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Diffusion in differentially rotating stars

P.A. Bernasconi

Observatoire de Genève, CH-1290 Sauverny, Switzerland (paolo.bernasconi@obs.unige.ch)

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Abstract. A linear stability analysis of rotating Boussinesq flows including shear, μ-gradients and radiation losses leads to modification of the classical Ledoux criterion for convective equilibrium. The role of shear in spreading the radial extent of the convective layers is briefly investigated, and we found that some effects may be expected in the latest nuclear evolutionary stages of massive stars, whose short timescales and rather steep μ-barriers concur to hinder angular momentum redistribution within their condensed cores. Also, *semiconvective shear zones* (Maeder 1996) are predicted in those radiative shells where the available shear energy is not sufficient to completely overturn the stable thermal gradient. We then provide a prescription to compute a modified diffusion coefficient for passive scalars in presence of shear and thermal conductivity.

Key words: stars: rotation – stars: interiors – convection – turbulence – diffusion

1. Introduction

Rotation and shear mixing are becoming increasingly important physical ingredients that are not to be omitted in the new run of evolutionary calculations. They are both believed to be responsible for the onset of macroscopic motions in the stellar plasma resulting in enhanced mixing of chemical elements and the maintenance of angular momentum transport phenomena. In recent years, in particular, shear mixing has been frequently invoked to explain the unexpected degree of enrichment in heavy elements in the atmospheres of massive MS stars (Herrero et al. 1992; Venn 1995). Also, mixing processes can be inferred to operate within the shear layer at the base of the convective envelope of the Sun, thus possibly explaining the shallower helium profile deduced by helioseismological data as compared to solar model solutions with helium settling (Gough et al. 1996). Unfortunately, the thresholds for shear induced instabilities and the magnitude of the ensuing transport phenomena are badly known, and the few works devoted to the study of their effects on stellar evolution calculations remain in an exploratory phase.

The problem of the observed enrichment in metals at the surface of fast rotating massive stars has been studied in a first paper by Meynet & Maeder (1996), who remark that the common understanding of the Richardson criterion proved of formidable efficacy in preventing any significant mixing in regions where large μ-gradients are generated in consequence of nuclear evolution. In that respect let us recall that the Richardson criterion, in its original form, imposes a threshold, namely $\frac{1}{4}(\frac{dU}{dz})^2 > g\varphi\nabla_\mu H_p^{-1}$ for shear mixing in a plane-parallel, radiative zone (U being the horizontal velocity; the other symbols are defined below). According to Maeder (1996), it is possible to solve the present theoretical discrepancies if one *supposes* that a fraction of the local energy excess available in radiative shear flows is degraded by turbulence. This working hypothesis implies that even in regions stable according to the Ledoux criterion, partial, turbulent mixing occurs within a fraction of the hydrogen burning timescale, determining the progressive erosion of the μ-barriers and the consequent He- and N- enrichments in fast rotating O-stars. Of course, the existence of *semiconvective shear zones*, where the Richardson number $Ri(\mu = 0) < 1/4 < Ri(\mu \neq 0)$, can only be assessed by confronting all their logical consequences with carefully selected observations.

As a matter of fact, turbulent transport in stellar radiation zones has a rather long history, which goes back to the first proposition of Schatzman (1969) to explain the chemical composition at the surface of the Sun and solar-like stars by a mild but efficient transport of matter. Since then, much theoretical work has been devoted to the study of its possible origin, soon recognizing that only shear instabilities may reach a turbulent state strong enough to actually mix the stellar material (Zahn 1983, 1990). In general, shear instabilities depend on the exact profile of U, the horizontal components are likely more vigorous and of larger extent, so that one should account for the three dimensional character of the motion field in deriving turbulent diffusivities that are necessarily anisotropic (Zahn 1991). Here, however, and more modestly in view of the aforementioned considerations on massive stars evolution, our purpose is just to demonstrate that shear induced semiconvection is in fact a natural prediction of a linear stability analysis of the basic hydrodynamic equations. Our approach is an extension of the classical work of Kato (1966) on chemically stratified Boussinesq mediums, when the inertial corrections are included in the force law equation. Additionally, we obtain a revised criterion for the onset of convection, and provide a way to compute the diffusion coefficient of scalar fields in presence of rotation, shear and thermal conductivity. These results, of modest computational effort and easy implementation in existing codes, could help in gaining some first order, deeper insight on massive stellar structure and evolution.

2. The linearized equations of motions

We shall consider the problem of thermal instability in a differentially rotating, incompressible flow stratified in cylindrical shells, and subject to axisymmetric perturbations. The unperturbed configuration, a gaseous star in hydrostatic and thermal equilibrium, is assumed to be axisymmetric, and we investigate the effects of differential rotation on the growing solutions perpendicular to the axis of rotation. For this purpose, we adopt a rotation law of the form $\Omega = \Omega(z)$, the axis of rotation being confused with the x-axis.

The potentially unstable region is considered to be sufficiently small compared to the stellar radius, rotating between two stable coaxial cylinders, so as to neglect both the Eulerian change in the gravitational acceleration $\boldsymbol{g}$ (i.e. the Poisson equation is omitted), as well as the density changes, except for small fluctuations yielding the buoyant force (Boussinesq limit). The linear perturbation counterparts of the coupled Navier-Stokes (with the inertial terms $\boldsymbol{\Omega} \wedge (\boldsymbol{\Omega} \wedge \boldsymbol{r}) + 2\boldsymbol{\Omega} \wedge \boldsymbol{v}$) and temperature equations then become, in the usual representation (cf. Appendix A; see also Chandrasekhar 1968 for the general procedure),

$$\begin{aligned} \frac{\partial}{\partial t}\nabla^2 w &= g\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\Gamma_T\theta - \frac{\varphi}{\mu_0}\mu - \frac{2\Omega z\varpi}{g}\right) \\ &+ \nu\nabla^4 w - 2\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\zeta + 2\varpi\frac{\partial\Omega}{\partial z} + 2\Omega\frac{\partial\varpi}{\partial z} \qquad (1) \\ \frac{\partial\zeta}{\partial t} &= \nu\nabla^2\zeta + 2\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}w \qquad (2) \\ \frac{\partial\theta}{\partial t} &= \beta w + \chi\nabla^2\theta \qquad (3) \end{aligned}$$

where θ is the fluctuation on the temperature, ζ and w are the z-component fluctuations on the vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla}\wedge\boldsymbol{u}$ and velocity $\boldsymbol{u}$; $\mu = \delta\mu_0$ is the fluctuation on the mean molecular weight μ_0, and $\boldsymbol{\varpi}(\boldsymbol{x},t) = (\varpi(\boldsymbol{x},t),0,0)$ is the fluctuation on the time independent angular frequency $\boldsymbol{\Omega} = (\Omega(z),0,0)$. ν is the kinematic viscosity, $\chi = K_{tot}/\rho c_p$ the thermometric conductivity (K_{tot} is the total - thermal plus radiative - conductivity), $\Gamma_T = -dln\rho/dT \equiv \delta/T$ is the coefficient of thermal expansion, $\alpha = \partial ln\rho/\partial lnp$, and $\beta = \partial\theta/\partial z = T(\nabla - \nabla')H_p^{-1}$ is the superadiabatic temperature gradient related to $\nabla - \nabla_{ad}$ by the well known identity (Cox 1968)

$$\nabla - \nabla' = \frac{\Gamma}{\Gamma+1}(\nabla - \nabla_{ad}) \qquad (4)$$

As usual, ∇' and ∇ are the average temperature gradients inside the perturbed fluid element and outside, in the local surroundings. For adiabatic motions $\nabla' = \nabla_{ad}$. The efficiency Γ can be expressed in terms of the Peclet number $Pe = 6\Gamma$ (Maeder 1995). Previous works on linear stability have often deliberately neglected non-adiabatic corrections, in spite of the fact that there resides precisely the physical reason for overstable convection in stars. Actually, the inclusion of radiative losses does not seriously compromise the mathematical tractability of the problem.

The closure relations on μ and ϖ are here obtained by imposing a vanishing Lagrangian derivative $D = \partial_t + u_i\partial_{x_i}$ on both the molecular weight and the specific angular momentum. The condition $D\mu_0 = 0$ is a very good approximation, since the effects of the diffusion of elements operate on a much longer timescale than the many other pertinent diffusion timescales in stellar interiors; setting $\nabla_\mu = \partial ln\mu_0/\partial lnp$, its linearized form takes the form

$$\frac{\partial\mu}{\partial t} - \frac{\mu_0}{H_p}\nabla_\mu w = 0 \qquad (5)$$

The second requirement is more demanding and may not always hold strictly (e.g. magnetic torques); with the help of the mass conservation equation, one finds

$$\frac{\partial\varpi}{\partial t} + \frac{w}{z^2}\frac{\partial(z^2\Omega)}{\partial z} = 0 \qquad (6)$$

which also expresses in cylindrical geometry the absence of meridional circulation. Though very crude and of limited applicability, Eq. 6 leads however to interesting insight on the role of shear as a destabilizing agent.

Eqs. (1)-(3), (5) and (6) are our final set of perturbation equations.

We will next assume normal-mode solutions for all the field variables in the linear stability analysis, and set

$$F = (w,\theta,\zeta,\mu,\varpi) = F_0 e^{nt+i(\boldsymbol{k}_\perp \boldsymbol{r}+k_3 z)} \qquad (7)$$

where $n(k)$ is the growth (or decay) rate of the instability. As noted by Sung (1977), this latter set of eigenfunctions may not be complete for the special problem in hand, and though sufficient conditions for instability remain meaningful, the validity of sufficient conditions for stability is an open question. Also, stellar interiors are inviscid to a very high degree, so let $\nu = 0$ in all our subsequent discussion. By successive elimination one can reduce the system of linearized forms into a single dispersion equation for the linear growth (Appendix B)

$$n^3 + a_2 n^2 + [a_1 + iA_1 - b_1]\,n + a_2\,[a_1 + iA_1] = 0 \qquad (8)$$

where small characters denote real parts, and capital letters the imaginary parts of the cubic coefficients

$$\begin{aligned} a_1(x) &= \frac{x}{1+x}\left[\tau_{dyn}^{-2}\varphi\nabla_\mu - 2\tau_{rot}^{-1}\tau_{shear}^{-1}\right] - \frac{2}{z^2k^2}\tau_{shear}^{-2} \\ A_1(x) &= -\frac{2}{(1+x)^{1/2}}\frac{1}{zk}\tau_{rot}^{-1}\tau_{shear}^{-1} \\ b_1(x) &= \frac{x}{1+x}\tau_{dyn}^{-2}\delta(\nabla - \nabla') \\ a_2 &= 2\tau_h^{-1} \qquad (9) \end{aligned}$$

In the equalities (9) we introduced the (unknown) eddy anisotropy factor $x = k_\perp^2/k_3^2$, and defined the heat diffusion, dynamical and rotation timescales as the quantities $\tau_h = (\chi k^2/2)^{-1}$, $\tau_{dyn} = (p/\rho g^2)^{1/2} = (H_p/g)^{1/2}$ and $\tau_{rot} = \Omega^{-1}$, respectively. Also, we made the following simplification for z sufficiently large

$$\frac{1}{z^2}\frac{\partial\Omega}{\partial z}\frac{\partial(z^2\Omega)}{\partial z} \approx \left(\frac{\partial\Omega}{\partial z}\right)^2$$

and defined

$$\tau_{shear}^{-2} = \left(z\frac{\partial\Omega}{\partial z}\right)^2 = \left(\frac{\delta U}{\delta z} - \frac{U}{z}\right)^2$$

δU is the difference between the expected tangential velocity due to rotation at a rate Ω, and that obtained by adding the shear velocity field. Then τ_{shear} is the time required by two adjacent shells moving at different angular velocities ($\delta\Omega$), to complete $\delta z/z$ relative rotations.

3. The conditions for shear instability and the diffusion coefficient in semiconvective zones

If there exists at least a purely real, positive solution for the growth rate $n(k)$, the stellar layer is said to be convectively unstable. If on the contrary there are no real, positive solutions but, however, one (or more) complex solutions $n_r + in_i$ with positive real part, then one speaks of overstable convection. Of the two natural timescales n_r^{-1} and n_i^{-1} characterizing these latter secularly unstable regions, the second one gives the phase velocity of the disturbances in the vertical direction, n_i/k_z. The resulting crossing time over the semiconvective thickness is several orders of magnitude shorter compared with the evolutionary timescale (Kato 1966), and should ensure that the amplitude of the disturbances can grow sufficiently for the medium to become turbulent [1]. As to the diffusion coefficient, it is suitably written out of the growing amplitude timescale n_r^{-1} as $D = \frac{1}{3}v_{eff}\ell_{mix} = \frac{2\pi}{3}n_r/k^2$ (Langer et al. 1983), where $v_{eff} = \ell_{mix}n_r/2\pi$, $\ell_{mix} = 2\pi/k$ in view of the definition adopted in (7) for the expansion in normal-mode.

If one is searching for at least a purely real, positive solution of a generic cubic $n^3 + c_2n^2 + c_1n + c_0 = 0$, with the c_i real coefficients, then it is easily verified that the following sufficient conditions hold

$$c_0 < 0\ ,\ c_2 < 0\ ,\ c_1 < 0 \tag{10}$$

[1] How much, and how fast the amplitude is likely to grow, are issues that cannot be tackled by the linear theory. We may remark however that the intensity of the reflected component at a boundary separating two media of different acoustic impedances $Z_i = \rho n_i k_z^{-1}$, is $I_r/I_i = (Z_1 - Z_2)^2/(Z_1+Z_2)^2$ (see for instance Pain 1993). Now, radiative boundaries can be assimilated to rigid walls since there the disturbances are rapidly damped and remain infinitesimal. For $Z_2 \gg Z_1$ then, where Z_1 is the acoustic impedance in the semiconvective zone, one finds $I_r \lesssim I_i$, i.e. the amplitude of the reflected component will possibly be higher than the initial amplitude if the disturbance grows even moderately during propagation.

If none of the previous inequalities is satisfied, the cubic may still possess complex solutions with positive real parts. Eq. (8) then splits into the couple

$$\begin{aligned} n_r^3 - 3n_rn_i^2 + a_2(n_r^2 - n_i^2) + (a_1 - b_1)n_r & \\ -A_1n_i + a_1a_2 &= 0 \\ 3n_r^2n_i - n_i^3 + 2a_2n_rn_i + A_1n_r & \\ +(a_1 - b_1)n_i + A_1a_2 &= 0 \end{aligned} \tag{11}$$

Note that if the complex (shear) term A_1 is set to zero, the algebra is simplified since we find in place of (11)

$$8n_r^3 + 8c_2n_r^2 + 2(c_1 + c_2^2)n_r + c_1c_2 - c_0 = 0 \tag{12}$$

which follows from the expression on the imaginary component $n_i^2 = 3n_r^2 + 2c_2n_r + c_1$. Applying conditions (10) on (12), we eventually obtain as criterion for the onset of overstable convection

$$\begin{vmatrix} c_1 & c_0 \\ 1 & c_2 \end{vmatrix} < 0 \tag{13}$$

3.1. The case with no shear

If we apply criterion (10) on the dispersion equation (8) with $\tau_{shear} \to \infty$, then we recover for the onset of convection the well known inequalities

$i)\quad \nabla_\mu < 0$ (Taylor)

$ii)\quad \nabla > \nabla' + \frac{\varphi}{\delta}\nabla_\mu$ (Ledoux)

(the solution $\chi < 0$ is rejected since unphysical). Semiconvective motions are excited wherever condition (13) is fulfilled, and leads to (Kato 1966)

$iii)\quad \nabla > \nabla'$ (Schwarzschild)

In this case, the overstable mode n_r is solution of (12) with frequency $\omega^2 = n_i^2$. We point out that all these latter criteria are independent from the accessible spectra in wavenumber k, and bear no explicit signature on the specific geometry of the problem.

In radiative regions stable according to the Ledoux criterion, we can recast Eq. (12) into a cubic for the quantity $y = n_r\tau_{dyn}$ in terms of the timescales ratio $f = \tau_{dyn}/\tau_h$,

$$\begin{aligned} y^3 + 2fy^2 + \left(f^2 - \frac{1}{4}\frac{x}{1+x}\delta\nabla_{tot}\right)y & \\ -\frac{1}{4}\frac{x}{1+x}f\delta(\nabla - \nabla') &= 0 \end{aligned} \tag{14}$$

where $\nabla_{tot} = \nabla - \nabla' - \frac{\varphi}{\delta}\nabla_\mu < 0$, $\nabla - \nabla' > 0$. In stars $f < 1$, and more often $f \ll 1$. However, if we were to solve for n_r on the hypothesis $f = 0$ (the adiabatic case), then one would lack of any growing solution ($n_r^2 < 0$), confirming the fact that semiconvection is really the outcome of the radiative leakages suffered by the perturbed fluid.

We now rely on the principle of maximum heat transport to select the anisotropy factor x which extremizes the growth rate, or equivalently y (Canuto & Hartke 1986). Let us then differentiate Eq. (14) with respect to x, and demand $dy/dx = 0$. One obtains

$$y = f\frac{\nabla' - \nabla}{\nabla - \nabla' - \frac{\varphi}{\delta}\nabla_\mu} \tag{15}$$

With the help of the expressions for y, f, and the identity (4), the diffusion coefficient becomes

$$D = \frac{\pi}{3}\chi\frac{\nabla - \nabla_{ad}}{\nabla_{ad} - \nabla + \frac{\varphi}{\delta}\frac{\Gamma+1}{\Gamma}\nabla_\mu} \tag{16}$$

in which the wavenumber k plays no role. Two more expressions must however be supplemented in order to close the system for the unknowns D, ∇ and Γ. These latter are (Maeder 1996)

$$D = 2\chi\Gamma \tag{17}$$

$$\nabla = \frac{\nabla_r + a_0\frac{\Gamma^2}{1+\Gamma}\nabla_{ad}}{1 + a_0\frac{\Gamma^2}{1+\Gamma}} \tag{18}$$

where $a_0 = \frac{1}{3}\frac{A}{V}\Lambda$, A and V being the mean area and volume of an unstable fluid cell, and $\Lambda^{-1} \approx k_0$ (the peak wavenumber in the turbulent energy spectrum $E(k)$, or ℓ_{mix}^{-1} in the framework of mixing length theories of convection). Within a numerical factor practically equal to unity ($\pi/3$), expression (16) for the diffusion coefficient is the same as that obtained by Maeder & Meynet (1996) from a totally different procedure. For a thorough discussion on the form taken by the solutions of the system (16)-(18), see Maeder (1996).

3.2. A new criterion for convective instability in shearing media

If we demand for a purely real growing solution, then the cubic (8) splits into the two equalities

$$\begin{aligned} n^3 + a_2 n^2 + \left[a_1 - \tau_{dyn}^{-2}\delta\frac{k_\perp^2}{k^2}(\nabla - \nabla')\right] n + a_2 a_1 &= 0 \\ (n + a_2)A_1 &= 0 \end{aligned}$$

The root $n = -a_2 = -\chi k^2$ is of no interest here. We next find in place of the condition for a Taylor instability

$$\nabla_\mu < max\frac{2}{\varphi}\left\{\frac{\tau_{dyn}^2}{\tau_{rot}\tau_{shear}} + \frac{1}{z^2 k_\perp^2}\frac{\tau_{dyn}^2}{\tau_{shear}^2}\right\}$$

Since $\min(k_\perp^2) = \alpha^{-1}\pi^2/z^2$, with α a geometrical factor of order unity, one gets

$$i)\ \nabla_\mu < \frac{\varphi_{shear,1}}{\varphi} + \frac{\varphi_{shear,2}}{\varphi} \tag{19}$$

where

$$\varphi_{shear,1} = 2\frac{\tau_{dyn}^2}{\tau_{rot}\tau_{shear}} \qquad \varphi_{shear,2} = \alpha\frac{2}{\pi^2}\frac{\tau_{dyn}^2}{\tau_{shear}^2}$$

Likewise, the previous Ledoux inequality modifies as

$$ii)\ \nabla > \nabla' + \frac{1}{\delta}\left(\varphi\nabla_\mu - \varphi_{shear,1} - \varphi_{shear,2}\right) \tag{20}$$

In the deep interior, and in terms of the Brunt-Väisälä frequency $N_{ad}^2 = \frac{g\delta}{H_p}[\nabla_{ad} - \nabla] + \frac{g\varphi}{H_p}\nabla_\mu \equiv N^2 + N_\mu^2$, Eq. (20) can be written in the most usual form

$$\frac{\left[N_{ad}^2 - 2\tau_{rot}^{-1}\tau_{shear}^{-1}\right]}{\tau_{shear}^{-2}} < \alpha\frac{2}{\pi^2} \tag{21}$$

In the limit of very large z, expression (21) reduces to $N_{ad}^2/(\frac{dU}{dz})^2 \equiv Ri < 2\alpha/\pi^2$, which is very near the same as the original Richardson criterion for plane-parallel, incompressible flows when the curvature is no longer important.

An interesting aspect of the inequalities (19)-(20) consists in the absence of corrective terms of the form $f(\tau_{rot}, \tau_{dyn})$. In other words, the centrifugal contribution only enters the equations coupled to the shear field. As a consequence, one does not recover in the limit $\tau_{shear} \to \infty$ the corresponding criteria [2] for the special case $\Omega = const$. In fact, if we were considering the star as a solid body, rotation would affect the hydrostatic equilibrium configuration by lowering the effective gravity, and the thermic properties only through modification of the spherical radiative gradient (Endal & Sofia 1976). As for the physical meaning of the new term appearing in Eq. 21, it can be understood by recalling that N_{ad}^2 is proportional to the mean buoyant acceleration over a distance H_p of an unstable element, caused by thermal effects $\propto \nabla_{ad} - \nabla$ and by chemical stratification $\propto \nabla_\mu$. In the same vein, the correction brought in by the inertial acceleration $r\Omega^2$ in the linear regime is $2r\Omega\frac{\partial\Omega}{\partial z}\delta z = 2\tau_{rot}^{-1}\tau_{shear}^{-1}H_p$.

Condition (20) on the temperature gradients offers an algebraic criterion for establishing convective instability in a shearing medium. The minus sign in the two right hand terms reminds us that differential rotation has a destabilizing effect, and that convective boundaries may thus spread beyond what is commonly assumed in standard calculations. This is certainly not the case for a slow rotator like the Sun, whose minimum timescale τ_{shear} in the shear layer at the base of the convection zone is of the order of 250 days, compared to rotation and dynamical timescales of 26 days and 1 hour, respectively, giving shear coefficients $\varphi_{shear,i} < 10^{-6}$. In contrast, massive MS stars are known to rotate faster, and their reduced evolutionary timescale weakens the effectiveness of the angular momentum transport mechanisms resulting in steeper angular velocity profiles. Rotational periods of a few days are common among massive field stars, and the effects of shear on convection boundaries may then become significant, especially in their final nuclear stages where one expects rapid structural changes on timescales equal to a fraction of the hydrogen burning stage.

To test the possible role of shear in fixing the radial extent of the unstable convective layers, we applied criterion (20) on the

[2] A similar analysis applied to the case of a uniform rotation law gives the conditions *i)* $\nabla_\mu < -\frac{\varphi_{rot}}{\varphi}$ *ii)* $\nabla > \nabla' + \frac{\varphi}{\delta}\nabla_\mu + \frac{\varphi_{rot}}{\varphi}$, where $\varphi_{rot} = 4(\tau_{dyn}/\tau_{rot})^2$.

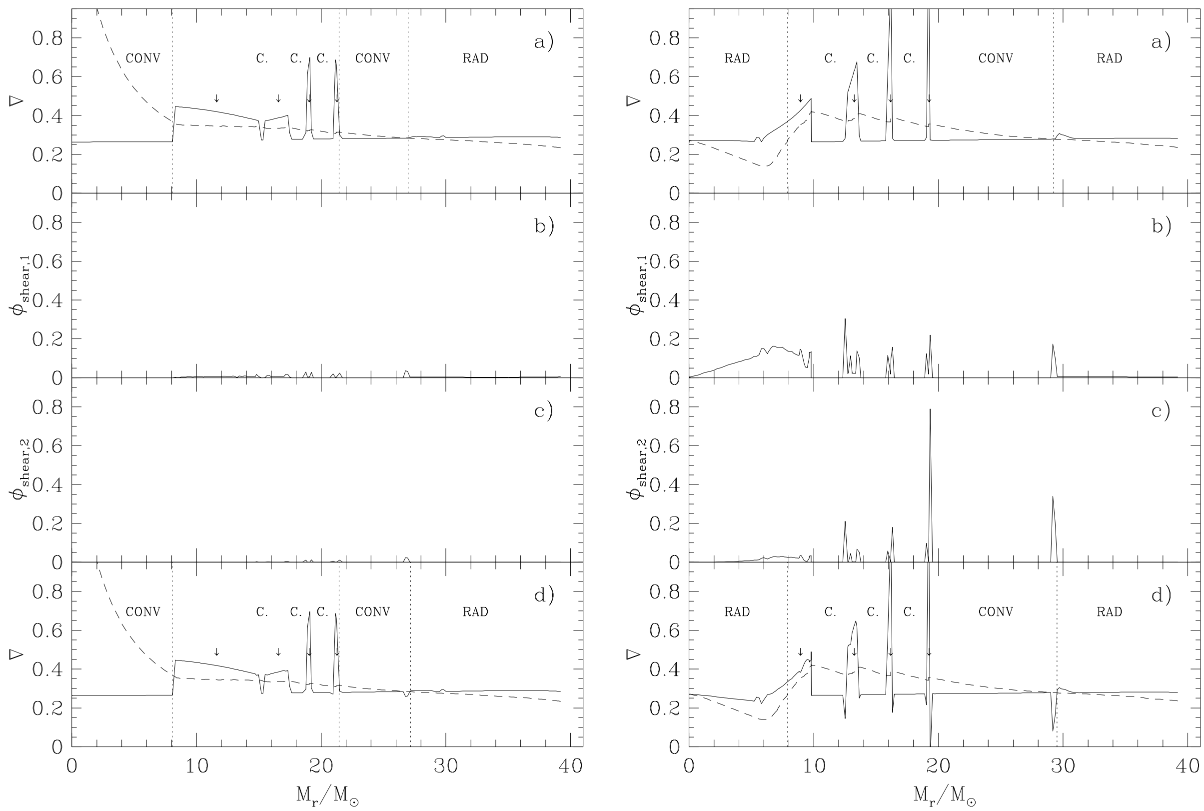

Fig. 1a–d. Profiles of various quantities inside a 40 $M_\odot$, rotating model nearing (left-hand side, $X_c = 0.19$) and at the end (right panel, $X_c = 0$) of the H-burning phase: **a** Profiles of $\nabla_{ad} + \frac{\varphi}{\delta}\nabla_\mu$ (solid line) and of ∇_{rad} (short dashed line). Arrows indicate semiconvective regions; **b** and **c** Profiles of the shear coefficients $\varphi_{shear,1}$ and $\varphi_{shear,2}$, respectively; **d** As for panel **a**, but where the solid line traces now the total effective gradient $\nabla_{ad} + \frac{\varphi}{\delta}\nabla_\mu - \frac{1}{\delta}(\varphi_{shear,1} + \varphi_{shear,2})$.

internal structure solution of a 40 $M_\odot$ star as obtained by Meynet & Maeder (1996). Their calculations account for both the hydrostatic corrections and first order effects introduced by rotational mixing. The starting model on the ZAMS is a uniformly rotating star with $\Omega/\Omega_c = 0.9$ (Ω_c is the break-up angular velocity at the surface) and metallicity $Z = 0.020$. During most of its MS lifetime, the angular velocity profile keeps sufficiently shallow so that no discernible differences are observed with respect to the standard analysis. However, nearing the end of the hydrogen burning phase the shearing terms $\varphi_{shear,i}$ grow sufficiently in the semiconvective regions and the radiative core slightly affecting the predicted convective boundaries. On Fig. 1, we present the radial profiles of the relevant physical quantities inside the models with central hydrogen mass fraction $X_c = 0.19$ and $X_c = 0$. In the more evolved model, the larger effect is observed at the upper boundary of the more external convective shell, where shear turns 0.3 $M_\odot$ into convective equilibrium. An additional correction consists in a general mild softening of the μ-barriers in what are actually semiconvective regions (see next section).

3.3. Semiconvective shear zones

The Richardson criterion imposes a necessary, but not sufficient condition on the horizontal velocity stratification for a radiative medium to become unstable to shear generated waves, which would then transfer energy vertically and thus modify the local entropy gradient. According to our linear stability analisys, a region that satisfies

$$\tau_{shear}^{-2} Ri(\mu \neq 0) = \tau_{shear}^{-2} Ri\ (\mu = 0) + \frac{g\varphi}{H_p}\nabla_\mu \ < \ \alpha \frac{2}{\pi^2}\left(z\frac{\partial\Omega}{\partial z}\right)^2$$

is convectively unstable, even if the Ledoux criterion is not verified. For an intermediate situation with $Ri(\mu \neq 0) < 2\alpha/\pi^2 < Ri(\mu = 0)$, the available shear energy is not sufficient to completely overturn the stable thermal gradient, but we can suspect that this excess energy may feed a turbulent field, and that some fraction of the turbulent kinetic energy production is used for mixing, the rest being dissipated by viscosity or transferred to global internal wave motions which do little mixing or none at all. Indeed, if it were not for the imaginary term A_1, criterion (13) for overstable convection would bear Schwarzschild inequality, as in many others situations (for example, if $\tau_{rot} \to \infty$), and one would thereby recover in a natural way the working hypothesis of Maeder in the regime $Ri(N_{ad}^2) < Ri < Ri(N^2)$. Actually, the shear term A_1 slightly weakens Schwarzschild inequality. This result should not come as a surprise since the modified Schwarzschild and Ledoux criteria must converge in the limit for a vanishing ∇_μ (see Eq. 20).

To exactly solve (11) for $n_r(x)$, with x such to maximize the growth rate, we must differentiate both equations with respect to x and assume $dn_r/dx = 0$. The resulting four expressions with Eqs. (4) and (18) provide six equations for the unknowns $n_r, n_i, \partial n_i/\partial x, x, \nabla$ and Γ. The case $A_1 = 0$ provides the initial guess for a Newton-Raphson algorithm.

A useful limiting case can be studied analytically. Note in fact that in the linear coefficients (9) there appear terms $\propto (kz)^{-1} \propto H_p/z$. As we approach the surface, H_p decreases and it becomes worthwhile to study the asymptotic solution $H_p/z \to 0$. One then recovers the Schwarzschild criterion for overstable convection, and a modified diffusion coefficient in presence of shear

$$D = \frac{\pi}{3}\chi \frac{\nabla - \nabla_{ad}}{\nabla_{ad} - \nabla + \frac{\varphi}{\delta}\frac{\Gamma+1}{\Gamma}(\nabla_\mu - \frac{\varphi_{shear,1}}{\varphi})} \tag{22}$$

Since the corresponding Ledoux stability criterion now reads $\nabla' - \nabla + \frac{\phi}{\delta}\nabla_\mu - \frac{\phi_{shear.1}}{\delta} > 0$ for semiconvection, the coefficient (22) is consistently positive defined even when the μ-barrier vanishes. Nearing the convective boundary one also obtains $D \to \infty$. At the radiative boundary, however, since A_1 is zero in this approximation, one should employ the classical Schwarzschild criterion to ensure $D > 0$. As anticipated, the role of shear in lowering the μ-barriers is also reflected in an increased magnitude for the diffusion coefficient. As a first approximation however (the plane parallel limit), the expected correction is rather limited.

4. Conclusion

Astrophysical turbulence is an intricate phenomenon specifically for the high Reynolds numbers encountered in stellar interiors that allow a wide spectra of eddies to coexist with large scale, coherent motions. The complex 3D character of the ensuing dynamical pattern goes well beyond the grasp of present day analytical modelling. In this paper, we bounded to a simple, ideal case to show that semiconvective shear zones as recently defined and suggested as a working hypothesis by Maeder (1996), are a natural prediction of standard perturbation theory. Of course, the extension of this result to real systems of more general geometry cannot be assured, also because of the possible influence of restoring forces other than buoyancy, inertial and radiative damping effects, as presently considered (e.g. magnetic fields). Indeed, diffusion is such a fragile transport mechanism that a wealth of secondary processes can easily modify its effects.

According to the linear analysis, overadiabatic stellar layers stable to the modified Ledoux criterion are semiconvective, whilst stellar regions stable to the Schwarzschild criterion can become convective for sufficiently large shears. Steep chemical gradients in rapidly rotating stellar interiors are no longer an insuperable obstacle for the products of stellar nucleosynthesis, and can diffuse to the surface sustained by a slightly increased diffusion coefficient and spatially reduced semiconvective and radiative stable layers. Differential rotation may thus appear as the only viable mechanism capable to contrast large μ-barriers, allowing for an appreciable enrichment in metals at the surface of fast rotating massive stars within their MS lifetime.

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Appendix A: derivation of the perturbation equations

Consider a vertically stratified medium $T = T(\lambda_j x_j)$ with unit vector $\lambda_j = (0,0,1)$, and similar dependences for the other pertinent physical quantities. The general Navier-Stokes equation for a fluid with constant coefficient of viscosity $\nu\rho$ (ν is the kinematic viscosity), in presence of chemical stratification and rotation reads

$$\begin{aligned}\frac{\partial v_k}{\partial t} + v_i\frac{\partial v_k}{\partial x_i} &= -\frac{1}{\rho}\frac{\partial p}{\partial x_k} + \nu\nabla^2 v_k - g\lambda_k + g\lambda_k\frac{\delta\rho}{\rho} \\ &\quad + \epsilon_{kji}\epsilon_{ilm}\Omega_j\Omega_l x_m + 2\epsilon_{kji}\Omega_j v_i\end{aligned} \tag{A1}$$

where ϵ_{ijk} is the Levi-Civita pseudotensor. The heat equation, assuming a spatially homogeneous thermometric conductivity χ, has the standard form

$$\frac{\partial T}{\partial t} + v_i\frac{\partial T}{\partial x_i} = \chi\nabla^2 T \tag{A2}$$

In Eq. A1 the Archimedian term accounting for μ-gradients will be written as

$$g\lambda_k\delta\rho = g\lambda_k\left(\frac{\partial\rho}{\partial\mu_0}\right)_{p,T}\delta\mu_0 = \frac{g\rho}{\mu_0}\varphi\delta\mu_0\lambda_k$$

where $\varphi = (\partial ln\rho/\partial ln\mu)_{p,T}$. In the Boussinesq approximation, one demands that any density contrast only build up as the consequence of moderate temperature fluctuations, then enabling us to take an equation of state of the form $\rho = \rho_0[1 - \Gamma_T(T - T_0)]$, where Γ_T is the coefficient of thermal expansion, and ρ_0, T_0 the average density and temperature of the surroundings. This condition implies a Mach number $v_{turb}/v_{sound} < 1$. If T, $v_k(=0)$, p, μ_0 and Ω are the temperature, velocity components, pressure, molecular weight and angular frequency in the unperturbed configuration, and $T' = T + \theta$, $v_k = u_k$, $p' = p + \delta p$, $\mu_0' = \mu_0 + \delta\mu$ and $\Omega' = \Omega + \varpi$ are the fluctuating quantities with θ, u_k, δp, $\delta\mu$ and ϖ lower order corrections, the perturbation equations in the linear limit become

$$\begin{aligned}\frac{\partial u_k}{\partial t} &= -\frac{1}{\rho}\frac{\partial\delta p}{\partial x_k} + \nu\nabla^2 u_k + \Gamma_T\theta g\lambda_k - \frac{g\varphi}{\mu_0}\mu\lambda_k \\ &\quad -2\epsilon_{kij}\Omega_i u_j + \epsilon_{kji}\epsilon_{ilm}\left(\Omega_j\varpi_l + \Omega_l\varpi_j\right)x_m\end{aligned} \tag{A3}$$

$$\frac{\partial\theta}{\partial t} = \chi\nabla^2\theta + \beta u_k\lambda_k \tag{A4}$$

We can eliminate the disturbing term $\delta p/\rho$ by applying the operator $\epsilon_{lmk}\frac{\partial}{\partial x_m}$ to Eq. A3. If we define $\omega_l = \epsilon_{lmk}\frac{\partial u_k}{\partial x_m}$ as the vorticity,

$$\frac{\partial\omega_l}{\partial t} = \nu\nabla^2\omega_l + \Gamma_T g\epsilon_{lmk}\lambda_k - \frac{g\varphi}{\mu_0}\lambda_k\epsilon_{lmk}\frac{\partial\mu}{\partial x_m}$$

$$-2\epsilon_{lmk}\epsilon_{kij}\Omega_i\frac{\partial u_j}{\partial x_m}+\epsilon_{lmk}\left(\Omega_m\varpi_k+\Omega_k\varpi_m\right)$$
$$+\epsilon_{lmk}\left\{\frac{\partial\Omega_n}{\partial x_m}\varpi_k x_n+\frac{\partial\Omega_k}{\partial x_m}\varpi_n x_n-2\frac{\partial\Omega_n}{\partial x_m}\varpi_n x_k\right.$$
$$\left.+(\varpi\Leftrightarrow\Omega)\right\} \quad \text{(A5)}$$

Acting once more on this latter equality with $\epsilon_{lij}\frac{\partial}{\partial x_j}$ and exploiting the well known identity $\epsilon_{lmk}\epsilon_{lij}=(\delta_{mi}\delta_{kj}-\delta_{mj}\delta_{ki})$ one next finds

$$\frac{\partial}{\partial t}\nabla^2 u_i = \nu\nabla^4 u_i-\Gamma_T g\left[\lambda_j\frac{\partial^2}{\partial x_j\partial x_i}-\lambda_i\frac{\partial^2}{\partial x_j^2}\right]\theta$$
$$+\frac{g\varphi}{\mu_0}\left[\lambda_j\frac{\partial^2}{\partial x_j\partial x_i}-\lambda_i\frac{\partial^2}{\partial x_j^2}\right]\mu-2\Omega_m\frac{\partial\omega_i}{\partial x_m}$$
$$-\left\{2\Omega_n\left(\frac{\partial^2\varpi_n}{\partial x_j^2}x_i-\frac{\partial^2\varpi_n}{\partial x_i x_j}x_j\right)-\Omega_i\left(\frac{\partial^2\varpi_n}{\partial x_j^2}+\frac{\partial\varpi_j}{\partial x_j}\right)\right.$$
$$+\Omega_j\left(\frac{\partial^2\varpi_n}{\partial x_i x_j}x_n-2\frac{\partial\varpi_j}{\partial x_i}-\frac{\partial\varpi_i}{\partial x_j}\right)+\frac{\partial\Omega_n}{\partial x_i}\frac{\partial\varpi_j}{\partial x_j}x_n$$
$$+\frac{\partial\Omega_j}{\partial x_i}\frac{\partial\varpi_n}{\partial x_j}x_n+2\left(\frac{\partial\Omega_n}{\partial x_j}\frac{\partial\varpi_n}{\partial x_j}x_i-\frac{\partial\Omega_n}{\partial x_i}\frac{\partial\varpi_n}{\partial x_j}x_j\right.$$
$$\left.\left.-\frac{\partial\Omega_n}{\partial x_j}\frac{\partial\varpi_i}{\partial x_j}x_n\right)+(\varpi\Leftrightarrow\Omega)\right\}$$
$$-\left\{\frac{\partial^2\Omega_j}{\partial x_i x_j}-\frac{\partial^2\Omega_i}{\partial x_j^2}+(\varpi\Leftrightarrow\Omega)\right\}\varpi_n x_n \quad \text{(A6)}$$

Finally, project expressions A5 and A6 onto λ_l and λ_i, respectively, and profit of the special geometry of the problem to simplify the numerous terms and obtain

$$\frac{\partial}{\partial t}\nabla^2 w = g\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\Gamma_T\theta-\frac{\varphi}{\mu_0}\mu\right)+\nu\nabla^4 w$$
$$+2\varpi\frac{\partial\Omega}{\partial z}+2\Omega\frac{\partial\varpi}{\partial z}-2\Omega z\nabla^2\varpi \quad \text{(A7)}$$
$$\frac{\partial\zeta}{\partial t} = \nu\nabla^2\zeta+2\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}w \quad \text{(A8)}$$

where $\zeta=\omega_i\lambda_i$ and $w=u_i\lambda_i$ are the z-components of the vorticity and the velocity. Eqs. A7, A8 and A4, supplemented by the closures 9 and 6, constitute the final set of perturbation equations.

Appendix B: the dispersion equation for the growth rate

Assume that the field variables can be expanded in normal modes

$$w=W(z)e^{nt+i\boldsymbol{k}_\perp\boldsymbol{r}} \qquad \theta=\Theta(z)e^{nt+i\boldsymbol{k}_\perp\boldsymbol{r}} \quad \text{(B1)}$$
$$\mu=M(z)e^{nt+i\boldsymbol{k}_\perp\boldsymbol{r}} \qquad \zeta=Z(z)e^{nt+i\boldsymbol{k}_\perp\boldsymbol{r}}$$
$$\varpi=O(z)e^{nt+i\boldsymbol{k}_\perp\boldsymbol{r}}$$

The operators appearing in the perturbation equations become

$$\partial_t \mapsto n(k) \qquad \nabla^2\mapsto\partial^2_{z^2}-k_\perp^2\equiv D^2-k_\perp^2$$
$$\nabla \mapsto D+ik_\perp \qquad \nabla^2\mapsto D^2-k_\perp^2\equiv\Delta$$
$$\boldsymbol{\Omega}\cdot\boldsymbol{\nabla} \mapsto i\boldsymbol{\Omega}_\perp\boldsymbol{k}_\perp+\Omega_3 D\equiv D_\Omega$$

and lead to the following system

$$n\Delta W = -gk_\perp^2\Gamma_T\Theta+gk_\perp^2\frac{\varphi}{\mu_0}M+2k_\perp^2\Omega zO$$
$$+\nu\Delta^2 W-2D_\Omega Z+2O\frac{\partial\Omega}{\partial z}+2\Omega DO \quad \text{(B2)}$$
$$(n-\nu\Delta)\,Z = 2D_\Omega W \quad \text{(B3)}$$
$$(n-\chi\Delta)\,\Theta = \beta W \quad \text{(B4)}$$
$$nM = \frac{\mu_0}{H_p}\nabla_\mu W \quad \text{(B5)}$$
$$nO = -\frac{1}{z^2}\frac{\partial(z^2\Omega)}{\partial z}W \quad \text{(B6)}$$

Eliminate Θ in B1 by operating with $(n-\chi\Delta)$ and using B3, eliminate Z by operating with $(n-\nu\Delta)$ and using B2, and finally eliminate both M and O by operating with n and using B4-B5. The resulting identity is further developed on the hypothesis that the vertical spatial variation of W is given by $\exp(ik_3 z)$, which implies

$$D\mapsto ik_3 \qquad \Delta\mapsto -k^2 \qquad D_\Omega\mapsto i\boldsymbol{k\Omega}$$

After a few algebraic manipulations one eventually obtains the desired dispersion equation

$$\left(n+\nu k^2\right)\left\{n\left(n+\chi k^2\right)\left(n+\nu k^2\right)-g\frac{k_\perp^2}{k^2}\Gamma_T\beta n\right.$$
$$+g\frac{k_\perp^2}{k^2}\frac{\varphi}{H_p}\nabla_\mu\left(n+\chi k^2\right)-2\frac{k_\perp^2}{k^2}\frac{\Omega}{z}\left(n+\chi k^2\right)\frac{\partial(z^2\Omega)}{\partial z}$$
$$\left.-\frac{2}{z^2k^2}\frac{\partial\Omega}{\partial z}\frac{\partial(z^2\Omega)}{\partial z}\left(n+\chi k^2\right)-2i\frac{k_3\Omega}{z^2k^2}\left(n+\chi k^2\right)\frac{\partial(z^2\Omega)}{\partial z}\right\}$$
$$+4n\left(n+\chi k^2\right)\frac{(\boldsymbol{k\Omega})^2}{k^2}=0 \quad \text{(B7)}$$

Setting $\nu=0$, $\boldsymbol{k\Omega}=k_\perp\Omega$ and rearranging, we recover the form (8) employed in our linear stability analysis.

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