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Theory of motion and ephemerides of Hyperion*

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Abstract. We present here a new theory of motion for Hyperion, the seventh major satellite of Saturn. The Hyperion's motion is defined like in TASS1.6 for the other satellites (Vienne & Duriez 1995), by the osculating saturnicentric orbital elements referred to the equatorial plane of Saturn and to the node of this plane in the mean ecliptic for J2000.0. These elements are expressed as semi-numerical trigonometric series in which the argument of each term is given as an integer combination of 7 natural fundamental arguments. These series collect all the perturbations caused by Titan on the orbital elements of Hyperion, whose amplitudes are larger than 1 km in the long-period terms and than 5 km in the short-period ones. However, the convergence of these series is so slow that, in spite of several hundreds of terms, their internal accuracy over one century is about 200 km only. These series have been constructed in such a way that the fundamental arguments and the amplitude of each term depend explicitly on 13 parameters (the twelve initial conditions of the motions of Titan and Hyperion and the mass of Titan). Taking also account of the perturbations from other satellites and Sun, we have fitted these series to 8136 Earth-based observations of Hyperion in the interval [1874-1985], giving a set of better values for these parameters. In particular the mass of Titan is found equal to $(237.399 \pm 0.005)\,10^{-6}$ (in units of the Saturn's mass) and we discuss this value in comparison with that $[(236.638 \pm 0.008)\,10^{-6}]$ obtained by Campbell & Anderson from their analysis of the Voyager missions to Saturn. The resulting fitted series allows us to produce new ephemerides for Hyperion. Their comparison to those from Taylor (1992) shows that, with the same set of observations and the same way to weight them, we obtain a root mean square (o−c) residual of $0''.156$ while the ephemerides of Taylor gives $0''.203$.

Key words: celestial mechanics – planets and satellites – Hyperion – ephemerides

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* The full Tables 3 to 8 of this paper are also available by anonymous FTP at cdsarc.u-strasbg.fr or ftp 130.79.128.5

1. Introduction

In this paper, we use the classical notations (a, n, e, i, Ω, ϖ, λ) for the osculating elements of the Saturn's satellites, referred to the centre of Saturn, to its equatorial plane and to the ascending node of this plane on the mean ecliptic for J2000.0; in the following, this reference frame will be called SSE (as Saturnicentric-Saturn-Equatorial) and it will be considered as inertial. We use also N, the mean mean motion, and the other variables: $p = n/N - 1$, $q = \lambda - Nt$, $z = e \exp\sqrt{-1}\,\varpi$ and $\zeta = \sin(i/2) \exp\sqrt{-1}\,\Omega$. Hence a fixed semi-major axis A is defined from $N^2A^3 = n^2a^3 = k^2(M_s + m)$ where m and M_s are the masses of the satellite and of Saturn and k the gaussian constant of gravitation. Then, a, A and p are connected by $a = A(1 + p)^{-2/3}$. They are the same variables in the same reference frame as those used in TASS, the '**T**héorie **A**nalytique des **S**atellites de **S**aturne' constructed by us for the other major satellites of Saturn (Vienne & Duriez, 1995). The indexes 6 and 7 used below with these variables refer to variables of Titan and Hyperion respectively. Other indexes refer to other major satellites ordered by increasing distances to Saturn; in particular, the index 9 refers to the Sun, considered as a far satellite of Saturn.

Among the Saturn's satellites, the motion of Hyperion is special, because it is involved in a resonance 3 : 4 with Titan: The combination of the mean angular velocities $3N_6 - 4N_7 + \langle\dot\varpi_7\rangle$ is zero[1], leading to a large libration of the angle $\theta = 3\lambda_6 - 4\lambda_7 + \varpi_7$:

$$\theta = 3(\lambda_6 - \lambda_7) - M_7 \approx 180^\circ - 36^\circ \sin\tau - 13^\circ \sin\varphi + \cdots$$

The argument τ represents the libration, while φ is the linear part of $(\varpi_6 - \varpi_7)$. Their periods are about $640^{\rm d}$ and $6850^{\rm d}$ respectively; the period of the synodic longitude, corresponding to the frequency $N_6 - N_7$, is about $64^{\rm d}$, that is almost a long period when compared to the orbital periods of Titan ($16^{\rm d}$) and Hyperion ($21^{\rm d}$).

This resonance is stabilised by the relatively high eccentricity of Hyperion ($e_7 = 0.104 + 0.024\,\cos\varphi + \cdots$). In effect, this high eccentricity and the high value of a_6/a_7 (≈ 0.85) make that the orbits of Titan and Hyperion are almost intersecting ones,

[1] elsewhere below, $\langle x\rangle$ stands for "mean value of x"

but the libration of θ around 180°shows also that the conjunctions of the two satellites occur only when the mean anomaly M_7 of Hyperion is near of 180°, that is near of the apocentre of its orbit, where the two orbits are the most distant, thus preventing too close approaches of the two satellites. However, the classical expansion of the disturbing function of Hyperion by Titan (in Laplace coefficients and Fourier series in the mean longitudes) does not converge sufficiently to be usable when one does not take account of this resonant feature. Instead, the perturbations must be expanded directly by Fourier series in θ or τ, as it was made numerically by Woltjer (1928) or analytically by Message (1989). Besides, the last theories of the Hyperion's motion (Dourneau 1993; Taylor 1992) are simply adjustments of the Woltjer's expressions, completed for Taylor by a better determination of the short-period perturbations, but still limited to terms larger than 120 km (ie. larger than 0.″02 as seen from the Earth). In these works, the resonance of Hyperion is studied "in the plane", neglecting the influence of the inclinations on the resonance, and adding simply some of the largest perturbations to take account of the motion of the orbital plane. Besides, Message (1996) proposes a new analytical representation of this last motion.

The present work resumes completely the interaction Titan-Hyperion in space and tries to represent the Hyperion's motion with the same accuracy as that obtained in TASS for the other satellites: Ultimately, we hope an accuracy as high as some kilometres on the satellites' positions in order to be able to take account of their observations by the next CASSINI mission.

To succeed in this work, we have adopted a numerical approach : We have made the numerical integration of the equations of motion, with initial conditions corresponding to the real resonant feature of the interaction "Titan-Hyperion"; the equations take also account of the secular perturbations coming from the other satellites, from the Sun and from the oblateness of Saturn. Then, from the resulting time-series, a frequency analyser reconstructs numerically a Fourier representation of the motions of Titan and Hyperion.

The time-span of the numerical integration has been chosen sufficiently long (about 1500 years) to allow the frequency analyser to discriminate the frequencies and the phases of the 7 natural fundamental arguments of the dynamic system concerned by the only interaction between Titan and Hyperion:

$\psi = (N_6 - N_7)\,t + \psi_0$	synodic, period:	63^{d}
$\tau = \dot\tau\,t + \tau_0$	libration	640^{d}
$\varpi_7^* = \langle\dot\varpi_7\rangle t + \varpi_{07}$	Hyperion's perisaturn	$-7\,041^{\mathrm{d}}$
$\varpi_6^* = \langle\dot\varpi_6\rangle t + \varpi_{06}$	Titan's perisaturn	$254\,935^{\mathrm{d}}$
$\Omega_7^* = \langle\dot\Omega_7\rangle t + \Omega_{07}$	Hyperion's node	$-55\,302^{\mathrm{d}}$
$\Omega_6^* = \langle\dot\Omega_6\rangle t + \Omega_{06}$	Titan's node	$-255\,037^{\mathrm{d}}$
$\Omega_0 = \;\; 0 \;\; + \Omega_{00}$	invariable plane	∞

The last argument is the node of the invariable plane of the Saturnian satellites system, inclined with an angle of 0.°32 on the equatorial plane of the planet and already present in the Titan's motion. The periods close to 700 years ($\approx 255\,000^{\mathrm{d}}$) come also from the observed Titan's motion. We note also that, while in the plane problem one of the arguments could be $\varphi \approx \varpi_6^* - \varpi_7^*$, in the space problem, we have to consider ϖ_6^* and ϖ_7^* separately.

The frequency analyser has been implemented in order to reconstruct the orbital elements of Titan and Hyperion as series of periodical terms, in such a way that we have been able to detect all long-period terms greater than 1 km, and all short-period terms exceeding 5 km. All their arguments have been identified as integer combinations of the 7 above fundamental arguments. Among these terms, we have found in particular new important long-period terms in the mean longitude of Hyperion, showing a not negligible influence of the inclinations on the resonance.

Then, by varying the mass of Titan and the twelve initial conditions of the numerical integration in the vicinity of nominal values, we have succeeded in computing a synthetic representation of the elements of Hyperion, giving a nominal solution plus its partial derivatives with respect to these thirteen parameters. Adding also the solar perturbations and the short-period perturbations coming from other satellites and from the Saturn's oblateness, these series allow to compute positions of Hyperion at any date, with their variations in function of the 13 parameters. Thus, using TASS1.6 to compute the positions of other satellites, we have adjusted these parameters on 8136 Earth-based observations of Hyperion in the interval [1874-1985], rejecting observations with (o−c) residuals larger than 1″. That gives a set of better values for the parameters, which may be adopted as new nominal values.

Hence, the whole procedure (numerical integration, frequency analysis, partial derivatives, adjustment of the parameters to observations) has been iterated until the corrections to the parameters become lesser than their standard error given by the least-square adjustment. Finally, after 4 iterations, we have obtained a new representation of the Hyperion's elements, and a new value of the mass of Titan: $(237.399 \pm 0.005)\ 10^{-6}$ in units of the Saturn's mass. However this value differs notably from that $[(236.638 \pm 0.008)\ 10^{-6}]$ obtained by Campbell & Anderson from their analysis of the Voyager missions to Saturn; that could comes from a lack of coherence between the present representation of the Hyperion's motion and the representations of other satellites given by TASS1.6, as discussed in the Sect. 3.3. The last adjustment to observations is finally summarised in the Table 9.

At last, the ephemerides of Hyperion obtained from these series have been compared to those from Taylor (1992). The differences on positions given by each theory, over one century, have been discussed in Duriez & Vienne (1997). We have also compared both theories to about 4000 recent observations of Hyperion (made after 1966): The (o−c) residuals computed from each theory and coming from the same set of observations are presented in the Table 10, in form of root-mean-square residuals of some major data sets of observations; it allows to appreciate the progress of our theory with respect to that of Taylor, but also the progress which are still necessary to represent the observations at best.

To preserve space, only the largest terms of this representation are given in the Tables 4 to 8. The full series are available by anonymous FTP from the server ftp.bdl.fr, in form of plain TEX and ASCII files including also the FORTRAN program to compute, from them, the osculating orbital elements of Hyperion. The full Tables 3 to 8 of this paper are also available by anonymous FTP at cdsarc.u-strasbg.fr or ftp 130.79.128.5

2. The interaction Titan-Hyperion

The interaction between Titan and Hyperion is given, strictly speaking, by the equations of the 3-body problem applied to Saturn-Titan-Hyperion. However, to obtain the correct values of the natural fundamental frequencies present in the observed motions, and in particular those coming from the secular motions, we must add to these equations the parts giving the secular influence of other satellites, of the Sun and of the Saturn's oblateness. We have chosen to use the Lagrange's equations for the variations of the osculating elements p, q, z et ζ, because the secular variations of these elements are well known analytically and hence, it is easy to add them to the principal part coming from the strict interaction Titan-Hyperion.

To have exact equations for this strict interaction, we have used the Lagrange's equations expressed in the closed form given by Chapront et al. (1975) for the variations of a, λ (or ϵ), z et ζ in the N-body problem. With our variables, the equations for p and q are simply $\frac{dp}{dt} = -\frac{3(1+p)}{2a}\frac{da}{dt}$ instead of $\frac{da}{dt}$, and $\frac{dq}{dt} = Np + \frac{d\varepsilon}{dt}$ instead of $\frac{d\lambda}{dt} = n + \frac{d\epsilon}{dt}$. We add to these equations the analytical expressions representing the secular variations of the corresponding elements which come from other satellites, from the Sun and from the Saturn's oblateness. These expressions are obtained to the fourth degree in the eccentricities and inclinations by the method elaborated for the planets by Duriez (1977, 1979), and to the second order in the m_i, J_2, J_4 and J_6 by the method described in Laskar (1985). From these expansions, we have retained the largest terms only, which are in fact terms depending on the variables z and ζ of Titan and Hyperion exclusively. They depend on the semi-major axes of other satellites, but these are considered as constant. Thus no variable related to other satellites exists in the equations, leading to an easier numerical integration, and the fundamental arguments in the interaction Titan-Hyperion remain the seven ones cited above. These secular variations are given as $(\dot{q}_6)^*$, $(\dot{z}_6)^*$, $(\dot{\zeta}_6)^*$, $(\dot{q}_7)^*$, $(\dot{z}_7)^*$ and $(\dot{\zeta}_7)^*$ in the following formulas (expressed in radian per day):

$$(\dot{q}_6)^* = 4.085063\,10^{-5} + 8.3022\,10^{-5}\, z_6\bar{z}_6 - 3.3227\,10^{-4}\,\zeta_6\bar{\zeta}_6$$
$$(\dot{z}_6)^* = 2.463958\,10^{-5}\, z_6$$
$$(\dot{\zeta}_6)^* = -2.463818\,10^{-5}\,\zeta_6$$
$$(\dot{q}_7)^* = 2.754399\,10^{-4} + 4.1090\,10^{-5}\, z_7\bar{z}_7 - 1.6448\,10^{-4}\,\zeta_7\bar{\zeta}_7$$
$$(\dot{z}_7)^* = 1.309076\,10^{-5}\, z_7 + 2.3927\,10^{-5}\, z_7^2\bar{z}_7$$
$$(\dot{\zeta}_7)^* = -1.309028\,10^{-5}\,\zeta_7$$

The coefficients of these terms are computed numerically using the values of the mean mean motions, of the satellites' masses and of J_2, J_4, J_6 given in TASS1.6 (issued from the adjustment to observations, of the theory of motion of all the satellites except Hyperion (Vienne & Duriez 1995)). The dependence of these terms with respect to the physical parameters is known explicitly. For example the coefficient of the first term in $(\dot{z}_7)^*$ is in fact :

$$1.309076\,10^{-5} + 1.1963\,10^{-5}\,\delta J_2/J_2 + 8.62\,10^{-7}\,\delta m_9/m_9 + 1.59\,10^{-7}\,\delta m_5/m_5 + 8.4\,10^{-8}\,\delta m_8/m_8 + \cdots$$

where the coefficients of δJ_2, δm_9, δm_5 and δm_8 show the contributions of J_2 and of the masses of Sun, Rhea and Iapetus into the first term.

Table 1. First and final values of the initial conditions of the numerical integration, ie. osculating elements of Titan and Hyperion at the Julian date 2418800.5 in our SSE reference frame. The standard errors (s.e.) come from the last adjustment of these values as explained in Sect 3.2

	first	final	s.e.
p_6	$-1.3927232\,10^{-4}$	$-1.3940119\,10^{-4}$	$7.3\,10^{-9}$
q_6	2.36978736	2.36992933	$6.6\,10^{-5}$
Re(z_6)	$-1.3726244\,10^{-2}$	$-1.3448636\,10^{-2}$	$1.5\,10^{-5}$
Im(z_6)	$2.5607472\,10^{-2}$	$2.5642512\,10^{-2}$	$1.0\,10^{-5}$
Re(ζ_6)	$-3.1166876\,10^{-3}$	$-3.5146556\,10^{-3}$	$8.1\,10^{-6}$
Im(ζ_6)	$9.8670731\,10^{-4}$	$3.9082453\,10^{-4}$	$9.1\,10^{-6}$
p_7	$2.9865815\,10^{-3}$	$2.5441298\,10^{-3}$	$1.1\,10^{-6}$
q_7	4.55700187	4.56312782	$3.9\,10^{-5}$
Re(z_7)	$2.5731133\,10^{-2}$	$2.5543410\,10^{-2}$	$1.9\,10^{-5}$
Im(z_7)	0.11424323	0.11528283	$1.2\,10^{-5}$
Re(ζ_7)	$-2.2571768\,10^{-3}$	$-2.16396910\,10^{-3}$	$1.6\,10^{-5}$
Im(ζ_7)	$5.9954034\,10^{-3}$	$6.10895764\,10^{-3}$	$1.7\,10^{-5}$

As central mass, we have used the mass of Saturn determined by Campbell & Anderson (1989) from their analysis of the Voyager missions: $M_s = 1/3498.790\ M_\odot$. For the perturbations of Titan by Hyperion, the mass of Hyperion has been taken equal to $3\ 10^{-8}\ M_s$, from Burns (1986). At last, before the first adjustment, the mass of Titan has been taken equal to $236.638\ 10^{-6}$ from Campbell & Anderson (1989).

2.1. Numerical integration

The initial conditions of the numerical integration of the Lagrange's equations are the osculating saturnicentric elements of Titan and Hyperion for a given date in our SSE reference frame. For Hyperion, before adjustments, these are issued from the position and velocity given by Harper et al. (1989) for the Julian date t_0 = 2418800.5 in the B1950 reference frame, converted in osculating elements for this date in the SSE reference frame; this date corresponds to the mid time-span in which Harper et al. have adjusted a numerical integration of Titan, Hyperion and Iapetus to 1874-1943 observations. For Titan, the initial conditions of the numerical integration are given by TASS1.6 in the form of saturnicentric osculating elements directly referred to the SSE reference frame. These first initial conditions are given in Table 1, as well as those obtained at the fourth iteration.

The mean mean motions N_6 and N_7 used in the equations $\frac{dq_6}{dt}$ and $\frac{dq_7}{dt}$ are those given by observations: At the first iteration, we had N_6 = 0.39040425667 rad/d from TASS1.6 and N_7 = 0.2953088612 rad/d from Taylor (1992). These values occur in the equations as: $\frac{dq}{dt} = Np + \frac{d\epsilon}{dt}$ and $\lambda = Nt + q$. Hence, to have $\langle \frac{d\lambda}{dt} \rangle = N$, the constants parts p_{o6} and p_{o7} of the variables p_6 and p_7 should be adjusted so that the mean value $\langle Np + \frac{d\epsilon}{dt} \rangle$ becomes zero for Titan and Hyperion. In fact, these constants have been automatically adjusted when comparing the results of the integration to observations, allowing also to correct accordingly the mean mean motions.

To integrate the equations, we have used a 10th order Adams method, with a predictor-corrector scheme, with a step size equal to 0.1 day, allowing an accuracy of 10^{-10} with only one corrector pass per step. This has been verified to be sufficient, all the more so since the integration of the Lagrange's equations produces only the perturbations of the elements.

We have adapted the running time of the integration and the sampling output time of the results in order to allow the best use of the frequency analysis. Preliminary integrations and frequency analysis have shown that, for terms larger than 2 km, the lowest frequencies existing in the dynamics of Titan-Hyperion correspond to periods of about 700 years and the highest frequencies to periods larger than 3 days. The two following subsections explain why and how we have adopted a running time of 1507 years with a sampling output time of $22.4^{\rm d}$ for low-pass filtered time-series able to give the long-period part of the motions, and a running time of 93 years with a sampling output time of $1.4^{\rm d}$ for the time-series able to give the short period part.

2.1.1. Frequency analysis

Let us consider a time-series of $2n+1$ values $\{S(t_i)\}$ representing a quasi-periodic function $S(t)$ with a sampling interval Δt over a total time-span $D = 2n\Delta t$ (we have: $t_i = t_0 + i\,\Delta t$ for $0 \le i \le 2n$). In theory of digital signal processing, such a time-series can be analysed in frequencies by discrete Fourier transform with the frequency resolution: $\Delta\omega = 2\pi/(n\,\Delta t)$; then, a representation of $S(t)$ may be obtained in the form:

$$\tilde{S}(t) = \sum_j S_j \ \exp\sqrt{-1}\,\omega_j t$$

where ω_j is within the Nyquist interval: $|\omega_j| \le \frac{1}{2}\omega_s$ with the cut-off frequency $\omega_s = 2\pi/\Delta t = n\Delta\omega$. To avoid aliased terms, ω_s must be chosen so that the power spectrum of $S(t)$ shows only negligible terms for frequencies higher (in absolute value) than $\frac{1}{2}\omega_s$. In short, to represent a time-series depending on terms whose periods are between $T_{\rm min}$ and $T_{\rm max}$, we must have:

$$\begin{cases} \dfrac{2\pi}{T_{\rm max}} \ge \ \Delta\omega \ = \dfrac{2\pi}{D/2} & \Longrightarrow \qquad D \ge \ 2\,T_{\rm max} \\ \dfrac{2\pi}{T_{\rm min}} \le \ \frac{1}{2}\omega_s \ = \dfrac{\pi}{\Delta t} & \Longrightarrow \qquad \Delta t \le \ \dfrac{T_{\rm min}}{2} \end{cases}$$

With $T_{\rm min}$ = 3 days and $T_{\rm max}$ = 700 years, we obtain: $D \ge$ 1400 years and $\Delta t \le 1.5^{\rm d}$, and then also: $2n \approx 341\,000$; now, such a value is too large to be used in practice with our program of frequency analysis. This program is an implementation of the method described in Laskar et al. (1992) and its particularities are shortly given in Appendix A. To bypass the difficulty to work a too large number of sampled values, we have separated the problem in two parts:

First, we apply a low-pass digital filter on the time-series produced by the numerical integration in such a way that $T_{\rm min}$ becomes about $45^{\rm d}$ instead of $3^{\rm d}$: With a time-span of 1507.2 years and a sampling interval $\delta t = 22.4^{\rm d}$, we have now a smaller value : $2n$ = 24576 (for technical reasons, $2n$ must be 3 times a power of 2, here 3×2^{13}); in return, the frequency analysis of the filtered series allows to represent only the long-period terms, with periods longer than $44.8^{\rm d}$. This cut-off period has been chosen because, in the Titan-Hyperion interaction, we have found that a large gap exists in the spectrum of the time-series, between 38 days and 45 days, where all terms have negligible amplitudes (much smaller than 1 km). This gap separates the synodic terms (ψ-terms) around the period of $64^{\rm d}$, from those of its second harmonic (2ψ-terms) around the period of $32^{\rm d}$.

Second, we keep also the not-filtered time-series over 93 years with a sampling interval $\delta t = 1.4^{\rm d}$ (corresponding to the same value $2n$ = 24576). We shall see below how that allows us to analyse the short-period terms of the motions, that is with periods between $2.8^{\rm d}$ and $38^{\rm d}$.

2.1.2. Digital filtering

The filtering method used in this work is the same as that used by Carpino et al. (1987) to integrate and to analyse the motions of outer planets over 10 millions years. In short, applying such a filter consists in transforming the initial time-series $S(t_i)$ with a sampling Δt, in an other $S_1(t'_k)$ with a sampling δt, by convolution with $2p+1$ coefficients $\{f_j\}_{j=-p..p}$:

$$S_1(t'_k) = \sum_{j=-p}^{p} f_j\ S(t_{k,j})$$

with $t'_k = t_0 + p\Delta t + k\,\delta t$ and $t_{k,j} = t'_k + j\,\Delta t$. Of course, δt is a multiple of Δt.

It is known that such a convolution conserves the phases if the number of coefficients f_j is odd; these f_j are computed by the algorithm FIR given in Rabiner & Gold (1975), for a given ripple ρ and a given attenuation α: These coefficients are determined in order to preserve (or multiply by $1 \pm \epsilon_0$ with $0 \le \epsilon_0 \le \rho$) the amplitude of the terms whose frequency is lower than $\omega_0 = 2\pi/T_0$, and to delete (or multiply by $0 \pm \epsilon_1$ with $0 \le \epsilon_1 \le \alpha$) the terms whose frequency is higher than $\omega_1 = 2\pi/T_1$. The filter is not defined between ω_0 and ω_1. The number $2p+1$ of the coefficients f_j depends on ρ, α, ω_0/ω_s and ω_1/ω_s; we have constructed filters in such a way that $\rho \approx 10^{-6}$, $\alpha \approx 10^{-7}$ and $p < 100$. With such a ripple, the amplitude of the largest terms present in the Hyperion's elements, are conserved with an accuracy of about 0.3 km.

We could have taken $T_0 = T_{\rm min} = 105^{\rm d}$ in order to filter only all long period terms (including harmonics of the libration up to 6τ). In fact, because of the gap cited above between $45^{\rm d}$ and $38^{\rm d}$, we have found better to take $T_0 = T_{\rm min} = 45^{\rm d}$ so that the filter preserves also the terms associated to the ψ-term but destroys its higher harmonics. However, to delete all terms whose

Table 2. Number of terms (N) found in the representations of p_7, q_7, z_7 and ζ_7. The root mean square residuals (RMSR) and the maximum errors (ME) between these representations and the numerical integration from which they are issued, are computed over 1507 years with a step-size of 22.4^{d} for the long-period part and over 93 years with a step-size of 1.4^{d} for the short-period part respectively

	long-period part (terms ≥ 1 km)			short-period part (terms ≥ 5 km)		
element	N	RMSR (km)	ME (km)	N	RMSR (km)	ME (km)
p_7	39	2.34	11.1	65	19.34	144.1
q_7	111	4.09	19.5	103	22.08	194.1
z_7	79	3.18	20.1	100	22.63	168.3
ζ_7	47	2.73	11.5	4	9.19	60.4

Table 3. Fundamental arguments of the Titan-Hyperion interaction. Each one is in the form : $\omega_k^* t + \varphi_k^*$ where t = (Julian Date) − 2 451 545.0. The columns $\varepsilon(T_k^*)$ and $\varepsilon(\varphi_k^*)$ give the estimated error on each period and on each phase, issued from the last fit of the theory to observations as explained in Sect. 3.2

k	argument	ω_k^* (rad/d)	φ_k^* (rad)	period (d)	$\varepsilon(T_k^*)$ (d)	$\varepsilon(\varphi_k^*)$ (rad)
1	ψ	0.098733765027	1.379026808	63.6377	0.00008	0.00011
2	τ	0.009810539955	1.803677249	640.4525	0.03454	0.00100
3	ϖ_7^*	−0.000892481124	3.382691058	−7040.1324	3.98970	0.00038
4	ϖ_6^*	0.000024646231	2.860542690	254934.9313	0.00504	0.00063
5	Ω_7^*	−0.000113616050	3.864510578	−55301.9163	15.97624	0.00537
6	Ω_6^*	−0.000024636367	6.141812995	−255037.0101	0.01695	0.00915
7	Ω_0	0.000000000000	3.221557438			0.00001

period is shorter than $T_1 = 38^{\mathrm{d}}$, with a length of filtering $2p$ sufficiently small, we have used a two-stage filter, defined by:

	first stage	second stage
Δt	1.4^{d}	11.2^{d}
T_0	45^{d}	45^{d}
T_1	15^{d}	38^{d}
δt	11.2^{d}	22.4^{d}
$2p$	132	178

With these values, both filters work with a ripple $\rho = 9\ 10^{-7}$ and an attenuation $\alpha = 9\ 10^{-8}$.

To limit the size of the output files, these two filters are applied as and when generating the time-series in the numerical integration: Every 11.2^{d}, we apply the 1st filter on the time-series with a sampling of 1.4^{d}. Next, every 22.4^{d}, we apply the 2nd filter on the just filtered time-series with the new sampling of 11.2^{d}; finally, the resulting filtered time-series $S_1(t'_k)$ are saved on files to be used later by the program of frequency analysis. In the same time, to anticipate the analysis of the short period terms, we save also in other files the not-filtered time-series $S(t_i)$ with the sampling of 1.4^{d}, but over one 93 years only.

The full computation over 1500 years with a step size of 0.1^{d} and the filtering described above lasts about 18 minutes on a DEC alphastation 200 at 166 MHz.

2.2. Synthesis of a semi-analytical representation

Coming from this filtered numerical integration, we obtain for each element of Titan and Hyperion, a time-series $S(t)$ with a sampling of 1.4^{d} over about one century, and a filtered time-series $S_1(t)$ with a sampling of 22.4^{d} over about 1500 years. All these time-series represent quasi-periodic function of time, except those for q_6 and q_7 which may include a residual linear part in t; for these elements, we obtain this linear part by fitting the time-series to a line by least-square; subtracting it from the initial time-series returns a new one representing now a quasi-periodic function of time, ready to be analysed in frequencies.

The frequency analysis of $S_1(t)$ produces a representation $\tilde{S}_1(t)$ of the long-period part of $S(t)$, with periods longer than 45^{d}. Hence, we are able to compute the new time-series: $S(t_i) - \tilde{S}_1(t_i)$, with the same sampling than $S(t_i)$ (1.4^{d}) over 93 y. Their frequency analysis produces a representation $\tilde{S}_2(t)$ of the short-period part of $S(t)$, with periods between 2.8^{d} and 38^{d}. However, because the time-span is reduced to 93 years only, it becomes impossible to separate two terms when their frequencies are distant by less than $4\pi/93\ \mathrm{y}^{-1}$; this is not catastrophic because the ψ-terms, analysed with the long-period ones up to the kilometre level, show that those terms with very close frequencies depend in fact on several longitudes of nodes, but are present between 1 and 5 km only; because the amplitudes of the other $k\psi$-terms are generally decreasing when k grows, the short-period terms depending on the longitudes of nodes are certainly lesser than 5 km also.

Finally, we have been able to represent the interaction Titan-Hyperion in the elements p, q, z and ζ of each satellite, in the form:

$$\tilde{S}(t) = \tilde{S}_1(t) + \tilde{S}_2(t)$$

Table 4. Series for p_7 : We give here only the largest terms; each one is in the form $\alpha\cos(\omega t+\varphi)$ with t = (Julian Date) − 2 451 545.0. Each argument is also identified as an integer combination of the seven fundamental arguments given in Table 3; the values of ω and φ are in fact computed from these combinations. The osculating semi-major axis is then $a_7 = A_7(1+p_7)^{-2/3}$ where A_7 is a kind of mean value of a_7, computed from the third Kepler's law : $A_7 = (k^2(M_s+m_7)/N_7^2)^{1/3}$ with k the gaussian constant, with $M_s = M_\odot/3498.790$ (from Campbell & Anderson 1989) $m_7 = 3\ 10^{-8}\ M_s$ (from Burns 1986) and with the mean mean motion N_7 = 0.2953088139 rad/d. The amplitudes in km represent $-(2/3)A_7p_7$ with A_7 = 1482333.4 km. The estimated error $\varepsilon(\alpha)$ on each amplitude comes from the fit of the theory to observations as explained in Sect. 3.2

n°	amplitude α (rad)	phase φ (deg)	frequency ω (rad/d)	period (d)	argument	amplitude (km)	$\varepsilon(\alpha)$ (km)
0	−0.0015747	0.000	0.0000000000			1556.14	0.53
1	0.0052692	103.343	0.0098105400	640.45 *	τ	−5207.14	5.30
2	−0.0009448	79.012	0.0987337650	63.64 *	ψ	933.67	0.26
3	−0.0006016	335.669	0.0889232251	70.66 *	$\psi-\tau$	594.52	0.61
4	0.0005148	182.356	0.1085443050	57.89 *	$\psi+\tau$	−508.75	0.52
5	−0.0001310	73.426	0.0107276673	585.70 *	$\tau-\varpi_7^*+\varpi_6^*$	129.44	0.23
6	0.0001186	133.260	0.0088934126	706.50 *	$\tau+\varpi_7^*-\varpi_6^*$	−117.21	0.21
...							
17	0.0000101	212.272	0.1076271776	58.38	$\psi+\tau+\varpi_7^*-\varpi_6^*$	−10.01	0.02
...							
39	−0.0000013	219.194	0.1084306889	57.95	$\psi+\tau+\Omega_7^*-\Omega_0$	1.33	0.01
40	0.0009846	158.025	0.1974675301	31.82 *	2ψ	−972.98	0.20
41	0.0005722	237.037	0.2962012951	21.21 *	3ψ	−565.48	0.15
42	0.0003064	316.050	0.3949350601	15.91 *	4ψ	−302.81	0.12
43	0.0002329	35.062	0.4936688251	12.73 *	5ψ	−230.13	0.07
44	0.0001830	114.074	0.5924025902	10.61 *	6ψ	−180.87	0.05
45	0.0001428	193.087	0.6911363552	9.09 *	7ψ	−141.13	0.04
46	0.0001031	272.099	0.7898701202	7.95	8ψ	−101.85	0.04
...							
80	0.0000154	282.831	1.3724621704	4.58	$14\psi-\tau$	−15.26	0.01
...							
104	0.0000055	317.893	1.8661309956	3.37	$19\psi-\tau$	−5.47	0.00

$$= \sum_i \alpha_i \exp\sqrt{-1}\,(\omega_i t+\phi_i)\ \ [+at] \tag{1}$$

with $|2\pi/\omega_j| \geq 2.8^{\rm d}$, and with a linear optional part $[+at]$ in q. In fact, for the real variables p and q, the series may be expressed with cosinus and sinus respectively instead of the exponential.

All terms larger than 1 km have been detected in the long-period part $\tilde{S}_1(t)$, and all those larger than 5 km in $\tilde{S}_2(t)$. The level of truncation is lower in the long-period part than in the short-period part in order to have negligible errors when computing $S(t_i)-\tilde{S}_1(t_i)$.

We show in Table 2 the internal accuracy of the representations $\tilde{S}_1(t)$ and $\tilde{S}_2(t)$, by comparing them to the time-series $S_1(t_i)$ and $S(t_i)-\tilde{S}_1(t_i)$ from which they are issued. We see that the root-mean-square residuals (RMSR) are between 2 and 5 times the level of truncation of the representations, but the maximum errors (ME) may be as large as 9 times the RMS residual for the short-period parts. This shows the badness of the convergence of these short-period parts, corroborated by the fact that half of the terms in these series are between 5 and 15 km. Thus, it seems that many smaller terms still exist whose global contribution may amount several hundreds of kilometres; besides, the maximum difference between positions computed from the elements given by the numerical integration and by their representation grows up to 284 km over 93 years. However, we consider that such an error is already sufficiently small to use these representations in a fit to Earth-based observations.

In the representations of p_7 or q_7, it is easy to identify by their frequency the terms associated to the fundamental arguments τ and ψ, while those associated to ϖ_7^* and ϖ_6^* are found directly in the series for z_7, and those associated to Ω_7^*, Ω_6^* and Ω_0 are in the series for ζ_7. The same frequencies are also found in the representations of p_6, q_6, z_6 and ζ_6. The values of these fundamental arguments are given in Table 3 by their frequency and their phases. These values are issued from the analysis corresponding to the last iteration, after adjustment to observations.

Hence, knowing these fundamental arguments (frequencies and phases), it is possible to represent the frequency and the phases of each other term, as the same integer combination of the 7 fundamental frequencies and of the 7 fundamental phases, in the form:

$$\omega_i t+\phi_i = j_1\psi + j_2\tau + j_3\varpi_7^* + j_4\varpi_6^* + j_5\Omega_7^* + j_6\Omega_6^* + j_7\Omega_0 \tag{2}$$

This identification is made easier by using the d'Alembert rules: Even degree in inclinations in the series for p, q and z, and odd degree for ζ; that imply in Eq. (2), the same parity for $j_5+j_6+j_7$. Furthermore, $j_3+j_4+j_5+j_6+j_7$ represents the characteristic of monomial as defined in (Laskar 1985); so it must be 0 in the series for p and q, and 1 in those for z and ζ.

Table 5. Series for q_7 : We give here only the largest terms; each one is in the form $\alpha \sin(\omega t + \varphi)$ with t = (Julian Date) − 2 451 545.0. Each argument is also identified as an integer combination of the seven fundamental arguments given in Table 3. The mean longitude is then $\lambda_7 = (4.3486836 \pm 0.0001432) + (0.2953088139 \pm 0.0000001286)\, t + q_7$. The amplitudes in km represent $A_7 q_7$ with A_7 = 1482333.4 km.

n°	amplitude α (rad)	phase φ (deg)	frequency ω (rad/d)	period (d)	argument	amplitude (km)	$\varepsilon(\alpha)$ (km)
1	0.1591300	103.343	0.0098105400	640.45 *	τ	235883.78	229.13
2	0.0040425	133.260	0.0088934126	706.50 *	$\tau + \varpi_7^* - \varpi_6^*$	5992.32	10.19
3	−0.0036745	73.426	0.0107276673	585.70 *	$\tau - \varpi_7^* + \varpi_6^*$	−5446.77	9.42
4	0.0018763	79.012	0.0987337650	63.64 *	ψ	2781.35	1.52
5	−0.0015590	335.669	0.0889232251	70.66 *	$\psi - \tau$	−2311.02	2.38
6	0.0015341	330.083	0.0009171274	6850.94 *	$-\varpi_7^* + \varpi_6^*$	2274.02	2.53
7	0.0011322	182.356	0.1085443050	57.89 *	$\psi + \tau$	1678.35	1.72
8	−0.0003899	140.182	0.0096969239	647.96	$\tau + \Omega_7^* - \Omega_0$	−577.96	3.88
9	0.0003851	66.505	0.0099241560	633.12	$\tau - \Omega_7^* + \Omega_0$	570.90	3.83
10	−0.0003603	323.161	0.0001136161	55301.92	$-\Omega_7^* + \Omega_0$	−534.05	3.23
11	−0.0003109	232.326	0.0791126851	79.42 *	$\psi - 2\tau$	−460.84	0.96
12	−0.0003037	310.029	0.0294316199	213.48 *	3τ	−450.16	1.33
13	−0.0001967	192.682	0.0000246364	255037.01	$-\Omega_6^* + \Omega_0$	−291.58	2.57
14	0.0001585	206.686	0.0196210799	320.23 *	2τ	234.91	0.46
15	0.0001444	48.131	0.0082528098	761.34	$\tau + 2\varpi_7^* - 2\Omega_7^*$	214.01	2.70
16	−0.0001337	285.699	0.1183548449	53.09 *	$\psi + 2\tau$	−198.12	0.41
17	0.0001354	332.863	0.0097215603	646.31	$\tau + \Omega_7^* - \Omega_6^*$	200.76	2.25
18	−0.0001344	233.823	0.0098995196	634.70	$\tau - \Omega_7^* + \Omega_6^*$	−199.26	2.23
19	−0.0001248	158.556	0.0113682701	552.69	$\tau - 2\varpi_7^* + 2\Omega_7^*$	−185.04	2.33
20	−0.0000934	55.212	0.0015577301	4033.55	$-2\varpi_7^* + 2\Omega_7^*$	−138.50	1.71
21	0.0000856	130.480	0.0000889797	70613.71	$-\Omega_7^* + \Omega_6^*$	126.87	1.33
22	0.0000722	43.509	0.0116447947	539.57	$\tau - 2\varpi_7^* + 2\varpi_6^*$	107.09	0.25
...							
47	−0.0000070	188.472	0.0104511427	601.20	$\tau - \varpi_7^* - \varpi_6^* + 2\Omega_7^*$	−10.35	0.13
...							
111	−0.0000007	248.860	0.1184684610	53.04	$\psi + 2\tau - \Omega_7^* + \Omega_0$	−1.02	0.01

Finally, in the representations of p_7, q_7, z_7 and ζ_7, we have found that all terms with a period longer than $45^{\rm d}$ and an amplitude greater than 1 km are identified with arguments in the form of Eq. (2) where: $|j_1| \le 1$, $|j_2| \le 5$, $|j_3| \le 3$, $|j_4| \le 2$, $|j_5| \le 3$, $|j_6| \le 2$ and $|j_7| \le 2$.

In the same way, all terms with a period between $2.8^{\rm d}$ and $38^{\rm d}$ and an amplitude greater than 5 km have been identified with arguments in the form:

$$j_1\psi + j_2\tau + j_3\varpi_7^* + j_4\varpi_6^* \qquad (3)$$

with: $2 \le |j_1| \le 21$, $|j_2| \le 3$, $|j_3| \le 2$ et $|j_4| \le 1$ in the series for p_7, q_7 and z_7, and in the form:

$$j_1\psi + j_3\varpi_7^* + j_5\Omega_7^* \qquad (4)$$

with: $2 \le |j_1| \le 3$, $|j_3| \le 2$ et $|j_5| \le 1$ in the series for ζ_7.

In spite of the great number of combinations, this identification can be made quickly and without ambiguity, helped by the constraint that, for any given argument (ω_i, ϕ_i), the same combination $(j_1, \cdots, j_7)$ of the 7 frequencies and of the 7 phases must occur to represent both this ω_i and this ϕ_i: The number j_1 is found at first as the integer part of $\omega_i/\dot{\psi}$, and then it suffices to investigate for the other j_k which minimise at the same time the difference between ω_i and the possible combinations of the 7 frequencies, and the difference between ϕ_i and the same combination of the 7 phases. To finish, the fundamental frequencies and phases being fixed to the values given by the frequency analysis, and the combinations of these 7 arguments being reconstructed for each term, we adjust the full set of amplitudes α_i^* by fitting the following new representation to $\{S(t_k)\}$ (the initial time-series coming from the numerical integration):

$$\tilde{S}^*(t) = \sum_i \alpha_i^* \exp\sqrt{-1}\left(\sum_{k=1}^{7} j_{ik}(\omega_k^* t + \phi_k^*)\right) \; [\, + a^* t] \qquad (5)$$

where $\omega_k^* t + \phi_k^*$ represent the 7 fundamental arguments. The amplitudes α_i^* are very close to the α_i of the representation (1), and we have verified that for each element, this last representation produces practically the same RMS residuals and the same maximum errors as those already given in Table 2 for the representation (1).

The frequency analysis of all the time-series and the identification of each argument in function of the fundamental arguments last about 90 minutes, that is 5 times longer than the numerical integration.

Table 5. (continued)

n°	amplitude α (rad)	phase φ (deg)	frequency ω (rad/d)	period (d)	argument	amplitude (km)	$\varepsilon(\alpha)$ (km)
112	0.0024777	158.025	0.1974675301	31.82 *	2ψ	3672.78	0.85
113	0.0011774	237.037	0.2962012951	21.21 *	3ψ	1745.35	0.43
114	0.0007098	316.050	0.3949350601	15.91 *	4ψ	1052.09	0.23
115	0.0004277	35.062	0.4936688251	12.73 *	5ψ	633.97	0.17
116	0.0002883	114.074	0.5924025902	10.61 *	6ψ	427.37	0.12
117	0.0002445	54.682	0.1876569901	33.48 *	$2\psi - \tau$	362.46	0.40
118	0.0002080	133.694	0.2863907551	21.94 *	$3\psi - \tau$	308.27	0.35
119	0.0001998	193.087	0.6911363552	9.09 *	7ψ	296.12	0.10
120	0.0001532	212.707	0.3851245202	16.31 *	$4\psi - \tau$	227.10	0.26
121	0.0001392	272.099	0.7898701202	7.95 *	8ψ	206.31	0.08
122	−0.0001347	340.380	0.3060118350	20.53 *	$3\psi + \tau$	−199.68	0.23
123	−0.0001169	261.368	0.2072780700	30.31 *	$2\psi + \tau$	−173.23	0.18
124	0.0001158	291.719	0.4838582852	12.99 *	$5\psi - \tau$	171.71	0.19
125	−0.0001110	59.393	0.4047456001	15.52 *	$4\psi + \tau$	−164.57	0.19
126	0.0000988	351.112	0.8886038852	7.07 *	9ψ	146.40	0.06
127	0.0000936	10.731	0.5825920502	10.78 *	$6\psi - \tau$	138.76	0.15
128	−0.0000879	138.405	0.5034793651	12.48 *	$5\psi + \tau$	−130.26	0.15
129	0.0000763	89.744	0.6813258152	9.22	$7\psi - \tau$	113.05	0.12
130	−0.0000742	217.418	0.6022131301	10.43	$6\psi + \tau$	−110.04	0.12
131	0.0000707	70.124	0.9873376503	6.36	10ψ	104.78	0.05
132	−0.0000695	187.942	0.1965504027	31.97	$2\psi + \varpi_7^* - \varpi_6^*$	−102.98	0.06
...							
166	−0.0000106	30.351	0.2765802152	22.72	$3\psi - 2\tau$	−15.67	0.03
...							
214	0.0000035	340.815	0.5835091776	10.77	$6\psi - \tau - \varpi_7^* + \varpi_6^*$	5.12	0.01

2.3. The series

2.3.1. Hyperion

We give in Tables 4 to 7, the series representing the elements p_7, q_7, z_7 and ζ_7 of Hyperion disturbed by Titan. These series are issued from the last iteration, fitted to observations.

To preserve space, these series are truncated, showing essentially the terms larger than 100 km. For each element, we give at first the long-period terms found in the part $\tilde{S}_1(t)$ and then the short-period ones in the part $\tilde{S}_2(t)$. In each part the terms are ordered by decreasing amplitudes. To quote the fall in amplitudes and the number of terms in those parts, we give also an intermediary term at the level of about 15 or10 km and the smallest term detected in each part. However, we give all the short-period terms in ζ_7 since they are only 4. This shows the very slow convergence of these Fourier series, since almost half of the terms in the long-period part are between 1 and 10 km, and almost half of the terms in the short-period part are between 5 and 15 km. Of course, to compute ephemerides of Hyperion, it is necessary to use the full series (available by FTP on the server of the Bureau des Longitudes in Paris) instead of these truncated series.

In Tables 4 and 5 for p_7 and q_7, we have noted with an asterisk the terms already present in a_7 and λ_7 of the theory of Taylor (1992), which is the previous most complete one (for other variables, the comparison is not immediate because we use the variable z while Taylor uses e and ϖ, and because the reference frame of Taylor is not our SSE reference frame but is connected to the ecliptic B1950). The amplitude of the libration (first term of q_7) is here equal to $9\overset{\circ}{.}1175 \pm 0\overset{\circ}{.}0088$, while Taylor finds $9\overset{\circ}{.}1278$, corresponding to a difference of about 270 km on the longitude of Hyperion, that is of the same order than the estimated error obtained on this amplitude. However, the period of the libration is correlated with this amplitude; the period (640.4525 ± 0.0345) given in the Table 3 differs significantly from that obtained by Taylor (640.3306 ± 0.0012), and this difference would produce a shift of more than 1600 km in the position of Hyperion after 10 years only! In fact, Taylor obtain this period by fitting its theory to only recent observations (made from 1967 to 1983), while our value comes from a fit to observations made over one century; besides, Dourneau (1993) finds 640.4473 ± 0.0048 from a fit of its theory over also one century.

We note also in our solution some important new terms, specially in q_7, which are related to the influence of the inclinations on the principal term of libration: We recognise them by the presence of τ and of longitudes of nodes in their arguments. Thus, the terms n° 8, 9, 15,17, 18, 19, 23 and 24 in Table 5, bring a total contribution larger than 2100 km.

Let us note also that Taylor (1992) includes in the mean longitude of Hyperion, an empirical term (l_ϕ sin ϕ (with the Taylor's notation) whose period is about 2128 days and amplitude about 750 km. We have not detected this term neither that in 3ζ (in the Taylor's notation again) whose period is about 2280

Table 6. Series for $z_7 = e_7 \exp\sqrt{-1}\,\varpi_7$: We give here only the largest terms; each one is in the form $\alpha \exp\sqrt{-1}\,(\omega t + \varphi)$ with t = (Julian Date) − 2 451 545.0. Each argument is also identified as an integer combination of the seven fundamental arguments given in Table 3. The amplitudes in km represent $A_7 z_7$ with A_7 = 1482333.4 km

n°	amplitude α (rad)	phase φ (deg)	frequency ω (rad/d)	period (d)	argument	amplitude (km)	$\varepsilon(\alpha)$ (km)
1	0.1030661	193.814	−0.0008924811	7040.13	ϖ_7^*	152778.39	46.77
2	0.0244818	163.897	0.0000246462	254934.93	ϖ_6^*	36290.25	20.63
3	−0.0025006	297.157	0.0089180588	704.55	$\tau + \varpi_7^*$	−3706.74	4.04
4	−0.0016531	90.471	−0.0107030211	587.05	$-\tau + \varpi_7^*$	−2450.48	2.65
5	−0.0011220	272.826	0.0978412839	64.22	$\psi + \varpi_7^*$	−1663.13	0.38
6	0.0007518	114.802	−0.0996262462	63.07	$-\psi + \varpi_7^*$	1114.43	0.39
7	0.0002580	218.145	−0.0898157062	69.96	$-\psi + \tau + \varpi_7^*$	382.46	0.43
8	−0.0001702	11.458	−0.1094367861	57.41	$-\psi - \tau + \varpi_7^*$	−252.33	0.28
9	−0.0001630	16.169	0.1076518239	58.37	$\psi + \tau + \varpi_7^*$	−241.69	0.27
10	0.0001502	169.483	0.0880307439	71.37	$\psi - \tau + \varpi_7^*$	222.66	0.25
11	0.0001081	267.240	0.0098351862	638.85	$\tau + \varpi_6^*$	160.20	0.25
12	0.0000856	223.731	−0.0018096085	3472.12	$2\varpi_7^* - \varpi_6^*$	126.94	0.18
13	−0.0000764	327.074	0.0080009315	785.31	$\tau + 2\varpi_7^* - \varpi_6^*$	−113.28	0.18
...							
35	0.0000067	248.061	−0.0907328336	69.25	$-\psi + \tau + 2\varpi_7^* - \varpi_6^*$	9.96	0.02
...							
79	−0.0000007	139.566	0.0889478713	70.64	$\psi - \tau + \varpi_6^*$	−1.02	0.00
80	0.0003778	35.789	−0.1983600112	31.68	$-2\psi + \varpi_7^*$	560.07	0.17
81	−0.0003775	351.839	0.1965750489	31.96	$2\psi + \varpi_7^*$	−559.65	0.16
82	−0.0003598	149.864	0.3940425790	15.95	$4\psi + \varpi_7^*$	−533.32	0.05
83	−0.0002928	70.851	0.2953088140	21.28	$3\psi + \varpi_7^*$	−434.02	0.09
84	0.0002217	316.777	−0.2970937762	21.15	$-3\psi + \varpi_7^*$	328.61	0.10
85	0.0001404	237.764	−0.3958275412	15.87	$-4\psi + \varpi_7^*$	208.10	0.08
86	0.0001283	139.132	−0.1885494712	33.32	$-2\psi + \tau + \varpi_7^*$	190.14	0.21
87	−0.0000984	292.446	−0.2081705511	30.18	$-2\psi - \tau + \varpi_7^*$	−145.91	0.16
88	0.0000930	158.752	−0.4945613063	12.70	$-5\psi + \varpi_7^*$	137.90	0.06
89	0.0000861	60.120	−0.2872832363	21.87	$-3\psi + \tau + \varpi_7^*$	127.61	0.13
90	−0.0000698	213.434	−0.3069043162	20.47	$-3\psi - \tau + \varpi_7^*$	−103.41	0.11
...							
130	0.0000107	5.438	−0.4749402263	13.23	$-5\psi + 2\tau + \varpi_7^*$	15.85	0.03
...							
179	0.0000034	40.934	0.2962259413	21.21	$3\psi + \varpi_6^*$	5.04	0.01

days and for which Taylor gives an amplitude of about 200 km. The term in ϕ is in fact a solar term, but we shall see below that its amplitude is only 35 km. The term in 3ζ which corresponds to our argument $3(\varpi_7^* - \varpi_6^*)$, seems to be lesser than 1 km since it has not been detected.

Concerning the short-period terms, Taylor (1992) gives those larger than 120 km; their amplitudes in the variables a_7 and λ_7 are rather close to the corresponding ones quoted in the present Tables 4 and 5. However, the internal accuracy of the present theory, illustrated in the Table 2, shows that the short-period terms are so numerous that our level of truncation at 5 km could be still insufficient.

2.3.2. Titan

The numerical integration has also given times-series for the elements of Titan disturbed by Hyperion. Their frequency analysis allows to represent these perturbations in function of the same 7 fundamental arguments. The following terms, larger than 1 km, should be added to the series of Titan published in TASS1.6:

in λ_6 :	$-0.0000222 \sin\tau$	amplitude:	27.1 km
in z_6 :	$-0.0000122 \exp\sqrt{-1}\,\varpi_7^*$		15.0 km
in ζ_6 :	$-0.0000009 \exp\sqrt{-1}\,\Omega_7^*$		2.3 km

They confirm the previous estimations given in (Duriez 1992), showing that the perturbations of Titan by Hyperion, issued essentially from the resonance, are not negligible at the level of the few kilometres expected in the CASSINI mission. The spatial observations of Titan should be able to detect at least the term in $\sin\tau$ with its period of 640^{d} , allowing then to determine the mass of Hyperion since this term is directly proportional to this mass.

2.4. Partial derivatives with respect to the parameters

The series described above represent the elements of Titan and Hyperion corresponding to a fixed choice of the 13 parameters: the 12 initial values of the elements of Titan and Hyperion and the mass of Titan. We call them “the nominal values”. To obtain

Table 7. Series for $\zeta_7 = \sin(i_7/2)\exp\sqrt{-1}\,\Omega_7$: We give here only the largest terms; each one is in the form $\alpha\exp\sqrt{-1}\,(\omega t + \varphi)$ with t = (Julian Date) − 2 451 545.0. Each argument is also identified as an integer combination of the seven fundamental arguments given in Table 3. The amplitudes in km represent $2A_7\zeta_7$ with A_7 = 1482333.4 km

n°	amplitude α (rad)	phase φ (deg)	frequency ω (rad/d)	period (d)	argument	amplitude (km)	$\varepsilon(\alpha)$ (km)
0	0.0049552	184.582	0.0000000000		Ω_0	14690.65	0.59
1	0.0059485	221.420	−0.0001136161	55301.92	Ω_7^*	17635.36	102.71
2	0.0015359	351.900	−0.0000246364	255037.01	Ω_6^*	4553.41	40.15
3	−0.0001497	166.208	−0.0016713462	3759.36	$2\varpi_7^* - \Omega_7^*$	−443.91	2.76
4	−0.0000491	136.291	−0.0007542188	8330.72	$\varpi_7^* + \varpi_6^* - \Omega_7^*$	−145.62	0.95
...							
15	0.0000036	118.077	−0.0099241560	633.12	$-\tau + \Omega_7^*$	10.57	0.07
...							
47	−0.0000004	321.983	0.0008924910	7040.05	$-\varpi_7^* + \varpi_6^* + \Omega_6^*$	−1.04	0.01
48	0.0000031	63.395	−0.1975811461	31.80	$-2\psi + \Omega_7^*$	9.33	0.05
49	−0.0000022	19.445	0.1973539140	31.84	$2\psi + \Omega_7^*$	−6.43	0.04
50	0.0000018	324.233	0.1957961839	32.09	$2\psi + 2\varpi_7^* - \Omega_7^*$	5.44	0.03
51	0.0000018	344.383	−0.2963149111	21.20	$-3\psi + \Omega_7^*$	5.28	0.03

the partial derivative of the solution with respect to a parameter c in the vicinity of the nominal values, we modify slightly the value of this only parameter (as $c+\delta c$), and then we make again the same computations, from the filtered numerical integration, the frequency analysis and the identification of arguments, to the last fit giving the elements in form of Eq. (5). Hence each fundamental argument which was $\omega_k^* t + \phi_k^*$ with the nominal values, is found slightly modified because of δc, becoming $(\omega_k^* + \delta_1\omega_k^*)t + \phi_k^* + \delta_1\phi_k^*$; each amplitude α_i^* of Eq. (5) is also slightly modified into $\alpha_i^* + \delta_1\alpha_i^*$. The same computations made with $c-\delta c$ produce other modifications of the fundamental arguments and of the amplitudes: $(\omega_k^* - \delta_2\omega_k^*)t + \phi_k^* - \delta_2\phi_k^*$ and $\alpha_i^* - \delta_2\alpha_i^*$. These two evaluations allow to compute a numerical derivative with respect to c for each fundamental frequency and phase, and for each amplitude, in the vicinity of their nominal value, approximated by $\frac{\delta f}{\delta c}$ in the form:

$$\frac{\partial f}{\partial c} \approx \frac{\delta f}{\delta c} = \frac{\delta_1 f - \delta_2 f}{2\delta c}$$

where f replaces any ω_k^*, ϕ_k^* or α_i^*. The values of the variations used for the δc are $3\,10^{-5}$ for δp_6 and $2\,10^{-4}$ for the other variables of Titan, $5\,10^{-5}$ for δp_7 and $2\,10^{-3}$ for the other variables of Hyperion, and 10^{-7} for $\delta(m_6/M_s)$. These variations produce each one a shift lesser than about 250 km on the position of Titan, and lesser than 3000 km on that of Hyperion.

After having computed such partial derivatives with respect to the 13 parameters, the final solution is now expressed as the following, instead of Eq. (5):

$$\tilde{S}^*(t, \Delta c_1, \cdots, \Delta c_{13}) = \sum_i \alpha_i^* \exp\sqrt{-1}\left(\sum_{k=1}^{7} j_{ik}(\omega_k^* t + \phi_k^*)\right)$$
$$[+a^* t] + \sum_{l=1}^{13}\left\{\sum_i \left(\frac{\delta\alpha_i^*}{\delta c_l} + \sum_{k=1}^{7}\sqrt{-1}\,j_{ik}\left(\frac{\delta\omega_k^*}{\delta c_l}t + \frac{\delta\phi_k^*}{\delta c_l}\right)\right) \times\right.$$
$$\left.\times \exp\sqrt{-1}\left(\sum_{k=1}^{7} j_{ik}(\omega_k^* t + \phi_k^*)\right)[+\frac{\delta a^*}{\delta c_l}t]\right\}\Delta c_l \qquad (6)$$

To preserve space, we cannot give here the full Tables of the partial derivatives; however, as an illustration, here is the expression of the amplitude of the libration in the mean longitude of Hyperion (first term in the Table 5, expressed here in km):

$$\begin{aligned}
&235883.78\\
&\quad -2407.21\,\Delta p_6 - 79.07\,\Delta q_6 - 195.51\,\Delta\Re(z_6)\\
&\quad +122.54\,\Delta\Im(z_6) + 5.34\,\Delta\Re(\zeta_6) - 2.83\,\Delta\Im(\zeta_6)\\
&\quad +2398.64\,\Delta p_7 + 106.88\,\Delta q_7 + 237.77\,\Delta\Re(z_7)\\
&\quad -148.61\,\Delta\Im(z_7) - 6.26\,\Delta\Re(\zeta_7) + 3.05\,\Delta\Im(\zeta_7)\\
&\quad -12350.5\,\Delta(m_6/M_s)
\end{aligned}$$

where all variations Δc_l are expressed in units of 10^{-4}. Finally, the 6 elements of Hyperion have been expressed in the form of Eq. (6). These expressions have been used to adjust the Δc_l in order that positions computed from these elements fit to observations.

3. Ephemerides of Hyperion

We want to compute ephemerides of Hyperion in order to compare and to adjust them to observations. For that, the elements of Hyperion expressed in form of Eq. (6) do not suffice, because the dynamic model of the interaction Titan-Hyperion should be completed by adding :

1. the periodic solar perturbations of elements,
2. the short-period perturbations coming from the other satellites and from the Saturn's oblateness.
3. the long-period terms depending on the variables z and ζ of other satellites (complements to $(\dot{q}_7)^*$ etc... considered above),

Table 8. Solar and short-period perturbations of Hyperion (terms larger than 20 km only, but the full series extent up to 5 km): These are expressed as $\alpha\cos(\omega t + \phi)$ for p_7, as $\alpha\sin(\omega t + \phi)$ for q_7 and as $\alpha\exp\sqrt{-1}\,(\omega t + \phi)$ for z_7 and ζ_7, with t = (Julian Date) − 2 451 545.0. We give also for each term, the combination of arguments from which the frequency and the phases are computed: λ_9, ϖ_9 and Ω_9 concern elements of the Sun in the saturnicentric Saturn-equatorial reference frame referred to the mean ecliptic for the J2000.0 epoch, λ_{oi} refers to the linear part of the mean longitude of the satellite i and other arguments to the fundamental arguments given in Table 3. Amplitudes in kilometres are computed like in Tables 4 to 7.

	amplitude α (rad)	phase φ (deg)	frequency ω (rad/d)	period (d)	argument	amplitude (km)
p_7	−0.0000268	123.098	1.0955449042	5.74	$\lambda_{o5} - \lambda_{o7}$	26.45
q_7	−0.0002989	112.955	0.0011679623	5379.61	$2\lambda_9 - 2\Omega_9$	−443.05
	−0.0002231	317.020	0.0005839811	10759.23	$\lambda_9 - \varpi_9$	−330.68
	−0.0000409	69.976	0.0017519434	3586.41	$3\lambda_9 - 2\Omega_9 - \varpi_9$	−60.67
	−0.0000233	94.436	0.0029528601	2127.83	$2\lambda_9 - 2\varpi_7^*$	−34.51
	0.0000175	155.935	0.0005839811	10759.23	$\lambda_9 + \varpi_9 - 2\Omega_9$	26.00
	−0.0000160	124.398	0.0020357649	3086.40	$2\lambda_9 - \varpi_7^* - \varpi_6^*$	−23.72
	−0.0000137	274.041	0.0011679623	5379.61	$2\lambda_9 - 2\varpi_9$	−20.28
	0.0000146	55.315	0.2962012606	21.21	$\lambda_{o7} - \varpi_7^*$	21.64
z_7	0.0001928	288.280	0.0020604112	3049.48	$2\lambda_9 - \varpi_7^*$	285.74
	0.0000823	318.242	0.0011433161	5495.58	$2\lambda_9 - \varpi_6^*$	122.02
	0.0000517	306.799	0.0002755134	22805.37	$2\lambda_9 - 2\Omega_9 + \varpi_7^*$	76.69
	0.0000327	245.300	0.0026443923	2376.04	$3\lambda_9 - \varpi_9 - \varpi_7^*$	48.46
	−0.0000163	150.864	−0.0003084677	−20369.02	$\lambda_9 - \varpi_9 + \varpi_7^*$	−24.20
	0.0000404	249.158	0.2953088117	21.28	λ_{o7}	59.92
ζ_7	−0.0001609	297.539	0.0011679623	5379.61	$2\lambda_9 - \Omega_9$	−477.05
	−0.0000247	227.564	−0.0005839811	−10759.23	$-\lambda_9 + \varpi_9 + \Omega_9$	−73.10
	0.0000247	141.604	0.0005839811	10759.23	$\lambda_9 - \varpi_9 + \Omega_9$	73.10
	−0.0000221	254.560	0.0017519434	3586.41	$3\lambda_9 - \varpi_9 - \Omega_9$	−65.62
	0.0000095	340.519	0.0005839811	10759.23	$\lambda_9 + \varpi_9 - \Omega_9$	28.12
	−0.0000091	71.629	−0.0011679623	−5379.61	$-2\lambda_9 + 3\Omega_9$	−27.05

The complement 3 should include essentially some long-period terms coming from Iapetus, with periods longer than 3200 years if we extrapolate them from looking at the influence of Iapetus on Titan in TASS1.6. Because we are at first interested by a representation of the motion of Hyperion fitted to observations over about one century, we have neglected this complement in the present work, but we have considered the two first ones.

3.1. Solar and other short-period perturbations

The solar perturbations must be added to the equations $\frac{dq_7}{dt}$, $\frac{dz_7}{dt}$ and $\frac{d\zeta_7}{dt}$. We have computed their analytical expansions in the same way as the secular variations $(\dot{q}_7)^*$, $(\dot{z}_7)^*$, etc... considered above, to the fourth degree in eccentricities and inclinations, in the form:

$$\sum_{k\neq 0} P_k(z_7, \bar{z}_7, \zeta_7, \bar{\zeta}_7, z_9, \bar{z}_9, \zeta_9, \bar{\zeta}_9)\exp\sqrt{-1}\,(k\lambda_9) \quad (7)$$

where P_k is a polynomial in the variables z, $\bar{z}$, ζ and $\bar{\zeta}$ of Hyperion and Sun. However, the variables z_9 and ζ_9 of the Sun must refer to our SSE reference frame, like the other variables related to the satellites. To transform the saturnicentric elements of Sun from the mean ecliptic J2000 to the SSE reference frame, we have used the position of the equatorial plane of Saturn given in the mean ecliptic J2000 by the angles:

$$i_a = 28\overset{\circ}{.}0512 \qquad \text{and} \qquad \Omega_a = 169\overset{\circ}{.}5291$$

from Campbell & Anderson (1989). Because we want to compute the solar perturbations over only one century, we have considered that the Sun moves on a keplerian orbit, and its elements in the SSE reference frame, are then given by the following expressions, issued from the secular variations of the Saturn's elements given in JASON84 (Simon & Bretagnon, 1984):

$$\lambda_9 = 1.065725361 + 0.005839811453\,(\mathrm{JD} - 2451545.0)$$
$$z_9 = 0.0555481521\,\exp\sqrt{-1}\,(1.815862521)$$
$$\zeta_9 = 0.2311331788\,\exp\sqrt{-1}\,(3.221596817)$$

The solar perturbations are obtained by integrating expressions like (7): It suffice at first to replace the variables relative to the Sun by the previous values, and z_7 and ζ_7 by the series given in Tables 6 and 7, and then to integrate term by term. In fact, only the two (resp. three) first term in z_7 (resp. ζ_7) have

Table 9. Root-mean-square (rms) and mean values of (o−c) residuals of the position of Hyperion obtained in the fit of the present theory (TASS) to observations of Hyperion (Saturn-Hyperion and intersatellite observations coming from the data sets given in the catalogue of Strugnell & Taylor (1990)). All (o−c) residuals larger than 1″.0 have been rejected. The rms and mean residuals are expressed in units of 0″.01; for each data type, the columns u and r give the number of used and rejected observations in the fit of TASS. The weight of each data set increases with the number of observations and with their global quality (measured by the rms residuals of the data set).

	data type 1				data type 2				weight
Data sets	u	r	rms	mean	u	r	rms	mean	%
1 USNO (1877-1887)	430	47	34.8	−2.7	184	58	54.1	29.3	5.1
3 USNO (1911)	270	24	29.1	−2.0	125	22	46.3	13.7	3.9
4 USNO (1929)	178	8	26.8	−3.2	87	7	41.7	9.0	2.8
6 Struve (1933)	149	9	30.3	3.6	74	5	37.5	10.3	2.3
9 Struve (1898)	446	26	33.2	7.1	232	2	31.5	−4.1	7.1
14 Soulie (1972)	81	5	38.4	−3.7	83	3	36.1	1.1	1.3
31 Pascu (1982)	50	0	26.4	−5.4	50	0	22.6	4.8	1.0
34 Mulholland & Shelus (1980)	115	13	24.8	4.1	117	11	22.4	−2.4	2.2
39 Taylor & Sinclair (1985)	159	9	29.1	−2.3	162	6	25.5	−1.1	3.0
47 Veillet & Dourneau (1992)	168	1	23.0	−5.6	169	0	13.8	3.5	4.8
48 Veillet & Dourneau (1992)	888	0	17.4	−1.0	888	0	9.6	−0.6	35.9
49 Veillet & Dourneau (1992)	84	0	24.6	3.6	84	0	14.1	−4.7	2.6
52 Dourneau et al. (1986)	399	1	17.3	−1.1	400	0	9.6	−1.8	16.6

been used in this substitution. The other terms would produce complements smaller than 5 km. The resulting solar perturbations larger than 20 km in q_7, z_7 and ζ_7 are given in the Table 8 in a form directly ready to use for ephemerides (the full series used for ephemerides include 46 terms larger than 5 km). The relatively high value of $|\zeta_9|$ produces the terms larger than 400 km in q_7 and ζ_7. In q_7, we note also the term with the argument $2\lambda_9 - 2\varpi_7^*$: It has the same period as the empirical term ($l_\phi \sin\phi$) in the Taylor's theory, but its amplitude is far lesser, as it was already said above.

In the same way, we have computed the analytic expansions of the short-period perturbations of elements coming from other satellites and from the oblateness of Saturn. The way to compute them is already described in Vienne & Duriez (1991). The major ones are also given in the Table 8. In q_7 and z_7, the terms in $\lambda_{o7} - \varpi_7^*$ and λ_{o7} are proportional to the coefficient J_2 of Saturn.

3.2. Adjustment to observations

We have made four successive adjustments of the above representation of the Hyperion's elements to the observations of Hyperion contained in the catalogue of Strugnell & Taylor (1990); for each fit, we have used exactly the same computer program as for the adjustment of the elements of other satellites in TASS1.6 (Vienne & Duriez, 1995), so we refer to this paper for all relating technical details; let us specify only that the way to weight now the equations of condition for Hyperion is the same as that chosen for Mimas in TASS1.6, because we think that Hyperion is probably as difficult to observe as Mimas. The final adjustment corresponds to modifications of the adjusted parameters which are of the same order than the standard error given by the least-square procedure.

From the catalogue, we have constructed 8136 equations of condition (including 3178 observations made before 1966); they concern modern photographic or old visual intersatellite observations of Hyperion, as well as old visual observations of Hyperion directly referred to Saturn; we have not considered the meridian observations which require a good theory of motion for Saturn and which should be used rather to improve the motion of Saturn itself, as suggested by Taylor et al. (1991). The intersatellite positions of Hyperion have been computed by using TASS1.6 for the other satellites, even for Titan. In effect, the only adjusted parameters are here the mass of Titan and the twelve initial values of the elements of Titan and Hyperion which appear in the partial derivatives presented above, but this does not mean that the motion of Titan has been also adjusted here with that of Hyperion: In fact, the values of the initial conditions of Titan correspond only to the model of the Titan-Hyperion interaction described above, but this model being less complete for Titan than TASS1.6, we have preferred to compute positions of Titan by TASS1.6, as well as for other satellites. Besides, Table 1 allows to compute the differences between the initial conditions of Titan given at first by TASS1.6 and those obtained at the final adjustment: they correspond to a difference smaller than 1000 km in the initial position of Titan, that is of the same order as the (o−c) of Titan in TASS1.6. On the contrary, the initial values of the elements of Hyperion have been rather modified by the successive adjustments, moving the initial position of Hyperion by more than 10 000 km from the first to the last adjustment. The standard errors on the initial values given in Table 1 for the last adjustment correspond to errors on position lesser than 80 km. In the same time, from the first to the last adjustment, the mass of Titan has notably changed, coming from 236.638 10^{-6} to (237.399 ± 0.005) 10^{-6} (in units of

Table 10. Root-mean-square and mean values of (o−c) residuals of the position of Hyperion computed by the present theory (TASS) and by that of Taylor (1992), for intersatellite observations of Hyperion made after 1966, with a rejection level of 1″.0. Melting all types of observations, the residuals are given for the best data sets of observations of the catalogue of Strugnell & Taylor (1990) and are expressed in units of 0″.01. The columns u and r give the number of used and rejected observations by TASS. The weights are the same as in Table 9.

			TASS		Taylor		weight
Data sets	u	r	rms	mean	rms	mean	%
14 Soulie (1972)	164	8	37.3	−1.3	35.9	−2.8	1.3
31 Pascu (1982)	100	0	24.5	−0.3	22.7	−0.5	1.0
34 Mulholland & Shelus (1980)	232	24	23.6	0.9	23.7	3.6	2.2
39 Taylor & Sinclair (1985)	321	15	27.3	−1.7	29.3	0.8	3.0
47 Veillet & Dourneau (1992)	337	1	18.4	−1.0	18.9	4.0	4.8
48 Veillet & Dourneau (1992)	1776	0	13.5	−0.8	19.4	1.2	35.9
49 Veillet & Dourneau (1992)	168	0	19.3	−0.6	23.6	−5.9	2.6
52 Dourneau et al. (1986)	799	1	13.4	−1.4	18.4	−6.1	16.6
total	3897	49					67.4
weighted average			15.6	−0.9	20.3	−0.7	

the Saturn's mass); the standard error on this mass ($0.005\ 10^{-6}$) seems very small, but we shall discuss this value in the next section.

The Tables 3 to 7 give the series resulting from the final adjustment, showing also the estimated errors on each term. These are computed by: $\varepsilon(\alpha) = \sum_{l=1}^{13} |\delta\alpha/\delta c_l|\ \varepsilon_l$ where α represents the amplitude of a term, and where the ε_l are the standard errors of the 13 parameters. The estimated errors on the fundamental frequencies and phases given in Table 3 are obtained by analogous expressions. The largest estimated error on the amplitudes is that of the libration in q_7, but it is of the same order than the maximum errors given in Table 2, which represent the internal accuracy of our theory.

We give in Table 9 the rms values and the mean values obtained in the last fit, for the (o−c) residuals of the best data-sets of observations, knowing that all observations with (o−c) residuals larger than 1″ have been rejected. In fact, we have at first fitted our theory with a rejection level fixed at 2″ and then we have made the final iteration with reject at 1″. The weight of each data-set used in the adjustments is roughly proportional to the number of observations used in this data-set, and inversely proportional to the rms residual obtained for it. Thus, the Table 9 concerns observations which "weight" altogether about 89% of all the data-sets.

The data-types 1 concern observations of right ascensions ($\Delta\alpha$ or $\Delta\alpha\cos\delta$) or of position angles ($s\cos p$ and $s\sin p$, so that p is weighted by the separation s computed from the theory at the time of observation of p), while the data-types 2 represent observations of declinations ($\Delta\delta$) or of angular separations (s). It is striking that the rms residuals are almost twice smaller for data-types 2 than for data-types 1 when looking at the most recent observations. This suggests that some progress should be still made to represent the motion of Hyperion in right ascension with the same accuracy as that in declination; if the motion in right ascension comes mainly from the motion in the orbital plane, the representation of the longitude λ_7 should be still perfectible.

The references 48 and 52 represent the best observations: (888 + 400) observations of each type, made by Veillet & Dourneau at the 1.5m telescope of ESO; they concern about one week of observations per year in 1981, 1982, 1984 and 1985. They give the best rms residuals and the best mean residuals. The present theory of Hyperion depends strongly on these observations, but represents also correctly the oldest ones which are more dispersed but have comparable mean residuals, specially for data-types 1. However, the mean residuals of old observations made at the USNO for data-type 2 (that is observations of intersatellite separations) reveal systematic large mean residuals which could come perhaps from a systematic error in the scaling calibration of such observations. Harper & Taylor (1994) had already observed such systematic large mean residuals with the old observations of intersatellite separations concerning all satellites, and suggested also errors of calibration.

3.3. Comparisons to previous works

We have compared our (o−c) residuals to those given by the theory of Taylor (1992) when using the same sets of observations. For that, we have computed the positions of Hyperion from the series published in Taylor (1992), and the positions of other satellites by using TASS1.6; of course, the positions of Hyperion from Taylor are transformed in order to be put in the same reference frame as TASS and thus, the comparison of the residuals given by TASS and by Taylor for the same intersatellite observations depends on the theories of Hyperion only. The results are presented in Table 10, where we show only the best sets of observations made after 1966 (because Taylor has fitted its theory to only recent observations). Melting the data types 1 and 2, we observe that the dispersion of the residuals is comparable for both theories with the oldest of these observations; this dispersion could represent in fact the accuracy of the corresponding observations. The most recent observations made by Dourneau and Veillet show better the progress brought by our theory. We have also made the same computations with the theory of Hyperion presented by Dourneau (1993), but the residuals are far more larger than ours or those of Taylor, because Dourneau did not include the short-period perturbations coming from Titan; thus, with the same sets of observations given in Table 10, the weighted average of the rms residuals of Dourneau amounts to $0''.382$ when Taylor gives $0''.203$ and our theory $0''.156$.

Concerning the mass of Titan, the new value obtained here: $(237.399 \pm 0.005)\,10^{-6}$, is rather incompatible with that obtained by Campbell & Anderson from analysis of the missions Voyager to Saturn: $(236.638 \pm 0.008)\,10^{-6}$. This difference (about 150σ !) could come in part from the old observations which have been used in our adjustment. In effect, we have also tried to adjust the present theory of Hyperion to observations made after 1966 only: We have obtained $(237.303 \pm 0.007)\,10^{-6}$ and the amplitude of libration in q_7 has fallen slightly from $9^\circ.117 \pm 0^\circ.009$ (in the present series) to $9^\circ.110 \pm 0^\circ.078$; the standard errors were somewhat larger, but the rms (o−c) residuals of the best observations were practically not changed. However, this last value of the mass of Titan is again incompatible with the two other cited above! In fact, these values of the mass of Titan obtained in the present work, are not strictly coherent with TASS1.6 because we have adjusted only Hyperion, using TASS1.6 for other satellites whereas TASS1.6 was adjusted without Hyperion; so we think that a correct value of the mass of Titan will be obtained only when we shall fit to observations together the theory of Hyperion and that of all other satellites, giving at the same time all the physical parameters of the Saturnian system. This work is still in progress.

This lack of coherence with TASS1.6 may be seen also when comparing the frequencies of ϖ_6^* and Ω_6^* obtained by frequency analysis in the present work (Table 3), with those obtained in TASS1.6 as proper frequencies associated to Titan and corresponding to periods of 256880.325^{d} and -256957.027^{d} respectively: These last values exceed ours by about 1930^{d} ; however, these differences would correspond to a shift of about 300 km on the position of Hyperion after one century (i.e. only $0''.05$ as seen from earth).

4. Conclusion

The present theory of motion of Hyperion has been constructed so that it gives the perturbations by Titan completely up to the level of 5 kilometres (and not truncated to some order in the masses) and it takes account of other perturbations coming from the Sun, from other satellites and from the oblateness of Saturn. After adjusting it on observations made over more than one century, we have obtained a representation which gives smaller residuals than previous theories, but progress should be still possible: improvement of the representation of the short-period perturbations by Titan, improvement of the model used to compute the secular perturbations coming from other satellites. These improvements are in progress, constructing now a new global representation of TASS, including Hyperion with all other satellites. The present theory of Hyperion should be a good basis for this work.

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Appendix A: features of the frequency analysis

The method used in this work is adapted from that described in Laskar et al. (1992). In short, given a time-series representing a complex quasi-periodic function of time $S(t)$, one determines accurately the frequency ω_1, the phase ϕ_1 and the amplitude α_1 of the term of greatest amplitude in $S(t)$; then, the term $\alpha_1 \exp\sqrt{-1}\,(\omega_1 t + \phi_1)$ is subtracted from the time-series; the resulting new time-series may be analysed again, to give the next term (α_2, ω_2, ϕ_2) of the representation, and so on in the order of decreasing amplitudes; each term found in this way, is used to construct a new element of the orthonormal basis of functions in which $S(t)$ is represented; the process is iterated up to the wanted truncation level. However, the process of orthonormalisation may fail if a new frequency is found too close to a previously determined one. We propose here a way to overcome this difficulty.

In fact, the way to determine each term depends on the sampling Δt of $S(t)$ and on the finite time-span $D = t_2 - t_1$ in which $S(t)$ is known.

In theory, the frequency ω of the greatest term corresponds to the maximum modulus of the spectrum:

$$f(\nu) = \lim_{T\to\infty} \frac{2}{T} \int_{-T}^{T} S(t) \exp\sqrt{-1}\,(-\nu t)\, dt \qquad (A1)$$

which may be represented by infinitely thin spectral lines with height equal to the modulus of the amplitude of each periodic term present in $S(t)$.

In practice, a numerical approximation of ω is obtained by computing the maximum modulus of

$$f^*(\nu) = \frac{1}{D} \int_{t_1}^{t_2} w(t) S^*(t) \exp\sqrt{-1}\,(-\nu t)\, dt \qquad (A2)$$

where $S^*(t)$ represents the sampled time-series and $w(t)$ a window function like the "Hann window": $w(t) = (1 - \cos(2\pi t/D))/2$. To locate roughly the frequency ω corresponding to the maximum amplitude, one determines at first the maximum of the power spectrum given by a FFT of the time-series. Then, by analysing the "shape" of $f^*(\nu)$ near of this approximate frequency, one obtains the accurate frequency corresponding to the local maximum of f^*; with this ω, one obtains also the numerical approximation: $\alpha \exp\sqrt{-1}\,\phi = f^*(\omega)$. One supposes here that the sampling interval is sufficiently small to not obtain aliased spectral lines. One supposes also that D is sufficiently large to allow to discriminate the longest period present in $S(t)$. In effect, resulting from Eq. (A2), the spectral resolution is $\Delta\omega = 4\pi/D$, because each spectral line is as large as $\Delta\omega$ and is accompanied on each side by a series of "ghost" spectral lines with decreasing amplitudes and distant each one by $\Delta\omega/2$ (like a figure of diffraction in theory of images). The Hann windows allows mainly to reduce at best these ghost lines; however, if one has two spectral lines corresponding to the frequencies ω_j and ω_k with $|\omega_j - \omega_k|$ close to $\Delta\omega$ and with comparable amplitudes $\alpha_j > \alpha_k$, the outline of each one is more or less deformed by its neighbour, so that the determination of the maximum modulus gives only an approximation of ω_j, α_j and ϕ_j. When we subtract from the time-series the term $\alpha_j \exp\sqrt{-1}\,(\omega_j t+\phi_j)$ coming from such a "deformed" line, a residual term may be still present near of the frequency ω_j; this residual term may be found later at the frequency ω'_j (close to ω_j and ω_k) if its amplitude α'_j is larger than the truncation level. There, the orthonormalisation proposed by Laskar fails.

In fact, to overcome this problem, it suffices to determine again the term at the frequency ω_j:

If the term at ω_k has been already determined before falling on the term at ω'_j, we add at first to the time-series the term $\alpha_j \exp\sqrt{-1}\,(\omega_j t + \phi_j)$ previously determined; because the term at ω_k has been already subtracted from the time-series, now the term at ω_j is no longer deformed and the frequency analysis of the new time-series gives a better determination of ω_j, and then of α_j and ϕ_j.

If the term at ω_k has not yet been determined when one falls on the term at ω'_j, adding again the term $\alpha_j \exp\sqrt{-1}\,(\omega_j t + \phi_j)$ previously determined produces lines as deformed as at the beginning, so that the frequency analysis fails. The solution is now to try an adjustment of this term by the least square method.

This procedure has allowed us to extract up to 120 terms from each of the time-series representing the elements of Hyperion, by using the frequency analysis, improved by several determinations for too close terms or by least square fits when it was necessary as explained above. Finally, it appears also that such a frequency analysis is less time-expensive and as accurate when one suppresses the construction of the orthonormal basis of functions, on the condition that one determines all terms up to a sufficient truncation level, and provided that one adjusts, at the end, all the terms to the initial time series by a least-square process.

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