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Boundary conditions and critical surfaces in astrophysical MHD winds

S.V. Bogovalov

Moscow Engineering Physics Institute, Kashirskoje Shosse 31, 115409 Moscow, Russia

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Abstract. Boundary conditions and the nature of the critical surfaces for the problem of stationary MHD outflows are formulated from the point of view of the causality principle. It is shown that the number of the boundary conditions which have to be specified on the surface of the object ejecting plasma is equal to the number of MHD waves outgoing normally from this surface. The boundary conditions for objects of different nature are formulated. It is shown that the spherically symmetric Parker solution of the pure hydrodynamical outflow of the plasma from the surface of the star with a given temperature and density of the plasma on the surface of the star is not the unique solution. To pick out the unique solution of this problem an additional condition defining the tangential components of the velocity of the plasma on the surface of the star must be specified. The nature and properties of the critical surfaces are studied and the rules to determine the critical surfaces in MHD flow are proposed. In the general case, the critical surfaces where the general solution of the problem is singular are placed on characteristics. The slow and the fast mode critical surfaces do not coincide strictly speaking with the classical critical surfaces.

Key words: MHD – solar wind – pulsars – ISM; jets and outflows – galaxies; jets

1. Introduction

Starting with the classical work of Parker (1958) MHD winds were studied in the framework of the phenomenology of a large variety of astrophysical objects. In particular the model of stationary axisymmetrical winds was used to analyze plasma outflows from AGN's (Blandford & Payne 1982, Camenzind 1986a, 1986b, Ferrari et al. 1985, 1986, Königl 1989, Mobarry & Lovelace 1986, Takahashi et.al. 1990 and Beskin & Par'ev 1993), pulsars (Michel 1969, Camenzind 1989, Beskin 1993, Sulkanen & Lovelace 1990), young stellar objects (Pelletier 1992, Pudritz & Norman 1986), Mundt et al. 1986, and ordinary stars (Weber & Davis 1967, Mestel 1968, Sakurai 1985).

To solve the problem of the stationary MHD outflow it is necessary to specify along with equations the boundary conditions on the surface of the object ejecting plasma. The question arises: how to do that correctly? Any stationary flow can be considered as the result of temporal evolution of the flow from some initial state. To describe the temporal evolution of the flow the Cauchy problem have to be formulated (Courant & Hilbert 1937). The solution of the Cauchy problem includes the specification of the equations, the boundary conditions and the initial state of the plasma. As a result the stationary solution depends in general on the equations describing the flow, on the boundary conditions and the history of the system. It is known that some partial differential equations have stationary solutions not depending on the history. Trivial example is the Poisson equation $\nabla\varphi = 0$: its solution depends only on boundary conditions. But in MHD configurations the solution can depend on the history of the system. A cold plasma placed in the closed cylinder with the uniform magnetic field is the trivial example. Let imagine the magnetic field directed along the axis of the cylinder and frozen in the upper and lower walls of the cylinder. A lot of new states can be obtained by rotating the lower wall of the cylinder: the new solutions depend on the angle of the rotation of the lower wall. The state of the plasma in the cylinder is related to the prehistory of the system.

The general solution of the stationary MHD flow depends on free functions (Courant & Hilbert 1937). Mathematically the dependence of the stationary solution on the history of the system means that the number of the free functions exceeds the number of the boundary conditions which can be specified for the flow. In this case a lot of solutions exists satisfying to the boundary conditions and the system appears underdefined. But the case when the system is overdefined and the number of boundary conditions exceeds the number of free functions often occurs in MHD flows. It happens, for example, when a supersonic flow interacts with an obstacle. In that case discontinuities appear in the solution giving an additional freedom level to the solution to satisfy to all boundary conditions.

Usually it is not difficult to distinguish these cases, but the problem becomes nontrivial when we have to deal with astrophysical winds. The difficulty is connected to the fact that the equation describing the stationary flow is the mixed-type equation. The general solution of this equation is singular on some surfaces and it is not clear whether the problem is well defined, underdefined or overdefined. To solve this point an analysis of the boundary conditions and singular surfaces of the flow is necessary.

Parker first pointed out the importance of the singular points in the spherically symmetric solutions for the hydrodynamical unmagnetized wind. These points are placed on the surface where the plasma velocity equals the local sound velocity. The number of the singular surfaces increases in the magnetized wind. Weber and Davis (1967) found two singular surfaces where the plasma velocity equals to the fast and slow magnetosound velocities. The surface where the plasma speed equals the Alfven speed is not singular in their solution. It was found later that this surface becomes a true singularity when the force balance accross the streamlines is consistenly included in the MHD equations.

This step was made by Sakurai (1985) for nonrelativistic and by Ardavan (1979) and Camenzind (1986b) for relativistic winds, who considered a plasma flow in the magnetic field affected by the moving plasma. In these works three critical surfaces are present, corresponding to the velocities of three MHD perturbations in the axisymmetric magnetized plasma: are the slow magnetosound surface (SMS), the Alfven surface (AS) and the fast magnetosound surface (FMS).

Along with that, some classes of self-similar flows were investigated, starting with the work by Blandford and Payne (1982) on winds from accretion disks. It was found that these solutions are singular not on the classical FMS but on the so called ”modified fast points”. A similar results was found by Tsinganos & Trussoni (1991) for outflow of nonpolytropic plasma and more general assumptions on the self-similarity of the wind.

The attempt to clarify the nature of the singularities in the MHD solutions was made in our previous paper (Bogovalov 1994). It was proved, following the causality principle, that the solution is singular on the so called fast magnetosound separatrix surface (FMSS). This surface is exactly the ”modified fast points” surface which has been found in self-similar solutions. In this paper we extend the analysis of singularities in the stationary MHD flows.

The plane of the paper is following. In Sect. 2 we give a short description of the equations defining the magnetosphere of the axisymmetric rotator. The boundary conditions are discussed in Sect. 3 from the causality principle point of view, while Sect. 4 is devoted to the detailed analysis of the pure hydrodynamical wind. In Sect. 5 we define and outline the properties of the critical surfaces in stationary MHD flows and in Sect. 6 the main results are discussed and summarized.

2. Equations of the stationary flow

The MHD equations governing the dynamics of a steady axisymmetric magnetized wind have been widely studied in several papers (see previous section). We outline now the general properties of the nonrelativistic MHD system (our consideration is independent on relativistic effects) following the formalism of Bogovalov (1994).

In the axisymmetric flow the magnetic field $\mathbf{H}$ can be expressed as a sum of the poloidal component $\mathbf{H_p}$ and the azimuthal component $\mathbf{H}_\varphi$. The poloidal magnetic field can be expressed as

$$\mathbf{H_p} = \frac{\nabla\psi \times \mathbf{e}_\varphi}{\rho}, \tag{1}$$

where ρ is the distance to the axis of rotation and $\mathbf{e}_\varphi$ is the unit vector corresponding to the rotation around the axis z. The function ψ is proportional to the full flux of the poloidal magnetic field through a surface at the radius ρ. In the frozen in approximation the relationship between the electric field $\mathbf{E}$ and the poloidal magnetic field is $\mathbf{E} = \rho\Omega/cq(\psi)\mathbf{H_p} \times \mathbf{e}_\varphi$ (Weber & Davis 1967), where Ω is the angular velocity of the central object. The function $q(\psi)$ is constant along the poloidal field line and describes their differential rotation.

The first equation defining the dynamics of the plasma along the poloidal field lines is the conserved specific energy flux

$$\frac{u^2}{2} + \mu + \phi - f(\psi)\rho\Omega q(\psi)H_\varphi = W(\psi). \tag{2}$$

The second equation is the conserved specific angular momentum flux

$$\rho\Omega u_\varphi - f(\psi)\rho\Omega H_\varphi = L(\psi). \tag{3}$$

Projection of the frozen-in condition on the electric field gives

$$\rho\Omega q H_p + u_p H_\varphi = u_\varphi H_p. \tag{4}$$

Here $u^2 = u_p^2 + u_\varphi^2$, u_p is the velocity of the plasma along a field line, u_φ is the azimuthal velocity of the plasma, μ is the enthalpy of the plasma per one particle, $\phi = -GM/r$ is the gravitational potential of the star, G is the gravitational constant, M is the mass of the central star, $f = H_p/(4\pi m n u_p)$, n is the density of the plasma, m is the mass of the particles. The functions $W(\psi)$ and $L(\psi)$, proportional to the energy and to the angular momentum flux per one particle, are constant along the field lines. Therefore they depend only on ψ. The last equation is the adiabatic condition

$$s = s(\psi), \tag{5}$$

where s is the entropy per particle (non polytropic flows were considered by Trussoni & Tsinganos 1993 and Sauty & Tsinganos 1994).

The classical critical surfaces appear in the solution of the system of Eqs. 2 - 5. The singularities are seen in the derivatives of variables on ρ along field lines, in particular the derivative of u_p on ρ along a field line is defined by the expression

$$u_p(U_m - u_p)\frac{\partial u_p}{\partial\rho} = \frac{\partial\phi}{\partial\rho}(u_p - fH_p) + \left[C_s^2(u_p - fH_p) + f u_p^2\frac{H_\varphi^2}{H_p}\right]\frac{\partial \ln H_p}{\partial\rho} - 2\Omega q f u_p H_\varphi - (u_p - fH_p)\frac{u_\varphi^2}{\rho}, \tag{6}$$

where $U_m = \nu(u_p - fH_p) + fH^2/H_p, \nu = (C_s/u_p)^2$ with C_s the adiabatic sound velocity. This derivative is singular on the fast mode surface (FMS), where the velocity of plasma equals the velocity u_f of fast MHD mode, and on the slow mode surface (SMS) where the velocity of the plasma equals the velocity u_{sl} of the slow MHD mode. The canonical conditions of regularity of the solution require that the right hand member of Eq. 6 goes to zero on the FMS and SMS. They are the classical critical surfaces.

The singularity of the general solution on the Alfven surface follows from the transfield equation for ψ (see Bogovalov 1994)

$$(u_p^2 - u_a^2)\{(u_p H_\rho^2 - H_p^2 U_m)\psi_{\rho\rho} + 2H_\rho H_z u_p \psi_{\rho z} + (u_p H_z^2 - H_p^2 U_m)\psi_{zz} + H_z H_p^2 \times \left(u_m + \frac{u_\varphi^2}{u_p}\right) + \frac{H_p^4 \nu}{u_p}\rho^2 \left[u_p^2 (\ln f)' + \frac{V' u_\varphi}{\rho \Omega q}\right] + \frac{H_p^2 c}{\Omega q u_p}(\mathbf{E}\cdot\nabla\phi) + \frac{\rho^2 k H_p^4 u_\varphi^2}{u_p}(\ln q)'\} = u_a^2 H_p \rho \Omega q H_\varphi [\frac{1}{(q\Omega)^2} \left(\frac{R H_p U_m}{f} - \frac{V' H_p}{u_p}\mathbf{U}\cdot\mathbf{H}\right) - (\ln q)' \frac{\rho H_p}{q\Omega} \left(cE + \frac{u_\varphi}{u_p}\mathbf{U}\cdot\mathbf{H}\right) - 2H_z] \quad (7)$$

where $V = W - Lq$, $R = W' + \rho\Omega H_\varphi (fq)'$, $\mathbf{U}\cdot\mathbf{H} = u_p H_p + u_\varphi H_p$ [the symbol $()'$ denotes the derivative with respect to ψ].

Eq. 7 is a mixed type equation: in the regions $0 < u_p < u_c$ and $u_{sl} < u_p < u_f$, it is elliptic, while in the regions $u_c < u_p < u_{sl}$ and $u_p > u_f$ it is hyperbolic, with u_c the cusp velocity (Polovin & Demutskii 1980):

$$u_c = \frac{u_a C_s}{\sqrt{u_a^2 (H/H_p)^2 + C_s^2}}, \quad (8)$$

where $u_a = H_p/\sqrt{4\pi m n}$. The presence of the general multiplier $(u_p - u_a)$ makes Eq. 7 singular on the alfvenic surface: the regularity condition requires that the right member of Eq. 7 vanishes on the AS (Sakurai 1985).

3. Boundary conditions

It is evident that we cannot assume arbitrary boundary conditions for the stationary problem and it is often unclear how unique is their choice. Both these questions have clear answers following from the causality principle.

According to the causality principle, the temporal evolution of any system is totally defined by the forces affecting on the system, by the boundary conditions and by the initial state of the plasma. Any real stationary configuration of the plasma can be obtained as the result of the temporal evolution of the plasma from some initial state. This is why among all possible choice of boundary conditions for a stationary solution we must select only those which ensure ”good” evolution of the system. By ”good” we mean that the Cauchy problem of the evolution of the system has unique solution at the accepted boundary conditions and this solution has not unphysical discontinuities on the boundaries during the evolution. In this way the stationary solution will satisfy to the causality principle. Physically this solution can be obtained in the result of the temporal evolution of the system. In other words, to obtain physically real solution of the stationary problem the boundary conditions of the time dependent problem must be taken.

To clarify the above statement let us consider the following simple equation in partial derivatives

$$\frac{\partial u}{\partial t} = -v\frac{\partial u}{\partial x}, \quad (9)$$

defined in the region $0 < x < x_0$, that describes the propagation of a perturbation in positive direction of the x axis. The stationary limit is $\partial u/\partial x = 0$ with solution $u = const$. To define this $const$ it is possible to specify the boundary condition either at $x = 0$ or at $x = x_0$: in principle both conditions are acceptable. It is evident however that boundary conditions cannot be specified simultaneously in the case under consideration. We do not consider it here.

Now let us look at the general solution of Eq. 9, that we can express in the form $u(t, x) = \phi(x - vt)$, where $\phi(\tau)$ is an arbitrary function that defines the temporal evolution of the system. The initial value $u(0, x) = u_0(x)$ provides the value of ϕ on the interval $0 < \tau < x_0$ while the boundary condition at $x = 0$, $u(t, 0) = z(t)$, specifies the value of ϕ in the interval $\tau < 0$. The Cauchy problem for Eq. 9 has unique solution if the boundary condition at $x = 0$ and the state of plasma at $t = 0$ are specified. In this example it is evident that the boundary condition at $x = x_0$ cannot be used: it does not define the temporal evolution of the plasma because violates the causality in the system. The solution formally obtained with this boundary condition cannot be obtained as the result of the physically real temporal evolution of the configuration.

To specify correctly the boundary conditions for the stationary problem it is necessary at least to define how the solution will evolve in time in the vicinity of the boundary under arbitrary perturbation. The solution will evolve in time without formation of unphysical discontinuities on the boundaries if the time dependent problem has unique solution in terms of small amplitude perturbations. In particular it must be correct in relation to all small amplitude perturbations in the form of arbitrary plane waves incident on the boundary. It means that the boundary conditions of the stationary solution are well specified if the problem of reflection of a small amplitude plane wave incident on the boundary at arbitrary angle has a unique solution. This principle is similar to the principle of ”evolutionarity” used by Achiezer, Lubarskii & Polovin (1958) and Kontorovich (1958) for investigation of the stability of the shock fronts in relation to their splitting.

The problem of the reflection of the MHD plane waves from the boundary is reduced to the solution of the system of linear equations in relation to the amplitudes of the waves outgoing from the boundary. The number of the equations equals to the number of independent boundary conditions. The system of the equations has unique solution if the number of independent boundary conditions (this is the number of equations) is equal to the number of outgoing waves (this is the number of unknown amplitudes) generated at the reflection of the plane wave from the boundary.

So, to define the number of boundary conditions which must be specified on the boundary it is necessary to account the number of the outgoing waves generated at the reflection of plane waves. It is of crucial importance that the number of the outgoing waves, generated at the reflection of any wave, does not depend on the angle of incidence and the type of the wave. In any case this number equals the number of the waves outgoing normally from the boundary. This statement, proved by Kontorovich (1958), allows to propose the simple rule how to define the number of boundary conditions which have to be specified on any boundary.

This conclusion is in agreement with the rule of the boundary condition specification given in the book of Landau & Lifshitz (1959, Sect. 104) for the pure hyperbolic hydrodynamical equations. According to it the number of the boundary conditions is equal to the number of the characteristics outgoing from the boundary. It is easy to make sure that the number of characteristics outgoing from the boundary equals the number of the waves outgoing normally from the boundary. The advantage of the rule formulated in this paper is that it is formulated from the more general point of view and can be used also for elliptical and mixed-type equations.

In the plasma, the number of the waves with the wave vector **k** is equal to the number of the independent physical parameters defining the state of the plasma. In the magnetized plasma these parameters are the density, the pressure, three components of the velocity and two components of the magnetic field. Totally the number of the parameters equals 7 (the magnetic field has only two independent vectors because it's components are connected by the equation $\nabla \cdot \mathbf{H} = 0$). Correspondingly we have 7 MHD waves: the entropy wave, the slow magnetosound, the fast magnetosound and the Alfven waves. (Achiezer 1974)

These waves are characterized by the wave vector, by a polarization and by the frequency in the rest frame of the plasma. The own frequency and the speed of the entropy wave are equal to zero: at the wave vector **k** there is only one entropy wave, that in the moving plasma propagates with the same speed. At the given wave vector **k** there are couples of the Alfven, slow and fast magnetosound waves corresponding to the own frequencies with different signs. These couples of waves have different signs of the own frequencies and of the own phase velocities of propagation.

In the moving plasma the frequency of the wave is modified as follows

$$\omega(\mathbf{k}) = \mathbf{k}\mathbf{U} + \omega_i(\mathbf{k}). \tag{10}$$

Here **U** is the velocity of the plasma, ω_i is the own frequency of the wave of type i (i = Alfven, slow, fast, entropy). A wave can be characterized by two velocities: by the phase velocity $\mathbf{v}_{ph} = (\omega/k)(\mathbf{k}/\mathbf{k})$ and the group velocity $\mathbf{v}_g = \partial\omega/\partial\mathbf{k}$. The direction of propagation of a perturbation is defined by the group velocity, therefore the outgoing waves are the ones with projection of the group velocity on the vector normal to the boundary and directed outwards. In MHD the phase and the group velocities are different: in general their projections on an arbitrary vector can have opposite signs. There is only one exception: since the frequency of any MHD wave can be expressed as $\omega = v_{ph}(\theta)k$, the group and phase velocities are connected by the relationship

$$\mathbf{v}_g = \mathbf{v}_{ph} + \delta v\, \mathbf{e}_\perp. \tag{11}$$

Here θ is the angle between **k** and the magnetic field, $\mathbf{e}_\perp$ is the unit vector perpendicular to the wave vector and δv is the difference between the group and the phase velocities. It follows from Eq. 11 that for the waves propagating perpendicular to the surface of the boundary, the projections of the phase and group velocities on the vector normal to the surface are equal to each other. For the wave moving perpendicular to the surface, the direction of propagation can be defined from the phase velocity .

Below we consider the boundary conditions on the stellar surface for two cases typical for astrophysics (in both cases the condition $v_a > C_s$ is assumed). The boundary conditions for the flow with the opposite unequality are considered partially in the next section and in the Sect. 5.

In the first case, we assume that the normal component of the plasma velocity on the stellar surface is lower than the velocity of the slow magnetosound wave (u_{sl}) (hot plasma) propagating normally to the surface, as typical for stellar winds. In this case the number of waves outgoing from the stellar surface is equal to 4: the entropy wave, the slow, the fast and the Alfven waves (Achiezer, Lubarskii & Polovin 1958). Correspondingly, the density and the temperature of the plasma, together with two components of the electric field parallel to the star surface ought to be prescribed on the star surface in this case.

The second case is found when the normal component of a cold plasma speed exceeds the velocity of the slow magnetosound waves propagating perpendicular to the surface. This is typical for the problem of the plasma ejection by radio pulsars. In this case the number of the outgoing from the star surface waves is equal to 5. Along with the slow mode wave with the wave vector directed from the boundary, the slow mode with the wave vector directed towards the boundary becomes outgoing, so that the total group velocities of these waves are directed outwards with respect to the boundary position. Contrary to the previous case, an additional boundary condition has to be added. From the physical point of view the normal component of the velocity of the plasma on the stellar surface has to be taken as the additional boundary condition. It can be shown in this case an acceptable solution exists in terms of small perturbations (Bogovalov 1992).

4. Hydrodynamical wind

It is interesting to notice that in a hydrodynamical wind, with the Alfven velocity is negligibly small with respect to the sound velocity and the plasma speed, the boundary conditions appears nontrivial. It follows from above that in such a case the specification of the density and the temperature of the plasma are not sufficient to pick out unique solution. We can see that by looking at the number of the waves outgoing from the stellar surface.

In hydrodynamics 5 parameters define the state of the plasma: the density, the temperature and the three components of the velocity. Correspondingly 5 types of perturbations exist in the nonmagnetized plasma: the entropy, two vorticity perturbations and two sound perturbations. The vorticity perturbations are polarized perpendicular to the wave vector in two orthogonal directions and propagate with the velocity of the plasma (Landau & Lifshitz, Sect. 82, 1959). If the normal component of the plasma velocity does not exceed the sound velocity on the surface of the star there are 4 outgoing waves from the surface of the star. Thence four boundary conditions must be specified on the boundary to obtain a unique solution satisfying to the causality principle. Among them the density and the temperature of the plasma are trivial, while the nature of others two is connected to the existence of two vorticity waves. From the vorticity conservation we deduce that the tangential components of the plasma velocity on the star surface are continuous. This allows to specify the values of the tangential components of the plasma velocity on the surface of the star. If the star rotates with the angular velocity Ω, the toroidal velocity of the plasma on the stellar surface will be equal $\Omega\rho$. The other component of the plasma velocity is defined by specific physical conditions under the surface of the star.

The specification of two components of the plasma velocity tangential to the surface of the star is necessary to ensure uniqueness of the solution of the time dependent Cauchy problem. It is clear shown in Landau & Lifshitz (1959 see the problem to the Sect. 82) that arbitrary perturbation in the moving gas is described by 5 waves including the vorticity waves. If we exclude the vorticity waves the causality is violated, and the Cauchy problem (and thence the stationary problem) has not a unique solution. This conclusion can be verified directly on the example of the Parker wind at $\Omega = 0$. Lets consider the solutions which differ a little from the spherically symmetric wind: in this case it is possible to solve the stationary problem following the theory of perturbations.

Since the poloidal field goes to zero it is convenient to introduce another flux function Φ, that follows from the condition $\nabla \cdot n\mathbf{u} = 0$:

$$n\mathbf{u} = \frac{\nabla\Phi \times \mathbf{e}_\varphi}{\rho}, \tag{12}$$

This function is proportional to the matter flux through the surface with the radius ρ.

It would be possible to obtain the transfield equation for Φ directly from the equations of motion and Maxwell equations, but it is more convenient to obtain this equation from Eq. 7 in the limint at $H_p \to 0$. Taking into account the relationship $4\pi mf\nabla\Phi = \nabla\psi$ at $H_p \neq 0$, in spherical coordinate system Eq. 7 becomes

$$\left[\left(\frac{u_r}{u}\right)^2 - \nu\right]\Phi_{rr} + 2\frac{u_r u_\theta}{u^2}\frac{\Phi_{r\theta}}{r} + \left[\left(\frac{u_\theta}{u}\right)^2 - \nu\right]\frac{\Phi_\theta}{r^2} + \nu n u_r \cos\theta - \left[1 + \left(\frac{u_r}{u}\right)^2\right] n u_\theta \sin\theta - \frac{1}{u^2}(\nabla\Phi\nabla\phi) + \nu n^2 (r\sin\theta)^2 W_\Phi' = 0. \tag{13}$$

at $H_p \to 0$, where we have assumed $\Omega = 0$. The perturbed solution can be expressed in the form

$$\Phi(\mathbf{r}) = \psi_0(1 - \cos\theta) + \xi, \tag{14}$$

where ξ is the perturbation of the spherically symmetric solution, which is given by the following equation

$$(1-\nu)\xi_{rr} - \frac{\nu}{r^2}\sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\xi}{\partial\theta}\right) + \frac{1}{r}\left(2 - \frac{GM}{u^2 r}\right)\xi_r = -\nu n^2 (r\sin\theta)^2 W_\Phi'. \tag{15}$$

where u, ν, n are taken from the unperturbed Parker solution. The classical (Parker) boundary conditions are

$$\delta n_0 = \delta T_0 = 0, \tag{16}$$

where n_0 and T_0 are the density and the temperature of the plasma on the surface of the star. In this solution the expressions $(1-\nu)$ and $[2 - GM/(u^2 r)]$ go to zero simultaneously on the critical surface of the radius r_c to cross it smoothly. The first order perturbation of the Bernoulli integral can be presented as follows

$$W(\Phi) = W_0(\Phi_0 + \xi) + \epsilon(\Phi_0), \tag{17}$$

where $W_0(\Phi)$ is the Bernoulli integral of the unperturbed spherically symmetrical Parker solution. Since W_0 is constant and $\Phi_0 = \psi_0(1-\cos\theta)$ is directly connected to the polar angle θ, the right hand part of Eq. 17 can be expressed as

$$-\nu n^2 r^2 \sin\theta\epsilon_\theta'/\psi_0. \tag{18}$$

The general solution of Eq. 17 can be expanded on eigenfunctions Y_m of the differential operator $\sin\theta\,\partial/\partial\theta[1/\sin\theta\,(\partial/\partial\theta)]$ with eigenvalues $-m(m+1)$ (Bogovalov 1992)

$$\xi(r,\theta) = \sum_m Z_m(r) Y_m(\theta). \tag{19}$$

The equation for the functions $Z_m(r)$ is

$$(1-\nu)\frac{d^2 Z_m}{dr^2} + \frac{1}{r}\left(2 - \frac{GM}{u^2 r}\right)\frac{dZ_m}{dr} + \frac{\nu m(m+1)}{r^2} Z_m = -\frac{\nu n^2}{\psi}\epsilon_m. \tag{20}$$

where ϵ_m are the coefficients of the expansion of the function $\sin\theta\,(\partial\epsilon/\partial\theta)$ on eigenfunctions $Y_m = \sin\theta\,[dP_m(\theta)/d\theta]$, with $P_m(\theta)$ the m-order Legendre polynomials.

Eq. 20 has a particular point on the critical surface of the Parker solution: its general solution can be expressed as the sum of two independent solutions

$$Z_m(r) = g_m(r) + s_m(r) \tag{21}$$

with $g_m(r)$ regular and $s_m(r)$ singular on the critical surface. The condition of the regularity of the solution picks out only functions g_m from the general solution, that satisfy to the condition

$$\frac{g_m(r_c)}{\epsilon_m} = -\frac{r_c^2 n_c}{m(m+1)u_c} \tag{22}$$

on the critical surface (where the variables are marked with the index c). Let make sure that it follows from condition (22) that the Parker criticality condition for the perturbed solution is satisfied automatically . The Bernoulli equation for the pertubed solution is as follows

$$u\delta u + \delta\mu = \epsilon(\theta). \tag{23}$$

Together with the relationship $r^2(\delta n u + n\delta u) = 1/\sin\theta\,(\partial\xi/\partial\theta)$ it gives the following equation

$$(u^2 - C_s^2)\frac{\delta u}{u} = \left[\epsilon - \frac{C_s^2}{nur^2\sin\theta}\frac{\partial\xi}{\partial\theta}\right]. \tag{24}$$

The vanishing of the right member of Eq. 24 on the critical surface is just the Parker criticality condition for the perturbed parameters: it is automatically fulfilled when Eq. 22 holds (we must remember that the expansion of ϵ on P_m does not contain constant terms with $m = 0$, see also Eq. 25).

The boundary conditions on the star surface, Eq. 16, and the Bernoulli equation, Eq. 23, give the relationship

$$\frac{g_m(r_0)}{\epsilon_m} = -\frac{r_0^2 n_0}{m(m+1)u_0}. \tag{25}$$

which selects a unique solution of Eq. 20 for arbitrary set of ϵ_m. But this is not unique solution of the problem as there is freedom in the choice of the set of ϵ_m . They are not defined by the classical Parker boundary conditions, so that there is a lot of solutions satisfying the given temperature and the density of the plasma.

This freedom is due to the unspecified tangential component of the plasma velocity (we have assumed $\Omega = 0$): to specify this value it is necessary to have information about plasma flow below the star surface. For example, if the plasma is unmagnetized everywhere in the star, the flow can be assumed as a potential one: in this case $\nabla \times \mathbf{u} = 0$ on the star surface (Landau & Lifshitz 1959). It is possible to make sure that this condition is followed by the condition $u_\theta = 0$ on the star surface which picks out unique solution. In more general case a production of the nonzero vorticity is possible in the atmosphere of the star. More in general, the specification of the boundary condition depends on the physics of the problem.

So, we have shown from the analysis of the boundary conditions that to select the unique solution of the pure hydrodynamical stationary problem it is necessary to specify the temperature, the density of the plasma and the tangential components of the velocity on the star surface.

5. Critical surfaces

In this section we come back to the consideration of the MHD flows. The analysis of the boundary conditions alone does not allow to make definite conclusion about the existence of the unique stationary solution. Below, following from the causality principle, we will analyse the properties of the stationary solution super fast magnetosound at the infinity, assuming that such solution does exist.

It is clear that no boundary conditions have to be specified in the infinity: no MHD wave can propagate upstream from infinity. This is why the boundary conditions specified a priori on the stellar surface together with Eqs. (2-7) allow in principle to determine the stationary solution if it exist and does not depend on the history. Note however that the stationary problem cannot be treated as the Cauchy problem, because the number of the boundary conditions that can be specified is less than the number of the parameters defining the state of the magnetized plasma on the star surface. These parameters are the density and pressure, the three components of the flow speed and the two components of the magnetic field, for a total number of 7. Subtracting the number of the boundary conditions, we are left with the 3 parameters if the plasma velocity on the the star surface does not exceed the slow mode speed, and with 2 parameters if it does. According to our assumptions on the existence of the unique stationary solution not depending on the history, there must exist additional conditions in the flow specifying the free parameters on the star surface. In some extent the problem is similar to a boundary value problem, the difference is only that we do not know a priori the boundary conditions on the outer boundary of the flow, and we even do not know where this outer boundary is placed.

Experience shows that the outer boundary conditions are the conditions of regularization of the stationary solution on some surfaces named ”critical surfaces”. We assume that the critical surfaces exist in the stationary problem: our goal is to find them. The basic property of these surfaces is that some boundary conditions can be applied on them. It means that the number of the boundary conditions specified on surfaces placed upstream or donwstream of the critical surface are different. This property allows to determine the critical surfaces in the stationary flow.

Let us introduce artificially some surface surrounding the star in a known stationary MHD flow. We can wonder how many boundary conditions satisfying the causality principle must be specified on this surface to reproduce the flow in the region from this surface to infinity. The answer can be deduced from the above discussion: this number equals the number of waves outgoing normally from the surface to the direction of infinity. Let’s now move this artificial boundary downstream the flow. Near a critical surface, the number of boundary conditions specified on a surface infinitesimally close, but upstream, to the critical surface is different from the number to be specified on an analogous surface, but placed downstream of the critical surface. This follows directly from the nature of the critical surface. Simultaneously it means that the number of waves outgoing normally

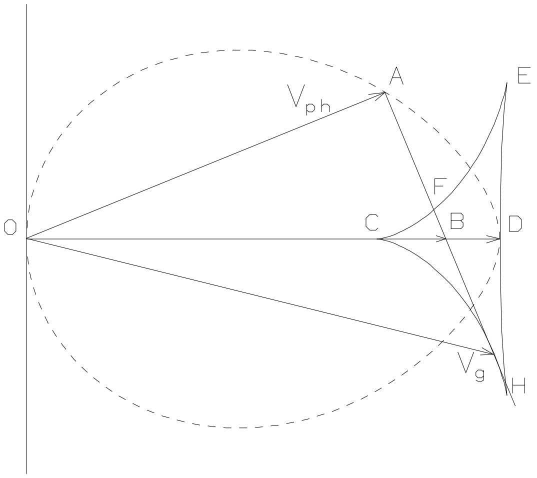

Fig. 1. Phase (dashed line) and Group (solid line) polars of the slow magnetosound waves. The poloidal velocity u_p is directed along the poloidal magnetic field. The line OA is perpendicular to some arbitrary surface (in particular critical one) where the equality $u_n = v_{ph}$ takes place. The velocity of plasma u_p (the vector OB) is directed along the magnetic field and lies in the interval between the points C and D. The vector OC is the cusp velocity vector, the vector OD is the slow mode velocity vector along the magnetic field. In this interval Eq. 7 is hyperbolic. The vector of the group velocity lies on the perpendicular drawn through the end of the vector of the phase velocity. This perpendicular is tangential to the assumed surface and is simultaneously the characteristics. At the plasma velocities satisfying the conditions $u_n > v_{ph}$ (see the part of the group polar FEH) and $u_p > v_{sl}$ waves exist with projection on the vector OA exceeding u_n.

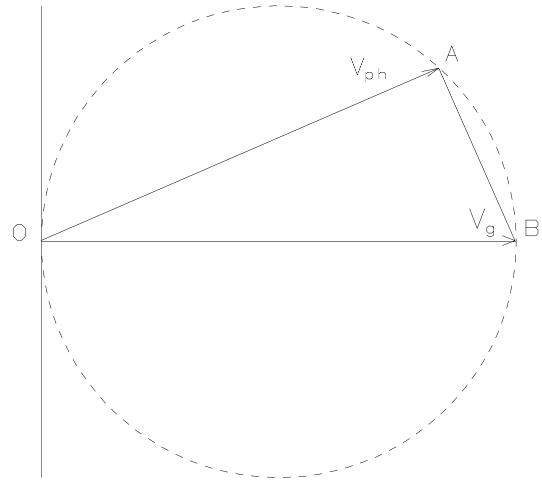

Fig. 2. Phase (dashed line in the form of a circle) and Group (point B) polars of the Afven waves. OA is the vector perpendicular to some arbitrary surface (in particular critical one) where the equality $u_n = v_{ph}$ takes place. In this case the plasma velocity is equal to the vector OB. If $u_n > v_{ph}$ there are no waves with projection of the group velocity on OA exceeding u_n.

from the surface to the direction of infinity differs in dependence on whether the surface is placed up or down the flow in relation to the critical surface.

In astrophysical winds the velocity of the plasma increases downstream in comparison with the MHD wave velocities. Therefore in the vicinity of the critical surface the following relationship holds place: $v_{wave} < u_n$ downstream from the critical surface and $v_{wave} > u_n$ upstream from the critical surface. On the critical surface we have

$$v_{wave} = u_n, \tag{26}$$

where v_{wave} is the phase velocity of one of types of the MHD waves propagating perpendicular to the surface, u_n is the normal to the surface component of the plasma speed.

It follows from Eq. 26 and the geometrical properties of the Friedrichs diagrams presented in Figs. 1 - 3 (Polovin & Demutskii 1980) that the slow and the fast mode critical surfaces are placed in the regions where Eqs. (7) is hyperbolic. It also follows from these figures that the critical surfaces coincide with one of slow and fast mode characteristics. According to Fig. 2 the Alfven critical surface coincides with the so called Alfven surface (AS), where the poloidal velocity of the plasma equals the local Alfven velocity.

The slow and the fast critical surfaces are the characteristics of Eq. 7, but the local property in Eq. 26 does not pick out a unique critical surface from the family of characteristics. An additional global property should be determined to select the critical surface from all the family of characteristics. This property was pointed out in the work (Bogovalov 1994): the characteristic where the critical surface is placed is the separatrix for the amount of characteristics of one family. One part of these characteristics propagates and stops on the sound surface where Eq. 7 changes type while another part of characteristics stops in a different region disconnected from the sound surface. This is why the characteristics which are the critical surfaces were named as slow magnetosound separatrix surface (SMSS) and fast magnetosound separatrix surface (FMSS) (Bogovalov 1994). In more detail, the structure of the characteristics and their separatrices are discussed by Bogovalov (1996) and Tsinganos (1996).

Three critical surfaces can then exist in the stationary MHD flow in general case. They are the slow mode critical surface coinciding with the SMSS, the Alfven mode critical surface coinciding with the AS and the fast mode critical surface coinciding with the FMSS.

The slow and the fast critical surfaces do not coincide in general with the classical slow and fast magnetosound surfaces (Weber & Davis 1967) found in the analysis of Eqs. 2-5 describing the dynamics of the plasma in prescribed poloidal magnetic fields. It is clear why. The classical surfaces are also critical but in relation to the signals propagating exactly along the field lines which are not perturbed. They are critical in relation to the artificially limited amount of MHD signals (Bogovalov 1996).

It is interesting to consider the physical sense of the fast, the Alfven and the slow critical surfaces from the causality point of view. The Friedrichs diagrams or polars for the phase and the group velocities of the MHD perturbations are presented

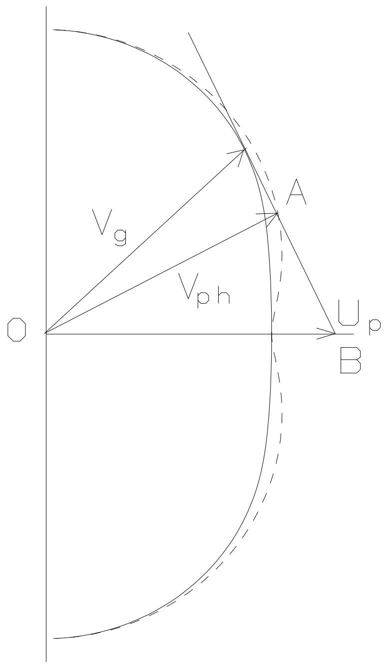

Fig. 3. Phase (dashed line) and Group (solid line) polars of fast magnetosound waves. The poloidal velocity is directed along the poloidal magnetic field. OA is the vector perpendicular to some arbitrary surface (in particular critical one) where the equality $u_n = v_{ph}$ takes place. In this case $u_p > v_f$ and Eq. 7 is hyperbolic. If $u_n > v_{ph}$ there are no waves with the projection of the group velocity exceeding u_n. The line BA is the characteristics.

in Figs. 1, 2, 3. They are drawn for the dimensionless sound velocity equal to 1 and for the Alfven velocity equal to 1.1. According to Eq. 12 the vector of the group velocity lies on the perpendicular drawn through the end of the vector of the phase velocity. The group polar is tangential to this perpendicular. It follows from this geometry that in the region down the flow from the fast critical surface there are no fast mode signals with the component of the group velocity exceeding the normal component of the plasma velocity. It means that no fast mode signal can propagate from the region down from the FMSS upstream through the FMSS. The region upstream the FMSS is causally disconnected in relation to the fast mode signals from the region down stream the FMSS (Bogovalov 1994). The Alfven critical surface has the same meaning for the Alfven waves: both these surfaces are horizons for the corresponding signals.

The slow magnetosound critical surface does not divide the flow on the regions with different causal relationship in relation to the slow mode signals. It is seen from Fig. 1 that when Eq. 26 is fulfilled on slow mode critical surface, slow mode signals exist which can propagate upstream the flow with the projection of the group velocity on the normal to the critical surface exceeding u_n. The waves with vectors of the group velocity laying on the part of the group polar FEH can propagate upstream the flow from the critical surface. Moreover, it is seen from this figure that even in the region downstream the classical slow magnetosound surface, where $u_p > v_{sl}$ and Eq. 7 is elliptical, there are slow mode signals which can propagate upstream, through the SMS and SMSS. The slow mode critical surface is not the horizon for the slow mode waves.

The physical meaning of the slow mode critical surface is related to the condition of reflection of the MHD waves. According to Kontorovich (1958) the number of reflected waves at arbitrary angle equals the number of the waves outgoing perpendicular from the surface. At first glance this statement looks not valid for the slow mode waves: from Fig. 1 we see that at different angles the number of outgoing slow mode waves can be different. All group velocities laying on the part of the group polar FCH have projection on the normal less then u_{ph}: these waves propagate downstream the critical surface. The waves with vectors of the group velocity laying on the group polar FEH can poragate upstream the critical surface. To make sure that the statement of Kontorovich is valid it is necessary to take into account all types of waves: the slow, the fast and the Alfven waves. It appears that the number of all types of reflected MHD waves does not change with angle: when a new wave of one type appears, the wave of a different type disappears. The physical sense of the slow mode critical surface is that the total number of outgoing at the reflection waves changes on the slow mode critical surface.

The perturbation of the flow by slow mode waves depends on its nature. The slow mode critical surface is not a horizon for arbitrary slow mode perturbation. But we analyze the stationary solution, therefore it is necessary to consider the perturbation of the flow from a small stationary obstacle placed in the region of hyperbolicity of Eq. 7 $v_c < u_p < v_{sl}$. The slow mode perturbation from small stationary obstacle is concentrated on the slow mode characteristics (Kulikovskii & Lubimov 1962). The presence of the separatrix characteristic means that the space is divided in two parts. The introduction of the stationary obstacle in one region does not perturb the flow in another region due to the slow mode signals (see the structure of the characteristics in the paper of Bogovalov 1994). The slow mode critical surface is the horizon for the slow mode signals produced only by the stationary obstacles, but it is not valid for arbitrary moving obstacles. As it was mentioned above, the presence of slow mode waves propagating through the SMSS means that moving obstacle can perturb the flow upstream the SMSS.

A qualitative analysis of the equations in the self-similar approximation confirms our conclusion: three types of critical surfaces exist, with the FMSS and the SMSS that do not coincide in general case with the classical critical surfaces. The singularity of the general stationary solutions on the FMSS have been found in (Blandford & Payne 1982), Lovelace 1991, Li 1992, Contopoulos 1994, Tsinganos & Trussoni 1991). The singularity on the SMSS have been found in the range of works by Tsinganos (1992, 1993) and Sauty & Tsinganos (1994). The Alfven critical surface coincides with the classical singular surface introduced by Sakurai (1985) only if the poloidal velocity of the plasma is parallel to the poloidal magnetic field. Recently Contopoulos (1996) analyzed the more general case with poloidal velocity not parallel the magnetic field. In this case the Alfven critical surface does not coincide with the classical Alfven surface: the

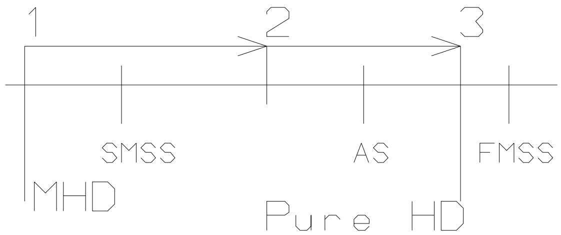

Fig. 4. The schematic diagramm of transitions of the boundary conditions from MHD to the pure hydrodynamical flow

Alfven critical surface conversely is localized where the projection of the poloidal velocity of plasma equals the projection of the group Alfven velocity on the direction perpedicular to the surface.

We discuss now how the problem of the boundary conditions changes when moving from MHD to the pure hydrodynamics. We assume that the magnetic field is small enough that the relation $u_a < C_s$ holds, and that the star does not rotate. Then we decrease the magnetic field to attain a full hydrodynamic regime. A schematic diagram of the transitions of the boundary conditions is presented in Fig. 4. In position 1 we assume that the star is surrounded by the slow mode critical surface: as discussed above, we have to specify in this case 4 independent boundary conditions. They are the temperature, the density and two components of the electric field parallel to the star surface. When the magnetic field decreases the slow mode critical surface goes under the star surface, so that a new boundary condition must be added. It can be one of the tangential componets of the plasma velocity. When the magnetic field becomes so small that the Alfven critical surface goes under the star surface we have to add the second tangential component of the plasma velocity as boundary condition. When finally the magnetic field goes to zero, the star is left surrounded only by the fast mode critical surface, where the fast mode velocity has become equal to the sound velocity. We are now in the hydrodynamical case and must specify the following boundary conditions: $T_0, n_0, E_t, E_\varphi, u_t, u_\varphi$. The index t means tangential component of a vector. The same conditions were obtained in Sect. 4 directly in the analysis of the hydrodynamical case (E_t and E_φ are equal to zero at $H = 0$).

The results obtained above allow us to takle the problem on the dependence of hydrodynamical winds on the history of the flow. Assuming that the plasma emerges from the star surface with sub slow magnetosound velocity, the free functions that define the steady solutions are the integrals $q(\psi), L(\psi), W(\psi), f(\psi), s(\psi), \rho E_\varphi$ (the last integral is usually not considered as free function since $\rho E_\varphi = 0$, but it must be added for fullness, see also Contopoulos 1996). We have to add to them the free functions which define the solution of the second order Eq. 7 (Courant & Hilbert 1937), They are ψ and the normal derivative $\partial\psi/\partial\mathbf{n}$ on the stellar surface: the specific choice of these free functions is not unique, but the number is unique. Now we see that the number of these free functions (8) is larger than the number of the boundary and critical conditions (7): T_0, n_0, E_t, E_φ and three critical conditions on the SMSS, AS and FMSS. It is easy to see that the boundary conditions cannot determine the normal component of the magnetic field on the star surface H_n which is proportional to the derivative of ψ along the surface of the star. But it can not be specified arbitrary: the temporal dependence of H_n is described by the equation

$$\frac{\partial H_n}{\partial t} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta E_\varphi) - \frac{\partial E_t}{\partial\varphi}\right] \tag{27}$$

which is the projection of the Maxwell equation $\partial\mathbf{H}/\partial t = \nabla\times\mathbf{E}$ on the vector normal to the star surface. The solution of this equation determines H_n: it depends on the boundary conditions and on the initial value of H_n at $t = 0$. To determine the unique solution of the problem it is necessary to add to the boundary conditions the value H_n which is the relic of the history of the system. In other words, the stationary flow, strictly speaking, depends on the history of the flow.

6. Discussion

We have outlined some general properties of the outflows solutions of the steady MHD equations. First of all it has been shown that, if the stationary solution exists, it has the same boundary conditions of the time dependent problem. Any other boundary condition does not ensure that the stationary solution obtained with them is physically acceptable. The number of boundary conditions fulfilling the causality principle is unique, and the specific kind of the boundary conditions is defined by the physics of the problem.

We have also seen that, if the steady solution exists, with critical surfaces where the problem is singular, the positions of these surfaces must coincide the the slow and fast mode separatrix characteristics, and on the Alfven surface.

These are the necessary, but not sufficient conditions for the existence of the solution with speed larger than the fast magnetosound velocity, as evident from the examples of self-similar solutions (Blandford & Payne 1982), Lovelace 1991, Li 1992, Contopoulos 1994, Tsinganos & Trussoni 1991). It is worth noticing that no self-similar solution passing smoothly through all critical surfaces have been obtained up to now. On the other hand numerical solutions show that a discontinuity exists on the FMS, that can be related to the singular structure of characteristics near the position where the flow velocity is perpendicular to the FMS (Bogovalov 1996). Further numerical and analytical investigations are necessary to clarify this problem.

The regularity conditions of the solution on the classical critical surfaces require that the right members of Eqs. 6 and 7 vanish on these surfaces. However, it is not possible at the moment to know the correct regularity conditions of the solution on the real critical surfaces. For a particular analytical solution the condition of the regularity every time arises naturally, as appear from our example with hydrodynamical flow in Sect. 4, with the regularity condition given by Eq. 20. However it is not clear how to formulate this condition for numerical solution of the stationary problem. This is why on the present day the most reliable way to obtain the stationary solution is to solve the nonstationary problem, as, for example, in Washimi & Shibata (1993), Washimi & Sakurai (1993), Bogovalov 1996). On the other hand it would be important to formulate the regularity conditions for the steady solution on the real critical surfaces directly from the stationary equations.

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