\partial}{\partial y} T_i(y)}{\sqrt{1-y^2}} \,,$$

$$(\mathsf{T}_\mathrm{o})_{ji} := \int_{-1}^{1} \mathrm{d}y \, \frac{T_j(y) T_i(y)}{\sqrt{1-y^2}} \,,$$

$$(\mathsf{T}_\mathrm{h})_{ji} := \int_{-1}^{1} \mathrm{d}y \, \frac{T_j(y)}{\sqrt{1-y^2}} \,,$$

$$(\mathsf{E}_\mathrm{t})_{ik} := (-1)^{i+1} \, \delta_{ik} \,. \tag{A48}$$

The matrix elements are straightforwardly calculated using the orthogonality relation for Chebyshev polynomials:

$$\int_{-1}^{1} \mathrm{d}y \, \frac{T_j(y) T_i(y)}{\sqrt{1-y^2}} = \begin{cases} 0 & : \quad i \neq j \\ \pi/2 & : \quad i = j \neq 0 \\ \pi & : \quad i = j = 0 \end{cases} \,, \tag{A49}$$

the expression for the derivative:

$$\frac{\mathrm{d}T_i(y)}{\mathrm{d}y} = \frac{-i\, y\, T_i(y) + i\, T_{i-1}(y)}{(1-y^2)} \,, \tag{A50}$$

and the recurrence relation

$$T_{i+1}(y) = 2\, y\, T_i(y) - T_{i-1}(y) \,. \tag{A51}$$

The boundary condition $\boldsymbol{q}_0 = 0$ is not explicitly included in the above matrix equation, but must be inserted 'by hand' into Eq. (A47). This is done by extending the matrix equation to $i, j = 0, \ldots, K$, and adding the equations

$$\sum_{i=0}^{K} C_{ji}\, \boldsymbol{q}_i = 0 \tag{A52}$$

with

$$(\mathsf{C})_{ji} = \begin{cases} 1 & : \quad j = K \quad \text{and} \quad i = 0 \\ 0 & : \quad \text{otherwise} \end{cases} \,. \tag{A53}$$

All other matrices have to be set to 0 for $j = K$. The matrices are then given by

$$(\mathsf{T}_\mathrm{d})_{ji} = \begin{cases} i\,\pi & : \quad i > j \quad \text{and} \quad i + j = \text{odd} \\ 0 & : \quad \text{otherwise} \end{cases} \,,$$

$$(\mathsf{T}_\mathrm{o})_{ji} = \begin{cases} 0 & : \quad i \neq j \quad \text{or} \quad j = K \\ \pi/2 & : \quad i = j \neq 0 \quad \text{and} \quad j \neq K \\ \pi & : \quad i = j = 0 \end{cases} \,,$$

$$(\mathsf{T}_\mathrm{h})_{ji} = \begin{cases} \pi & : \quad j = 0 \\ 0 & : \quad \text{otherwise} \end{cases} \,. \tag{A54}$$

It is useful to write Eq. (A47) in a different way. It is a vector equation in which the elements are themselves vectors, multiplied by matrices. This form of matrix multiplication is just the outer product of matrices, denoted by $\otimes$. The $K+1$ vectors of dimension $N+1$ can be written as a common vector in a $(K+1) \cdot (N+1)$-dimensional product-space:

$$\underline{\boldsymbol{q}} := \begin{pmatrix} \boldsymbol{q}_0 \\ \boldsymbol{q}_1 \\ \vdots \\ \boldsymbol{q}_K \end{pmatrix} \,. \tag{A55}$$

Note that the vector $\underline{\boldsymbol{q}}$ does not have an index but includes *all* expansion coefficients of the angular and spatial dependence of $J(\mu, \tau)$. Eq. (A47) now reads

$$\left[\frac{1}{\tau_0} (\mathsf{E}_\mathrm{t}\, \mathsf{T}_\mathrm{d}) \otimes \mathsf{Z}\right] \underline{\boldsymbol{q}} = \left[\mathsf{C} \otimes \mathbb{1} + \mathsf{T}_\mathrm{h} \otimes (\mathsf{M} + \tilde{\mathsf{M}}) + (\mathsf{E}_\mathrm{t}\, \mathsf{T}_\mathrm{o}) \otimes \mathsf{M} - \mathsf{T}_\mathrm{o} \otimes \tilde{\mathsf{M}}\right] \underline{\boldsymbol{q}} \,. \tag{A56}$$

Let us denote the matrix on the left hand side, which does not depend on any parameter, by

$$\underline{\underline{\mathsf{A}}} := (\mathsf{E}_\mathrm{t}\, \mathsf{T}_\mathrm{d}) \otimes \mathsf{Z} \,, \tag{A57}$$

and the right hand matrix, which depends on α and Θ (due to $\omega^{\pm}(\alpha, \Theta)$ in M and $\tilde{\mathsf{M}}$) by $\underline{\underline{\mathsf{D}}}(\alpha, \Theta)$. Eq. (A56) reads with these definitions:

$$\left[\frac{1}{\tau_0} \underline{\underline{\mathsf{A}}} - \underline{\underline{\mathsf{D}}}(\alpha, \Theta)\right] \underline{\boldsymbol{q}} \equiv \underline{\underline{\mathsf{F}}}(\alpha, \Theta, \tau_0)\, \underline{\boldsymbol{q}} = 0 \,. \tag{A58}$$

This defines an algebraic eigenvalue problem for all of the expansion coefficients $\underline{\boldsymbol{q}}$.

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