Astron. Astrophys. 323, 652–655 (1997)

ASTRONOMY AND ASTROPHYSICS

Time-dependent self-similar solution of the vertical structure of a thin accretion disk

H.-E. Fröhlich

Astrophysikalisches Institut Potsdam, An der Sternwarte 16, D-14482 Potsdam, Germany

Received 22 November 1996 / Accepted 9 January 1997

Abstract. A self-similar solution for the vertical structure of a thin accretion disk – apart from thermal equilibrium – is described. The main assumptions made are that the energy loss is due to radiation and that throughout the disk the Rosseland mean of the opacity can be fitted by the same power-law dependency on density and temperature.

Key words: accretion, accretion disks – hydrodynamics

1. Introduction

Being concerned with the radiative cooling of the primordial solar nebula, Lin & Papaloizou (1980) have described an homologously contracting solution for its vertical structure. They found this solution assuming the opacity to depend on some power of temperature.

Here an homology solution will be presented for the more general case of a radiative accretion disk with an internal heat source and power-law dependencies of the opacity on temperature *and* density.

In the accretion disk case such an analytical approach is easier and even more justified than in stellar structure theory, where homology relations have been used extensively in order to make conceivable famous empirical relations (e.g. Kippenhahn & Weigert 1994).

A self-similar flow solution, even if its strict applicability is restricted, provides insight into the process of thermal adjustment and allows testing out numerical codes.

Especially in understanding dwarf novae, where the recurrent outbursts are attributed to an instability, but in other sites in disk astrophysics too, knowledge *how* a disk behaves apart from thermal equilibrium is desired (cf. e.g. Mineshige & Osaki 1983; Meyer 1984; Osaki 1996; Lin & Papaloizou 1996). Perhaps one can be guided by an homology consideration to an appropriate relaxation prescription. Whether a self-similar evolving vertical structure is what nature prefers is, of course, quite a different matter.

In what follows the vertical structure of a thin Keplerian disk is considered, which is always in mechanical, but not necessarily in thermal equilibrium. Viscously generated heat, which does not do work, is radiated away vertically. The energy transport be due to radiation, with the diffusion approximation being adopted. One should emphasize that the departure from thermal equilibrium may be large, as long as the opacity law (and, of course, the thermo-dynamical properties of the disk matter) do not change.

For the disk to be in mechanical equilibrium or, what is the same, to prevent the bulk velocity to become supersonic within a few scale-heights, the thermal time-scale has to exceed the dynamical one, which is fulfilled for a low viscosity parameter, i.e. $\alpha \ll 1$.

Different from Mineshige & Osaki (1983), where the z-dependence of the energy flux is prescribed somewhat artificially, here the viscous heat source is presumed to go with the pressure.

This paper aims at compiling relevant scaling relations for an accretion disk, not necessarily in thermal equilibrium, and for homologous contraction and expansion, respectively.

It remains to be seen whether the self-similar solution, found for an idealized opacity prescription, proves indeed attractive and appropriate to describe thermal relaxation in reality.

2. Basic equations

The z-dependent quantities ρ, p, T, and u_z have their usual meanings. The specific entropy is denoted by s. In the thin disk approximation the acceleration is proportional to the height z above the mid-plane. The viscous heat generation is proportional to the pressure, with α being the famous Shakura-Sunyaev (1973) viscosity parameter. The radiative flux is denoted by F, κ is the opacity, and σ Stefan's constant. The basic equations are:

$$\frac{\partial p}{\partial z} = -\rho\, \Omega_{\rm K}^2\, z, \qquad (1)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_z)}{\partial z} = 0, \qquad (2)$$

$$\frac{\partial s}{\partial t} + u_z \frac{\partial s}{\partial z} = \frac{1}{\rho T}\left(\frac{9}{4}\,\alpha\, p\, \Omega_{\rm K} - \frac{\partial F}{\partial z}\right), \quad (3)$$

$$\frac{\partial T}{\partial z} = -\frac{3\,F\,\kappa\,\rho}{16\,\sigma\,T^3}. \quad (4)$$

It is further essential that the equation of state $p(\rho, T)$ as well as the opacity $\kappa(\rho, T)$ are power laws of density and temperature:

$$p = \frac{\mathscr{R}}{\mu}\rho\,T, \quad \kappa = \kappa_0\,\rho^n\,T^q, \quad (5)$$

with $\mathscr{R}$ and μ being gas constant and molecular weight, respectively. Replacing in Eq. (3) the entropy by that for an ideal gas, $s = c_v\,\ln(p/\rho^\gamma)$, with $\gamma = c_p/c_v$ the ratio of the (constant) specific heats, one obtains for the thermal pressure:

$$\frac{\partial p}{\partial t} + u_z \frac{\partial p}{\partial z} + \gamma\, p\, \frac{\partial u_z}{\partial z} = (\gamma - 1)\left(\frac{9}{4}\,\alpha\, p\, \Omega_{\rm K} - \frac{\partial F}{\partial z}\right). \quad (6)$$

Now the dimension-less coordinate $\xi = z/z_0(t)$ is introduced. Heights are therefore expressed in units of a time-dependent vertical scale:

$$z_0(t) = \sqrt{p_{\rm c}(t)/\rho_{\rm c}(t)}/\Omega_{\rm K} = \sqrt{\mathscr{R}\,T_{\rm c}/\mu}/\Omega_{\rm K}. \quad (7)$$

Its time derivative is $\dot{z}_0$. The mid-plane values for pressure p, density ρ and temperature T are labelled with 'c'.

(A thorough inspection of the vertical momentum Equation (1), including the inertia term, would reveal the validity of the assumption of hydrostatic equilibrium, viz. $\ddot{z}_0 \ll \Omega_{\rm K}^2\, z_0$.)

Looking for a self-similar solution we try:

$$p(z,t) = p_{\rm c}(t) \cdot f_p(\xi), \quad (8)$$
$$\rho(z,t) = \rho_{\rm c}(t) \cdot f_\rho(\xi), \quad (9)$$
$$u_z(z,t) = u_0(t) \cdot \xi, \quad (10)$$
$$F(z,t) = p_{\rm c}(t)\, z_0(t)\, \Omega_{\rm diss}(t) \cdot f_F(\xi), \quad (11)$$
$$T(z,t) = T_{\rm c}(t) \cdot f_T(\xi). \quad (12)$$

Inserting this into Eqs. (1 - 4), one realizes that for arbitrary density profiles $u_0 = \dot{z}_0$ holds. From the Eq. (2) of continuity follows then $\dot{\rho}_{\rm c}/\rho_{\rm c} = -\dot{z}_0/z_0$. The relative rates of change for pressure and temperature mid-plane values can be expressed in terms of $\dot{z}_0/z_0$ too, obeying the Definition (7) and the ideal gas Law (5), viz. $\dot{p}_{\rm c}/p_{\rm c} = \dot{z}_0/z_0$ and $\dot{T}_{\rm c}/T_{\rm c} = 2\,\dot{z}_0/z_0$, respectively.

The Set (1 - 4) turns into a new set of ordinary differential equations for the shape functions f_p, $f_{\rm F}$ and $f_{\rm T}$ ($f_\rho = f_p/f_T$):

$$\frac{d\,f_p}{d\,\xi} = -f_\rho\,\xi, \quad (13)$$

$$\frac{d\,f_F}{d\,\xi} = f_p, \quad (14)$$

$$\frac{d\,f_T}{d\,\xi} = -K \cdot f_{\rm F} f_\rho^{n+1} f_T^{q-3}, \quad (15)$$

with $K = 3\,\Omega_{\rm diss}\, p_{\rm c} z_0^2 \kappa_{\rm c} \rho_{\rm c}/(16 \sigma T_{\rm c}^4)$ (16)

and $\Omega_{\rm diss} = \frac{9}{4}\,\alpha\,\Omega_{\rm K} - \left(\frac{\gamma+1}{\gamma-1}\right) \cdot \frac{\dot{z}_0}{z_0}.$ (17)

The opacity $\kappa_{\rm c}$ ($= \kappa_0\, \rho_{\rm c}^n\, T_{\rm c}^q$) is the mid-plane opacity.

Any departure from thermal equilibrium can be described formally by an effective $\alpha_{\rm eff}(t) > 0$, which is the reason why a self-similar solution in the time-dependent case exists at all:

$$\alpha_{\rm eff} = \alpha - \frac{4}{9}\left(\frac{\gamma+1}{\gamma-1}\right)\frac{1}{\Omega_{\rm K}}\,\frac{\dot{z}_0}{z_0}. \quad (18)$$

In the limiting case $\alpha_{\rm eff} \to 0$ all the viscous heat is needed for blowing up the disk.

Provided K is known the Set (13 - 15) can be integrated numerically with a Runge-Kutta method, obeying the boundary conditions $f_p(0) = f_T(0) = 1$ and $f_{\rm F}(0) = 0$.

The constancy of K yields a differential equation describing the evolution of the scale-height $z_0(t)$:

$$\frac{d}{dt}\left(\frac{\dot{z}_0}{z_0}\right) = -2(3 + \frac{n}{2} - q)\,\frac{\dot{z}_0}{z_0}\left(\frac{9}{4}\,\frac{\gamma-1}{\gamma+1}\,\alpha\Omega_{\rm K} - \frac{\dot{z}_0}{z_0}\right), \quad (19)$$

With the abbreviations $A_0 = \dot{z}_0(t_0)/z_0(t_0)$, $A_1 = 2(3 + n/2 - q)$ and $A_2 = \frac{9}{4}\,\frac{\gamma-1}{\gamma+1}\,\alpha\Omega_{\rm K}$, the solution reads:

$$\frac{z_0(t)}{z_0(t_0)} = {\rm e}^{A_2(t-t_0)}\left[\frac{A_0}{A_2} + \left(1 - \frac{A_0}{A_2}\right){\rm e}^{A_1\,A_2(t-t_0)}\right]^{-\frac{1}{A_1}}. \quad (20)$$

The Lin-Papaloizou solution (1980) holds for no heating, i.e. $A_2 \to 0$, and $n = 0$.

Having computed the shape functions f_p, f_ρ and f_T, one gets by integrating through the whole disk ($-\infty < \xi < \infty$, if the disk extents to infinity) three constants, all of order unity, which depend on opacity law and parameter K only,

$$a_1 = \frac{\int_{-\infty}^{\infty} f_p\, d\,\xi}{\int_{-\infty}^{\infty} f_\rho\, d\,\xi}, \quad (21)$$

$$a_2 = \frac{1}{2}\int_{-\infty}^{\infty} f_\rho\, d\,\xi, \quad (22)$$

$$a_3 = \frac{\int_{-\infty}^{\infty} f_\rho^{n+1} f_T^q\, d\,\xi}{\int_{-\infty}^{\infty} f_\rho\, d\,\xi}. \quad (23)$$

They relate integrated quantities with mid-plane values:

$$\Pi = \int_{-\infty}^{\infty} p\, dz = a_1 \frac{\mathscr{R}}{\mu}\,\Sigma\, T_{\rm c} = 2\, a_1\, a_2\, p_{\rm c}\, z_0, \quad (24)$$

$$\Sigma = \int_{-\infty}^{\infty} \rho\, dz = 2\, a_2\, \rho_{\rm c}\, z_0, \quad (25)$$

$$\tau = \int_{-\infty}^{\infty} \rho\,\kappa\, dz = a_3\, \kappa_{\rm c}\, \Sigma. \quad (26)$$

Expressing in Eq. (26) $\kappa_{\rm c} = \kappa_0 \rho_{\rm c}^n T_{\rm c}^q$ (with the help of Eqs. (7), (24), and (25)) in terms of Σ and Π, one gets for the optical depth

$$\tau(\Sigma, \Pi) = \frac{a_3\,\kappa_0\,\left(\mu/\mathscr{R}\right)^q}{2^n\, a_1^{q-n/2}\, a_2^n}\,\Sigma^{1+3n/2-q}\,\Pi^{q-n/2}\,\Omega_{\rm K}^n. \quad (27)$$

The dimension-less flux is

$$f_F(\infty) = \int_0^\infty f_p\, d\xi = a_1\, a_2. \quad (28)$$

The Definition (16), together with Eqs. (11), (25), (26) and (28), relates the effective temperature, i.e. the flux $\sigma T_{\rm eff}^4 = F(\infty)$, with the mid-plane temperature $T_{\rm c}$ via the optical depth τ:

$$T_{\rm eff}^4 = \frac{32\, a_1\, a_2^2\, a_3\, K}{3}\, \frac{T_{\rm c}^4}{\tau}. \quad (29)$$

Now K is to be specified. We restrict ourself to the case of an asymptotically flat temperature profile, i.e. the pressure drops Gaussian-like for $\xi \to \infty$.
In order to fix K one can make use of Eddington's approximation (1926) for a grey atmosphere: $T(\infty)^4 = 17\, T_{\rm eff}^4/32$. Replacing in Eq. (29) the effective temperature by this constraint, one obtains the disk's optical depth in terms of K:

$$\tau(K) = \frac{17\, a_1\, a_2^2\, a_3\, K}{3} \left(\frac{T_{\rm c}}{T(\infty)}\right)^4. \quad (30)$$

Remember that the constants a_i as well as the temperature ratio $T_{\rm c}/T(\infty)$ are functions of K. K depends therefore in a complicated way on the optical depth. Moreover, in the case of Kramers' opacity ($n = 1$ and $q = -3.5$) K approaches 0.2687... for $\tau \to \infty$. For $K = 0.26$ the optical depth τ proves already larger than 10. Hence, K does not depend sensitively on the optical depth, provided the latter is large.
By the way, in that limiting case ($K = 0.2687$) the constants a_i have the following values: $a_1 = 0.884, a_2 = 1.295$ and $a_3 = 1.104$.
It has been proven that the vertical structure in that case is dynamically stable according to the Schwarzschild criterion.

3. The time evolution of the vertically integrated pressure

Integrating Eq. (6) over z, one gets:

$$\dot{\Pi} + (\gamma - 1)\, \Pi\, \frac{\dot{z}_0}{z_0} = (\gamma - 1) \left(\frac{9}{4}\, \alpha\, \Pi\, \Omega_{\rm K} - 2\, F(\infty)\right), \quad (31)$$

where $2\, F(\infty)$ is equal $\Omega_{\rm diss}\Pi$. With the Definition (17) for $\Omega_{\rm diss}$ the scaling

$$\frac{\dot{\Pi}}{\Pi} = 2\, \frac{\dot{z}_0}{z_0} \quad (32)$$

results. This allows one to substitute the $\dot{z}_0/z_0$ term in the energy equation, getting eventually

$$\dot{\Pi} = 2\, \frac{\gamma - 1}{\gamma + 1} \left(\frac{9}{4}\, \alpha\, \Pi\, \Omega_{\rm K} - 2\, F(\infty)\right). \quad (33)$$

Expressing the radiative energy loss $F(\infty) = \sigma T_{\rm eff}^4$ with the help of Eqs. (24), (29) and (27) in terms of Σ and Π, one arrives at the looked for relation

$$\dot{\Pi} = \frac{9}{2} \left(\frac{\gamma - 1}{\gamma + 1}\right) \alpha\, \Pi\, \Omega_{\rm K} \left[1 - \left(\Pi/\Pi_{\rm eq}\right)^{3+\frac{n}{2}-q}\right], \quad (34)$$

with the integrated equilibrium pressure

$$\Pi_{\rm eq} = a_1\, \frac{\mathscr{R}}{\mu}\, \Sigma \left[\frac{3^3 \alpha \kappa_0 \left(\mathscr{R}/\mu\right)^{1-\frac{n}{2}} \Sigma^{2+n} \Omega_{\rm K}^{1+n}}{2^{8+n} a_2^{2+n} \sigma K}\right]^{\frac{1}{3+n/2-q}}.$$

Because generally the surface density Σ evolves only slowly (with the viscous time-scale), $\dot{\Pi}$ will be essentially a function of Π alone. For fixed Σ the time-scale, $\Pi/\dot{\Pi}$, for thermal settling is (assuming $\gamma = 5/3$)

$$t_{\rm eq} = \frac{8}{9 \alpha \Omega_{\rm K}} \left[1 - \left(\Pi/\Pi_{\rm eq}\right)^{3+\frac{n}{2}-q}\right]^{-1}. \quad (35)$$

Two comments should be made:
First, if $3 + n/2 - q < 0$, a thermal run-away arises. A disk, somewhat too hot, would get even hotter due to the further reduced ability to radiate the heat away (cf. Piran 1978).
Second, near the equilibrium $t_{\rm eq}$ exceeds the thermal time-scale $t_{\rm th} = (8\alpha\Omega_{\rm K}/9)^{-1}$ by far. It is the deviation $\Pi - \Pi_{\rm eq}$ that decays roughly with the thermal time-scale:

$$\Pi - \Pi_{\rm eq} \simeq \left(\Pi_0 - \Pi_{\rm eq}\right)\, {\rm e}^{-(3+\frac{n}{2}-q)\,(t-t_0)/t_{\rm th}}. \quad (36)$$

Thermal equilibrium provided, i.e. $\sigma\, T_{\rm eff}^4 = \frac{9}{8}\alpha\Pi\Omega_{\rm K}$, the effective temperature depends on surface density according to

$$\sigma\, T_{\rm eff}^4(\Sigma) = \frac{9}{8}\, \alpha\, \Omega_{\rm K}\, a_1\, \frac{\mathscr{R}}{\mu}\, \Sigma\, T_{\rm c}, \text{ with} \quad (37)$$

$$T_{\rm c}(\Sigma) = \left[\frac{3^3 \alpha \kappa_0 \left(\mathscr{R}/\mu\right)^{1-\frac{n}{2}} \Sigma^{2+n} \Omega_{\rm K}^{1+n}}{2^{8+n} a_2^{2+n} \sigma K}\right]^{\frac{1}{3+n/2-q}}.$$

If $(5+3n/2-q)(3+n/2-q) < 0$, the disk will be viscously unstable because then d $\log(\sigma\, T_{\rm eff}^4)/$d $\log(\Sigma) < 0$ (cf. Lightman, Eardley 1974). This might happen (at least for $n < -2$) even if the disk is stable in the thermal mode, i.e. $3 + n/2 - q > 0$. Hence, a disk undergoing partial ionisation is not necessarily the reason for this kind of instability to occur.
For $n > 0$ and $q < 3$ a disk in thermal equilibrium is in any case stable.

4. For sake of completeness: surface density scaling for a stationary disk

For a steady-state disk (with accretion rate $\dot{M}$ and around a star of mass M), where all the viscous heat is radiated away locally, the radial run of the flux, $\sigma T_{\rm eff}^4(r)$, is for the reason of energy conservation (cf. Frank, King & Raine 1992):

$$\sigma T_{\rm eff}^4 = \frac{3\, G\, M \dot{M}\, f}{8\, \pi\, r^3}, \text{ with } f = \left(1 - \sqrt{r_{\rm in}/r}\right). \quad (38)$$

G denotes the gravitational constant and at $r_{\rm in}$ the disk meets the star. Setting this equal to Eq. (37) results in the radial run of the surface density:

$$\Sigma(r) = \left(\frac{\dot{M}\Omega_{\rm K} f}{3\pi\alpha a_1}\, \frac{\mu}{\mathscr{R}}\right)^{x/y} \left(\frac{2^{8+n} a_2^{2+n} \sigma K}{3^3 \alpha \kappa_0 \left(\mathscr{R}/\mu\right)^{1-\frac{n}{2}} \Omega_{\rm K}^{1+n}}\right)^{1/y}, \quad (39)$$

with $x = 3 + n/2 - q$ and $y = 5 + 3n/2 - q$. Because the angular velocity goes with $\Omega_{\rm K} \propto r^{-3/2}$, the surface density scales like

$$\Sigma(r) \propto f^{\frac{3+n/2-q}{5+3n/2-q}} \; r^{-\frac{6-3n/2-3q}{10+3n-2q}} \; . \qquad (40)$$

Acknowledgements. I am grateful to my colleague D. Elstner for reading an earlier version of the manuscript and offering helpful suggestions. Moreover, I am much obliged an anonymous referee for pointing out the Lin-Papaloizou solution to me.

References

Eddington A.S., 1926, The Internal Constitution of the Stars. Cambridge Univ. Press, Cambridge
Frank J., King A., Raine D., 1992, Accretion Power in Astrophysics. Cambridge Univ. Press, Cambridge
Kippenhahn R., Weigert A., 1994, Stellar Structure and Evolution. Springer, Berlin
Lightman A.P., Eardley D.M., 1974, ApJ 187, L1
Lin D.N.C., Papaloizou J.C.B., 1980, MNRAS 191, 37
Lin D.N.C., Papaloizou J.C.B., 1996, ARA&A 34, 703
Meyer F., 1984, A&A 131, 303
Mineshige S., Osaki Y., 1983, PASJ 35, 377
Osaki Y., 1996, PASP 108, 39
Piran T., 1978, ApJ 221, 652
Shakura N.I., Sunyaev R.A., 1973, A&A 24, 337

This article was processed by the author using Springer-Verlag LaTeX A&A style file *L-AA* version 3.