Astron. Astrophys. 324, 829–834 (1997)

ASTRONOMY
AND
ASTROPHYSICS

The dynamics of twisted accretion disc around a Kerr black hole

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Received 26 December 1996 / Accepted 27 January 1997

Abstract. We derive the system of equations describing the time dependent twisted accretion disc around a Kerr black hole (twist equations). The post-Newtonian corrections to the equations of motion and effects of the gravitomagnetic force are taken into account. We show that introduction of appropriate coordinate system greatly simplifies the derivation of equations and allows to consider sufficiently large but smoothly changing with distance and time inclination angles. The twist equations obtained by different authors in previous works are limiting cases of our general system.

Key words: accretion, accretion discs

1. Introduction

Accretion discs around compact objects are often used to explain several different astrophysical phenomena. Accretion discs can explain spectra and energetics of compact X-ray binaries, quasars, and active galactic nuclei. Sometimes there are astrophysical situations when the accretion disc has to be twisted (see for example Gerend & Boynton 1976; Katz 1972,1980; Kumar 1986, 1987; Petterson 1975,1977b; van den Heuvel et al.; Crosa & Boynton 1980; Shakura 1972; Rees 1978 a,b). A twisted accretion disc around a compact object can appear when disc rings sufficiently far from the compact object have some inclination with respect to the compact object symmetry plane. For example, the symmetry plane can be misaligned with respect to the galactic plane in the case of a supermassive central black hole in AGN's and QSO's or with the orbital plane in the case of a binary system. The disc plane itself may be inclined to the binary orbital plane due to effects of precession of the rotational axis of the second companion (Gerend & Boynton 1976; Katz 1972,1980; Petterson 1975, 1976b; van den Heuvel et al.; Crosa & Boynton 1980; Shakura 1972). In this paper we consider an accretion disc around a central rotating black hole.

The investigation of twisted accretion discs was initiated by Bardeen and Peterson (1975). They concluded that the gravitational influence of a central Kerr black hole forces the disc to lay in the equatorial plane of the hole, and motion of matter at distances $r < r_{BP}$ ($r_{BP} \approx \delta^{-4/3} r_g$, δ is the disc opening angle, r_g is the gravitational radius of the black hole) is axially symmetric. Petterson (1977a, 1978) proposed a covariant description of twisted discs. Distortions of the disc are described by two Euler angles β (the inclination angle) and γ (the rotation angle) depending, in general, on time t and radius r and a convenient curved (twisting) coordinate system is introduced. The hydrodynamical equations describing motion of matter are written in a non holonomic basis connected with the curved coordinates and this basis is used to obtain equations describing distortions of the disc in terms of the angles β and γ. These equations are called the twist equations. Hatchett et al. (1981) noticed that the twist equations become linear after introduction of a new variable $\mathbf{W} = \beta \exp i\gamma$ and that the initial twist equations introduced by Bardeen and Petterson do not conserve angular momentum. Papaloizou and Pringle (1983, Paper I) showed that to obtain a self consistent system of the twist equations conserving the angular momentum it is necessary to include density and velocity perturbations. They derived the twist equations in cylindrical coordinates considering the disc twist as a perturbed z-component of velocity v^z, in this case $\mathbf{W}$ reduces to $\overline{\frac{iv^z}{v^\varphi}}$, (overline denotes complex conjugation). From these equations it follows that r_{BP} decreases with decreasing viscosity α ($r_{BP} \sim \alpha^{2/3}$) (this property is connected with a resonance contribution of terms describing density and velocity perturbations, see below) and previous estimates qualitatively hold only for $\alpha \sim 1$. Ivanov and Illarionov (1997, Paper II) showed that the stationary twist equation is applicable when $\beta > \delta$, for β and γ sufficiently smoothly changing with r. Also in Paper II relativistic post-Newtonian corrections to the equations of motion were considered allowing to generalize the twist equation of Paper I for the case of low viscosity stationary twisted disc around a Kerr black hole. It was shown that for sufficiently small $\alpha < \delta^{4/5}$, and therefore for small characteristic radii r_{BP}, the shape of a twisted disc is determined by relativistic corrections and these corrections lead to oscillatory dependence of the inclination angle of the stationary disc on the distance r. The nature of this oscillatory regime and stationary solutions of the modified twist equations were also analyzed in Paper I. For sufficiently small r such os-

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cillations may destroy the disc or lead to effective increase of the viscosity by creating shear flow instability and additional turbulization of the accreting flow (Paper II; Kumar & Coleman 1993).

Not only the shape of a stationary twisted disc but also its evolution essentially depends on the value of α. When α is large the disc evolution is determined by a diffusion type equation (Paper I; Kumar 1990), describing relaxation to a stationary configuration with characteristic decay time $\approx \alpha\delta^{-2}t_K$ (t_K is the Keplerian time scale of motion around the central source). When $\alpha < \delta$ new wave type degrees of freedom appear and the twist evolution equation becomes a wavelike type (Papaloizou & Lin 1995). The twist wave can propagate distortions over the disc with the sound velocity $v_s \approx \delta v_\phi$ ($v_\phi = r/t_K$) and little dissipation.

In this paper we generalize the results obtained earlier and derive the system of time dependent twist equations, in the twisting coordinate system, directly from the viscous fluid hydrodynamical equations. We take into account the influence of a central Kerr black hole and twist wave degrees of freedom on the structure of the disc. We show that the structure of a time dependent accretion disc with a constant viscosity parameter α can be described, in general, by two complex variables **W**, and **A**, where **A** characterizes the small elliptical deviations of disc particle orbits from a circle. Analysis of our system shows that for large viscosity the general system effectively reduces to one twist equation in the form obtained earlier (Paper I, Kumar 1990), while in the case of low viscosity it consists of two time dependent coupled equations describing the evolution of the disc twist and the evolution of disc particle orbits. This system of equations may be considered as describing propagation of twist waves in the disc.

In our paper we use the following general assumptions:

1. The gradient $r\frac{\partial W}{\partial r} \ll 1$, and the velocity perturbation $v \ll v_s$, where v_s is the speed of sound.
2. Disc configuration is changing with time in a characteristic time scale $t_* \gg t_K$.
3. We consider only thin discs $\delta \ll 1$.
4. We assume that the viscosity is sufficiently small $\alpha < 1$.
5. The rotating black hole is described by the Kerr metric and we consider distortions of the disc sufficiently far from the hole so that the relativistic effects are considered as effective corrections to the equations of motion.
6. We assume that the disc corotates with the central black hole (the rotational parameter $a > 0$).

According to the assumptions $(1-6)$ we derive the system of twist equations in the leading order in small parameters $t_K/t_*, \delta, r_g/r$. We retain all terms in the decomposition in small parameter α to be able to consider also the case of sufficiently large value of α. We use the natural system of units $G = c = 1$ and assume the summation convention, the Latin indices run from 0 to 3, the Greek indices run from 1 to 3.

2. The coordinate system and the main equations

Following Petterson (1977a, 1978) we use a curved coordinate system in which the hydrodynamical equations assume the simplest possible form. We write down all components of the geometrical quantities in the linear approximation only. In our derivation we shall use only the essential parts of these quantities (one can check that the other parts appear in the twist equations in the next order). The essential parts we denote by the index *.

The transition to the curved coordinate system (τ, r, ψ, ξ) from the global Cartesian coordinates (t, x, y, z) (the z-axis coincides with the black hole angular momentum) is

$$\begin{pmatrix} \tau \\ r\cos\psi \\ r\sin\psi \\ \xi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\gamma & \cos\gamma & 0 \\ 0 & -\sin\gamma & \cos\gamma & \beta \\ 0 & \beta\sin\gamma & -\beta\cos\gamma & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}, \quad (1)$$

where $\beta(t, r)$ and $\gamma(t, r)$ denote the Euler angles.

In our calculations instead of β and γ we use $\Psi_1 = \beta\cos\gamma$, $\Psi_2 = \beta\sin\gamma$, and $\phi = \psi + \gamma$ instead of ψ to consider the degenerate case $\beta = 0$ (Paper II). We project all vectors and tensors onto a non holonomic orthonormal basis $\mathbf{e}_\tau$, $\mathbf{e}_r$, $\mathbf{e}_\phi$, $\mathbf{e}_\xi$

$$\mathbf{e}_\tau = \frac{\partial}{\partial\tau}^* + \xi U\frac{\partial}{\partial r} - \frac{\xi}{r}\frac{\partial U}{\partial\phi}\frac{\partial}{\partial\phi} - rU\frac{\partial}{\partial\xi},$$
$$\mathbf{e}_r = (1+\xi W)\frac{\partial}{\partial r}^* + \frac{\xi}{r}\frac{\partial W}{\partial\phi}\frac{\partial}{\partial\phi} - rW\frac{\partial}{\partial\xi}^*,$$
$$\mathbf{e}_\phi = \frac{1}{r}\frac{\partial}{\partial\phi}^*, \quad (2)$$
$$\mathbf{e}_\xi = \frac{\partial}{\partial\xi}^*,$$

where $U = \dot\beta\sin\psi - \dot\gamma\beta\cos\psi = \dot\Psi_1\sin\phi - \dot\Psi_2\cos\phi$, $W = \beta'\sin\phi - \gamma'\beta\cos\phi = \Psi'_1\sin\phi - \Psi'_2\cos\phi$, dot denotes $\frac{\partial}{\partial\tau}$, and prime denotes $\frac{\partial}{\partial r}$. To perform covariant differentiation in the basis (2) one should use the following connection coefficients Γ_{ikl} (taking into account the symmetry $\Gamma_{ikl} = -\Gamma_{kil}$)

$$\Gamma_{\phi r\tau} = \frac{\xi}{r}\frac{\partial U}{\partial\phi}, \quad \Gamma_{\phi rr} = \frac{\xi}{r}\frac{\partial W}{\partial\phi}^*, \quad \Gamma_{\phi r\phi} = \frac{1}{r}^*,$$

$$\Gamma_{\xi r\tau} = U, \quad \Gamma_{\xi rr} = W, \quad \Gamma_{\xi\phi\tau} = \frac{\partial U}{\partial\phi}^*, \quad \Gamma_{\xi\phi r} = \frac{\partial W}{\partial\phi}^*. \quad (3)$$

With the help of (3) the covariant derivative of a vector is

$$v^i_{;j} = \mathbf{e}_j(v^i) + \Gamma^i_{lj}v^l, \quad (4)$$

and a covariant tensor divergence is

$$t^j_{i\,;j} = \mathbf{e}_j(t^j_i) + \Gamma^j_{ij}t^k_i - \Gamma^k_{ij}t^j_k. \quad (5)$$

All tetrad indices can be raised or lowered with the help of the Minkowski metric.

The hydrodynamical equations projected onto the basis (2) should have a covariant form and reduce to the continuity equation

$$(\varrho v^i)_{;i} = 0, \quad (6)$$

and the Navier-Stokes equation

$$\varrho(v^j v^i{}_{;j} + a^i) = \varrho\Phi^{,i} - p^{,i} - t^{ij}{}_{;j} + \varrho\mathscr{F}^i, \quad (7)$$

where ϱ, p, v^i denote respectively the density, the pressure, and the 4-velocity. [1] $\Phi = \frac{M}{r}$ is the Newtonian potential and M is the black hole mass. The viscous stress tensor t^i_j has only spatial components different from zero

$$t^{\alpha}_{\beta} = -\eta(v^{\alpha}{}_{;\beta} + v_{\beta}{}^{;\alpha} - \frac{2}{3}\delta^{\alpha}_{\beta} v^{\gamma}{}_{;\gamma}), \tag{8}$$

where η is the coefficient of dynamical viscosity. A^i denotes post-Newtonian corrections to the convective term $v^j v^i{}_{;j}$, in the form calculated in Paper I:

$$A^r = \frac{M}{r}(\frac{3}{2}\frac{\partial}{\partial\phi}v^r_1 + v^{\phi}_1),$$

$$A^{\phi} = \frac{M}{r}(\frac{3}{2}\frac{\partial}{\partial\phi}v^{\phi}_1 - \frac{5}{4}v^r_1). \tag{9}$$

where the index $_1$ means that we consider the perturbed parts of velocity components only (see next section). $\mathscr{F}^i$ is the gravitomagnetic force (see, for example Thorne et al. 1986), in our case it has only one relevant component

$$\mathscr{F}^{\xi} = \frac{4v^{\phi}aM^2}{r^3}\beta\sin\psi = \frac{4v^{\phi}aM^2}{r^3}[\Psi_1\sin\phi - \Psi_2\cos\phi], \tag{10}$$

where a is the black hole rotation parameter.

3. Time dependent twist equations

Now we derive the system of equations which determines the time evolution of a twisted accretion disc. Following Paper I we separate the density ϱ, pressure P, and velocity v^i into even and odd parts with respect to the coordinate ξ and we denote them by an index $_0$ (even) and $_1$ (odd), so

$$\varrho = \varrho_0 + \varrho_1, \quad P = P_0 + P_1, \quad v^{\alpha} = v^{\alpha}_0 + v^{\alpha}_1. \tag{11}$$

We assume that the odd functions are small (they are proportional to $r\frac{\partial\beta}{\partial r}$) and they depend on ϕ through $\sin(\phi + \phi_0)$.

Substituting (11) into the continuity equation (6) and the equations of motion (7), and separating the terms with different dependence on ξ and ϕ we obtain:

1. A system of equations for functions with the index $_0$ (background system) which is identical with the standard system of equations describing a flat accretion disc (Paper I, Paper II). As it was mentioned in the Introduction, in twisted accretion discs the effective coefficient of turbulent viscosity α_{turb} and disc opening angle δ can depend on the distance. Therefore in this section we treat the viscosity and the disc half height as arbitrary functions and we derive the system of twist equations in the most general form. We assume near Keplerian motion so

$$v^{\phi}_0 = \Omega r, \quad \Omega = t_K^{-1} = \frac{M^{1/2}}{r^{3/2}}, \tag{12}$$

and use the standard relations

$$\frac{1}{\varrho_0}\frac{\partial P_0}{\partial\xi} = \frac{\partial\Phi}{\partial\xi} = \frac{\xi}{r}\frac{\partial\Phi}{\partial r} = -\frac{\xi}{r}\frac{{v^{\phi}_0}^2}{r}. \tag{13}$$

[1] Note, that in our formalism we consider the 4-velocity in the classical limit $(1, v^{\alpha})$, taking into account the relativistic effects as corrections $(A^i, \mathscr{F}^i)$ to the equations of motion.

We also assume that the radial drift velocity is small $v^r_0 < \delta^2 v^{\phi}_0$.

2. A system of equations for quantities with index $_1$ determining the density and velocity perturbations and the equation describing the disc twist. From the continuity equation we have (Petterson 1977a, 1978)

$$\Omega\frac{\partial}{\partial\phi}\varrho_1 + \frac{\varrho}{r}\frac{\partial}{\partial\phi}v^{\phi}_1 + \frac{1}{r}\frac{\partial}{\partial r}(\varrho_0 v^r_1 r) - \varrho_0\Omega\frac{\partial W}{\partial\phi}\xi = 0. \tag{14}$$

The r-component of the equation of motion gives

$$\varrho_0\{\dot{v}^r_1 + \Omega\{\frac{\partial v^r_1}{\partial\phi} - 2v_{\phi 1} + \frac{M}{r}(\frac{3}{2}\frac{\partial v^r_1}{\partial\phi} + v^{\phi}_1)\}\}$$

$$= \varrho_0\xi W\frac{\partial\Phi}{\partial r} - \frac{\partial t^{r\xi}}{\partial\xi} \tag{15}$$

and the ϕ-component gives

$$\varrho_0\{2\frac{\partial\dot{v}^{\phi}_1}{\partial\phi} + \Omega\{\frac{\partial v^r_1}{\partial\phi} - 2v^{\phi}_1 - \frac{M}{r}(3v^{\phi}_1 + \frac{5}{2}\frac{\partial v^r_1}{\partial\phi})\}\}$$

$$= -2\frac{\partial}{\partial\xi}\frac{\partial}{\partial\phi}t^{\phi\xi} + 2r\frac{\partial W}{\partial\phi}\frac{\partial t^{r\phi}}{\partial\xi}, \tag{16}$$

where we use the explicit form of the relativistic corrections (9). The relevant components of the viscous stress tensor are

$$t^{r\phi} = -\eta r\frac{\partial}{\partial r}\Omega,\ t^{r\xi} = -\eta(\frac{\partial v^r_1}{\partial\xi} + v^{\phi}_0\frac{\partial W}{\partial\phi}),\ t^{\phi\xi} = -\eta\frac{\partial v^{\phi}_1}{\partial\xi}. \tag{17}$$

The Eqs. (15 − 17) for a given background system determine the perturbed velocities v^{ϕ}_1, v^r_1.

Neglecting in (15, 16) corrections $\propto t_K/t_*, \alpha, M/r$ one gets $\Omega\{\frac{\partial v^r_1}{\partial\phi} - 2v^{\phi}_1\} = \xi W\frac{\partial\Phi}{\partial r}$ from (15), but $\frac{\partial v^r_1}{\partial\phi} = 2v^{\phi}_1$ from (16). The nonzero projection of pressure gradient $\nabla_r P = -\rho_0\xi W\frac{\partial\Phi}{\partial r}$ (see Paper II for the discussion of this term) leads to a resonance and the system (15 − 16) cannot be solved in the leading order (Paper I). Therefore we retain next order corrections which allow us to solve (15 − 16). The relative significance of correction terms of different kind will be discussed below (see Sect. 5).

To get the equation determining the disc twist we use the ξ-component of (8)

$$2\varrho_0 v^{\phi}_0\{\frac{\partial U}{\partial\phi} + \frac{\partial W}{\partial\phi}v^r_0\} = \varrho_1\frac{\partial\Phi}{\partial\xi} - \frac{\partial P_1}{\partial\xi} - \frac{1}{r}\frac{\partial}{\partial r}(rt^{r\xi})$$

$$-\frac{\partial W}{\partial\phi}t^{r\phi} - \frac{1}{r}\frac{\partial t^{\xi\phi}}{\partial\phi} - \frac{\partial}{\partial\xi}t^{\xi\xi} + \varrho_0\mathscr{F}^{\xi}, \tag{18}$$

where P_1 is the perturbed pressure, and factor two in front of the brackets on the left hand side comes from the nonzero ξ-component of the velocity (Hatchett et al.)

$$v^{\xi} = r(U + Wv^r_0). \tag{19}$$

Now, using the Eqs. (14 − 16) and integrating the ξ-component of the Newtonian gravity force we express it in terms of perturbed velocities and components of the stress tensor

$$\int d\xi\varrho_1\frac{\partial\Phi}{\partial\xi} = \int d\xi\xi\{-\frac{1}{r^2}\frac{\partial}{\partial r}(r\varrho_0 v^{\phi}_0\frac{\partial v^r_1}{\partial\phi})\}$$

$$+\frac{1}{r}\{-t^{r\xi}+\frac{\partial t^{\xi\phi}}{\partial\phi}-r\frac{\partial W}{\partial\phi}t^{r\phi}\}, \tag{20}$$

where we assume that the components of the stress tensor are integrated over ξ. We also neglected in Eq. (20) terms proportional to $\dot{v}_1^{\phi}, \dot{v}_1^{r}$. These terms are $\frac{1}{\Omega t_*}$ times smaller than the leading order terms.

Taking into account (20) and integrating (18) over ξ we obtain the final equation for the twist dynamics in the form

$$2\{\frac{\partial U}{\partial\phi}\Sigma v_0^{\phi}+\frac{\partial W}{\partial\phi}(\Sigma v_0^{\phi}v_0^{r}+t^{r\phi})\}=-\frac{1}{r^2}\frac{\partial}{\partial r}\{\int d\xi(\xi\varrho_0 v_0^{\phi}\frac{\partial v_1^r}{\partial\phi})$$

$$+r^2t^{r\xi}\}+\Sigma\mathscr{F}^{\xi}, \tag{21}$$

where the surface density $\Sigma=\int d\xi\varrho_0$, the perturbed pressure term and the gradient of stress tensor in the ξ direction vanish after integration.

The system of Eqs. (15, 16) and (21) determines the dynamics of the twist and velocity perturbations for a given background model. In comparison with analogous system obtained in Paper I in cylindrical coordinates, our system is applicable for $\beta>\delta$ and it also takes into account relativistic corrections.

4. The isothermal α-disc

To obtain the twist equations in a more explicit form we should specify the background model. Since we are interested in dynamical effects of the disc twist we do not consider the detailed disc structure and use only simple model (following, for example Shakura & Sunyaev 1973). Following earlier works (Paper I; Kumar & Pringle 1985; Kumar 1990) we choose stationary α-model with the coefficient of kinematic viscosity $\nu=\eta/\varrho$, which does not depend on the height ξ and with isothermal density distribution with height

$$\varrho_0=\varrho_*(r)\exp(-\frac{\xi^2}{2\xi_*(r)^2}), \tag{22}$$

for which the system (15, 16, 21) greatly simplifies, here $\xi_*(r)$ denotes the disc half height (more realistic polytropic models have been discussed by Kumar (1988)). To determine ϱ_* we use the mass conservation low

$$\Sigma\ v_0^r\ r=-\frac{\dot{M}}{2\pi}, \tag{23}$$

where the mass transfer rate $\dot{M}$ does not depend on the distance r. We also use the conservation principle for the z-component of angular momentum

$$\Sigma v_0^{\phi}v_0^{r}+t_{\phi}^{r}=-\frac{\dot{M}}{2\pi}\frac{M^{1/2}r_*^{1/2}}{r^2}, \tag{24}$$

where r_* is the radius of marginally stable orbit and the integration constant is fixed by the "no torque" boundary condition $t_{\phi}^{r}(r_*)=0$. We use the standard α prescription for the viscosity (Shakura & Sunyaev 1973)

$$\nu=\alpha\frac{\xi_*^2}{r}v_0^{\phi}, \tag{25}$$

and assume that far from the black hole ξ_* is approximately linear with r

$$\xi_*=\delta r(1-(\frac{r_*}{r})^{1/2})^{1/2}, \tag{26}$$

where $\delta\approx$ const and the factor in the brackets is used to get nonsingular v_0^r at r_* (Kumar & Pringle 1985) (this approximation is quite reasonable from the astrophysical point of view). The Eqs. $(17, 24-26)$ give the explicit expression for v_0^r in the form

$$v_0^r=-\frac{3}{2}\alpha\delta^2v_0^{\phi}. \tag{27}$$

Using the Eqs. (22, 25) we can rewrite the Eqs. (15, 16) with the help of the following ansatz for perturbed velocities

$$v_1^{\phi}=\xi(A_1\sin\phi+A_2\cos\phi), \tag{28}$$

and

$$v_1^{r}=\xi(B_1\sin\phi+B_2\cos\phi). \tag{29}$$

With the adopted assumptions the set of Eqs. (15, 16) reduces to

$$\mathbf{B}-2i\mathbf{A}=\frac{\alpha(i+4\alpha)}{E(\alpha,r)}v_0^{\phi}\mathbf{W}', \tag{30}$$

$$\Omega^{-1}(2\dot{\mathbf{A}}-i\dot{\mathbf{B}})=2iC(\alpha,r)\mathbf{A}-(1+2i\alpha-3\alpha^2)v_0^{\phi}\mathbf{W}', \tag{31}$$

where $\mathbf{A}=A_2+iA_1$, $\mathbf{B}=B_2+iB_1$, $\mathbf{W}=\Psi_1+i\Psi_2=\beta\exp i\gamma$, and

$$E(\alpha,r)=2i\alpha-\alpha^2+6\frac{M}{r}. \tag{32}$$

With the help of $(22-27)$, (30, 31) we also rewrite the Eq. (21) in the form

$$\dot{\mathbf{W}}=\frac{\delta^2}{2r}\{\frac{\partial}{\partial r}(r^2(1-(\frac{r_*}{r})^{1/2})\mathbf{F}(\mathbf{W}',\dot{\mathbf{A}},\dot{\mathbf{B}})$$

$$+3\alpha r_*^{1/2}M^{1/2}\mathbf{W}'\}+\frac{2aM^2}{r^3}\mathbf{W}, \tag{33}$$

where

$$\mathbf{F}(\mathbf{W}',\dot{\mathbf{A}},\dot{\mathbf{B}})=\frac{(i(1+6\alpha^2)-2\alpha)v_0^{\phi}\mathbf{W}'+i\Omega^{-1}(2\dot{\mathbf{A}}-i\dot{\mathbf{B}})}{E(\alpha,r)}. \tag{34}$$

According to algebraic relation between $\mathbf{A}$ and $\mathbf{B}$ (30) the dynamics of twisted discs can be always described by two variables $\mathbf{W}$ and $\mathbf{A}$ (or $\mathbf{B}$).

The Eqs. $(28-30)$ have simple interpretation. Assuming smooth dependence of $\mathbf{A}$ and $\mathbf{B}$ on r, the direct integration of (29) gives

$$r=r_0(1+\frac{\xi|\mathbf{B}|}{v_0^{\phi}}\cos(\Omega t+\arccos\frac{Im\mathbf{B}}{|\mathbf{B}|})), \tag{35}$$

where $r=r(t)$ is the trajectory of a disc particle slightly deviating from a circle of radius r_0. Thus the particle trajectories in the twisted discs are ellipses with small eccentricity $e=-\frac{\xi|\mathbf{B}|}{v_0^{\phi}}$ and the main axis is oriented at $\phi_0=\arccos\frac{Im\mathbf{B}}{|\mathbf{B}|}$. Using Eqs. (28, 29) we can express the time change of the proper angular momentum along particle trajectory $\dot{L}^{\xi}=(r^2\dot{\phi})=\dot{L}_1\sin\Omega t+\dot{L}_2\cos\Omega t$ in the form $\dot{\mathbf{L}}=\dot{L}_2+i\dot{L}_1=\frac{v_0^{\phi}\xi}{2}(\mathbf{B}-2i\mathbf{A})$. The Eq. (29) implies that the angular momentum is changing along trajectory due to viscous interaction $\propto \alpha$ between disc particles.

The Eq. (31) determines the perturbed velocities. Let us note that when relativistic corrections are neglected in the limit $\alpha \rightarrow 0$, $v_1^r, v_1^\phi \sim \frac{1}{\alpha}$ contrary to previous conclusions (Kumar 1986; Kumar & Coleman 1993) that $v_1^r, v_1^\phi \sim \frac{1}{\alpha^2}$.

To conserve the angular momentum the system $(30-34)$ must have the divergence free form. The total time change of the angular momentum $\dot{\boldsymbol{L}} = \int d^3\boldsymbol{r}\dot{l}$ must be converted into a surface integral in the absence of an external force (l is the angular momentum density). Taking into account that the angular momentum is mainly determined by the Keplerian part of the velocity $l \approx \rho_0 v_0^\phi r \mathbf{e}_\xi$ and integrating the angular momentum density over ϕ and ξ we find that $\frac{d}{dr}(i\dot{L}_x - \dot{L}_y) \propto \Sigma r^2 v_0^\phi \dot{\mathbf{W}}$. Therefore, to conserve the x and y-components of angular momentum (z-component is conserved according to (24)) the Eq. (33) must have a form $\dot{\mathbf{W}} = \frac{1}{\Sigma v_0^\phi r^2}\frac{\partial}{\partial r}\mathbf{K}(r,t) + \mathbf{F}_{ext}$, where $\mathbf{F}_{ext}$ -external force term (in our case the gravitomagnetic term - the last term in Eq. (33)). With the help of Eqs. (23, 27) one can easily check that Eq. (33) does have the right form.

The presence of gravitomagnetic term leads to the exchange of angular momentum between the disc and the black hole. The angular momentum conservation principle requires that the black hole has to change the direction of its rotation but in a very long time scale (in comparison with t_*) (see Rees 1978a).

If one neglects the time dependent term $\propto (2\dot{\mathbf{A}} - i\dot{\mathbf{B}})$ in (34) and relativistic correction $\propto \frac{M}{r}$ in (32) (this can be done for $\alpha > \delta^{4/5}$, see next section) the Eq. (33) reduces to the twist equation obtained in Paper I in the form presented by Kumar (1990)

$$\frac{\partial \mathbf{W}}{\partial T} + \frac{3}{4}x^5\frac{\partial \mathbf{W}}{\partial x} = \frac{1}{8}x^5 f(\alpha)\frac{\partial}{\partial x}\{(1-x)\frac{\partial \mathbf{W}}{\partial x}\} + 2a(\frac{M}{r_*})^2\delta^2 x^6 \mathbf{W}, \tag{36}$$

where two new variables $T = \delta^2\frac{1}{r_*}(\frac{M}{r_*})^{1/2}t$, $x = (\frac{r}{r_*})^{-1/2}$ have been introduced, and $f(\alpha) = \frac{i(1+6\alpha^2)-2\alpha}{2i\alpha-\alpha^2}$.

The system $(30-34)$ assumes very simple form in the low viscosity limit $\alpha \ll 1$.[2] In this case the Eq. (30) reduces to the relation

$$\mathbf{B} = 2i\mathbf{A}, \tag{37}$$

from (31) we have

$$\dot{\mathbf{A}} = -\frac{M}{4r^2}\mathbf{W}_{,r} - (\alpha - i\frac{3M}{r})\Omega\mathbf{A}, \tag{38}$$

and from (33), neglecting the correction term $(1-(\frac{r_*}{r})^{1/2})$, we have

$$\dot{\mathbf{W}} = -\frac{\delta^2}{r}\frac{\partial}{\partial r}(r^2\mathbf{A}) + 2i\frac{aM^2}{r^3}\mathbf{W}. \tag{39}$$

The first term and the term proportional to α in (38) describe correspondingly the influence of pressure and viscosity on the particle motion in the disc, the term proportional to M/r describes relativistic precession of the principal axes of the ellipse. The first term in (39) describes the perturbed ξ component of the Newtonian gravitational force $F_{grav} = \int \varrho_1 \frac{\partial \Phi}{\partial \xi} d\xi$, and the last term represents the gravitomagnetic force. In the stationary case the Eqs. $(37-39)$ reduce to the equation obtained in Paper II. When one neglects the last term in (39), which describes the influence of the gravitomagnetic force and the relativistic term in (38), and introduces Fourier transforms, one gets the equation describing the propagation of waves with dissipation, obtained by Papaloizou & Lin (1993).

5. Discussion

In this section we discuss the relative role of different terms in the system of twist Eqs. $(30-34)$. For estimates we assume below that $r\frac{\partial}{\partial r}\{\mathbf{W},\mathbf{A},\mathbf{B}\} \sim \{\mathbf{W},\mathbf{A},\mathbf{B}\}$ and the rotation parameter of the black hole $a \sim 1$.

At first we consider the case of large viscosity ($\alpha \sim 1$). To estimate the characteristic time of relaxation t_r we neglect the term $2\dot{\mathbf{A}} - i\dot{\mathbf{B}}$ in (31) and also in (34), small relativistic correction term $\sim \alpha\frac{M}{r}$ in (32), and the gravitomagnetic force term in (33). In this case we obtain $t_r \approx \alpha\delta^{-2}t_K$ (Paper I), and therefore our approximation is valid for $\alpha \gg \delta, \frac{M}{r}$. Hence in the case of large viscosity we can use the approximate Eq. (36).

Stationary solutions of Eq. (36) were discussed by Kumar and Pringle (1985) and non stationary solutions have been thoroughly analyzed by Kumar (1990). In the stationary case solutions of (36) describe exponentially decreasing angle $\beta \sim \exp(-(\frac{r}{r_{BP}})^{-3/4})$ with decreasing r and corotating with the central black hole angle $\gamma \sim (\frac{r}{r_{BP}})^{-3/4}$ (Bardeen & Petterson 1975; Kumar & Pringle 1985; Paper II). The alignment scale $r_{BP} \sim \alpha^{2/3}\delta^{-4/3}M$ can be obtained from (36) (or from (38, 39)).

When $\alpha < \frac{M}{r}$ relativistic corrections dominate the evolution and they determine the form of stationary solutions. In Paper II it was shown that in the limit $\alpha \rightarrow 0$ the stationary solution describes strongly oscillating angle $\beta(r)$

$$\beta \sim (\frac{r}{r_{rel}})^{-1/8}|\cos((\frac{r}{r_{rel}})^{-5/4} + \frac{\pi}{20})|, \tag{40}$$

where $r_{rel} \approx \delta^{-4/5}M$, and $\gamma \approx$ const. For sufficiently small $r \leq r_f \approx \delta^{-3}\beta^{8/3}(\infty)M$ ($\beta(\infty)$) is the disc inclination angle at large distances) the solution (40) is unstable with respect to shear instability. From the condition $\alpha < M/r_{rel}$ we see that when $\alpha < \delta^{4/5}$ the solution (40) approximately describes the stationary solution.

To get qualitative picture of evolution of the twisted disc we temporarily neglect in (38, 39) the viscosity and relativistic corrections. In this case the system (38, 39) reduces to one equation

$$\ddot{W} = \frac{\delta^2 M}{r}\frac{\partial^2}{\partial r^2}W, \tag{41}$$

which describes propagation of waves with local velocity $\sim v_s = \delta v_K$ (Papaloizou & Lin 1993) and slowly decreasing amplitude (with increasing r).

Effects of viscosity additionally damp the wave amplitude. Relativistic effects cause precession of principal axes of elliptical trajectories and precession of the disc inclination angle

[2] Note, that the different estimates of viscosity (see for example Meyer & Meyer-Hofmeister 1984; Canuto et al.) typically give $\alpha \approx 10^{-2}$.

during propagation of waves. To estimate importance of these effects we compare characteristic time of the Einstein precession $t_E \sim \frac{r}{M} t_K$ and the Lense-Thirring precession $t_{LT} \sim \frac{r^3}{M^2}$ with the time of wave propagation from the radius r, $t_{wave} \approx \frac{r}{v_s}$. From the conditions $t_E < t_{wave}$ and $t_{LT} < t_{wave}$ it follows that the post-Newtonian correction and the gravitomagnetic force are important for wave propagation correspondingly at scales $r < \delta^{-1}M$ and $r < \delta^{-3/2}M$.

Finally we discuss the range of applicability of the twist equation. Formally our approach is valid in the linear approximation when $rW' < 1$. However solutions become unstable with respect to shear instabilities for much smaller gradients.

We assume that the shear instability appears when velocity perturbation at the disc half height exceeds the sound velocity

$$v_1^r(\xi_*, r) \sim v_1^\phi(\xi_*, r) \geq v_s(\xi_*, r). \quad (42)$$

Introducing the complex quantity $\mathbf{v}_1^\phi = \xi \mathbf{A}$, which characterizes the velocity perturbations, we rewrite the Eq. (38) in the form

$$\frac{\mathbf{v}_1^\phi}{v_s}(\alpha - \frac{i3M}{r}) + \Omega^{-2}\dot{\mathbf{A}} = -\frac{r}{4}\mathbf{W}_{,r}. \quad (43)$$

From the previous discussion it follows that for $\alpha > \delta$ terms proportional to $\frac{M}{r}$ and $\dot{\mathbf{A}}$ in (43) can be neglected. Using (42) and (43) we see that the shear instability appears when $\frac{|\mathbf{v}_1^\phi|}{v_s} \geq \alpha^{-1} r \frac{\partial}{\partial r}\beta$. Solutions of our equations remain stable for $r\frac{\partial}{\partial r}\beta \leq \alpha$, when $\alpha \geq \delta$. When $\alpha \leq \delta$, assuming that $\dot{\mathbf{A}} \sim \mathbf{A}/t_{wave}$ the shear instability appears when $\frac{|\mathbf{v}_1^\phi|}{v_s} \geq \max\{\frac{r}{M}, \delta^{-1}\} r\frac{\partial}{\partial r}\beta$. In the low viscosity limit solutions are stable only when $r\frac{\partial}{\partial r}\beta \leq \max\{\frac{M}{r}, \delta\}$.

6. Conclusions

We derive the general time dependent twist equations for twisted accretion disc around a Kerr black hole. The equations derived by previous authors can be obtained from our system in the different limiting cases. Our equations show that twist dynamics strongly depends on the viscosity law. For the large value of viscosity we approximately have diffusive type equation which describes the relaxation to a stationary solution and nonstationary twisted distortion decays at place of its birth. For the case of small viscosity the twist waves propagate in the disc. These waves may carry the distortion from distant perturber to the inner part of accreting flow, where the main bulk of radiation comes from. The effect of changing of disc luminosity due to twist waves might possibly explain the observed long-term periodicity of some compact objects. In the low viscosity case the twist dynamics strongly depends on the imposed boundary conditions. When the inner boundary condition is determined at scales comparable with r_f, the modification of structure of accretion disc should be taken into account. This modification lies beyond the scope of our paper. Note, that if the modification causes only a change of the viscosity law, the disc dynamics will be described by our system, but with effective viscosity parameter and disc opening angle depending on scale.

Although we explicitly derive the twist equation taking into account the influence of a Kerr black hole only, our formalism can be easily extended to more complicated astrophysical situations. In particular it can describe effects of a central star cluster (in the case of a supermassive black hole in AGN's and QSO's) or the binary companion, or influence of radiation pressure on a twisted disc. These effects lead to additional corrections to the gravitational potential or redefinition of the external force term $\mathscr{F}_\xi$ (Kumar 1986; Petterson 1977).

We do not consider here solutions of our equations. In general this can be done numerically for specified initial and boundary conditions. We are going to obtain these solutions in our future work.

Acknowledgements. We are grateful to A. F. Illarionov and D. I. Novikov for valuable discussions. We also thank M. Abramowicz for useful conversations. P. B. I. thanks N. Copernicus Astronomical Center for hospitality. This research project has been supported in part by the Russian Foundation of Fundamental Research (Grants 93-02-2929 and 95-02-06063), and KBN grant 2P304-003-05.

References

Bardeen, J. M., Petterson, J. A., ApJ 195, L65, 1975
Canuto V. M., I. Goldman, O. Hubickyi, ApJ 280, L55, 1984
Crosa, L., Boynton, P. E., ApJ 235, 999, 1980
Eggleton, P. P., Pringle, J. E., ApJ 288, 275, 1985
Gerend, D., Boynton, P. E., ApJ. 209, 562, 1976
Hatchett, S., Begelman, M., Sarazin, C. ApJ 247, 677, 1981
Ivanov P. B., Illarionov A. F., "Radial Oscillations of Inclination Angle of Accretion Disc around Kerr Black Hole" Göteborg University preprint ASTRO-1995-04/ITP 95-09, MNRAS, in press (Paper II)
Katz, J. I., Nature 246, 87, 1972
Katz, J. I., ApJ 236, L127, 1980
Kumar, S., Pringle, J. E., MNRAS 213, 435, 1985
Kumar, S., MNRAS 223, 225, 1986
Kumar, S., MNRAS 225, 823, 1987
Kumar, S., MNRAS 233, 33, 1988
Kumar, S., MNRAS 245, 670, 1990
Kumar, S., Coleman, C. S., MNRAS 260, 323, 1993
Meyer F., Meyer-Hofmeister E., A&A 128, 420, 1984
Papaloizou, J., Pringle, J. E., MNRAS 202, 1181, 1983 (Paper I)
Papaloizou, J. C. B., Lin, D. N. C., ApJ 438, 841, 1995
Petterson, J. A., ApJ 201, L61, 1975
Petterson, J. A., ApJ 214, 550, 1977 (a)
Petterson, J. A., ApJ 216, 827, 1977 (b)
Petterson, J. A., ApJ 226, 253, 1978
Rees, M., Nature 275, 516, 1978 (a)
Rees, M., Phys. Scripta 17, 193, 1978 (b)
Thorne K. S. et al., 1986, In: Black Holes: The Membrane paradigm, eds. K. S. Thorne, R. Price, D. MacDonald, Yale University Press, New Haven
Shakura, N. I., AZh 49, 921, 1972
Shakura, N. I., Sunyaev, R. A., A& A 24, 337, 1973
van den Heuvel, E. P. J., Ostriker, J. P., Petterson, J. A., A&A 81, L7, 1980

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