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Correlated gravitational lensing of the cosmic microwave background

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Abstract. Cosmological inhomogeneities gravitationally deflect radiation propagating from distant sources, transforming the spatial and angular correlation functions of intrinsic source properties. For a gaussian distribution of deflections (e.g. from a primordial gaussian density perturbation spectrum or from the central limit theorem) we calculate the probability distributions for geodesic deviations. If the intrinsic variable is also gaussian, e.g. the large scale velocity flow field or cosmic microwave background temperature anisotropies, then distributions and correlation functions of the observed image sky properties can be obtained. Specialising to CMB temperature fluctuations we rederive simply the influence of independent gravitational lensing on the anisotropy angular correlation function and calculate the new effect of lensing correlated with the anisotropies, e.g. arising from the same primordial gravitational perturbation field. Characteristic magnitudes and scales are given in terms of the density power spectrum. The correlated deflection-temperature effect is shown to be negligible.

Key words: gravitational lensing – cosmic microwave background – cosmology: theory

1. Introduction

The origin and development of structure in the universe needs to be probed by means of observations of distant sources. Not only the direct properties of those sources, however, but also the characteristics induced in the radiation propagating through the gravitational inhomogeneities of the structure provide clues to the nature and formation of structure. Within linear theory, i.e. when the deviations from homogeneity are still perturbations, there are simple and intimate relations between the important cosmological descriptors of the matter, such as the large scale velocity flow field or the energy density fluctuations, and of radiation propagation characteristics such as temperature anisotropies (redshifts) and gravitational lensing deflections (momentum changes). These interrelationships constrain theories of structure formation.

This paper concentrates on the statistical effects of the inhomogeneities on observations, developing an analytic approach that requires only that the physical processes behind the observable and lensing quantities be gaussian in nature. This avoids sensitivity of results to a specific geometry or density perturbation model and is both physically relevant and surprisingly powerful. Although here we are primarily concerned with temperature anisotropies in the cosmic microwave background radiation (CMB), the method is easily generalised to investigations of the velocity flow field or density perturbations.

Before primordial CMB anisotropies were first detected by the Cosmic Background Explorer satellite (COBE; Smoot et al. 1992) there was much discussion in the literature concerning the effects of inhomogeneities and their gravitational lensing on the anisotropies (Blandford & Narayan 1992 provide a review with references). More recently Fukushige, Makino, and Ebisuzaki (1994) investigated this by calculating scattering with an N-body code, and Tomita (1996) considered the effects of superhorizon scale inhomogeneities. For the statistical distribution of the lensing deflections, in particular the correlation of neighboring lines of sight, crucial to the key property of geodesic deviation, various assumption have been made, ranging from a diffusion approximation to nearest neighbor correlations.

No previous work, however, dealt with the case where the deflections are not superimposed independently on the temperature fluctuations but are physically correlated with them. For example, consider a hot spot in the CMB. This corresponds to a region where the primordial gravitational potential deviates strongly from the mean but that potential also determines the strength of the lensing caused by the primordial density field from that region. Thus one can imagine that extremes of the temperature field are preferentially strongly lensed relative to the milder deviations, leading to a significant distortion of the intrinsic temperature correlation function.

This paper investigates the properties of cosmologically important gaussian fields including the possibility of such a cross correlation. The joint probability formalism allows for concise derivations of familiar results, e.g. beam smearing, from gravitational lensing while showing new effects from the cross correlation. By estimating the magnitude of these on anisotropy observations we can probe large scale structure in the universe by placing constraints on the density power spectrum P_k. The conditional probability formalism also points out ways to test the underlying gaussian nature of the primordial cosmological perturbations.

In Sect. 2 we review the mathematical basis and derive properties of joint and conditional probability distributions for the variables entering the problem – the relative deflections of the null geodesics and the intrinsic source sky characteristics. These are combined in Sect. 3 to form expressions for the observable correlation functions, especially under the condition of coherence between the source and propagation conditions. Section 4 relates the mathematical results to the underlying physics in terms of the density fluctuation power spectrum and Sect. 5 presents quantitatively the effects on CMB anisotropy measurements.

2. Probability distributions

In an imperfectly homogeneous universe radiation observed at some angular position $\boldsymbol{\phi}$ on the sky may have been deflected during propagation to us from a source whose true position is $\boldsymbol{\psi}$. By true position we mean that position at which the source would appear if the inhomogeneity (gravitational lens) were smoothed out. A general line of sight has

$$\boldsymbol{\phi} = \boldsymbol{\Psi} + \boldsymbol{\theta} \tag{1}$$

for a sky projected deflection angle $\boldsymbol{\theta}$, with $\boldsymbol{\theta}$ varying along different directions. The central question for observations in a universe with inhomogeneities is how to relate an intrinsic function of the source position to the observed function of image position.

In order to investigate this in as model independent a way as possible, we assume that the gravitational lensing deflection and the physical source variables of interest individually have d dimensional gaussian distributions with zero mean (no preferred direction). Then

$$p(\boldsymbol{v}) = [\pi(2\sigma^2/d)]^{-d/2} e^{-v^2/(2\sigma^2/d)}, \tag{2}$$

where the variance $\sigma^2 = \langle v^2 \rangle$. One justification for adopting a gaussian is that if the variable, e.g. projected deflection $\boldsymbol{\theta}$, is compounded out of many elements, e.g. isolated spatial deflections at different distances from the observer (many inhomogeneities along the line of sight), then the central limit theorem leads to a gaussian distribution. Another rationale enters if the underlying physical process has a gaussian nature, generated by quantum fluctuations for example. One case would be gaussian density perturbations which create gaussian distributions in lensing deflection, velocity fluctuations, and temperature anisotropies in the linear regime. This relation is discussed further in Sect. 4.

Since absolute shifts in the source position are undetectable we must compare at least two light rays, which requires knowing the joint probability of deflections along lines of sight 1,2,.... More generally one can consider the variables $\boldsymbol{v}_i$ to denote different physical processes, such as density or temperature fields. Using the characteristic function approach (Cramér 1957) the joint probability of occurence is found to be

$$p(\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n) = (2\pi/d)^{-nd/2} M^{-d/2} \times \exp\Big\{-\frac{d}{2M}\sum_{i,j} M_{ij}\boldsymbol{v}_i\cdot\boldsymbol{v}_j\Big\}, \tag{3}$$

where M and M_{ij} are the determinant and cofactors, respectively, of the covariance matrix composed of elements $m_{ij} = \langle \boldsymbol{v}_i \cdot \boldsymbol{v}_j \rangle$. Angle brackets denote integration over the joint probability distribution.

In the case of two sky variables ($n = d = 2$), for example comparing deflections of two rays, (3) simplifies to

$$p(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = \left[\pi^2\sigma_1^2\sigma_2^2(1-b^2)\right]^{-1} \times e^{-[(\theta_1^2/\sigma_1^2)-2b(\boldsymbol{\theta}_1\cdot\boldsymbol{\theta}_2/\sigma_1\sigma_2)+(\theta_2^2/\sigma_2^2)]/(1-b^2)}, \tag{4}$$

where the correlation coefficient $b = \langle \boldsymbol{\theta}_1 \cdot \boldsymbol{\theta}_2 \rangle / \sigma_1\sigma_2$. This distribution is normalized, and integration over one variable recovers the gaussian distribution for the other variable. As $b \to 0$ (incoherence or unrelated variables) the distribution separates into two independent gaussians while as $b \to 1$ (total coherence) it approaches a delta function $\delta(\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1)$.

One can imagine physical situations in which the variances σ_1, σ_2 of the two variables $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2$ are not identical even though they correspond to the same physical process, e.g. gravitational deflection. For example if the universe were not homogeneous then lines of sight through different patches of the universe might have different statistical properties. Even with homogeneity, consider sources at different redshifts. Because of the differing path lengths the cumulative gravitational deflections and hence variances σ_1, σ_2 would be unequal, as is evident from the explicit physical expressions (34) and (36). If we concentrate on the source being the CMB, however, the variances of deflections along any lines of sight are equal: $\langle\theta_1^2\rangle = \langle\theta_2^2\rangle = \sigma_\theta^2$.

For other physical situations and gaussian processes, such as the large scale velocity field, evaluation out to different distances can be of interest, e.g. windowing different volumes of the density field or slicing redshift surveys. Then, for example, one can find probability distribution functions such as for the ratio of the variable value in sample 1 to that in sample 2, $k = (\theta_1/\sigma_1)/(\theta_2/\sigma_2)$:

$$p(k) = 2(1-b^2)\, k^{-1}(k+k^{-1})\, [(k+k^{-1})^2 - 4b^2]^{-3/2}. \tag{5}$$

As expected, the differential probability $p(k)\, dk$ is invariant under transforming k to its reciprocal. One can also calculate angular distributions, e.g. $p(\mu)$, where $\mu = \boldsymbol{\theta}_1 \cdot \boldsymbol{\theta}_2/\theta_1\theta_2$. But this is of more use when dealing with three dimensional vector variables such as $\boldsymbol{v}_R$, the velocity flow field out to some survey depth R.

Of particular physical interest is the difference between a quantity evaluated along two different lines of sight, e.g. the relative deflection or deviation $\boldsymbol{D} = \boldsymbol{\theta}_2 - \boldsymbol{\theta}_1$. Similarly one can investigate the coherence of the variables, e.g. $\beta = \boldsymbol{\theta}_1 \cdot \boldsymbol{\theta}_2$. Both are important to the physics. From (1) the observed separation of two rays $\boldsymbol{\Phi} = \boldsymbol{\phi}_2 - \boldsymbol{\phi}_1$ is

$$\boldsymbol{\Phi} = \boldsymbol{\psi} + \boldsymbol{D}, \tag{6}$$

where $\boldsymbol{\psi} = \boldsymbol{\Psi}_2 - \boldsymbol{\Psi}_1$ is the intrinsic (true) separation and the separations can represent either the angular distance between two sources or the distance within a source (e.g. source size). Thus $\boldsymbol{D}$ is involved in the transformation of field separations and the quantities that depend on them. The coherence variable β measures how independent the deflections are. Physically, we expect that neighboring rays have strongly correlated deflections while lines of sight far apart have independent deflections.

To find the statistical distribution of the deviation and coherence one changes variables in (4) to, say, $\boldsymbol{D}$, β, and $h = \theta_2^2 - \theta_1^2$, employing the Jacobian of the transformation:

$$d^4P(\boldsymbol{D},\beta,h) = \mathscr{J}^{-1}\left(\frac{\boldsymbol{D},\beta,h}{\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}\right) p(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)$$
$$= (\pi\sigma_\theta)^{-2}Q^{-1}e^{-D^2/Q}e^{-2(1-b)\beta/Q} \times(D^4+4\beta D^2-h^2)^{-1/2}\; d^2\boldsymbol{D}\,d\beta\,dh, \quad (7)$$

with $Q = \sigma_\theta^2(1-b^2)$. From this new joint distribution one derives the individual distributions by integrating over the other variables:

$$p(\boldsymbol{D}) = \left[2\pi\sigma_\theta^2(1-b)\right]^{-1}e^{-D^2/2\sigma_\theta^2(1-b)}\,, \quad (8)$$

$$p(\beta) = \begin{cases}\sigma_\theta^{-2}e^{-2\beta/\sigma_\theta^2(1+b)}, & \beta\geq 0,\\ \sigma_\theta^{-2}e^{2\beta/\sigma_\theta^2(1-b)}, & \beta\leq 0,\end{cases} \quad (9)$$

$$p(h) = \left[2\sigma_\theta^2\sqrt{1-b^2}\right]^{-1}e^{-|h|/\sigma_\theta^2\sqrt{1-b^2}}\,. \quad (10)$$

Equations (8–10) illustrate that the distribution of a linear combination of gaussian variables is itself gaussian (Cramér 1957); note only $\boldsymbol{D}$ is such a linear function. The magnitude, or modulus, of a gaussian distributed vector quantity is Rayleigh distributed,

$$p(D) = 2\langle D^2\rangle^{-1}D\,e^{-D^2/\langle D^2\rangle}. \quad (11)$$

We can find the moments of our chosen variables:

$$\langle\boldsymbol{D}\rangle = 0 = \langle\boldsymbol{D}^{2n+1}\rangle\,;\;\; \langle D^2\rangle = 2\sigma_\theta^2(1-b)\,; \quad \langle D^{2n}\rangle = n!\,\langle D^2\rangle^n, \quad (12)$$

$$\langle\beta\rangle = b\sigma_\theta^2\,; \quad \langle\beta^n\rangle = 2^{-(n+1)}n!\,\sigma_\theta^{2n}\left[(1+b)^{n+1}+(-)^n(1-b)^{n+1}\right], \quad (13)$$

and verify that they have the expected behavior in the limits $b = 0$ and $b = 1$. The above use of relative and coherence variables, Jacobians, and moments is equally applicable to variables besides the ray deflection.

Note that since by (6) the observed separation on the sky depends only on the intrinsic separation and $\boldsymbol{D}$, we can write correlations of position dependent quantities using only $p(\boldsymbol{D})$ and not the full $p(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)$. For example for any function f only of the relative field positions, e.g. separation (which is all that can enter in a globally homogeneous universe),

$$\int d^2\boldsymbol{\theta}_1\int d^2\boldsymbol{\theta}_2\,f(\boldsymbol{\theta}_2-\boldsymbol{\theta}_1)\,p(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)$$
$$= \int d^2\boldsymbol{A}\int d^2\boldsymbol{D}\,f(\boldsymbol{D})\,p(\boldsymbol{D},\boldsymbol{A})$$
$$= \int d^2\boldsymbol{D}\,f(\boldsymbol{D})\,p(\boldsymbol{D}), \quad (14)$$

where, say, $\boldsymbol{A} = (\boldsymbol{\theta}_1+\boldsymbol{\theta}_2)/2$.

We can immediately see that for the transformation of an intrinsic sky function C_0 of only the source separation ψ into an observed function C of the image separation Φ,

$$C(\Phi) = \int d^2\boldsymbol{D}\int d^2\boldsymbol{\psi}\,C_0(\psi)\,p(\boldsymbol{D})\,\delta(\boldsymbol{\Phi}-\boldsymbol{\psi}-\boldsymbol{D})$$
$$= \int d^2\boldsymbol{\psi}\,C_0(\psi)\,p(\boldsymbol{D}=\boldsymbol{\Phi}-\boldsymbol{\psi})$$
$$= (\pi S^2)^{-1}\int d^2\boldsymbol{\psi}\,C_0(\psi)\,e^{-(\boldsymbol{\Phi}-\boldsymbol{\psi})^2/S^2}$$
$$= 2S^{-2}\int_0^\infty d\psi\,\psi\,C_0(\psi)\,e^{-(\psi^2+\Phi^2)/S^2}\,I_0(2\psi\Phi/S^2), \quad (15)$$

where I_0 is a modified Bessel function and $S^2 = \langle D^2\rangle$.

This recreates Eq. (24) of Wilson & Silk (1981) for beam smearing, or scattering due to the observer and hence independent of source and path properties. In this case S is just a constant corresponding to the dispersion of the telescope beam response. The formalism epitomized by (15) also provides a compact method for deriving Eq. (19) of Linder (1990a) for the independent lensing of the microwave background temperature anisotropy correlation function. Here the path properties do enter but are uncorrelated with the intrinsic source characteristics observed. Then S will be a function of path length and field separation Φ, due to the different physical conditions along different lines of sight.

The peak value of the correlation function, at $\Phi = 0$, will be diluted by a factor $1 + S^2/(2\Theta^2)$, where Θ is the characteristic scale, or coherence angle, of the observed quantity (cf. Eq. 18). Any oscillations in the angular behavior $C_0(\psi)$ will be damped by the coarse graining of the integral. Observers seek to have beam sizes much smaller than the anisotropy scale of interest, $S \ll \Theta$, in order to avoid this smearing. In the independent lensing case the zero lag correlation is unaffected since at $\Phi = 0$ the lines of sight are identical, forcing $S = 0$ ($b = 1$). Smearing does enter at larger angles though (cf. Linder 1990a, Figs. 2 and 3), its importance going as $S^2/\Theta^2 \sim (\sigma_\theta/\Theta)^2(1-b)$.

However, (15) only holds if the lensing process is independent of the observed process, i.e. detected characteristic, so that the separation into $C_0(\psi)\,p(\boldsymbol{D})$ is possible. Section 3 provides more detailed discussion of this point and treats the correlated case, correcting (15) to (24), (25). Meanwhile let us continue examining the necessary variables.

The other gaussian variable to enter our correlated microwave background calculation is the fractional temperature anisotropy $t_i = \Delta T(\boldsymbol{\phi}_i)/T$. This is a scalar variable ($d = 1$) on the sky. Hence its probability function is

$$p(t_1,t_2) = (2\pi\sigma_t^2\sqrt{1-B^2})^{-1}e^{-(t_1^2-2Bt_1t_2+t_2^2)/2\sigma_t^2(1-B^2)}, \quad (16)$$

where the variance $\sigma_t^2 = \langle t^2 \rangle$ and the correlation parameter $B = \langle t_1 t_2 \rangle / \sigma_t^2$. In a more explicit notation,

$$B\sigma_t^2 = \left\langle \frac{\Delta T}{T}(\phi_1)\,\frac{\Delta T}{T}(\phi_2) \right\rangle_{\Phi=|\phi_1-\phi_2|} = C(\Phi), \tag{17}$$

the familiar angular anisotropy correlation function. In this notation,

$$\begin{aligned} \sigma_t^2 &= \langle t^2 \rangle = C(0) \\ B &= \langle t_1 t_2 \rangle / \sigma_t^2 = C(\Phi)/C(0) \\ \Theta &\equiv [-C''(0)/C(0)]^{-1/2} = (-d^2 B(0)/d\Phi^2)^{-1/2}, \end{aligned} \tag{18}$$

relating the correlation function C and the coherence angle or curvature Θ to the probability function characteristics σ_t and B. Because the temperature is a random process evaluated at the same last scattering surface regardless of line of sight, the variance σ_t^2 is not dependent on direction (cf. Eq. 31).

While B or $C(\Phi)$ gives the temperature anisotropy correlations averaged over the sky, i.e. the rms value, one can also calculate the entire probability distribution of the correlations from (16). For the coherence variable $c = t_1 t_2$ this is

$$p(c) = \left(\pi\sigma_t^2\sqrt{1-B^2}\right)^{-1} K_0\left[|c|/\sigma_t^2\left(1-B^2\right)\right] \times e^{cB/\sigma_t^2(1-B^2)}, \tag{19}$$

where K_0 is a modified Bessel function. This is important for computing the cosmic variance or intrinsic error due to the fact that we observe only a single universe, i.e. one realization of the probability distribution. While the mean $\langle c \rangle = \sigma_t^2 B = C$, the variance is

$$\sigma_c^2 = \langle c^2 \rangle - \langle c \rangle^2 = \sigma_t^4(1+B^2). \tag{20}$$

Note that the distribution of c is not gaussian.

Observations of the CMB sky temperature can be related to C according to the geometry of the fields, i.e. the number of telescope beams and the angles between them. This can be made explicit by defining the differential variable $r = t_2 - t_1$ with gaussian distribution

$$p(r) = [4\pi\sigma_t^2(1-B)]^{-1/2} e^{-r^2/4\sigma_t^2(1-B)}. \tag{21}$$

A two-beam experiment with throw Φ measures the variance

$$\langle r^2 \rangle = 2\sigma_t^2(1-B_\Phi) = 2\,[C(0) - C(\Phi)], \tag{22}$$

while a three-beam one symmetric about the center field measures the variance

$$\begin{aligned} \langle R^2 \rangle &= \langle (r_a - r_b)^2 \rangle = 2\langle r^2 \rangle (1 - b_{ab}) \\ &= 2\sigma_t^2\,(3 - 4B_\Phi + B_{2\Phi}) \\ &= 6C(0) - 8C(\Phi) + 2C(2\Phi), \end{aligned} \tag{23}$$

where indices a, b represent the two pairs of fields. Note the similarity of form between (22) and (23): $\sigma_t^2 = \langle t^2 \rangle \to \langle r^2 \rangle$, $B\sigma_t^2 = \langle t_1 t_2 \rangle \to \langle r_a r_b \rangle = b_{ab}\langle r^2 \rangle$.

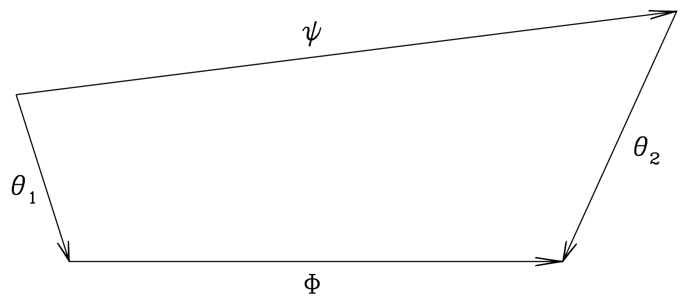

Fig. 1. The geometry of the intrinsic field separation vector $\boldsymbol{\psi}$, the deflection vectors $\boldsymbol{\theta}$ for the two lines of sight, and the observed separation vector $\boldsymbol{\Phi}$ is shown in the plane of the sky

Rather than obtaining only the variance one could, if desired, compute other quantities such as the conditional probability of obtaining a certain value in one set of fields given that a fixed value was observed in other fields. This could be useful in that discrepancies between observations and probability theory could be a possible probe of nongaussianness in the temperature fluctuation distribution since only gaussian processes are fully determined by the variance.

In summary, for the physical situation of gravitational inhomogeneity deflection of microwave background radiation we are interested in the mathematical correlations between two two dimensional gaussian variables – one (the sky deflection $\boldsymbol{\theta}$) a vector and one (the temperature fluctuations t) a scalar. We will see that it is in fact convenient to decompose the vector deflection into arbitrarily oriented but orthogonal components, thus giving a total of three variables per field, all magnitudes.

3. Cross correlations

The expression for the observed CMB temperature anisotropy correlation function will involve both the intrinsic temperature correlations and the conditional probability distribution of deflections given that the intrinsic fields have certain temperature fluctuations, i.e. values of the gravitational potential field.

Mathematically, the intrinsic correlation function is defined by the integral of the field temperatures weighted by the joint probability of obtaining those particular values in each field,

$$C_0(\psi) = \int dt_1 \int dt_2\, p(t_1, t_2)\, t_1 t_2, \tag{24}$$

where $\boldsymbol{\psi}$ is the field separation. To include the gravitational deflections one simply maps the intrinsic sky coordinates onto the observed ones because quasistatic gravitational lensing induces no redshift, i.e. the photon energy and hence temperature is unaffected. The geometry of this mapping is illustrated in Fig. 1.

Now to calculate the observed correlation function one employs Bayes' theorem and simply adds up all the ways in which the fixed observed field separation $\boldsymbol{\Phi}$ can be generated from the various intrinsic field separations $\boldsymbol{\psi}$. That is, one sums over all possible quadrilaterals with one fixed side, weighted by the probability for achieving that shape, i.e. obtaining deflections of the necessary magnitudes and directions, given the physical conditions represented by t_1, t_2.

So the observed correlation function becomes

$$C(\Phi) = \int d^2\boldsymbol{D} \int d^2\boldsymbol{\psi} \int dt_1 \times \int dt_2\, p(t_1,t_2)\, t_1 t_2\, p(\boldsymbol{D}|t_1,t_2)\, \delta^{(2)}(\boldsymbol{\Phi} - \boldsymbol{\psi} - \boldsymbol{D}), \quad (25)$$

where $\boldsymbol{D} = \boldsymbol{\theta}_2 - \boldsymbol{\theta}_1$ is the relative deflection mapping $\boldsymbol{\psi} \to \boldsymbol{\Phi}$, the delta function enforces closure of the quadrilateral, and $p(\boldsymbol{D}|t_1,t_2)$ is the conditional probability. This is a direct translation of the above words into an equation. For example the phrase "given the physical conditions" indicates that the probability function of all the variables does not range over the whole parameter space but is constrained by some prior knowledge or condition; hence one uses the conditional probability distribution to reflect this. Now conditional probability is calculated by the full probability divided by the probability to achieve the set condition. Hence we can convert the product of the temperature probability function $p(t_1,t_2)$ and the conditional deflection probability function given those temperature values $p(\boldsymbol{D}|t_1,t_2)$ to a joint probability over all the variables $p(\boldsymbol{D},t_1,t_2)$.

However, the expression is still not that useful for comparison with observations as it requires an integration over the ensemble of probability realizations, yet we have only a single universe to work with. The solution is to employ the ergodic hypothesis which states that we can substitute an integration over the whole sky for an ensemble average. Thus we can replace integration over all the values which the random process could take on at a location by integration over all evaluation locations.

The final form of the observed correlation function is then

$$\begin{aligned} C(\Phi) &= \int d^2\boldsymbol{\psi} \int dt_1 \int dt_2\, t_1 t_2\, p(\boldsymbol{D}{=}\boldsymbol{\Phi}-\boldsymbol{\psi}, t_1, t_2) \\ &= \int d^2\boldsymbol{\psi} \int d^2\boldsymbol{\psi}_1\, t(\boldsymbol{\psi}_1)\, t(\boldsymbol{\psi}_1+\boldsymbol{\psi})\, p(\boldsymbol{D}{=}\boldsymbol{\Phi}-\boldsymbol{\psi}, t_1, t_2), \quad (26) \end{aligned}$$

enforcing $\boldsymbol{\psi}_2 = \boldsymbol{\psi} - \boldsymbol{\psi}_1$ in the second line. Only in the case where the deflection process is independent of the temperature fluctuations does the joint probability separate into a product of $p(\boldsymbol{D})\, p(t_1,t_2)$, necessary to obtain the explicit appearance of the intrinsic correlation function $C_0(\psi)$ in (15).

To obtain the joint probability function between the gaussian variables of the ray deflections and the sky temperatures, it is convenient to decompose $\boldsymbol{D}$ into orthogonal components. This both puts all variables on an equal dimensional footing and makes the correlation $\langle D_x D_y \rangle = 0$. By using the relative variable $\boldsymbol{D} = \boldsymbol{\theta}_2 - \boldsymbol{\theta}_1$ we no longer explicitly need the deflection correlation parameter $\beta = \boldsymbol{\theta}_1 \cdot \boldsymbol{\theta}_2$. In Sect. 4 we find that only the component of $\boldsymbol{D}$ along the field separation has a nonvanishing correlation with the field temperatures, so we choose our decomposition parallel and perpendicular to this axis.

Now the joint probability function for the variables $D_\parallel, D_\perp, t_1, t_2$ follows from (3) via the covariance matrix,

$$\mathbf{M}(D_\parallel, D_\perp, t_1, t_2) = \begin{pmatrix} s^2 & 0 & -bs\sigma & bs\sigma \\ 0 & s^2 & 0 & 0 \\ -bs\sigma & 0 & \sigma^2 & B\sigma^2 \\ bs\sigma & 0 & B\sigma^2 & \sigma^2 \end{pmatrix}. \quad (27)$$

One can read off from the matrix that s^2 is the variance of each component of the deflection $\boldsymbol{D}$, σ^2 is now the variance of the temperature fluctuations, B is the temperature correlation parameter, and b is now the cross correlation parameter between the gravitational deflection and the temperature fluctuation.

The joint probability distribution is then

$$\begin{aligned} p(D_\parallel, D_\perp, t_1, t_2) &= (2\pi s\sigma)^{-2}(1+B)^{-1/2}(1-B-2b^2)^{-1/2} \\ &\quad \times \exp\{-E/[2s^2\sigma^2(1+B)(1-B-2b^2)]\}, \\ E &= s^2(1-b^2)(t_1^2+t_2^2) \\ &\quad - 2s^2(B+b^2)\, t_1 t_2 - 2bs\sigma(1+B)\, D_\parallel (t_2 - t_1) \\ &\quad + \sigma^2(1-B^2)\, D_\parallel^2 + \sigma^2(1+B)(1-B-2b^2)\, D_\perp^2. \end{aligned} \quad (28)$$

Carrying out the integrations specified in (26) yields the observed CMB temperature anisotropy correlation function

$$\begin{aligned} C(\Phi) &= (2\pi s^2)^{-1} \int d^2\boldsymbol{\psi}\, C_0(\psi)\, e^{-|\boldsymbol{\Phi}-\boldsymbol{\psi}|^2/2s^2} \\ &\quad + (2\pi s^2)^{-1} \int d^2\boldsymbol{\psi}\, b^2\, C_0(0) \\ &\quad \times \left[1 - \{(\boldsymbol{\Phi}-\boldsymbol{\psi})\cdot\hat{\psi}\}^2/s^2\right] e^{-|\boldsymbol{\Phi}-\boldsymbol{\psi}|^2/2s^2} \\ &\equiv C_{ind} + C_{cor}. \end{aligned} \quad (29)$$

The first term is proportional to $B = C_0(\psi)/\sigma^2$ and is the contribution to the observed temperature correlations when the gravitational lensing is independent of the temperature anisotropies, i.e. there is no cross correlation. This is precisely the old result derived in Linder (1990a) and mentioned in (15). The second term, involving b, represents the new effect of including correlations between the gravitational deflection field and the temperature fluctuation field, for example due to both originating from the same primordial gravitational potential perturbation.

The two crucial ingredients are thus the correlations B [or $C_0(\psi)$] and b. The former is a scalar function of field separation, i.e. independent of orientation, and from its definition one expects the latter to vary with angle as $\hat{\Phi}\cdot\hat{\psi}$; both are verified in the next section. Then the angular part of the integrals can be carried out without knowing their explicit form:

$$\begin{aligned} C(\Phi) &= s^{-2} e^{-\Phi^2/2s^2} \int_0^\infty d\psi\, \psi\, [C_0(\psi)\, I_0(\Phi\psi/s^2) \\ &\quad + C_0(0)\, \beta^2(\psi,\Phi)\, Y]\, e^{-\psi^2/2s^2} \\ Y &= [(\Phi^{-1} + 2\Phi s^{-2})\,\psi \\ &\quad + (2\Phi + 4s^2\Phi^{-1})\,\psi^{-1} + 6s^4\Phi\,\psi^{-3}]\, I_1(\Phi\psi/s^2) \\ &\quad - [s^{-2}\psi^2 + (1+\Phi^2 s^{-2}) + 3s^2\psi^{-2}]\, I_0(\Phi\psi/s^2), \end{aligned} \quad (30)$$

where I_ν are modified Bessel functions and $\beta = b/(\hat{\Phi}\cdot\hat{\psi})$.

The prescriptions for the variable variances and correlation parameters are given by the physics underlying the random processes and can be written in terms of the gravitational potential perturbations, or more conventionally the primordial density fluctuation power spectrum, as in the next section.

4. Density fluctuation spectrum

The mathematical basis of the statistical moment and correlation analysis relied on the gaussian nature of the observational characteristics. This implied that all the physics resided in the variance and cross correlation of the characteristic variables. Now we investigate those physical processes to obtain expressions for these quantities in terms of the underlying mechanism – fluctuations in the cosmological density field – expressed in terms of the density power spectrum.

First consider the source property of the cosmic microwave radiation background temperature anisotropies. The temperature fluctuations, or shifts in the photon energies, are induced by variations in the gravitational potential at the last scattering surface, $t = \phi/3$, (variations along the photon path contribute negligibly due to their quasistatic nature). In turn, the potential can be related to the density perturbations via the Poisson equation, reading $\phi_k = (3/2)H_0^2\, k^{-2}\delta_k$ in Fourier space. Thus the variance of the temperature fluctuations can be written in terms of the density power spectrum $P_k = |\delta_k|^2$ as

$$\sigma^2 = \langle t^2\rangle = C_0(0) = (8\pi^2)^{-1}H_0^4\int_0^\infty dk\,k^{-2}P_k. \tag{31}$$

The correlation parameter is

$$B = \langle t_1t_2\rangle/\langle t^2\rangle = \int_0^\infty dk\,k^{-2}\frac{\sin kL\Phi}{kL\Phi}P_k\Big/\int_0^\infty dk\,k^{-2}P_k, \tag{32}$$

where $L = 2H_0^{-1}[1-(1+z)^{-1/2}]$ is the comoving distance to the last scattering surface at redshift z. (The small angle approximation $\Phi \ll 1$ is used throughout.) From (18) the coherence angle is then

$$\Theta_t^2 = 3L^{-2}\int_0^\infty dk\,k^{-2}P_k\Big/\int_0^\infty dk\,P_k, \tag{33}$$

or $\Theta_t \approx 1/(kL)$.

For the ray deflection variables, the physical basis is gravitational lensing due to the density fluctuations along the line of sight. Following Linder (1990b) the geodesic equation gives

$$\boldsymbol{\theta} = 2L^{-1}\int_0^L dx\int_0^x dx'\,\boldsymbol{\nabla}_\perp\phi, \tag{34}$$

i.e. the deflection is caused by gradients in the gravitational potential. The equation of geodesic deviation governs the relative deflection, or deviation, of neighboring rays:

$$\boldsymbol{D} = \boldsymbol{\theta}_2 - \boldsymbol{\theta}_1 = 2L^{-1}\int_0^L dx\int_0^x dx'\,x'\,(\boldsymbol{\Phi}\cdot\boldsymbol{\nabla}_\perp)\,\boldsymbol{\nabla}_\perp\phi. \tag{35}$$

Converting the gravitational potential to the density power spectrum as before, by $\phi = (2\pi)^{-3/2}\int d^3k\,\phi_k e^{i\mathbf{k}\cdot\mathbf{x}} = (3/2)(2\pi)^{-3/2}H_0^2\int d^3k\,k^{-2}\delta_k e^{i\mathbf{k}\cdot\mathbf{x}}$, one then carries out the derivatives and integrations in (34) and (35). To calculate a variance one does the angle and ensemble averages as represented by the angle brackets to obtain for the variance of a single ray deflection

$$\sigma_\theta^2 = \langle\theta^2\rangle = (3/2\pi)H_0^4L\int_0^\infty dk\,k^{-1}P_k, \tag{36}$$

and for the variance of the ray bundle deviation

$$S^2 = \langle D^2\rangle = 2s^2 = 2\sigma_\theta^2\,(1-b_\theta), \tag{37}$$

in agreement with (12). The deflection correlation parameter comes from (35)-(37):

$$\begin{aligned} b_\theta\sigma_\theta^2 &= \int_0^\infty dk\,k^{-1}P_k\int d^2\boldsymbol{\Omega}\,f(\boldsymbol{\Omega},\boldsymbol{\Omega}',kL) \\ &= (3/2\pi)H_0^4L\int_0^\infty dk\,k^{-1} \\ &\quad\times\left[1-(kL\Phi)^2/40+\mathscr{O}(kL\Phi)^4\right]P_k, \end{aligned} \tag{38}$$

where the angular variables $\boldsymbol{\Omega},\boldsymbol{\Omega}'$ of $\boldsymbol{k}$ relative to the two lines of sight are related by a rotation of the axes by angle Φ. For example the direction cosine of the wave vector with respect to the second line of sight is $\mu' = \mu\cos\Phi + \sqrt{1-\mu^2}\sin\Phi\cos\beta$ with μ the cosine with respect to the first line of sight and β the azimuthal angle in the first frame.

Unfortunately the general evaluation of (38) is not analytically amenable but the second line of the equation shows the behavior for angles $\Phi \ll (kL)^{-1}$. This is sufficient to find the coherence angle and other interesting physical parameters like the gravitational lensing ray crossing variable $\mathscr{D}^2 = \lim_{\Phi\to 0}\langle D^2\rangle/\Phi^2 = (3/40\pi)H_0^4L^3\int dk\,k\,P_k$. When $\mathscr{D}^2 > 1$ then the Jacobian of the sky-image map goes singular, rays cross, and the image appears grossly distorted or multiple. The coherence angle of the lensing deflections is defined analogously to (33),

$$\Theta_\theta^2 = 20L^{-2}\int_0^\infty dk\,k^{-1}P_k\Big/\int_0^\infty dk\,kP_k, \tag{39}$$

or $\Theta_\theta \approx 1/(kL)$.

For the cross correlation parameter b one takes $t = \phi/3$ and (35) for $\boldsymbol{D}$, both involving Fourier transforms of ϕ_k and hence δ_k, and carries out the same ensemble average calculational process. This gives

$$\begin{aligned} bs\sigma = \langle D_\parallel t\rangle &= -(3/4\pi^2)H_0^4\Phi\alpha\int_0^\infty dk\,k^{-2}P_k \\ &\quad\times[1-(1/2)k^2L^2\psi^2+\mathscr{O}(kL\psi)^4], \end{aligned} \tag{40}$$

with $\alpha = \hat{\Phi}\cdot\hat{\psi}$ and again the general behavior unamenable. In the limit of small field separation $s \sim \Phi$ (see Eqs. 37, 38) so

$$b \approx -(120/\pi)^{1/2}\alpha L^{-3/2}\left[\int_0^\infty dk\,k^{-2}P_k\Big/\int_0^\infty dk\,kP_k\right]^{1/2}. \tag{41}$$

For the larger angle behavior one can turn to the more physical intuitive (though not more calculationally tractable) expression

$$b^2 = \left[\langle D^2t^2\rangle - \langle D^2\rangle\langle t^2\rangle\right]/(2s^2\sigma^2) = \langle D^2t^2\rangle/\langle D^2\rangle\langle t^2\rangle - 1, \quad (42)$$

which follows from the joint probability function. This shows that indeed b vanishes as the deflections and temperature fluctuations become uncorrelated.

To obtain specific quantitative results for the observed microwave background temperature anisotropy correlation function requires becoming more model dependent, choosing a particular density power spectrum. We illustrate the orders of magnitude for the variables and the behavior of the resulting correlation function by adopting a cold dark matter spectrum normalized to the detected microwave background quadrupole anisotropy, $P_k = 1.8\times10^{-9}(a_2/2.2\times10^{-5})^2 H_0^{-4}\, k/[1+l_1k+l_2k^{3/2}+l_3k^2]^2$, where l_1, l_2, l_3 are length constants (Davis et al. 1985). The results are $\sigma = 1.1\times10^{-5}$, $\Theta_t = 14'$, $\sigma_\theta = 3'$, $\Theta_\theta = 5'$, and $b = -2\times10^{-4}$. This value for b is at $\Phi = 0$ but its maximum at finite Φ is only a few times larger. In terms of approximating the coherence angles by $(kL)^{-1}$, the characteristic wavenumber is $k \approx (15\,\mathrm{h}^{-1}\,\mathrm{Mpc})^{-1}$.

5. Conclusion

We have investigated the behavior of the observed CMB temperature anisotropy correlation function in a universe with density inhomogeneities causing a gaussian light deflection field, both correlated with and independent of the perturbations giving rise to the temperature fluctuations. The ratio of the correlated effect to the independent one is of order the cross correlation parameter squared, b^2. While one might expect that the deflection and temperature be strongly coupled, $b \approx 1$ (cf. Eq. 42), this is found not to be true here. In the case of gravitational lensing of photons from the last scattering surface the correlation influence on CMB anisotropies is negligible because one loses the path length "resonance" of the deflections and temperature fluctuations together tracing the gravitational potential field along the entire line of sight. Such a dependence would increase the correlation parameter by a factor of $\approx kL \approx 10^{2.5} - 10^{3.5}$, making the effect significant. Due to the quasistatic nature of the potential along the path, however, this does not give rise to observable temperature fluctuations. Exceptions occur only in localized events such as cluster formation (Rees & Sciama 1968) or hot cluster cores (Sunyaev & Zel'dovich 1970), which do not offer the opportunity for path resonance or gaussian statistics.

Although the correlation of the two density perturbation derived variables considered here can be neglected, the formalism of calculating probability distributions and covariances of gaussian fluctuations is powerful. Insights garnered here into the methods and relations may prove useful in other cosmological applications such as analyzing large scale velocity flows.

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References

Blandford R.D., Narayan R., 1992, ARA&A 30, 311
Cramér H., 1957, Mathematical Methods of Statistics, Princeton University Press, Princeton
Davis M., Efstathiou G., Frenk C.S., White S.D.M., 1985, ApJ 292, 371
Fukushige T., Makino J., Ebisuzaki T., 1994, ApJ 436, L107
Linder E.V., 1990a, MNRAS 243, 353
Linder E.V., 1990b, MNRAS 243, 362
Rees M.J., Sciama D.W., 1968, Nat 217, 511
Smoot G.F., et al., 1992, ApJ 396, L1
Sunyaev R.A., Zel'dovich, Ya.B., 1970, ASS 7, 3
Tomita K., 1996, ApJ 461, 507
Wilson M.L., Silk J., 1981, ApJ 243, 14